

Point-Interval-Valued Sets: Aggregation and Construction

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Abstract. The concept of a point-interval-valued set (PIV set) is proposed as a tool for summary characterization of data from a two-way table. A PIV set is an L-fuzzy set whose membership labels can be numbers as well as special subintervals from the unit interval. Two relations of partial order of PIV sets are introduced and corresponding operations of union and intersection are studied. Aggregation of PIV sets by bounded t-norms is suggested.

1 Introduction

In decision making, the choice of the best object (product, individual, method, etc.) with respect to a given set of criteria is often based on evaluation of the degree of compatibility of each object with respect to each criterion by a group of experts. Usually the degrees of compatibility are numbers from the unit interval, where 1 represents full compatibility, while 0 represents no compatibility. Then evaluations of objects from a set $X = \{x_1, \dots, x_n\}$ by experts $E_1, \dots, E_m$ can be described by a two-way table

$$\tau = \{\delta_{ij} : \delta_{ij} \in [0, 1], i = 1, \dots, n, j = 1, \dots, m\}. \quad (1)$$

Numerous methods and aggregation functions for a summary characterization of τ were proposed, see e.g., [1–3, 14] and references therein. In this contribution, a new summary characterization of τ is introduced as a mapping φ on X satisfying the following properties: if all experts agree that x_i has low compatibility with a criterion C , then $\varphi(x_i) \in [0, 0.5]$, if all experts agree that x_i has high compatibility with C , then $\varphi(x_i) \in (0.5, 1]$, and if some experts assume that there is a low compatibility between x_i and C while other think that the compatibility is high, then $\varphi(x_i) = [a, b]$, where $0 \leq a \leq 0.5 \leq b \leq 1, a \neq b$. A special consideration needs to be given to the evaluation $\delta_{ij} = 0.5$. In applications, 0.5 can be viewed as the largest small value or the smallest large value. This is the reason why 0.5 needs to be handled separately, as a middle value. Then the mapping φ

discussed above is a point-interval-valued set on X (PIV set) which is formally introduced in Definition 1.

Definition 1. Let X be a universe of discourse and $D = \{[a, b] : 0 \leq a \leq 0.5 \leq b \leq 1, a \neq b\}$. A point-interval-valued set (PIV set) on X is any mapping $\varphi : X \rightarrow [0, 1] \cup D$.

Note that a fuzzy set $f : X \rightarrow [0, 1]$, as well as a shadowed set on X introduced by Pedrycz [12] as any mapping $S : X \rightarrow \{0, [0, 1], 1\}$, are examples of PIV sets (see also [13] for computing with shadowed sets). Further in this paper it is assumed that X is a finite set and $\Gamma(X)$, $F(X)$ and $\Omega(X)$ denote the set of all PIV sets on X , the set of all fuzzy sets on X and the set of all shadowed sets on X , respectively.

Recall that in the theory of fuzzy sets [5, 10, 15] for $f, g \in F(X)$, operations of union and intersection are defined pointwise for all $x \in X$ by

$$(f \cup g)(x) = \max\{f(x), g(x)\}, \quad (f \cap g)(x) = \min\{f(x), g(x)\}. \quad (2)$$

In the case of shadowed sets $A, B \in \Omega(X)$, the union and the intersection are defined in Tables 1 and 2, respectively.

Table 1. Union of shadowed sets

$A \setminus B$	0	$[0, 1]$	1
0	0	$[0, 1]$	1
$[0, 1]$	$[0, 1]$	$[0, 1]$	1
1	1	1	1

Table 2. Intersection of shadowed sets

$A \setminus B$	0	$[0, 1]$	1
0	0	0	0
$[0, 1]$	0	$[0, 1]$	$[0, 1]$
1	0	$[0, 1]$	1

Operations in Tables 1 and 2 are isomorphic with the logic connectives in three-valued logic [7, 11], in particular, Łukasiewicz logic.

The intent of this contribution is to accomplish the following tasks: First, to find operations of union and intersection of PIV sets which in the case of fuzzy sets correspond to the union and intersection of fuzzy sets, and, in the case of shadowed sets correspond to the union and intersection of shadowed sets. Then, to find operations of union and intersection of PIV sets such that the union and the intersection of membership labels of PIV sets expressed by intervals correspond to the union and intersection of intervals of real numbers. In applications, researchers can choose from a vast variety of aggregation functions [4, 6]. Among them, triangular norms (t-norms) and triangular conorms (t-conorms) are the most popular [8, 9]. The next task of this contribution is to extend t-norms and t-conorms used in aggregation of fuzzy sets to t-norms and t-conorms suitable in aggregation of PIV sets. The final task is to propose a method for deriving a PIV set from data given by $\tau = \{\delta_{ij} : \delta_{ij} \in [0, 1], i = 1, \dots, n, j = 1, \dots, m\}$. Note that majority of mathematical propositions in this contribution can be proved by simple algebraic manipulations. For this reason and because of the limited space, the proofs are omitted.

2 Operations on PIV Sets, Case I

The set of all possible labels of PIV sets on X is the set $\Lambda = [0, 1] \cup D$. First, a relation of partial order on Λ will be introduced.

Definition 2. Consider $\lambda_1, \lambda_2 \in \Lambda$. Then λ_1 s-precedes λ_2 , denoted by $\lambda_1 \leq_s \lambda_2$, if one of the following holds:

- (1) $\lambda_1, \lambda_2 \in [0, 1]$ and $\lambda_1 \leq \lambda_2$,
- (2) $\lambda_1 \in [0, 0.5]$ and $\lambda_2 \in D$,
- (3) $\lambda_1 \in D$ and $\lambda_2 \in (0.5, 1]$,
- (4) $\lambda_1 = [b, c] \in D, \lambda_2 = [p, q] \in D$ and $(b \leq p, c \leq q)$,
- (5) $\lambda_1 = [b, 0.5] \in D, \lambda_2 = 0.5$,
- (6) $\lambda_1 = 0.5, \lambda_2 = [0.5, q] \in D$.

Theorem 1. The relation $\leq_s$ is reflexive, antisymmetric and transitive and for all $\lambda \in \Lambda$ we have $0 \leq_s \lambda \leq_s 1$.

For $\lambda_1, \lambda_2 \in \Lambda$, there is exactly one $\lambda \in \Lambda$ such that with respect to the relation $\leq_s, \lambda = \inf(\lambda_1, \lambda_2) = \lambda_1 \wedge_s \lambda_2$. Operation $\wedge_s$ is defined in Table 3, where $u_1 = \min\{b, p\}, v_1 = \min\{c, q\}$,

$$u_2 = \begin{cases} [p, 0.5] & \text{if } p < 0.5, \\ 0.5 & \text{if } p = 0.5, \end{cases} \quad v_2 = \begin{cases} [b, 0.5] & \text{if } b < 0.5, \\ 0.5 & \text{if } b = 0.5. \end{cases}$$

Table 3. Relation $\wedge_s$ on Λ

$\lambda_2 \setminus \lambda_1$	$a < 0.5$	$[b, c] \in D$	$d > 0.5$	0.5
$r < 0.5$	$\min\{a, r\}$	r	r	r
$[p, q] \in D$	a	$[u_1, v_1]$	$[p, q]$	u_2
$t > 0.5$	a	$[b, c]$	$\min\{t, d\}$	0.5
0.5	a	v_2	0.5	0.5

For $\lambda_1, \lambda_2 \in \Lambda$, there is exactly one $\lambda \in \Lambda$ such that with respect to the relation $\leq_s, \lambda = \sup(\lambda_1, \lambda_2) = \lambda_1 \vee_s \lambda_2$. Operation $\vee_s$ is defined in Table 4, where $u_3 = \max\{b, p\}, v_3 = \max\{c, q\}$,

$$u_4 = \begin{cases} [0.5, q] & \text{if } q > 0.5, \\ 0.5 & \text{if } q = 0.5, \end{cases} \quad v_4 = \begin{cases} [0.5, c] & \text{if } c > 0.5, \\ 0.5 & \text{if } c = 0.5. \end{cases}$$

Corollary 1. $L_1 = (\Lambda, \leq_s, 0, 1)$ is a bounded lattice.

Recall that a mapping $f : X \rightarrow L$, where L is a lattice is called an L -fuzzy set on X .

Table 4. Relation $\vee_s$ on Λ

$\lambda_2 \setminus \lambda_1$	$a < 0.5$	$[b, c] \in D$	$d > 0.5$	0.5
$r < 0.5$	$\max\{a, r\}$	$[b, c]$	d	0.5
$[p, q] \in D$	$[p, q]$	$[u_3, v_3]$	d	u_4
$t > 0.5$	t	t	$\max\{t, d\}$	t
0.5	0.5	v_4	d	0.5

Corollary 2. *PIV sets on X are L_1 -fuzzy sets on X .*

For PIV sets from $\Gamma(X)$, a relation of partial order can be defined pointwise, using the relation $\leq_s$.

Definition 3. Let $f, g \in \Gamma(X)$. Then f is s -included in g , denoted by $f \subset_s g$, if for all $x \in X$,

$$f(x) \leq_s g(x). \quad (3)$$

Theorem 2. *PIV sets on X are partially ordered by the relation $\subset_s$. For all $g \in \Gamma(X)$, $\emptyset \subset_s g \subset_s X$.*

The corresponding operations of s -intersection and s -union of PIV sets can be derived from operations $\wedge_s$ and $\vee_s$, respectively.

Definition 4. Let $f, g \in \Gamma(X)$. Then the s -intersection of f and g , denoted by $f \cap_s g$, is defined for all $x \in X$ by $f(x) \wedge_s g(x)$.

Observe that when f and g are fuzzy sets then $f \cap_s g = f \cap g$ given by (2). At the same time, when f and g are shadowed sets then $f \cap_s g = f \cap g$, where $\cap$ is defined in Table 2.

Theorem 3. *The following properties are satisfied for $g, h, k \in \Gamma(X)$:*

- (1) *commutativity:* $g \cap_s h = h \cap_s g$,
- (2) *associativity:* $g \cap_s (h \cap_s k) = (g \cap_s h) \cap_s k$,
- (3) *idempotency:* $g \cap_s g = g$,
- (4) *boundary conditions:* $g \cap_s \emptyset = \emptyset$ and $g \cap_s X = g$.

Based on the label $g(x)$ of a PIV set g defined on X , each object $x \in X$ can be classified as an object with low, high or neither low nor high (middle) compatibility with the concept described by g . Let $L(g) = \{x \in X : g(x) < 0.5\}$ be the low compatibility region of g , $M(g) = \{x \in X : g(x) \in D \cup \{0.5\}\}$ be the middle compatibility region of g and $H(g) = \{x \in X : g(x) > 0.5\}$ be the high compatibility region of g . The relationship between compatibility regions of PIV sets g, h and the compatibility regions of their s -intersection is given in Theorem 4.

Theorem 4. Let $g, h \in \Gamma(X)$. Then

- (1) $L(g \cap_s h) = L(g) \cup L(h)$,
- (2) $M(g) \cap M(h) \subset M(g \cap_s h) \subset M(g) \cup M(h)$,
- (3) $H(g \cap_s h) = H(g) \cap H(h)$.

Note that $M(g \cap_s h) = X \setminus ((L(g) \cup L(h)) \cup (H(g) \cap H(h)))$.

Definition 5. Let $f, g \in \Gamma(X)$. Then the s -union of f and g , denoted by $f \cup_s g$, is defined for all $x \in X$ by $f(x) \vee_s g(x)$.

When f and g are fuzzy sets then $f \cup_s g = f \cup g$ given by (2). At the same time, when f and g are shadowed sets then $f \cup_s g = f \cup g$, where $\cup$ is defined in Table 1.

Theorem 5. The following properties are satisfied for $g, h, k \in \Gamma(X)$:

- (1) commutativity: $g \cup_s h = h \cup_s g$,
- (2) associativity: $g \cup_s (h \cup_s k) = (g \cup_s h) \cup_s k$,
- (3) idempotency: $g \cup_s g = g$,
- (4) boundary conditions: $g \cup_s \emptyset = g$ and $g \cup_s X = X$.

Theorem 6. Let $g, h \in \Gamma(X)$. Then

- (1) $L(g \cup_s h) = L(g) \cap L(h)$,
- (2) $M(g) \cap M(h) \subset M(g \cup_s h) \subset M(g) \cup M(h)$,
- (3) $H(g \cup_s h) = H(g) \cup H(h)$.

Note that $M(g \cup_s h) = X \setminus ((L(g) \cap L(h)) \cup (H(g) \cup H(h)))$.

Theorem 7. Let $g, h, k \in \Gamma(X)$. Then

- (1) $g \cap_s (h \cup_s k) = (g \cap_s h) \cup_s (g \cap_s k)$,
- (2) $g \cup_s (h \cap_s k) = (g \cup_s h) \cap_s (g \cup_s k)$,
- (3) $M(g \cap_s h) \cup M(g \cup_s h) = M(g) \cup M(h)$.

Corollary 3. $(\Gamma(X), \subset_s, \emptyset, X)$ is a distributive bounded lattice.

3 Operations on PIV Sets, Case II

In this section, another operation of partial order on $\Lambda = [0, 1] \cup D$ will be introduced and then extended to $\Gamma(X)$.

Definition 6. Consider $\lambda_1, \lambda_2 \in \Lambda$. Then λ_1 p -precedes λ_2 , denoted by $\lambda_1 \leq_p \lambda_2$, if one of the following holds:

- (1) $\lambda_1, \lambda_2 \in [0, 0.5]$ and $\lambda_1 \geq \lambda_2$,
- (2) $\lambda_1, \lambda_2 \in [0.5, 1]$ and $\lambda_1 \leq \lambda_2$,
- (3) $\lambda_1 \in [0, 1]$ and $[b, c] = \lambda_2 \in D$ and $b \leq \lambda_1 \leq c$,
- (4) $\lambda_1 = [b, c] \in D$ and $\lambda_2 = [p, q] \in D$ and $p \leq b \leq c \leq q$.

Theorem 8. *The relation $\leq_p$ is reflexive, antisymmetric and transitive and for all $\lambda \in \Lambda$ we have $0.5 \leq_p \lambda \leq_p [0, 1]$.*

Corollary 4. *The relation $\leq_p$ is partial order on Λ with the least element 0.5 and the largest element $[0, 1]$.*

Note that the number 0.5 is the smallest element from all numbers in $[0, 1]$ and the interval $[0, 1]$ is the largest interval from all intervals in D .

For $\lambda_1, \lambda_2 \in \Lambda$, there is exactly one $\lambda \in \Lambda$ such that, with respect to the relation $\leq_p$, $\lambda = \inf(\lambda_1, \lambda_2) = \lambda_1 \wedge_p \lambda_2$. Operation $\wedge_p$ is defined in Table 5, where

$$u_1 = \begin{cases} [\max\{b, p\}, \min\{q, c\}] & \text{if } \max\{b, p\} < \min\{q, c\}, \\ 0.5 & \text{if } \max\{b, p\} = 0.5 = \min\{q, c\}. \end{cases}$$

Table 5. Relation $\wedge_p$ on Λ

$\lambda_2 \setminus \lambda_1$	$a \leq 0.5$	$[b, c] \in D$	$d \geq 0.5$
$r \leq 0.5$	$\max\{a, r\}$	$\max\{r, b\}$	0.5
$[p, q] \in D$	$\max\{a, p\}$	u_1	$\min\{d, q\}$
$t \geq 0.5$	0.5	$\min\{t, c\}$	$\min\{t, d\}$

For $\lambda_1, \lambda_2 \in \Lambda$, there is exactly one $\lambda \in \Lambda$ such that with respect to the relation $\leq_p$, $\lambda = \sup(\lambda_1, \lambda_2) = \lambda_1 \vee_p \lambda_2$. Operation $\vee_p$ is defined in Table 6, where $u_2 = \min\{b, p\}$, $v_2 = \max\{c, q\}$ and

$$u_3 = \begin{cases} [r, d] & \text{if } r \neq d, \\ 0.5 & \text{if } r = d, \end{cases} \quad v_3 = \begin{cases} [a, t] & \text{if } a \neq t, \\ 0.5 & \text{if } a = t. \end{cases}$$

Table 6. Relation $\vee_p$ on Λ

$\lambda_2 \setminus \lambda_1$	$a \leq 0.5$	$[b, c] \in D$	$d \geq 0.5$
$r \leq 0.5$	$\min\{a, r\}$	$[\min\{r, b\}, c]$	u_3
$[p, q] \in D$	$[\min\{a, p\}, q]$	$[u_2, v_2]$	$[p, \max\{d, q\}]$
$t \geq 0.5$	v_3	$[b, \max\{c, t\}]$	$\max\{t, d\}$

Corollary 5. $L_2 = (\Lambda, \leq_p, 0.5, [0, 1])$ is a bounded lattice.

For PIV sets from $\Gamma(X)$, a relation of partial order can be defined pointwise, using the relation $\leq_p$.

Definition 7. Let $f, g \in \Gamma(X)$. Then f is p -included in g , denoted by $f \subset_p g$, if for all $x \in X$,

$$f(x) \leq_p g(x). \quad (4)$$

Theorem 9. *PIV sets on X are partially ordered by the relation $\subset_p$. For all $g \in \Gamma(X)$,*

$$0.5 \subset_p g \subset_p [0, 1].$$

Definition 8. Let $f, g \in \Gamma(X)$. Then the p -intersection of f and g , denoted by $f \cap_p g$, is defined for all $x \in X$ by

$$f(x) \wedge_p g(x). \quad (5)$$

Note that when $f(x) = [b, c] \in D$ and $g(x) = [r, q] \in D$ then $f(x) \wedge_p g(x) = [b, c] \cap [r, q]$.

Theorem 10. *The following properties are satisfied for $g, h, k \in \Gamma(X)$:*

- (1) *commutativity:* $g \cap_p h = h \cap_p g$,
- (2) *associativity:* $g \cap_p (h \cap_p k) = (g \cap_p h) \cap_p k$,
- (3) *idempotency:* $g \cap_p g = g$,
- (4) *boundary conditions:* $g \cap_p 0.5 = 0.5$ and $g \cap_p [0, 1] = g$.

Then compatibility regions of $g \in \Gamma(X)$ are: the low compatibility region $L(g) = \{x \in X : g(x) < 0.5\}$, the high compatibility region $H(g) = \{x \in X : g(x) > 0.5\}$ and the middle compatibility region $M(g) = X \setminus (L(g) \cup H(g))$ consists of two parts, $M_1(g) = \{x \in X : g(x) = 0.5\}$ and $M_2 = \{x \in X : g(x) \in D\}$.

Theorem 11. *Let $g, h \in \Gamma(X)$. Then*

- (1) $L(g) \cap L(h) \subset L(g \cap_p h) \subset L(g) \cup L(h)$,
- (2) $H(g) \cap H(h) \subset H(g \cap_p h) \subset H(g) \cup H(h)$,
- (3) $M_1(g) \cap M_1(h) \subset M_1(g \cap_p h) \subset M_1(g) \cup M_1(h)$,
- (4) $M_2(g) \cap M_2(h) = M_2(g \cap_p h)$.

Definition 9. Let $f, g \in \Gamma(X)$. Then the p -union of f and g , denoted by $f \cup_p g$, is defined for all $x \in X$ by

$$f(x) \vee_p g(x). \quad (6)$$

Note that when $f(x) = [b, c] \in D$ and $g(x) = [r, q] \in D$ then $f(x) \vee_p g(x) = [b, c] \cup [r, q]$.

Theorem 12. *The following properties are satisfied for $g, h, k \in \Gamma(X)$:*

- (1) *commutativity:* $g \cup_p h = h \cup_p g$,
- (2) *associativity:* $g \cup_p (h \cup_p k) = (g \cup_p h) \cup_p k$,
- (3) *idempotency:* $g \cup_p g = g$,
- (4) *boundary conditions:* $g \cup_p 0.5 = g$ and $g \cup_p [0, 1] = [0, 1]$.

Theorem 13. *Let $g, h \in \Gamma(X)$. Then*

- (1) $L(g \cup_p h) = L(g) \cap L(h)$,
- (2) $H(g \cup_p h) = H(g) \cap H(h)$,
- (3) $M_1(g \cup_p h) = M_1(g) \cap M_1(h)$,
- (4) $M_2(g \cup_p h) \supset M_2(g) \cap M_2(h)$.

Theorem 14. *Let $g, h, k \in \Gamma(X)$. Then*

- (1) $g \cap_p (h \cup_p k) = (g \cap_p h) \cup_p (g \cap_p k),$
- (2) $g \cup_p (h \cap_p k) = (g \cup_p h) \cap_p (g \cup_p k).$

Corollary 6. *$(\Gamma(X), \subset_p, 0.5, [0, 1])$ is a distributive bounded lattice.*

In applications, the primary use of PIV sets is to characterize compatibility of objects from X with a concept defined on X by labels small, large or neither small nor large, described by numbers or intervals. The preference of researchers for s -inclusion or p -inclusion (and corresponding aggregation functions) of PIV sets depends on properties of PIV sets on lattices $(\Gamma(X), \subset_s, \emptyset, X)$ and $(\Gamma(X), \subset_p, 0.5, [0, 1])$.

4 Aggregation of PIV Sets by T-norms

T-norms and t-conorms applied to aggregation of fuzzy sets can be extended to t-norms and t-conorms on the lattice $(\Gamma(X), \subset_s, \emptyset, X)$ or on the lattice $(\Gamma(X), \subset_p, 0.5, [0, 1])$. An extension based on the notion of the bounded t-norm and the bounded t-conorm is proposed in this section.

Definition 10. Let T be a t-norm on $[0, 1]$ and $\delta \in [0, 1]$. Then the δ -bounded t-norm T is the mapping $T_{(\delta)} : [\delta, 1] \times [\delta, 1] \rightarrow [\delta, 1]$ such that for all $(x, y) \in [\delta, 1]^2$

$$T_{(\delta)}(x, y) = \max\{\delta, T(x, y)\}. \quad (7)$$

Theorem 15. $T_{(\delta)}$ is a commutative, associative and nondecreasing function satisfying

$$T_{(\delta)}(x, 1) = x \quad \text{and} \quad T_{(\delta)}(x, \delta) = \delta \quad (8)$$

for all $x \in [\delta, 1]$.

Any t-norm T is the δ -bounded t-norm T when $\delta = 0$. Obviously, $T_{min,(\delta)} = T_{min}$ for all $\delta \in [0, 1]$.

Definition 11. Let S be a t-conorm on $[0, 1]$ and $\delta \in [0, 1]$. Then the δ -bounded t-conorm S is the mapping $S_{(\delta)} : [0, \delta] \times [0, \delta] \rightarrow [0, \delta]$ such that for all $(x, y) \in [0, \delta]^2$

$$S_{(\delta)}(x, y) = \min\{\delta, S(x, y)\}. \quad (9)$$

Theorem 16. $S_{(\delta)}$ is a commutative, associative and nondecreasing function satisfying

$$S_{(\delta)}(x, 0) = x \quad \text{and} \quad S_{(\delta)}(x, \delta) = \delta \quad (10)$$

for all $x \in [0, \delta]$.

Any t-conorm S is the δ -bounded t-conorm S when $\delta = 1$. Obviously, $S_{max,(\delta)} = S_{max}$ for all $\delta \in [0, 1]$.

Theorem 17. Consider a t -norm T on $[0, 1]$ and the lattice $(\Gamma(X), \subset_s, \emptyset, X)$. Let $\varphi_T : \Gamma(X) \times \Gamma(X) \rightarrow \Gamma(X)$ be defined for all $f, g \in \Gamma(X)$ in Table 7, where

$$u_1 = \begin{cases} [T(b, p), T_{(0.5)}(c, q)] & \text{if } T(b, p) < T_{(0.5)}(c, q), \\ 0.5 & \text{if } T(b, p) = T_{(0.5)}(c, q) = 0.5, \end{cases}$$

$$u_2(s) = \begin{cases} [T(s, 0.5), 0.5] & \text{if } T(s, 0.5) < 0.5, \\ 0.5 & \text{if } T(s, 0.5) = 0.5. \end{cases}$$

Then φ_T is a t -norm on $(\Gamma(X), \subset_s, \emptyset, X)$,

Table 7. T-norm on $\Gamma(X)$, Case I

$g \setminus f$	$f(x) < 0.5$	$f(x) = [b, c]$	$f(x) > 0.5$	0.5
$g(x) < 0.5$	$T(f(x), g(x))$	$g(x)$	$g(x)$	$g(x)$
$g(x) = [p, q]$	$f(x)$	u_1	$g(x)$	$u_2(p)$
$g(x) > 0.5$	$f(x)$	$f(x)$	$T_{(0.5)}(f(x), g(x))$	0.5
0.5	$f(x)$	$u_2(b)$	0.5	0.5

When $T = T_{min}$ then $\varphi_T(f, g) = f \cap_s g$.

Theorem 18. Consider a t -conorm S on $[0, 1]$ and the lattice $(\Gamma(X), \subset_s, \emptyset, X)$. Let $\varphi_S : \Gamma(X) \times \Gamma(X) \rightarrow \Gamma(X)$ be defined for all $f, g \in \Gamma(X)$ in Table 8, where

$$u_3 = \begin{cases} [S_{(0.5)}(b, p), S(c, q)] & \text{if } S_{(0.5)}(b, p) < S(c, q), \\ 0.5 & \text{if } S_{(0.5)}(b, p) = S(c, q) = 0.5, \end{cases}$$

$$u_4(s) = \begin{cases} [0.5, S(s, 0.5)] & \text{if } S(s, 0.5) > 0.5, \\ 0.5 & \text{if } S(s, 0.5) = 0.5. \end{cases}$$

Then φ_S is a t -conorm on $(\Gamma(X), \subset_s, \emptyset, X)$.

Table 8. T-conorm on $\Gamma(X)$, Case I

$g \setminus f$	$f(x) < 0.5$	$f(x) = [b, c]$	$f(x) > 0.5$	0.5
$g(x) < 0.5$	$S_{(0.5)}(f(x), g(x))$	$f(x)$	$f(x)$	0.5
$g(x) = [p, q]$	$g(x)$	u_3	$f(x)$	$u_4(q)$
$g(x) > 0.5$	$g(x)$	$g(x)$	$S(f(x), g(x))$	$g(x)$
0.5	0.5	$u_4(c)$	$f(x)$	0.5

When $S = S_{max}$ then $\varphi_S(f, g) = f \cup_s g$.

Theorem 19. Consider a t -norm T on $[0, 1]$, its dual t -conorm S and the lattice $(\Gamma(X), \subset_p, 0.5, [0, 1])$. Let $\rho_T : \Gamma(X) \times \Gamma(X) \rightarrow \Gamma(X)$ be defined for all $f, g \in \Gamma(X)$ in Table 9, where

$$u_1 = \begin{cases} [S_{(0.5)}(b, p), T_{(0.5)}(q, c)] & \text{if } S_{(0.5)}(b, p) < T_{(0.5)}(q, c), \\ 0.5 & \text{if } S_{(0.5)}(b, p) = 0.5 = T_{(0.5)}(q, c). \end{cases}$$

Then ρ_T is a t -norm on $(\Gamma(X), \subset_p, 0.5, [0, 1])$.

Table 9. T-norm on $\Gamma(X)$, Case II

$g \setminus f$	$f(x) \leq 0.5$	$f(x) = [b, c]$	$f(x) \geq 0.5$
$g(x) \leq 0.5$	$S_{(0.5)}(f(x), g(x))$	$S_{(0.5)}(g(x), b)$	0.5
$g(x) = [p, q]$	$S_{(0.5)}(f(x), p)$	u_1	$T_{(0.5)}(f(x), q)$
$g(x) \geq 0.5$	0.5	$T_{(0.5)}(g(x), c)$	$T_{(0.5)}(f(x), g(x))$

When $T = T_{min}$ then $\rho_T(f, g) = f \cap_p g$.

Theorem 20. Consider a t -conorm S on $[0, 1]$ and the lattice $(\Gamma(X), \subset_p, 0.5, [0, 1])$. Let $\rho_S : \Gamma(X) \times \Gamma(X) \rightarrow \Gamma(X)$ be defined for all $f, g \in \Gamma(X)$ in Table 10, where

Table 10. T-conorm on $\Gamma(X)$, Case II

$g \setminus f$	$f(x) \leq 0.5$	$f(x) = [b, c]$	$f(x) \geq 0.5$
$g(x) \leq 0.5$	$T(f(x), g(x))$	$[T(g(x), b), c]$	u_3
$g(x) = [p, q]$	$[T(f(x), p), q]$	$[u_2, v_2]$	$[p, S(f(x), q)]$
$g(x) \geq 0.5$	u_3	$[b, S(c, g(x))]$	$S(f(x), g(x))$

$$u_2 = T(b, p), \quad v_2 = S(c, q), \quad u_3 = \begin{cases} [f(x), g(x)] & \text{if } f(x) \neq g(x), \\ 0.5 & \text{if } f(x) = g(x). \end{cases}$$

Then ρ_S is a t -conorm on $(\Gamma(X), \subset_p, 0.5, [0, 1])$.

When $S = S_{max}$ then $\rho_S(f, g) = f \cup_s g$.

5 Construction of PIV Sets

Different methods for construction of PIV sets from available data can be proposed. A simple approach based on a preselected aggregation function Ag is outlined below.

Assume a fuzzy set f on a finite universal set U and an aggregation function Ag such that for $(\delta_1, \dots, \delta_n) \in [0, 1]^n$

$$\min(\delta_1, \dots, \delta_n) \leq Ag(\delta_1, \dots, \delta_n) \leq \max(\delta_1, \dots, \delta_n).$$

Let $W_1(f) = \{u \in U : f(u) \leq 0.5\}$, $W_2(f) = \{u \in U : f(u) \geq 0.5\}$, $a = Ag(f(u), u \in W_1(f))$ and $b = Ag(f(u), u \in W_2(f))$. Then the summary characterization of f based on Ag can be expressed by

$$S_{Ag}(f) = \begin{cases} a & \text{if } W_2(f) = \emptyset, \\ b & \text{if } W_1(f) = \emptyset, \\ [a, b] & \text{if } W_1(f) \neq \emptyset, W_2(f) \neq \emptyset \text{ and } a \neq b, \\ 0.5 & \text{if } a = b = 0.5. \end{cases} \quad (11)$$

The task is to find a summary characterization of data τ (see formula (1)). In our illustrative example τ may represent evaluations of objects from X by experts from $E = \{E_j, j = 1, \dots, m\}$. Then the following method can be used:

Method 1:

Step 1: For each $i = 1, \dots, n$, construct fuzzy set f_i on E such that for $E_j \in E$: $f_i(E_j) = \delta_{ij}$. Then choose aggregation function Ag and create $S_{Ag}(f_i)$.

Step 2: Construct PIV set φ on X such that for $x_i \in X$, $\varphi(x_i) = S_{Ag}(f_i)$.

Example 1. Suppose that data in Table 11 represent evaluations of objects $x_1, \dots, x_5$ by experts $E_1, \dots, E_6$ with respect to criterion C_1 . Let Ag be the arithmetic mean. Then, using Method 1, summary characterizations of Table 11 can be obtained by PIV set φ_{C_1} with labels presented in Table 12.

Table 11. Evaluations with respect to criterion C_1

X	x_1	x_2	x_3	x_4	x_5
E_1	0.5	0.8	1.0	0.2	0.7
E_2	0.3	0.4	0.9	0.0	0.6
E_3	0.1	0.6	0.8	0.1	0.4
E_4	0.2	0.5	0.7	0.1	0.4
E_5	0.1	0.7	0.8	0.3	0.3
E_6	0.4	0.4	0.9	0.4	0.8

Table 12. Summary of evaluations with respect to criterion C_1

X	x_1	x_2	x_3	x_4	x_5
φ_{C_1}	[0.267, 0.5]	[0.433, 0.65]	0.85	0.183	[0.367, 0.7]

6 Conclusion

PIV sets allow evaluation of compatibility of objects from a set X with a concept defined on X in terms (labels) that can be interpreted as either small or large (described by either a small or a large number from the unit interval), or neither small nor large (described by 0.5 or a subinterval from the unit interval covering 0.5). Description of the label “neither small nor large” by an interval gives researchers more flexibility in decision making under uncertainty. A method for a simple summary characterization of a family of finite fuzzy sets by a PIV set was suggested.

From the theoretical point of view, relations s -inclusion and p -inclusion of PIV sets were introduced and corresponding operations of union and intersection were defined. A restriction of t -norms (t -conorms) to bounded t -norms (t -conorms) was proposed and used in construction of t -norms (t -conorms) suitable for aggregation of PIV sets.

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