

Preface

The motion of fluids from the smaller to the large scales is described by a complex interplay between the momentum equations and the equations describing the thermodynamics of the system under consideration. The emerging motion comprises several scales, ranging from microscales, to planetary scales, often linked by non-trivial self-similar scalings. At the same time, the motion of classic fluids is described by a specific branch of continuum mechanics. It comes thus natural that one would like to describe the rich phenomenology of the fluid and geophysical fluid motion in a systematic way from first principles, derived by continuum mechanics. One of these first principles is given by Hamilton's principle, which allows to obtain the equations of motion through a variational treatment of the system.

A famous call for the need of a systematic derivation of the equations of Fluid and Geophysical Fluid Dynamics lies in the memorandum sent by the mathematician John von Neumann to Oswald Veblen, written in 1945 and here reported in the Introduction to Chap. 3. The quote reads: *“The great virtue of the variational treatment [...] is that it permits efficient use, in the process of calculation, of any experimental or intuitive insight [...]. It is important to realize that it is not possible, or possible to a much smaller extent, if one performs the calculation by using the original form of the equations of motion—the partial differential equations. [...] Symmetry, stationarity, similitude properties [...] applying such methods to hydrodynamics would be of the greatest importance since in many hydrodynamical problems we have very good general evidence of the above-mentioned sort about the approximate aspect of the solution, and the refining of this to a solution of the desired precision is what presents disproportionate computational difficulties [...].”* (see reference to von Neumann (1963) of Chap. 3). While sadly von Neumann intended to make practical use of such a treatment to study the aftershocks created by nuclear explosions, the quote still summarizes some of the most important features of the variational method: *“Symmetry, stationarity, similitude properties”*. With these properties, von Neumann clearly had in mind the self-similar structure of fluid flows (*“similitude”*), which is indeed the feature that allows us to study different scales of motion through a proper rescaling of the system; he probably had in mind also the study of the stability of the system under consideration

(“stationarity”); but he mentions also one of the most important results from field-theory that is the study of what he calls with the word “*symmetry*”. Continuous symmetries in mechanical systems have in fact the property to be related to conserved quantities, as it is well known by probably the most beautiful theorem in mathematical physics, the celebrated “*Noether’s Theorem*”. In the specific case of fluid dynamics, the continuum hypothesis is associated to a specific symmetry that is the particle relabelling symmetry. Application of Noether’s Theorem results in the fundamental conservation of vorticity in fluids, which is itself linked to the conservation of circulation and of potential vorticity, all quantities that have primary importance in a huge number of applications, ranging from fluids, geophysical fluids, plasmas and astrophysical fluids. It is from the particle relabelling symmetry and Noether’s Theorem that one sees that the conservation of vorticity is a fundamental property of the system and does not emerge just from skilful manipulation of the partial differential equations describing the dynamics.

The aim of this book is to go through the development of these concepts.

In Chap. 1, we give a résumé of the aspects of Fluid and Geophysical Fluid Dynamics, starting from the continuum hypothesis and then presenting the governing equations and the conservation of potential vorticity as well as energy and enstrophy, in various approximations.

In Chap. 2, we review the Lagrangian formulation of dynamics starting from Hamilton’s Principle of First Action. In the second part of the chapter, Noether’s Theorem is presented both for material particles and for continuous systems such as fluids.

In this way, Chap. 1 will serve as an introduction to Fluid and Geophysical Fluid Dynamics to students and researchers of subjects such as physics and mathematics. Chapter 2 will instead serve as an introduction to analytical mechanics to students of applied subjects, such as engineering, climatology, meteorology and oceanography.

In Chap. 3, we first introduce the Lagrangian density for the ideal fluid. The equations of motion are rederived using Hamilton’s principle first in the Lagrangian and then in the Eulerian frameworks. The relationship between the two frameworks is thus revealed from the use of canonical transformations. Noether’s Theorem is then applied to derive the conservation laws corresponding to the continuous symmetries of the Lagrangian density. Particular attention will be given to the particle relabelling symmetry, and the associated conservation of vorticity.

In Chap. 4, we extend the use of Hamilton’s principle to continuously stratified fluids and to uniformly rotating flows. Different sets of approximated equations, which constitute different commonly used approximation in Geophysical Fluid Dynamics, are considered, as well as the form taken by the conservation of potential vorticity in each of them. Finally, the variational methods are applied to study some selected topics of wave dynamics.

Technical derivations of equations that might interrupt the flow of the reading are reported in a number of appendices.

This book should be considered as an elementary introduction. Bibliographical notes at the end of each chapter will guide the reader to more advanced treatments of the subject.

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