Chapter 2
Mechanics, Symmetries and Noether's Theorem

Abstract The Lagrangian description of mechanics allows to derive the equations of motion from a variational principle based on conserved quantities of the system. In the first part of this chapter, the Lagrangian formulation of dynamics and the properties of the Lagrangian operator are synthetically reviewed starting from Hamilton's Principle of First Action. In the second part of the chapter, the important link between continuous symmetries of the Lagrangian operator and conserved quantities of the system is introduced through Noether's Theorem. The proof of the Theorem is reported both for material particles and for continuous systems such as fluids.

Keywords Classical mechanics · Particle mechanics · Continuum mechanics · Variational principle · Symmetry · Noether's Theorem · Lagrangian dynamics · Hamiltonian dynamics · Canonical transformations

2.1 Introduction

In this chapter, we will make a short review of the Lagrangian formalism of classical mechanics, both for systems with a finite number of degrees of freedom, i.e. for systems of point particles, and for systems with an infinite number of degrees of freedom, i.e. for continuous systems. The description for point particles can be applied, for example, to study the advection of a passive tracer in a fluid flow, while the description for continuous systems provides a framework to derive the equations of motion for fluids.

The Lagrangian formalism is based on a variational principle and it introduces a number of useful advantages to the study of the dynamics. In particular, the Lagrangian formulation allows to establish an important link between the symmetries of the resulting Lagrangian operator and the conservation laws of the system through the formulation of Noether's Theorem. As it will be seen in the following chapters, this link provides an important and powerful concept and instrument to analyse the system, giving a physical base for the conservation of energy through the invariance under time translations and for the conservation of vorticity through the

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particle relabelling symmetry, which is a unique feature for fluids. This fact holds even for a system resulting from approximate equations, if the conservation laws concerning the original system are preserved. The reader familiar with Geophysical Fluid Dynamics will recognize the immediate usefulness of this property, given that Geophysical Fluid Dynamics relies on approximations, some of which were introduced in Chap. 1.

2.2 Hamilton's Principle of Least Action

Consider a system of N point particles with masses $(m_1, \ldots, m_N)$ and positions $(r_1(t), \ldots, r_N(t))$ at time t. In the following, the shorter notation m_i and $\mathbf{r}_i(t)$, with $i = 1, \ldots, N$, will be often used. The point particles move in a potential V. Unless specified, we will often take the potential as function of position only, say $V(\mathbf{r}_i(t))$. Each particle moves following Newton's second law. While this system is generally described by $3N$ degrees of freedom, the presence of constraints in the system can act to make the coordinates not independent: if the constraints are expressed as $f(\mathbf{r}_1, \ldots, \mathbf{r}_k, t) = 0$, which take also the name of *holonomic* constraints, then only $n = 3N - k$ independent coordinates can be determined in terms of *generalized coordinates* $q_i(t)$ that form a parametric representation of the nonindependent positions $\mathbf{r}_i(t)$. Notice that the $q_i(t)$ must not be orthogonal coordinates. The space defined by the generalized coordinates is also known as *configuration space*. The description provided by Newton's second law is based on a differential principle. The motion of the point particles in the configuration space can be equivalently described using Hamilton's Principle of Least Action, which is based on the minimization of an action functional and which provides an integral description of the evolution of the system, i.e. a description depending on the entire path of the system in configuration space. Consider the motion of the system in the time interval $[t_1, t_2]$, and assume that the position of the system at the extremes t_1 and t_2 is fixed. Let T be the kinetic energy of the system. Hamilton's Principle of Least Action thus states (see, e.g. [4]):

Theorem 2.1 *Out of all possible paths by which the system point could travel from its position at time t_1 to its position at time t_2, it actually travels along that path for which the integral*

$$I = \int_{t_1}^{t_2} (T - V)\, dt \tag{2.1}$$

is an extremum, whether a minimum or maximum.

The function $L = T - V$ takes the name of *Lagrange function*, or simply *Lagrangian*.

Given $T = T(\dot{q}_1, \ldots, \dot{q}_n)$ and $V = V(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$, where n is the total number of generalized coordinates $q_i(t)$, the Lagrangian is a function of the kind

$$L = L(q_i, \dot{q}_i, t), \ (i = 1, \ldots, n) . \tag{2.2}$$

Thus, according to the Principle of Least Action, the task is to find an extremum of the functional

$$I = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt , \tag{2.3}$$

in the space $C^2_{[t_1,t_2]}$ of twice differentiable functions $q_i(t)$ such that

$$q_i(t_1) = q_i(t_2) = 0 . \tag{2.4}$$

Functions $q_i(t)$ can be conceived as being labelled by a parameter l and given by

$$q_i(t,l) = q_i(t,0) + l\eta_i(t) , \tag{2.5}$$

where $\eta_i(t) \in C^2_{[t_1,t_2]}$ are arbitrary functions that satisfy

$$\eta_i(t_1) = \eta_i(t_2) = 0 . \tag{2.6}$$

Substitution of (2.5) into (2.3) yields

$$I(l) = \int_{t_1}^{t_2} L[q_i(t,0) + l\eta_i(t), \dot{q}_i(t,0) + l\dot{\eta}_i(t), t]dt . \tag{2.7}$$

Equation (2.7) shows that (2.3) is actually a function of the free parameter l, so the variation δI of $I(l)$ is given by

$$\delta I = \frac{\partial I}{\partial l}dl = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial l}dl + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial l}dl \right) dt . \tag{2.8}$$

Because of (2.5), $\partial q_i/\partial l = \eta_i$, $\partial \dot{q}_i/\partial l = \dot{\eta}_i$, and therefore, Eq. (2.8) take the form

$$\begin{aligned}
\delta I &= \sum_{i=1}^{n} \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\eta_i + \frac{\partial L}{\partial \dot{q}_i}\dot{\eta}_i \right) dtdl \\
&= \sum_{i=1}^{n} \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i}\eta_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\eta_i \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right)\eta_i \right] dtdl \\
&= \sum_{i=1}^{n} \left\{ \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \eta_i dt + \left[\frac{\partial L}{\partial \dot{q}_i}\eta_i \right]_{t_1}^{t_2} \eta_i \right\} dl
\end{aligned}$$

$$= \sum_{i=1}^{n} \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] \eta_i dt dl \ , \tag{2.9}$$

where the identity,

$$\frac{\partial L}{\partial \dot{q}_i}\dot{\eta}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\eta_i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\eta_i \tag{2.10}$$

has been used. The arbitrariness of functions $\eta_i(t)$ leads us to conclude that

$$\delta I = 0 \iff \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \ (i = 1, \dots, n) \ .$$

In other words, Hamilton's Principle of Least Action implies the *Euler–Lagrange equations*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \ (i = 1, \dots, n) \ , \tag{2.11}$$

and vice versa.

In particular, if $T = \sum_{i=1}^{n} m_i \dot{q}_i^2/2$ and $V = V(q_1, \dots, q_n)$, the Euler–Lagrange equations take the form of Newton's second law, that is

$$m_i \ddot{q}_i^2 + \frac{\partial V}{\partial q_i} = 0, \ (i = 1, \dots, n) \ , \tag{2.12}$$

so Newton's second law is an extremal for the action

$$I = \int_{t_1}^{t_2} \left[\left(\sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i^2 \right) + V(q_1, \dots, q_n) \right] dt \ . \tag{2.13}$$

In (2.10),

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \left(\dot{q}_i \frac{\partial}{\partial q_i} + \ddot{q}_i \frac{\partial}{\partial \dot{q}_i} \right) \ , \tag{2.14}$$

and the presence of $\ddot{q}_i(t)$ in (2.14) explains why the space of twice differentiable functions $q_i(t)$ is the framework in which the variational problem $\delta I = 0$ is posed. Because (2.11) is thus a set of n second-order equations, the problem needs the specification of $2n$ initial conditions for q_i and $\dot{q}_i$ to be specified either at t_1 or t_2.

The Euler–Lagrange equations and the action principle show one of the benefits of the Lagrangian formulation of dynamics, that is that it is possible to derive the equations of motion from the knowledge of two scalars, T and V, rather than from all forces acting on the system.

2.3 Lagrangian Function, Euler–Lagrange Equations and D'Alembert's Principle

The Lagrangian function and the Euler–Lagrange equations can be derived also in a different way than present in the previous section, making use of two fundamental principles, namely the *virtual work's* and *D'Alembert's* principles. The second of them, of dynamical nature, is just an extension of the first, which is instead of statistical nature.

The virtual work's principle is applied to a system of point particles in an equilibrium, so that the sum of the forces applied to the point particle i

$$\mathbf{F}_i = 0 \tag{2.15}$$

is zero. Further, consider the *virtual displacement* $\delta\mathbf{r}_i$ of the point particle i as the displacement given by the infinitesimal change of the configuration of the entire system following the forces and the constraints associated with the system itself, occurring while time is held constant. The virtual displacement differs from real displacements in the way that real displacements imply a temporal evolution of the forces and constraints, which instead does not take place in the virtual case.

Equation (2.15) implies $\mathbf{F}_i \cdot \mathbf{r}_i = 0$, and thus,

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0\,, \tag{2.16}$$

where the sum is intended as over all the point particles of the system. The sum of the forces acting on point particle i can be separated in two classes: the applied forces $\mathbf{F}_i^{(a)}$ and the forces of constraint $\mathbf{f}_i$. Under this partition, (2.16) yields

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0\,. \tag{2.17}$$

In the framework of systems in which the net virtual work of the forces of constraint is zero, i.e. $\mathbf{f}_i \cdot \delta\mathbf{r}_i = 0$, (2.17) can be written as

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i = 0\,. \tag{2.18}$$

Equation (2.18) expresses the *virtual work's* principle. It should be noted that (2.18) does not apply, for example, in the presence of dissipative forces. While (2.18) refers here to a steady system, it could be easily applied to unsteady systems substituting (2.15) with Newton's law $\mathbf{F}_i - \dot{\mathbf{p}}_i = 0$. Under this assumption, (2.16) yields

$$\sum_i (\mathbf{F}_i - \mathbf{p}_i) \cdot \delta\mathbf{r}_i = 0\,, \tag{2.19}$$

and, once again under the assumption that the net virtual work of the forces of constraint is zero, (2.19) yields

$$\sum_i \left(\mathbf{F}_i^{(a)} - \mathbf{p}_i\right) \cdot \delta \mathbf{r}_i = 0 \,. \tag{2.20}$$

Equation (2.20) is *D'Alembert's principle*, which states that every state of the motion can be considered as a state in mechanical equilibrium.

The introduction of the Lagrangian function follows from a formal development of (2.20) making use of the generalized coordinates q_j in place of the vectors $\mathbf{r}_i$, following the transformation equations

$$r_i = r_i\left(q_1, \ldots, q_{3N-k}, t\right), \; i = 1, \ldots, N \,, \tag{2.21}$$

where N is the number of point particles and k is the number of holonomic constraints. From (2.21), one gets

$$\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \,, \tag{2.22}$$

$$\mathbf{v}_i = \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \,. \tag{2.23}$$

Using (2.22), the first term of (2.20) yields, omitting the superscript,

$$\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_i \sum_j \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \,. \tag{2.24}$$

If $Q_j = \sum_i \mathbf{F}_i \cdot \partial \mathbf{r}_i / \partial q_j$ is the j component of a generalized force, (2.24) becomes

$$\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_j Q_j \delta q_j \,. \tag{2.25}$$

If the forces derive from a scalar potential V, so that $F_i = -\nabla_i V(q_j)$, the j component of the generalized force for conservative systems can be written as

$$Q_j = -\sum_i \nabla_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j} \,. \tag{2.26}$$

Using (2.22), the second term of (2.20) yields instead

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_j \left(\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\right) \delta q_j \,. \tag{2.27}$$

Because the generalized coordinates are independent, (2.20), (2.25) and (2.27) yield

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} - Q_j = 0 \,, \tag{2.28}$$

which is equivalent to

$$\sum_i \left[m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_j} \right] - Q_j = 0 \,. \tag{2.29}$$

Remembering that $\dot{\mathbf{r}}_i = \mathbf{v}_i$ and using

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \,,$$

and

$$\frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \right) = \frac{\partial \mathbf{v}_i}{\partial q_j} \,,$$

which come from (2.23), Eq. (2.29) can be written as

$$\sum_i \left[\frac{d}{dt} \left(m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right) - m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j} \right] - Q_j = 0 \,,$$

or

$$\sum_i \left[\frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \frac{m_i \mathbf{v}_i^2}{2} - \frac{\partial}{\partial q_j} \frac{m_i \mathbf{v}_i^2}{2} \right] - Q_j = 0 \,, \tag{2.30}$$

where $T = \sum_i m_i \mathbf{v}_i^2/2$ is the total kinetic energy of the system. With (2.26), (2.30) yields

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial}{\partial q_j}(T - V) = 0 \,. \tag{2.31}$$

Because $\partial V / \partial \dot{q}_j = 0$, and introducing the Lagrangian function

$$L = T - V \,, \tag{2.32}$$

Equation (2.31) takes the final form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \,, \; i = 1, \ldots, 3N - k \,, \tag{2.33}$$

which are the Euler–Lagrange equations (2.11).

It should be noted that (2.33) is valid only if dissipative forces are not present and if the potential is independent from the velocities. For notable exception, such as the Lorentz force and Rayleigh's dissipation, see e.g. [4]. Finally, it should be also noted that (2.33) involves energy terms as it derives from D'Alembert's principle, which, in dimensional form, expresses an energy balance.

2.4 Covariance of the Lagrangian with Respect to Generalized Coordinates

As mentioned in the Introduction, one of the advantages of the action principle is that it is covariant with respect to a change of generalized coordinates.

To see this, reconsider Eq. (2.11) and assume that $\{Q_k\}_{k=1,\ldots,n}$ is another set of generalized coordinates. Then, $q_i = f_i(Q_1, \ldots, Q_n)$, that is, in short,

$$q_i = f_i(Q_k) \ , \ \frac{\partial f_i}{\partial Q_k} \neq 0 \ \forall i, \ \forall k \ . \tag{2.34}$$

Hence,

$$\dot{q}_i = \sum_{k=1}^{n} \frac{\partial f_i}{\partial Q_k} \dot{Q}_k \ . \tag{2.35}$$

Conversely, if $\hat{f}$ is the inverse coordinate transformation, i.e. $Q_i = \hat{f}_i(q_1, \ldots, q_n)$, one has

$$\dot{Q}_i = \sum_{k=1}^{n} \frac{\partial \hat{f}_i}{\partial q_k} \dot{q}_k \ . \tag{2.36}$$

Quantities that transform under change of coordinate as (2.36) are called *covariant vectors*. It should be noted that traditionally covariant (and their correspondent contravariant) vectors are indicated with the use of subscripts and superscripts. In this book, we will, however, not employ this notation.

Based on (2.34) and (2.35), the Lagrangian (2.2) transforms as

$$L(q, \dot{q}, t) = L\left[f(Q), \nabla_Q f \cdot \dot{\mathbf{Q}}, t\right] = \tilde{L}(Q, \dot{Q}, t) \ , \tag{2.37}$$

where

$$q = q_1 \ldots q_n \ , \tag{2.38a}$$

$$Q = Q_1 \ldots Q_n \ , \tag{2.38b}$$

$$\nabla_Q f \cdot \dot{\mathbf{Q}} = \sum_{k=1}^{n} \frac{\partial f}{\partial Q_k} \dot{Q}_k \ , \tag{2.38c}$$

In the following, we will assume the position $i = 1, \ldots, n$. Starting from (2.37), the quantities $\partial\tilde{L}/\partial Q_k,\ \partial\tilde{L}/\partial\dot{Q}_k,\ d/dt(\partial\tilde{L}/\partial\dot{Q}_k)$ can be evaluated as follows: the first two quantities are

$$\begin{aligned}
\frac{\partial\tilde{L}}{\partial Q_k} &= \frac{\partial}{\partial Q_k}L(q_i,\dot{q}_i,t)\\
&= \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial Q_k} + \frac{\partial L}{\partial\dot{q}_i}\frac{\partial\dot{q}_i}{\partial Q_k}\\
&= \frac{\partial L}{\partial q_i}\frac{\partial f_i}{\partial Q_k} + \frac{\partial L}{\partial\dot{q}_i}\frac{\partial}{\partial Q_k}\sum_j\frac{\partial f_i}{\partial Q_j}\dot{Q}_j\\
&= \frac{\partial L}{\partial q_i}\frac{\partial f_i}{\partial Q_k} + \frac{\partial L}{\partial\dot{q}_i}\sum_j\frac{\partial^2 f_i}{\partial Q_k\partial Q_j}\dot{Q}_j\ ,
\end{aligned} \tag{2.39}$$

and

$$\frac{\partial\tilde{L}}{\partial\dot{Q}_k} = \frac{\partial L}{\partial\dot{q}_i}\frac{\partial\dot{q}_i}{\partial\dot{Q}_k} = \frac{\partial L}{\partial\dot{q}_i}\frac{\partial}{\partial\dot{Q}_k}\sum_j\frac{\partial f_i}{\partial Q_j}\dot{Q}_j = \frac{\partial L}{\partial\dot{q}_i}\frac{\partial f_i}{\partial Q_k}\ . \tag{2.40}$$

The time derivative of (2.40) gives

$$\begin{aligned}
\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{Q}_k} &= \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right)\frac{\partial f_i}{\partial Q_k} + \frac{d}{dt}\left(\frac{\partial f_i}{\partial Q_k}\right)\frac{\partial L}{\partial\dot{q}_i}\\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right)\frac{\partial f_i}{\partial Q_k} + \frac{\partial L}{\partial\dot{q}_i}\sum_j\frac{\partial^2 f_i}{\partial Q_k\partial Q_j}\dot{Q}_j\ .
\end{aligned} \tag{2.41}$$

From (2.39), one has

$$\frac{\partial L}{\partial\dot{q}_i}\sum_j\frac{\partial^2 f_i}{\partial Q_k\partial Q_j}\dot{Q}_j = \frac{\partial\tilde{L}}{\partial Q_k} - \frac{\partial L}{\partial q_i}\frac{\partial f_i}{\partial Q_k}\ , \tag{2.42}$$

and substitution of (2.42) into (2.41) implies

$$\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{Q}_k} - \frac{\partial\tilde{L}}{\partial\dot{Q}_k} = \frac{\partial f_i}{\partial Q_k}\left(\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i} - \frac{\partial L}{\partial q_i}\right)\ . \tag{2.43}$$

Finally, because of the second equation of (2.34) and (2.11), Eq. (2.43) yields

$$\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{Q}_k} - \frac{\partial\tilde{L}}{\partial\dot{Q}_k} = 0\ . \tag{2.44}$$

Equation (2.44) retains openly the same form as (2.11), so the covariance of (2.11) under (2.34) is proved.

2.5 Role of Constraints

As stated in Sect. 2.2, the presence of constraints in the system can introduce mutual dependencies between the generalized coordinates. To see how the equations of motion can be derived from Hamilton's principle even in the presence of constraints, consider first the general problem of finding the extrema of a function $\phi(x_1, \dots, x_n)$ that is not subject to constraints. The extrema can thus be located at the points where $\nabla\phi = 0$. In the presence of constraints determined by m equations $f_\alpha(x_1, \dots, x_n) = 0$, $\alpha = 1, \dots, m$, the problem can be solved finding the extrema of the auxiliary function

$$F(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = \phi(x_1, \dots, x_n) + \sum_{\alpha=1}^{m} [\lambda_\alpha f_\alpha(x_1, \dots, x_n)] \ , \tag{2.45}$$

where

$$\lambda_\alpha, \ \alpha = 1, \dots, m \ , \tag{2.46}$$

are the *indeterminate Lagrange multipliers* of the system. The problem of finding the extrema of the function $\phi(x_1, \dots, x_n)$ subject to constraints is thus turned into the problem of finding the extrema of the auxiliary function $F(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m)$ in the absence of constraints.

Consider now a system which is described by the Lagrangian $L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)$ subject to m constraints that we assume can be expressed in the form

$$f_\alpha(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) = 0, \ (\alpha = 1, \dots, m) \ . \tag{2.47}$$

In analogy with the previous example, the derivation of the equations of motion can thus be obtained from Hamilton's Principle of Least Action as

$$\delta \int_{t_1}^{t_2} L_c dt = 0 \ , \tag{2.48}$$

where

$$L_c = L + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha \ . \tag{2.49}$$

Notice that

$$\lambda_\alpha = \frac{\partial L_c}{\partial f_\alpha} \ . \tag{2.50}$$

Hamilton's principle (2.48) thus leads to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i, \ (i = 1, \dots, n) \ , \tag{2.51}$$

where $F_i,\ i = 1, \ldots, n$, are the *generalized forces*

$$F_i = \sum_{\alpha=1}^{m} \left\{ \lambda_\alpha \left[\frac{\partial f_\alpha}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial f_\alpha}{\partial \dot{q}_i} \right) \right] \right\},\ (i = 1, \ldots, n)\ . \tag{2.52}$$

Equations (2.47) and (2.51) constitute thus a system of $n+m$ equations in $n+m$ variables describing a system under the action of the generalized forces (2.52) exerted by constraints.

2.6 Canonical Variables and Hamiltonian Function

Consider the Euler–Lagrange equations (2.11), and assume that the potential V is a function only of the position of the point particles and not of the generalized velocities. Then,

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial V}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i} \sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i^2 = m_i \dot{q}_i = p_i\ . \tag{2.53}$$

Equation (2.53) defines thus the *generalized* or *conjugate momentum*

$$p_i = \frac{\partial L}{\partial \dot{q}_i}\ . \tag{2.54}$$

The insertion of (2.54) in (2.11) gives thus the equation for the time evolution of p_i

$$\dot{p}_i = \frac{\partial L}{\partial q_i}\ . \tag{2.55}$$

It is important to notice that as the generalized coordinates q_i are not Cartesian, the conjugate momentum p_i does not necessarily correspond to the linear momentum. The pair of generalized variables (q_i, p_i) takes also the name of *canonical variables*. Notice that (2.54) and (2.55) allow to write the differential of the Lagrangian

$$dL = \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right) + \frac{\partial L}{\partial t} dt\ , \tag{2.56}$$

in the form

$$dL = \sum_{i=1}^{n} (\dot{p}_i dq_i + p_i d\dot{q}_i) + \frac{\partial L}{\partial t} dt\ , \tag{2.57}$$

which will be useful for the derivation of the canonical form of the equations of motion.

Consider now the case in which the Lagrangian does not depend on a given generalized coordinate q_j. In that case, the coordinate is called *cyclic* and (2.11) yields

$$\frac{\partial L}{\partial q_j} = 0 \;\Rightarrow\; \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0\,. \tag{2.58}$$

With (2.54), the r.h.s of (2.58) implies

$$\frac{dp_j}{dt} = 0\,, \tag{2.59}$$

which shows that in the case in which the generalized coordinate q_j is cyclic, the corresponding conjugate momentum p_j is a constant of the motion. Conversely, if the conjugate momentum p_j is a conserved quantity of the system, the Lagrangian L does not depend on the corresponding generalized coordinate q_j.

The Lagrangian formulation of mechanics here developed depends on the set of coordinates $(q_i, \dot{q}_i, t)$. It is possible, however, to build a different formulation aiming at describing the equations of motion in terms of first-order equations in function of the canonical coordinates (q_i, p_i, t). The new formulation can be built through a Legendre transform of $L(q_i, \dot{q}_i, t)$ that defines the function

$$H(q, p, t) = \sum_i \dot{q}_i p_i - L(q_i, \dot{q}_i, t)\,, \tag{2.60}$$

which takes the name of *Hamiltonian function*, or simply *Hamiltonian*. A definition and some of the mathematical properties of the Legendre transform are reported in Appendix C. One should notice the important difference in the dependent variables between the Hamiltonian function $H(q, p, t) = H(q_1, \ldots, q_n, p_1, \ldots, p_n, t)$ and the Lagrangian $L(q, \dot{q}, t) = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, q_n, t)$.

Notice that (2.60) yields, equivalently,

$$L(q_i, \dot{q}_i, t) = \sum_i \dot{q}_i p_i - H(q, p, t)\,, \tag{2.61}$$

which will sometimes be used.

2.7 Hamilton's Equations

The differential of (2.60) is

$$dH = \sum_i (\dot{q}_i dp_i + p_i d\dot{q}_i) - dL\ , \tag{2.62}$$

so that, using (2.57),

$$dH = \sum_i (\dot{q}_i dp_i - \dot{p}_i dq_i) - \frac{\partial L}{\partial t} dt\ . \tag{2.63}$$

Equation (2.63) can be compared with the differential of the Hamiltonian obtained from the chain rule

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt\ . \tag{2.64}$$

Direct comparison of (2.63) and (2.64) gives the *canonical equations*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\ , \tag{2.65a}$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i}\ , \tag{2.65b}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}\ , \tag{2.65c}$$

the first two of which, i.e.

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\ , \tag{2.66a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}\ , \tag{2.66b}$$

take the name of *Hamilton's equations*.

Remark 2.1 In comparison with the Euler–Lagrange equations (2.11), which leads to a set of n second-order equations, Hamilton's equations are $2n$ first-order ordinary differential equations for $q_i(t)$ and $p_i(t)$. The initial value problem can be solved specifying the initial conditions $q_i(0)$ and $p_i(0)$. For geometric reasons, one should notice that Hamiltonian dynamics take place in even dimensional spaces.

Hamilton's equations (2.66a), (2.66b) can be derived, in full analogy with the method of Sect. 2.2, from the Least Action Principle by finding an extremum of the functional

$$I(l) = \int_{t_1}^{t_2} \left[\sum_{i=1}^{n} \dot{q}_i p_i - H(q, p, t) \right] dt\ , \tag{2.67}$$

where

$$q_i = q_i(t,0) + l\eta_i(t),\;\; p_i = p_i(t,0) + l\varphi_i(t)\;, \tag{2.68}$$

and

$$\eta_i(t_1) = \eta_i(t_2) = \varphi_i(t_1) = \varphi_i(t_2) = 0\;. \tag{2.69}$$

In fact, according to (2.67),

$$\begin{aligned}
\delta I &= \frac{\partial I}{\partial l} dl \\
&= \int_{t_1}^{t_2} \left[\sum_{i=1}^{n} \left(\frac{\partial \dot{q}_i}{\partial l} p_i + \dot{q}_i \frac{\partial p_i}{\partial l} - \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial l} - \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial l} \right) dl \right] dt \\
&= \int_{t_1}^{t_2} \left[\sum_{i=1}^{n} \left(\dot{\eta}_i p_i + \dot{q}_i \varphi_i - \frac{\partial H}{\partial q_i}\eta_i - \frac{\partial H}{\partial p_i}\varphi_i \right) dl \right] dt \\
&= \int_{t_1}^{t_2} \left\{ \sum_{i=1}^{n} \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \varphi_i + \dot{\eta}_i p_i - \frac{\partial H}{\partial q_i}\eta_i \right] dl \right\} dt \\
&= \int_{t_1}^{t_2} \left\{ \sum_{i=1}^{n} \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \varphi_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right)\eta_i \right] dl \right\} dt + [p_i \eta_i]_{t_1}^{t_2}\, dl\;.
\end{aligned} \tag{2.70}$$

Because of the arbitrariness of functions η_i, φ_i and (2.69), from (2.70) one concludes that

$$\delta I = 0 \iff \left(\dot{q}_i = \frac{\partial H}{\partial p_i} \text{ and } \dot{p}_i = -\frac{\partial H}{\partial q_i} \right) \forall i = 1, \ldots, n\;. \tag{2.71}$$

Finally, from the last equality of (2.53) and the definition of Lagrangian

$$H(q, p, t) = \sum_{i=1}^{n} \dot{q}_i p_i - (T - V)\;, \tag{2.72}$$

so that, if the potential is a function only of the generalized coordinates, from (2.53),

$$H = T + V = E\;, \tag{2.73}$$

so that the Hamiltonian is the total energy of the system.

In matrix form, Hamilton's equations (2.66a), (2.66b) can be written as

$$\begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix}. \tag{2.74}$$

The matrix

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{2.75}$$

takes the name of *symplectic*, or sometimes *co-symplectic*, matrix. The term "symplectic" was introduced by the mathematician Hermann Weyl; it has origins from ancient Greek and it means "intertwined", as it clearly combines the variables q_i and p_i. It should be noted that $\mathbf{J}$ is antisymmetric that means that its transpose is its negative, i.e.

$$\mathbf{J}^T = -\mathbf{J} . \tag{2.76}$$

Further, it can be easily seen that the inverse of $\mathbf{J}$ is its transpose, so that

$$\mathbf{J}^T = \mathbf{J}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathbf{J} . \tag{2.77}$$

The properties

$$\mathbf{J}\mathbf{J}^T = \mathbf{J}^T\mathbf{J} = 1 , \tag{2.78a}$$

$$\mathbf{J}^2 = 1 , \tag{2.78b}$$

$$\det \mathbf{J} = 1 , \tag{2.78c}$$

follow directly from (2.77) and (2.76).

The relation (2.75) offers a trivial geometrical interpretation of Hamilton's equations in terms of the symplectic matrix. Consider in fact the rotation of coordinates in the plane, which can be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix} , \tag{2.79}$$

where $\mathbf{R}$ is the two-dimensional orthogonal rotation matrix, $\mathbf{R} \in SO(2)$, described in 1.116 and reported here again for convenience

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} . \tag{2.80}$$

It is visible thus that $\mathbf{J}$ is the matrix corresponding to a rotation through $\pi/2$ in the clockwise direction and, following (2.74), the flow in phase space equals the gradient of H rotated by the same angle. Further, one can see that a point in the phase space moves with the speed $(\dot{q}_i^2 + \dot{p}_i^2)^{1/2} = |\nabla H|$.

The previous discussion can be generalized observing that the canonical equations (2.66a), (2.66b) are not symmetric, due to the negative sign in the equation for $\dot{p}_i$ that is missing in the equation for $\dot{q}_i$. The two equations can be written in symmetric form introducing the symplectic notation defining the vector $\boldsymbol{z}$ so that

$$z_i = q_i \ , \tag{2.81a}$$

$$z_{i+n} = p_i \ , \tag{2.81b}$$

with $i = 1, \ldots, n$. In matrix form, $\boldsymbol{z}$ can be written as

$$\boldsymbol{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \\ z_{n+1} \\ \vdots \\ z_{2n} \end{pmatrix} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} . \tag{2.82}$$

In the same way, define the vectors

$$\left(\frac{\partial H}{\partial \boldsymbol{z}}\right)_i = \frac{\partial H}{\partial q_i} \ , \tag{2.83}$$

$$\left(\frac{\partial H}{\partial \boldsymbol{z}}\right)_{i+n} = \frac{\partial H}{\partial p_i} \ , \tag{2.84}$$

with $i = 1, \ldots, n$. Once again, in matrix form,

$$\frac{\partial H}{\partial \boldsymbol{z}} = \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_n} \\ \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_n} \end{pmatrix} . \tag{2.85}$$

The canonical equations (2.65c) can thus be rewritten using (2.82) and (2.85) as

$$\dot{\boldsymbol{z}} = \mathbf{J} \frac{\partial H}{\partial \boldsymbol{z}} \ , \tag{2.86}$$

where $\mathbf{J}$ is a $2n \times 2n$ squared matrix made by the four blocks composed by two $n \times n$ null matrices and two identity matrices $\mathbf{I}$ composed as

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} . \tag{2.87}$$

If $n = 2$, $\mathbf{J}$ reduces to (2.75). At the next order,

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} , \tag{2.88}$$

and so forth.

Remark 2.2 The symplectic form of Hamilton's equations allows for a geometric formulation of mechanics, which has many important features that are subject of current research. While the description of the dynamics on the symplectic manifold is an essential part of the study of classical dynamics and of mathematical physics, a description of the motion in local coordinates is to be preferred when explicit quantitative results are wanted. In this book, we will follow this second route. For a description of classical dynamics on the manifold, with attention also to infinite dimensional systems and to fluid dynamics, the reader is referred, for example, to [2, 9, 13].

2.8 Canonical Transformations and Generating Functions

In this section, we want to define what are the conditions to transform a set of canonical coordinates into a new set of canonical coordinates. To do so, consider the transformations of the kind

$$Q_i = Q_i(q, p, t) \ , \ i = 1, \ldots, n \ , \tag{2.89a}$$
$$P_i = P_i(q, p, t) \ , \ i = 1, \ldots, n \ , \tag{2.89b}$$

where $q = q_1, \ldots, q_n$ and $p = p_1, \ldots, p_n$ are canonical coordinates. The dependent variables Q_i and P_i are also canonical coordinates *provided that* there exists some function $K(Q, P, t)$ such that

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} \ , \ \dot{Q}_i = \frac{\partial K}{\partial P_i} \ . \tag{2.90}$$

The relationship between (q_i, p_i) and (Q_i, P_i) is based on the variational principles $\delta \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = 0$ and $\delta \int_{t_1}^{t_2} L(Q, \dot{Q}, t)dt = 0$ which, according to (2.61), take the form

$$\delta \int_{t_1}^{t_2} [\dot{q}_i p_i - H(q, p, t)]\, dt = 0 \ , \tag{2.91}$$

and

$$\delta \int_{t_1}^{t_2} \left[\dot{Q}_i P_i - H(Q, P, t)\right] dt = 0 \ , \tag{2.92}$$

respectively. We stress that if (Q_i, P_i) are canonical coordinates, then the simultaneous validity of (2.91) and (2.92) holds true. In turn, this request is realized if the integrands of (2.91) and (2.92) differ, at most, by the total derivative of an arbitrary function of both the old and the new canonical coordinates, say $F = F(q, p, Q, P, t)$. In fact, the difference between the integrals in (2.91) and in (2.92) is given by

$$\int_{t_1}^{t_2} \frac{dF}{dt} dt = F(q, p, Q, P, t_2) - F(q, p, Q, P, t_1) = 0 \, ,$$

as the phase space coordinates have zero variations at the end points. Apart from the time variable, the number of independent variables of $F = F(q, p, Q, P, t)$ is not $4n$; in fact, relationships (2.89a), (2.89b) reduce those independent to $2n$, so that only the following possibilities are allowed:

$$F = F_1(q, Q, t) \, , \; F = F_2(q, P, t) \, , \; F = F_3(p, Q, t) \, , \; F = F_4(p, P, t) \, , \tag{2.93}$$

where, for instance, $F_1(q, Q, t) = F_1(q_1, \ldots, q_n, Q_1, \ldots, Q_n, t)$, and so on. We now proceed to evaluate

$$\dot{q}_i p_i - H = \dot{Q}_i P_i - K + \frac{dF_1}{dt} \, , \tag{2.94}$$

keeping in mind that, in this particular case (i.e. $F = F_1$), q and Q are independent. Equation (2.94) yields

$$\dot{q}_i p_i - H = \dot{Q}_i P_i - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i \, , \tag{2.95}$$

and, owing to the independence between q and Q, we have

$$p_i = \frac{\partial F_1}{\partial q_i} \, , \tag{2.96a}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} \, , \tag{2.96b}$$

$$K = H + \frac{\partial F_1}{\partial t} \, . \tag{2.96c}$$

Equation (2.96a) can be solved to give

$$Q_i = Q_i(q, p, t) \, , \tag{2.97}$$

that is (2.89a). Once (2.97) is known, Eq. (2.96b) can be used to single out (2.89b). Moreover, Eq. (2.96c) connects the new and the old Hamiltonian functions. The method involving the generating functions F_2, F_3 and F_4 is analogous, with the following choices of F_2, F_3 and F_4 allowing to carry out the computations explicitly in the same way:

$$F_2(q, P, t) = F_1(q, Q, t) - P_i Q_i \ , \tag{2.98}$$

with $\partial F_2/\partial Q_i = 0$ because of (2.96b),

$$F_3(q, P, t) = F_1(q, Q, t) - p_i q_i \ , \tag{2.99}$$

with $\partial F_3/\partial q_i = 0$ because of (2.96a), and

$$F_4(q, P, t) = F_1(q, Q, t) + P_i Q_i - p_i q_i \ , \tag{2.100}$$

with $\partial F_4/\partial q = 0$ because of (2.96a) and $\partial F_4/\partial Q = 0$ because of (2.96b). Substitution of (2.98) into (2.94) yields

$$p_i = \frac{\partial F_2}{\partial q_i} \ , \tag{2.101a}$$

$$Q_i = \frac{\partial F_2}{\partial P_i} \ , \tag{2.101b}$$

$$K = H + \frac{\partial F_2}{\partial t} \ . \tag{2.101c}$$

Equation (2.101a) can be solved to give

$$P_i = P_i(q, p, t) \ , \tag{2.102}$$

that is to say (2.89b). Once (2.102) is known, Eq. (2.101b) can be used to single out (2.89a). The connection between the Hamiltonians is expressed by (2.101c). Substitution of (2.99) into (2.94) yields

$$q_i = -\frac{\partial F_3}{\partial p_i} \ , \tag{2.103a}$$

$$P_i = -\frac{\partial F_3}{\partial Q_i} \ , \tag{2.103b}$$

$$K = H + \frac{\partial F_3}{\partial t} \ . \tag{2.103c}$$

Equation (2.103a) can be solved to give

$$Q_i = Q_i(q, p, t) \ , \tag{2.104}$$

that is to say (2.89a). Once (2.104) is known, Eq. (2.103b) can be used to single out (2.89b). The connection between the Hamiltonians is here expressed by (2.103c). Substitution of (2.100) into (2.94) yields

$$q_i = -\frac{\partial F_4}{\partial p_i} \,, \tag{2.105a}$$

$$Q_i = \frac{\partial F_4}{\partial P_i} \,, \tag{2.105b}$$

$$K = H + \frac{\partial F_4}{\partial t} \,. \tag{2.105c}$$

Equation (2.105a) can be solved to give

$$P_i = P_i(q, p, t) \,, \tag{2.106}$$

that is to say (2.89b). Once (2.106) is known, Eq. (2.105b) can be used to single out (2.89a). The connection between the Hamiltonians is now expressed by (2.105c).

2.8.1 Phase Space Volume as Canonical Invariant: Liouville's Theorem and Poisson Brackets

The results from the previous section can be easily extended to the symplectic formalism. Consider a system described by a set of canonical coordinates under (2.86). Recalling the definition of covariant vectors from Sect. 2.4, in this section we will show that the transformation (2.89a), (2.89b) is covariant.

Defining

$$\zeta_i = Q_i(z_j) \,, \tag{2.107a}$$

$$\zeta_{i+n} = P_i(z_j) \,, \tag{2.107b}$$

$(i, j = 1, \ldots, n)$, if K is the Hamiltonian in the new coordinates, satisfying

$$H(\boldsymbol{z}) = K(\boldsymbol{\zeta}) \,, \tag{2.108}$$

the equation of motions must satisfy

$$\dot{\boldsymbol{\zeta}} = \mathbf{J}\frac{\partial K}{\partial \boldsymbol{\zeta}} \,. \tag{2.109}$$

At the same time,

$$\dot{\zeta}_i = \frac{\partial \zeta_i}{\partial z_j}\dot{z}_j, \;\; (i, j = 1, \ldots, n) \,, \tag{2.110}$$

which shows that $\boldsymbol{z}$ transforms covariantly. Equation (2.110) thus yields

$$\begin{aligned}\dot{\zeta}_i &= \frac{\partial \zeta_i}{\partial z_j} J_{jk} \frac{\partial H}{\partial z_k} \\ &= \frac{\partial \zeta_i}{\partial z_j} J_{jk} \frac{\partial K}{\partial \zeta_l} \frac{\partial \zeta_l}{\partial z_k} \\ &= \mathbf{M}_{il} \frac{\partial K}{\partial \zeta_l} \ ,\end{aligned} \tag{2.111}$$

$(i, j, k, l = 1, \ldots, n)$, where

$$\mathbf{M}_{il} = \frac{\partial \zeta_i}{\partial z_j} J_{jk} \frac{\partial \zeta_l}{\partial z_k} \ , \tag{2.112}$$

is the transformed symplectic operator. These last relationships show that **J** transforms as a *contravariant* (hence the name co-symplectic) tensor with rank 2. Comparison between (2.111), (2.112) and (2.110) states that the transformation of coordinates is canonical, i.e., it preserves the form of Hamilton's equations, if

$$\left(\frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{z}}\right) \mathbf{J} \left(\frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{z}}\right)^T = \mathbf{J} \ , \tag{2.113}$$

or, equivalently, if

$$\left(\frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{z}}\right)^T \mathbf{J} \left(\frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{z}}\right) = \mathbf{J} \ . \tag{2.114}$$

An important consequence of the these results is that the phase space defined by the canonical coordinates (q_i, p_i) has the property that the volume

$$V = \int dq_1 \ldots dq_n dp_1 \ldots dp_n \tag{2.115}$$

is a canonical invariant, i.e., it is invariant under a canonical transformation of coordinates. To demonstrate this property, indicate the infinitesimal volume in the original set of coordinates with

$$dq_1 \ldots dq_n dp_1 \ldots dp_n \ , \tag{2.116}$$

and with

$$dQ_1 \ldots dQ_n dP_1 \ldots dP_n \ , \tag{2.117}$$

the volume in a set of transformed coordinates (2.89a), (2.89b), under a canonical transformation. As well known from multivariate calculus, the two infinitesimal volumes are connected by the determinant of the Jacobian matrix, i.e., the determinant of a symplectic matrix that is equal to 1. This is an important result that states that in conservative systems, element of volumes in the phase space is conserved. The consequences of this result are wide and have applications in the different fields of mathematics and physics, ranging from chaotic dynamics to statistical mechanics. One of them that has consequences that have parallels with fluid dynamics is given by Liouville's theorem. If $\rho(q_1,\ldots,q_n,p_1,\ldots,p_n)dq_1\ldots dq_n dp_1\ldots dp_n$ is the probability that a trajectory of the system in phase space is in the infinitesimal volume $dq_1\ldots dq_n dp_1\ldots dp_n$, where $\rho(q_1,\ldots,q_n,p_1,\ldots,p_n)$ is the probability density function, the invariance of the volume yields

$$\begin{aligned}\frac{d\rho}{dt} &= \frac{\partial\rho}{\partial t}+\sum_{i=1}^{n}\left(\frac{\partial\rho}{\partial q_i}\dot{q}_i+\frac{\partial\rho}{\partial p_i}\dot{p}_i\right)\\ &= \frac{\partial\rho}{\partial t}+\sum_{i=1}^{n}\left(\frac{\partial\rho}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial\rho}{\partial p_i}\frac{\partial H}{\partial q_i}\right)\\ &= 0\,. \end{aligned}\tag{2.118}$$

Equation (2.118) has the same form of fluid dynamics equation for the conservation of density (1.6), or, in general, for the evolution of a passive tracer stirred by a flow characterized by a stream function that in (2.118) corresponds to the Hamiltonian.

In (2.118), the object

$$\{\rho,H\}=\sum_{i=1}^{n}\left(\frac{\partial\rho}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial\rho}{\partial p_i}\frac{\partial H}{\partial q_i}\right)\tag{2.119}$$

is called a *canonical Poisson bracket*. For a system in equilibrium, $\partial\rho/\partial t=0$, and (2.118) yields $\{\rho,H\}=0$, which is satisfied if $\rho=\rho(H)$, i.e. if ρ is a function of the energy of the system.

In general, given two functions $F(q_1,\ldots,q_n,p_1,\ldots,p_n)$ and $G(q_1,\ldots,q_n,p_1,\ldots,p_n)$, a canonical Poisson bracket is defined as

$$\{F,G\}=\sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)\,.\tag{2.120}$$

By the chain rule of differentiation, the time evolution of one of the functions, say F, can be written as

$$\frac{dF}{dt}=\sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{dq_i}{dt}+\frac{\partial F}{\partial p_i}\frac{dp_i}{dt}\right)\,.\tag{2.121}$$

Using Hamilton's equations (2.66a), (2.66b), Eq. (2.121) gives

$$\frac{dF}{dt}=\sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right)\,,\tag{2.122}$$

so that, using (2.120), (2.122) yields

$$\frac{dF}{dt} = \{F, H\} \tag{2.123}$$

that shows that the evolution of a generic function can be determined from the Poisson bracket of the function with the Hamiltonian of the system. Notice that (2.123) is in agreement with the definition of the canonical equations (2.66a), (2.66b),

$$\frac{dq_i}{dt} = \{q_i, H\} = \frac{\partial H}{\partial p_i}\ , \tag{2.124}$$

$$\frac{dp_i}{dt} = \{p_i, H\} = -\frac{\partial H}{\partial q_i}\ . \tag{2.125}$$

Relation (2.123) shows also that

$$\{F, H\} = 0 \iff \frac{dF}{dt} = 0\ , \tag{2.126}$$

i.e. if a function F commutes with the Hamiltonian H, it is an invariant of the motion of the system. In particular, the invariance of the Hamiltonian is trivially proved, as

$$\{H, H\} = 0\ , \tag{2.127}$$

in agreement with the energy conservation of the system. The Poisson bracket is an important geometric object in mechanics, and it is worth showing its algebraic properties:

Theorem 2.2 *Given the functions f, g, h, the Poisson bracket satisfies the following properties (the proofs are reported in Appendix I):*

1. *Self-commutation*

$$\{f, f\} = 0\ . \tag{2.128}$$

2. *Skew-symmetry*

$$\{f, g\} = -\{g, f\}\ . \tag{2.129}$$

3. *Distributive property*

$$\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}\ . \tag{2.130}$$

   where $\alpha, \beta \in \mathbb{R}$.

4. *Associative property*

$$\{fg, h\} = f\{g, h\} + \{f, h\}h\ . \tag{2.131}$$

5. *The Jacobi identity*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \,. \tag{2.132}$$

These properties define a nonassociative Lie algebra.

In a similar way as was done at the beginning of this section, it is possible to express the Poisson bracket in the new set of coordinates. Given two functions $f(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and $g(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and indicating with $F(Q_1, \ldots, Q_n, P_1, \ldots, P_n)$ and $G(Q_1, \ldots, Q_n, P_1, \ldots, P_n)$ the corresponding functions in the transformed coordinates, the Poisson bracket is

$$\begin{aligned}\{f, g\} &= \left(\frac{\partial f}{\partial \boldsymbol{z}}\right)^T \mathbf{J} \left(\frac{\partial g}{\partial \boldsymbol{z}}\right) \\ &= \left(\frac{\partial f}{\partial \boldsymbol{\zeta}} \frac{\partial \boldsymbol{\zeta}}{\partial \boldsymbol{z}}\right)^T \mathbf{J} \left(\frac{\partial g}{\partial \boldsymbol{\zeta}} \frac{\partial \boldsymbol{\zeta}}{\partial \boldsymbol{z}}\right) \\ &= \left(\frac{\partial f}{\partial \boldsymbol{\zeta}}\right)^T \mathbf{J} \left(\frac{\partial g}{\partial \boldsymbol{\zeta}}\right) ,\end{aligned} \tag{2.133}$$

where, in the last passage, we have used property (2.114). The relation (2.133) shows that the Poisson bracket is a canonical invariant of the system.

2.8.2 Casimir Invariants and Invertible Systems

A special case of (2.126) is given by a function C that commutes with every other functions F,

$$\{F, C\} = 0, \ \forall F \,. \tag{2.134}$$

In this case, the function C takes the name of *Casimir invariant*, as (2.123) with $F = H$ trivially implies

$$\{C, H\} = 0, \ \Rightarrow \ \frac{dC}{dt} = 0 \,. \tag{2.135}$$

Writing (2.134) in symplectic form and considering that F is an arbitrary function yield

$$\mathbf{J} \frac{\partial C}{\partial \boldsymbol{z}} = \mathbf{0} \,. \tag{2.136}$$

Relation (2.136) implies that $\partial C/\partial \boldsymbol{z}$ belongs to the kernel of $\mathbf{J}$ and has important consequences linked to the invertibility of the dynamics of the system. In fact, if $\mathbf{J}^{-1}$ exists, i.e., the Hamiltonian formulation is invertible, (2.136) implies

$$\frac{\partial C}{\partial \boldsymbol{z}} = \mathbf{0}, \ \Rightarrow \ C = \text{const} \,. \tag{2.137}$$

In this case, the Casmir function is said to be *trivial*. If instead $\mathbf{J}^{-1}$ does not exists, i.e. the Hamiltonian formulation is not invertible, there exists at least one Casimir that is nontrivial. In this case, the dynamics is invariant under the summation of a Casmir to the Hamiltonian; in fact,

$$\mathbf{J}\frac{\partial(H+C)}{\partial \mathbf{z}} = \mathbf{J}\frac{\partial H}{\partial \mathbf{z}} + \mathbf{J}\frac{\partial C}{\partial \mathbf{z}} = \mathbf{J}\frac{\partial H}{\partial \mathbf{z}} = \dot{\mathbf{z}} \,. \tag{2.138}$$

The geometric interpretation of (2.138) relies in the idea that the motion takes place on hypersurfaces where $\partial C/\partial \mathbf{z} = \mathbf{0}$, which are also called *symplectic leaves*. Each of these surfaces is a regular Hamiltonian phase space where the dynamics are regulated by H.

Remark 2.3 As remarked at the end of Sect. 2.7, in the remaining of this book we will not make use of the Poisson bracket. It is, however, difficult to overestimate the importance that this object covers in modern mechanics, making it thus deserving of this very short introduction.

2.9 Noether's Theorem for Point Particles

In this section, we will study the important link between continuous symmetries and conserved quantities, which is explicated by Noether's Theorem, using the symmetries of the Lagrangian operator. The theorem will be first exposed in its formulation for point particles and will then be extended to continuous systems, which include the case of fluids.

Consider a system of point particles, Noether's Theorem thus states that

Theorem 2.3 *If the Lagrangian function is invariant under a continuous and infinitesimal transformation of its spatial and temporal variables, the transformation defines a scalar quantity that is a constant of motion of the system.*

The proof of the theorem will be divided into a mathematical preliminary and it will then be linked to the physics of the system.

2.9.1 Mathematical Preliminary

We start by defining the functional

$$I = \int_{t_1}^{t_2} L\left(q_i, \dot{q}_i, t\right) dt \,, \tag{2.139}$$

and the transformations

$$t' = t + \delta t\ , \tag{2.140a}$$

$$q_i'(t') = q_i(t) + \delta q_i(t)\ , \tag{2.140b}$$

$$\dot{q}_i'(t') = \dot{q}_i(t) + \delta \dot{q}_i(t)\ . \tag{2.140c}$$

The quantities δt, $\delta q_i(t)$, $\delta \dot{q}_i(t)$ are arbitrary differentiable functions of time whose higher order amplitudes are negligible. Notice that these arbitrary differentiable functions can eventually be a constant. In the examples that will follow the proof, different forms of $\delta t,\ \delta q_i(t)$ and $\delta \dot{q}_i(t)$ will be introduced.

The transformations in (2.140a)–(2.140c) generate, through (2.139), a functional variation δI defined as

$$\delta I = \int_{R'} L\left(t', q_i', \dot{q}_i'\right) dt' - \int_R L\left(t, q_i, \dot{q}_i\right) dt\ , \tag{2.141}$$

where $R = [t_1, t_2]\,,\ R' = \left[t_1', t_2'\right]$ and $\int_R dt = \int_{R'} dt'$. Equation (2.141) shows that the functional variation δI is the difference between (2.139) calculated after and before the transformations (2.140a)–(2.140c). As it is shown in Appendix D, Eq. (2.141) yields, at first order, the functional variation in the form

$$\delta I = \int_R \left\{ \frac{D}{Dt}\left[\left(L - \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\delta t + \frac{\partial L}{\partial \dot{q}_i}\delta q_i\right] + \left[\frac{\partial L}{\partial q_i} - \frac{D}{Dt}\frac{\partial L}{\partial \dot{q}_i}\right](\delta q_i - \dot{q}_i \delta t)\right\} dt\ , \tag{2.142}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \dot{q}_i \frac{\partial}{\partial q_i} + \ddot{q}_i \frac{\partial}{\partial \dot{q}_i}\ . \tag{2.143}$$

If the integrand in (2.139) is the Lagrangian $L = T - V$, and using the initial and final conditions $\delta t(t_1) = \delta t(t_2) = 0\ ,\ \ \delta q_i(t_1) = \delta q_i(t_2) = 0$, due to Hamilton's principle, the equations of motion (see Sect. 2.2)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{2.144}$$

hold. In this (physically relevant) case, (2.142) yields

$$\delta I = \int_R \frac{D}{Dt}\left[\left(L - \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\delta t + \frac{\partial L}{\partial \dot{q}_i}\delta q_i\right] dt\ . \tag{2.145}$$

2.9.2 Symmetry Transformations and Proof of the Theorem

Among the transformations (2.140a)–(2.140c), of particular interest are the transformations denoted as

$$t' = t + \delta_S t \ , \tag{2.146a}$$

$$q_i'(t') = q_i(t) + \delta_S q_i(t) \ , \tag{2.146b}$$

$$\dot{q}_i'(t') = \dot{q}_i(t) + \delta_S \dot{q}_i(t) \ , \tag{2.146c}$$

for which the Lagrangian function transforms under two conditions, namely

$$L'\left(t', q_i', \dot{q}_i'\right) dt' = L\left(t, q_i, \dot{q}_i\right) dt \ , \tag{2.147a}$$

$$L'\left(t', q_i', \dot{q}_i'\right) = L\left(t', q_i', \dot{q}_i'\right) + \frac{D}{Dt'}\left(\delta_S \Omega\right) \ . \tag{2.147b}$$

Equation (2.147a) is valid because the functional (2.139) is a scalar and, under the action of the transformations (2.140a)–(2.140c) (and (2.146a)–(2.146c) as a special case) must transform as a scalar. Equation (2.147b), instead, expresses the covariance of Lagrangian function specifically under the *symmetry transformations* (2.146a)–(2.146c). This means that the functional dependence of the Lagrangian on the space–time coordinates remains unaltered under transformations (2.146a)–(2.146c), apart from the term $D/Dt' \left(\delta_S \Omega\right)$. The presence of this additional term can be explained observing that the equations of motion (2.144) are invariant under the divergence transformation

$$L \to L + \frac{D}{Dt}\left[\delta_S \Omega(t, q_i)\right] \ , \tag{2.148}$$

where $\partial \Omega / \partial \dot{q}_i = 0$, as demonstrated in Appendix E. The strategy of the proof will now be to substitute the transformations (2.146a)–(2.146c) in (2.147a) and (2.147b) and to compare the results in order to find what kind of constraints (2.146a)–(2.146c) pose on the equations of motion (2.144).

Substitution of (2.146a)–(2.146c) in (2.147a) yields

$$L'\left(t + \delta_S t, q_i + \delta_S q_i, \dot{q}_i + \delta_S \dot{q}_i\right) dt' = L\left(t, q_i, \dot{q}_i\right) dt \ , \tag{2.149}$$

while (2.147b) yields

$$\begin{aligned} & L'\left(t + \delta_S t, q_i + \delta_S q_i, \dot{q}_i + \delta_S \dot{q}_i\right) dt' \\ & = L\left(t + \delta_S t, q_i + \delta_S q_i, \dot{q}_i + \delta_S \dot{q}_i\right) dt' + \frac{D\left(\delta_S \Omega\right)}{Dt} dt' \ . \end{aligned} \tag{2.150}$$

Equating (2.149) and (2.150) yields

$$L\left(t + \delta_S t, q_i + \delta_S q_i, \dot{q}_i + \delta_S \dot{q}_i\right) dt' - L\left(t, q_i, \dot{q}_i\right) dt + \frac{D\left(\delta_S \Omega\right)}{Dt} dt' = 0 \ . \tag{2.151}$$

Because

$$dt' = dt\left[1 + \frac{d}{dt}\left(\delta_S t\right)\right] \ , \tag{2.152}$$

and because $\delta_S \ll 1$, ignoring terms of higher order, one gets

$$\frac{D}{Dt'}\left(\delta_S \Omega\right) dt' = \frac{D}{Dt}\left(\delta_S \Omega\right) dt \ . \tag{2.153}$$

Starting from (2.153), we thus pose two goals:

- To derive a test function that allows to prove the invariance of the Lagrangian function upon a continue and infinitesimal symmetry transformation in the class represented by (2.146a)–(2.146c);
- To obtain the correspondent conserved quantity.

In reference to the first goal, Eq. (2.151) can be rewritten using (2.152) and (2.153), so that

$$L\left(t + \delta_S t, q_i + \delta_S q_i, \dot{q}_i + \delta_S \dot{q}_i\right)\left[1 + \frac{d}{dt}\left(\delta_S t\right)\right] - L\left(t, q_i, \dot{q}_i\right) dt = -\frac{D}{Dt}\left(\delta_S \Omega\right) \ . \tag{2.154}$$

At first order, (2.154) yields

$$\left[\delta_S t \frac{\partial}{\partial t} + \delta_S q_i \frac{\partial}{\partial q_i} + \delta_S \dot{q}_i \frac{\partial}{\partial \dot{q}_i} + \frac{d\left(\delta_S t\right)}{dt}\right] L\left(t, q_i, \dot{q}_i\right) = -\frac{D}{Dt}\left(\delta_S \Omega\right) \ . \tag{2.155}$$

The test consists thus in applying the operator between square brackets on the l.h.s. of (2.155) to the assigned Lagrangian, in order to verify if it produces the term on the r.h.s.. Notice that the function $\delta_S \Omega\left(t, q_i\right)$ is arbitrary and may be null.

In reference to the second goal, consider the time integration of (2.151), i.e.

$$\int_{R'} L\left(t + \delta_S t, q_i + \delta_S q_i, \dot{q}_i + \delta_S \dot{q}_i\right) dt' - \int_R L\left(t, q_i, \dot{q}_i\right) dt + \int_R \frac{D}{Dt}\left(\delta_S \Omega\right) dt = 0 \, . \tag{2.156}$$

The first two terms in (2.156) are the functional variation (2.141) that, because of Hamilton's principle, must have the form (2.145), yielding

$$\int_R \frac{D}{Dt}\left[\left(L - \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\delta_S t + \frac{\partial L}{\partial \dot{q}_i}\delta_S q_i + \delta_S \Omega\right] dt = 0 \, . \tag{2.157}$$

Because (2.157) is independent on the integration interval, from the same equation it is possible to derive the conservation law

$$\frac{D}{Dt}\left[\left(L - \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\delta_S t + \frac{\partial L}{\partial \dot{q}_i}\delta_S q_i + \delta_S \Omega\right] = 0 \, , \tag{2.158}$$

or

$$\left(L - \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\delta_S t + \frac{\partial L}{\partial \dot{q}_i}\delta_S q_i + \delta_S \Omega = const \ . \tag{2.159}$$

To summarize: every continuous and differential transformation (2.146a)–(2.146c) that turns (2.155) into an identity produces a conserved quantity in the form of (2.159).

2.9.3 Some Examples

As a preamble notice that in classical mechanics for point particles, the Lagrangian has usually the shape

$$L = \sum_i \frac{m_i}{2}\left[(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2\right] - V\left(\sum_{i\neq j} r_{ij}\right) , \tag{2.160}$$

where, using Cartesian coordinates, $r_{ij} = \left[\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2 + \left(z_i - z_j\right)^2\right]^{1/2}$.

2.9.3.1 Invariance for Translations of Amplitude l Along the x-axis

In terms of equation (2.146a)–(2.146c), this symmetry transformation is

$$t' = t , \tag{2.161a}$$

$$x_i' = x_i + l, \quad y_i' = y_i, \quad z_i' = z_i , \tag{2.161b}$$

$$\dot{x}_i' = \dot{x}_i, \quad \dot{y}_i' = \dot{y}_i, \quad \dot{z}_i' = \dot{z}_i , \tag{2.161c}$$

where $\delta_S q_i = l$ if $q_i = x_i$ and $\delta_S q_i = 0$ if $q_i \neq x_i$. It is visible that, because $x_i - x_j = (x_i + l) - (x_j + l)$, the distance r_{ij} between two particles does not change for translations, so that the Lagrangian (2.160) does not change as well. It is, however, possible to proceed formally, recurring to (2.155), that for this case yields

$$\sum_k \frac{\partial L}{\partial x_k} = -\frac{D\Omega}{Dt} . \tag{2.162}$$

The sum in (2.162) is applied to the space derivatives with respect to x for all the particles of the system, so that

$$\sum_k \frac{\partial L}{\partial x_k} = -\sum_i \frac{\partial V}{\partial x_i} - \sum_j \frac{\partial V}{\partial x_j} = -\sum_{i\neq j}\left(\sum_i \frac{\partial V}{\partial x_i} + \sum_j \frac{\partial V}{\partial x_j}\right) = 0 . \tag{2.163}$$

Equation (2.163) shows that (2.155) is satisfied for $\Omega = const$. According to (2.159), the corresponding conserved quantity is

$$\sum_k \frac{\partial L}{\partial \dot{x}_k} = \sum_k m_k \dot{x}_k \ . \tag{2.164}$$

The term on the r.h.s. of (2.164), i.e. the conserved quantity, is the total (i.e. of the entire system) linear momentum along the x-axis.

2.9.3.2 Invariance for Time Translations of Amplitude τ

In terms of equation (2.146a)–(2.146c), the symmetry transformation is

$$t' = t + \tau \ , \tag{2.165a}$$
$$x_i' = x_i, \quad y_i' = y_i, \quad z_i' = z_i \ , \tag{2.165b}$$
$$\dot{x}_i' = \dot{x}_i, \quad \dot{y}_i' = \dot{y}_i, \quad \dot{z}_i' = \dot{z}_i \ , \tag{2.165c}$$

where $\delta_S t = \tau$. Equation (2.160) shows that the Lagrangian does not depend explicitly on time and thus (2.155) follows immediately from the invariance $\partial L/\partial t = 0$ with $\Omega = const$. As a consequence, (2.159) yields

$$L - m_i (\dot{x}_i)^2 = const \ . \tag{2.166}$$

Because $m_i (\dot{x}_i)^2 = 2T$, (2.166) is the conservation of the total (kinetic plus potential) energy of the system,

$$T + V = const \ . \tag{2.167}$$

2.9.3.3 Invariance for Rotations of Amplitude α Around the z-axis

In terms of equation (2.146a)–(2.146c), the transformation is

$$t' = t \ , \tag{2.168a}$$
$$x_i' = x_i + \alpha y_i, \quad y_i' = y_i - \alpha x_i, \quad z_i' = z_i \ , \tag{2.168b}$$
$$\dot{x}_i' = \dot{x}_i + \alpha \dot{y}_i, \quad \dot{y}_i' = \dot{y}_i - \alpha \dot{x}_i, \quad \dot{z}_i' = \dot{z}_i \ , \tag{2.168c}$$

so that

$$\delta_S q_i = \alpha y_i \quad \text{if} \quad q_i = x_i \ , \tag{2.169a}$$
$$\delta_S q_i = -\alpha x_i \quad \text{if} \quad q_i = y_i \ , \tag{2.169b}$$
$$\delta_S q_i = 0 \quad \text{if} \quad q_i = z_i \ . \tag{2.169c}$$

It should be noted that in this case the quantities $\delta_S q_i$ are not constants, but they are the product of an infinitesimal parameter α by a coordinate that is a function of time. The same considerations are valid for the $\delta_S \dot{q}_i$. Notice also that the infinitesimal parameter α derives from the finite rotation

$$x' = \cos\alpha x + \sin\alpha y \ , \tag{2.170a}$$

$$y' = -\sin\alpha x + \cos\alpha y \ , \tag{2.170b}$$

after the truncation at first order of the expansion in α of the trigonometrical functions, following the well-known approximations $\cos\alpha \approx 1$ and $\sin\alpha \approx \alpha$. Using (2.169a)–(2.169c), Eq. (2.155) yields

$$\left[\alpha\left(y_i\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial y_i}\right) + \alpha\left(\dot{y}_i\frac{\partial}{\partial \dot{x}_i} - \dot{x}_i\frac{\partial}{\partial \dot{y}_i}\right)\right](T - V) = -\frac{D}{Dt}\left(\delta_S \Omega\right) \ . \tag{2.171}$$

Because

$$\left(\dot{y}_i\frac{\partial}{\partial \dot{x}_i} - \dot{x}_i\frac{\partial}{\partial \dot{y}_i}\right)T = m_i\left(\dot{y}_i\dot{x}_i - \dot{x}_i\dot{y}_i\right) = 0 \ ,$$

and

$$\left(y_i\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial y_i}\right)V = 0 \ ,$$

as it is verified from the identity

$$\frac{\partial}{\partial x_i}\left(x_m - x_n\right) = \frac{\partial}{\partial y_i}\left(y_m - y_n\right) = \delta_{im} - \delta_{in} = 0 \ , \tag{2.172}$$

where δ_{kl} is a Kronecker delta, then (2.171) is immediately verified for $\Omega = const$, for all values of α. As a consequence, (2.159) takes the form

$$\delta_S q_i \frac{\partial L}{\partial q_i} = const \ , \tag{2.173}$$

or, using (2.169a)–(2.169c),

$$\frac{\partial}{\partial x_i}\left(x_m - x_n\right) = \frac{\partial}{\partial y_i}\left(y_m - y_n\right) = \delta_{im} - \delta_{in} = 0 \ , \tag{2.174}$$

that is

$$m_i\left(y_i\dot{x}_i - x_i\dot{y}_i\right) = const \ . \tag{2.175}$$

Equation (2.175) expresses the conservation of the total angular momentum about the z-axis.

2.9.3.4 Invariance for a Translation with Constant Velocity Along the x-axis

In terms of (2.146a)–(2.146c), the transformation is

$$t' = t \ , \tag{2.176a}$$

$$x_i' = x_i - t\delta v, \quad y_i' = y_i, \quad z_i' = z_i \ , \tag{2.176b}$$

$$\dot{x}_i' = \dot{x}_i - \delta v, \quad \dot{y}_i' = \dot{y}_i, \quad \dot{z}_i' = \dot{z}_i \ , \tag{2.176c}$$

so that $\delta_S q_i = -t\delta v$ if $q_i = x_i$ else $\delta_S q_i = 0$; $\delta_S \dot{q}_i = -\delta v$ if $q_i = x_i$ else $\delta_S \dot{q}_i = 0$. With this form for the transformation, (2.155) yields

$$\left[\sum_i \left(-t\delta v \frac{\partial}{\partial x_i} - \delta v \frac{\partial}{\partial \dot{x}_i}\right)\right] L = -\frac{D}{Dt}\left(\delta_S \Omega\right) \ . \tag{2.177}$$

Because $\sum_i \partial V/\partial x_i = 0$, (2.177) simplifies to

$$\sum_i \frac{\partial T}{\partial \dot{x}_i} = \frac{D\Omega}{Dt} \ , \tag{2.178}$$

that is

$$\sum_i m_i x_i = \frac{D\Omega}{Dt} \ . \tag{2.179}$$

Thus, if

$$\Omega = \sum_i m_i x_i \ , \tag{2.180}$$

Equation (2.177) becomes an identity. Given (2.180), Eq. (2.159) determines the conserved quantity

$$\sum_i \left(-t\frac{\partial T}{\partial \dot{x}_i} + m_i x_i\right) = const \ , \tag{2.181}$$

that is

$$\sum_i m_i x_i - t\sum_i m_i \dot{x}_i = const \ . \tag{2.182}$$

If

$$x_c = \frac{\sum_i m_i x_i}{\sum_i m_i} \ , \tag{2.183}$$

is the coordinate of the centre of mass of the system about the x-axis, and using (2.164), then (2.182) describes the motion of the centre of mass, that, as it is well known, is uniform in the absence of forces that are external to the system.

2.9.3.5 Damped Oscillator

The following example is drawn from [15] and reconsidered here in full details. It is noticeable because, although the single-particle system taken into account does not conserve neither energy nor momentum, Noether's Theorem allows to derive a conservation law which cannot emerge from the Euler–Lagrange equation alone. The starting point is the following Lagrangian governing a damped oscillator

$$L = \frac{1}{2}\left[m\dot{x}^2 - kx^2\right]\exp\left(\frac{bt}{m}\right) . \tag{2.184}$$

Hence, the evolution is given by the Euler–Lagrange equation (2.11) which yields the ODE

$$m\ddot{x} + b\dot{x} + kx = 0 . \tag{2.185}$$

In Eq. (2.185), $b\dot{x}$ ($b > 0$) is the damping term while kx ($k > 0$) is the restoring force. The first of them obviously prevents energy conservation; in fact, (2.185) implies

$$m\dot{x}\ddot{x} + b\dot{x}\dot{x} + kx = 0 .$$

that is to say, after time integration on (t_1, t_2)

$$E(t_2) - E(t_1) + b\int_{t_1}^{t_2}(\dot{x})^2 dt = 0 , \tag{2.186}$$

where $E = (m/2)(\dot{x})^2 + (k/2)x^2$ is total energy of the oscillating particle. Equation (2.186) openly shows that kinetic energy is not conserved because of the energy sink $-b\int_{t_1}^{t_2}(\dot{x})^2 dt$. The restoring force prevents linear momentum conservation. In fact, time integration of (2.185) with $x(t_1) = x(t_2) = 0$ immediately gives

$$p(t_2) - p(t_1) + k\int_{t_1}^{t_2} x dt = 0 , \tag{2.187}$$

where $p = m\dot{x}$ is the linear momentum of the particle. According to Eq. (2.187), the linear momentum is not conserved because of the sink $-k\int_{t_1}^{t_2} x dt$. After these preliminaries, consider again (2.184). The Lagrangian conserves its form under the space–time infinitesimal transformation

$$t' = t + \varepsilon , \tag{2.188a}$$

$$x' = x - \varepsilon\left(\frac{bx}{2m}\right) , \tag{2.188b}$$

where positive powers of ε are hereafter neglected with respect to the first one. Thus, (2.188a), (2.188b) can be reverted to give

$$t = t' - \varepsilon \; , \tag{2.189a}$$

$$x = x' + \varepsilon \left(\frac{bx'}{2m} \right) \; , \tag{2.189b}$$

Substitution of (2.189a), (2.189b) into (2.184) proves invariance. In fact,

$$\begin{aligned} L &= \frac{1}{2}\left[m\dot{x}'^2 - kx'^2\right]\left(1+\varepsilon\frac{b}{2m}\right)^2 \exp\left(\frac{b}{m}(t'-\varepsilon)\right) \\ &= \frac{1}{2}\left[m\dot{x}'^2 - kx'^2\right]\left(1+\varepsilon\frac{b}{m}\right)\exp\left(\frac{bt'}{m}\right)\exp\left(-\varepsilon\frac{b}{m}\right) \\ &= \frac{1}{2}\left[m\dot{x}'^2 - kx'^2\right]\left(1+\varepsilon\frac{b}{m}\right)\left(1-\varepsilon\frac{b}{m}\right)\exp\left(\frac{bt'}{m}\right) \\ &= \frac{1}{2}\left[m\dot{x}'^2 - kx'^2\right]\exp\left(\frac{bt'}{m}\right) \; , \end{aligned}$$

that is to say

$$L(t, x, \dot{x}) = L(t', x', \dot{x}') \; . \tag{2.190}$$

Invariance (2.190) leads to a conserved quantity according to Noether's Theorem. Transformations (2.188a), (2.188b) allow us to identify

$$\delta_s t = \varepsilon \; , \;\; \delta_s q = \delta_s x = \varepsilon \left(\frac{b}{2m} \right) x \; , \tag{2.191}$$

so the conserved quantity of the general theory (2.159) takes here the form

$$L - \frac{\partial L}{\partial \dot{x}}\dot{x} - \left(\frac{b}{2m} x \right) \frac{\partial L}{\partial \dot{x}} = const \; . \tag{2.192}$$

Finally, substitution of (2.184) into (2.192) produces the explicit version of the conserved quantity, that is to say

$$\left[m(\dot{x})^2 + kx^2 + bx\dot{x}\right]\exp\left(\frac{bt}{m}\right) = const \; . \tag{2.193}$$

In turn, the constant at the r.h.s. of (2.193) can be singled out by substituting the general integral of the ODE (2.185) into the l.h.s of the same equation. The exponential damping of the oscillator compensates exactly the exponential factor at the l.h.s. of (2.193).

2.10 Lagrangian Formulation for Fields: Lagrangian Depending on a Scalar Function

The Lagrangian formulation for point particles reviewed in the previous sections can be easily expanded for systems with an infinite number of degrees of freedom, i.e. for continuous systems such as fluids. In this section, the formulation of Lagrangian dynamics will be specifically introduced making use of a scalar function, i.e. a stream function, ψ. Later in the chapter, the formulation will be presented for Lagrangian functions depending on vector functions.

In a continuous system, the independent variables are the time t and the space coordinates $x,\ y,\ z$, while the dependent variables that will be considered here are the current function ψ and its first derivatives, e.g. $\partial\psi/\partial t$, etc. The notation

$$(t, x, y, z) = (q_0, q_1, q_2, q_3) \tag{2.194}$$

and its abbreviation

$$(q_0, q_1, q_2, q_3) = q \tag{2.195}$$

will be used when convenient; for example, we will use the notation $\int_R d(q) = \int_R \prod_{k=0}^{3} dq_k$, where $R \subset \mathbb{R}^4$ is the integration domain. Using this notation yields

$$\psi\ (t, x, y, z) = \psi\ (q_0, q_1, q_2, q_3) = \psi\ (q)\ , \tag{2.196}$$

while the partial derivatives are expressed with the index that identifies the variable with respect to which the derivative is taken, e.g.

$$\frac{\partial\psi}{\partial q_k} = \psi_k\ ,\ \frac{\partial^2\psi}{\partial q_k \partial q_l} = \psi_{kl}\ . \tag{2.197}$$

Using this notation, the Lagrangian L is a function of all the independent and dependent variables and it will be indicated as

$$L\ (q_k, \psi, \psi_i)\ , \tag{2.198}$$

where $k = 0, 1, 2, 3$ and $i = 0, 1, 2, 3$.

The symbol D/Dq_k generalizes the Lagrangian derivative with time as the only independent variable, and it is defined as

$$\frac{D}{Dq_k} = \frac{\partial}{\partial q_k} + \psi_k \frac{\partial}{\partial\psi} + \psi_{kl} \frac{\partial}{\partial\psi_l}\ . \tag{2.199}$$

An important observation comes from looking at the functional

$$I[\psi] = \int_R L(q_k, \psi, \psi_i)\, d(q)\ . \tag{2.200}$$

From (2.200), it is visible that the Lagrangian $L(q_k, \psi, \psi_i)$ does not have the same dimensions as the Lagrangian function used in the discrete formulation. The Lagrangian function used in continuous systems takes also the name of *Lagrangian density*. In fluid dynamics, the Lagrangian density is analogous to the Lagrangian function defined for point particles, i.e., it is defined as $L = T - V$ where now T and V represent the kinetic and potential energy densities. As it will be seen in the next chapter, in fluid dynamics the kinetic energy density is expressed as the kinetic energy of the parcels that fill up the continuum that constitutes the fluid. The potential energy density is instead given by the contribution of the internal energy, linked to the thermodynamics property of the fluid, and the external potential.

Analogously to the discrete case, the request

$$\delta I = 0 \tag{2.201}$$

allows to derive the equations of motion for the system. To do so, start by labelling the functions $\psi(q),\ \psi_i(q)$ by the label l

$$\psi(q, l) = \psi(q, 0) + l\phi(q)\ , \tag{2.202a}$$

$$\psi_i(q, l) = \psi_i(q, 0) + l\phi_i(q)\ , \tag{2.202b}$$

where $\phi(q),\ \phi_i(q) \in C^2_R$ are arbitrary functions satisfying $\phi = 0,\ \phi_i = 0$ on the boundary of the domain ∂R. Then, up to leading order in l, (2.201) corresponds to

$$\begin{aligned} \delta I &= \int_R \left[L(q_k, \psi + l\phi, \psi_i + l\phi_i) - L(q_k, \psi, \psi_i)\right] d(q) \\ &= \int_R \left[L(q_k, \psi, \psi_i) + \frac{\partial L}{\partial \psi} l\phi + \frac{\partial L}{\partial \psi_i} l\phi_i - L(q_k, \psi, \psi_i)\right] d(q) \\ &= l \int_R \left(\frac{\partial L}{\partial \psi}\phi + \frac{\partial L}{\partial \psi_i}\phi_i\right) d(q)\ , \end{aligned} \tag{2.203}$$

where Einstein's summation over repeated indices has been used.

Equation (2.203) yields

$$\delta I = 0 \iff \int_R \left(\frac{\partial L}{\partial \psi}\phi + \frac{\partial L}{\partial \psi_i}\phi_i\right) d(q) = 0\ . \tag{2.204}$$

Because the functions ϕ depend only on q,

$$\frac{\partial \phi}{\partial q_i} = \frac{D\phi}{Dq_i}\ , \tag{2.205}$$

holds, so that (2.204) yields

$$\delta I = 0 \iff \int_R \left(\frac{\partial L}{\partial \psi} \phi + \frac{\partial L}{\partial \psi_i} \frac{D\phi}{Dq_i} \right) d(q) = 0 \,. \tag{2.206}$$

Splitting the integrals in (2.206), it is possible to see that the second integral can be rewritten as

$$\int_R \frac{\partial L}{\partial \psi_i} \frac{D\phi}{Dq_i} d(q) = \int_R \left[\frac{D}{Dq_i} \left(\frac{\partial L}{\partial \psi_i} \phi \right) - \phi \frac{D}{Dq_i} \frac{\partial L}{\partial \psi_i} \right] d(q) \,, \tag{2.207}$$

so that (2.206) yields

$$\delta I = 0 \iff \int_R \left[\phi \left(\frac{\partial L}{\partial \psi} - \frac{D}{Dq_i} \frac{\partial L}{\partial \psi_i} \right) + \frac{D}{Dq_i} \left(\frac{\partial L}{\partial \psi_i} \phi \right) \right] d(q) = 0 \,. \tag{2.208}$$

Consider now the integral of the second term in the square brackets of (2.208),

$$\int_R \frac{D}{Dq_i} \left(\frac{\partial L}{\partial \psi_i} \phi \right) d(q) \,. \tag{2.209}$$

To simplify the notation, let

$$P_i = \frac{\partial L}{\partial \psi_i} \phi \,, \tag{2.210}$$

with boundary conditions

$$P_i(\partial R) = 0 \,. \tag{2.211}$$

In the simplified notation, (2.209) is written as

$$\int_R \frac{DP_i}{Dq_i} d(q) = \int_R \frac{DP_i}{Dq_i} dq_i \prod_{k \neq i} dq_k \,. \tag{2.212}$$

Considering the integral

$$\int_R \frac{DP_i}{Dq_i} dq_i \,, \tag{2.213}$$

it is possible to see that

$$\frac{DP_i}{Dq_i} dq_i = \frac{\partial P_i}{\partial q_i} dq_i + \frac{\partial P_i}{\partial \psi} \frac{\partial \psi}{\partial q_i} dq_i + \frac{\partial P_i}{\partial \psi_l} \frac{\partial \psi_l}{\partial q_i} dq_i = dP_i \,, \tag{2.214}$$

so that (2.213) yields

$$\int_R \frac{DP_i}{Dq_i} dq_i = \int_R dP_i = 0 \,. \tag{2.215}$$

Using (2.215), (2.208) reduces to

$$\delta I = 0 \iff \int_R \left[\phi \left(\frac{\partial L}{\partial \psi} - \frac{D}{Dq_i} \frac{\partial L}{\partial \psi_i} \right) \right] d(q) = 0 \,. \tag{2.216}$$

Because the functions ϕ are arbitrary, (2.216) yields

$$\frac{D}{Dq_i} \frac{\partial L}{\partial \psi_i} - \frac{\partial L}{\partial \psi} = 0 \,, \tag{2.217}$$

which are the Euler–Lagrange equations for continuous systems.

In (2.217), it is possible to define

$$\frac{D}{Dq_i} \frac{\partial L}{\partial \psi_i} - \frac{\partial L}{\partial \psi} = -\frac{\delta \mathscr{L}}{\delta \psi} \,, \tag{2.218}$$

where

$$\mathscr{L} = \int_{R_V} L dV \,, \tag{2.219}$$

where $dV = dq_1 dq_2 dq_3$ and $R_V \subset \mathbb{R}^3$. The r.h.s. of (2.218) defines the *functional derivative* of the functional $\mathscr{L}$. For a definition of functional derivatives, see Appendix F.

2.10.1 Hamiltonian for Scalar Fields

Analogously to the point-particle formulation for dynamics, from the Lagrangian density for scalar fields it is possible to define a Hamiltonian density from a Legendre transform of the Lagrangian density, i.e., in analogy with (2.61)

$$H = \frac{\partial L}{\partial \dot{\psi}} \dot{\psi} - L \,. \tag{2.220}$$

From (2.217), we define the canonical momentum density as

$$\pi = \frac{\partial L}{\partial \dot{\psi}} \,, \tag{2.221}$$

where we have underlined the role of the time derivative using the dot symbol. From (2.220), the Hamiltonian density of the system can thus be defined as

$$H = \pi \dot{\psi} - L \,. \tag{2.222}$$

It should be noted that, from the definition of the canonical momentum density (2.221), the Hamiltonian density (2.222) singles out the time variable from the spatial variables. This is different from the Lagrangian formulation in which the space and time variables are instead treated in the same way.

Defining the Hamiltonian as the integral of the Hamiltonian density over the spatial volume,

$$\mathscr{H} = \int_{R_V} H dV \ . \tag{2.223}$$

Hamilton's equations are defined as

$$\dot{\psi} = \frac{\delta \mathscr{H}}{\delta \pi} \ , \ \dot{\pi} = -\frac{\delta \mathscr{H}}{\delta \psi} \ . \tag{2.224}$$

The comparison between (2.224) and (2.66a), (2.66b) shows that in the case of fields the partial derivative is replaced by a functional derivative, as a straight consequence of the fact that $\mathscr{H}$ is indeed a functional.

Also for the case of scalar fields, one can introduce the concept of canonical transformations and generating functionals. Because the definition of these is essentially the same for scalar and vector fields, a complete exposition will be delayed to Sect. 2.12.2.

2.11 Noether's Theorem for Fields with the Lagrangian Depending on a Scalar Function

2.11.1 Mathematical Preliminary

As for the proof for point particles, we start the proof of Noether's Theorem for fields with the Lagrangian depending on a scalar function, with a mathematical preliminary. Given the functional (2.200) and the infinitesimal transformations of the independent variables

$$q_k' = q_k + \delta q_k \ , \tag{2.225}$$

where δq_k have the same meaning of the $\delta q_k(t)$ in the discrete system. The transformations (2.225) change the domain of integration from R into a different integration domain R'. Using (2.225), it is possible to define the dependent variables

$$\psi'(q') = \psi(q) + \delta\psi(q) \ , \tag{2.226a}$$

$$\psi_k'(q') = \psi_k(q) + \delta\psi_k(q) \ , \tag{2.226b}$$

where $q' = \left(q_o', q_1', q_2', q_3'\right)$ indicates the set of independent variables defined by (2.225). Using (2.225) and (2.226a), (2.226b), it is possible to introduce the

Lagrangian density $L\left(q'_k, \psi', \psi'_i\right)$ and the variation

$$\delta I = \int_{R'} L\left(q'_k, \psi', \psi'_i\right) d(q') - \int_R L\left(q_k, \psi, \psi_i\right) d(q) \, . \tag{2.227}$$

Taking into account the infinitesimal behaviour of δq_k in (2.225), the change of variables implies

$$d(q') = \left[1 + \frac{\partial\left(\delta q_k\right)}{\partial q_k}\right] d(q) \, , \tag{2.228a}$$

$$d(q) = \left[1 - \frac{\partial\left(\delta q_k\right)}{\partial q_k}\right] d(q') \, , \tag{2.228b}$$

where the quantities in the square brackets indicate the Jacobian of the transformation (2.225) and its inverse. As shown in Appendix G, from the expansion of (2.227) it is possible to derive, at first order,

$$\begin{aligned} \delta I = \int_R \Bigg[& \frac{D}{Dq_k}\left(L\delta q_k - \frac{\partial L}{\partial \psi_k}\psi_l \delta q_l + \frac{\partial L}{\partial \psi_k}\delta\psi\right) \\ & + \left(\frac{\partial L}{\partial \psi} - \frac{D}{Dq_k}\frac{\partial L}{\partial \psi_k}\right)\left(\delta\psi - \psi_l \delta q_l\right)\Bigg] d(q) \, . \end{aligned} \tag{2.229}$$

In the hypothesis that along the boundary of R hold the conditions $\delta q_k = 0, \ \delta\psi = 0$, the request of stationarity of (2.200) yields the validity of (2.217), so that (2.229) simplifies into

$$\delta I = \int_R \frac{D}{Dq_k}\left(L\delta q_k - \frac{\partial L}{\partial \psi_k}\psi_l \delta q_l + \frac{\partial L}{\partial \psi_k}\delta\psi\right) d(q) \, . \tag{2.230}$$

2.11.2 Linking Back to the Physics

Consider a particular case of the transformations (2.225) and (2.226a), (2.226b), represented by

$$q'_k = q_k + \delta_S q_k \, , \tag{2.231a}$$

$$\psi'(q') = \psi(q) + \delta_S \psi(q) \, , \tag{2.231b}$$

$$\psi'_k(q') = \psi_k(q) + \delta_S \psi_k(q) \, . \tag{2.231c}$$

Equations (2.231a), (2.231b) transform the Lagrangian density as

$$L'\left(q', \psi', \psi'_k\right) d(q') = L\left(q, \psi, \psi_k\right) d(q) \,, \tag{2.232a}$$

$$L'\left(q', \psi', \psi'_k\right) = L\left(q', \psi', \psi'_k\right) + \frac{D}{Dq'_k}\left(\delta_S \Omega_k\right) \,. \tag{2.232b}$$

Equation (2.232a) shows that (2.200) is a scalar that is invariant under (2.225) and (2.231a), (2.231b), while (2.232b) expresses the covariance of the Lagrangian density under the transformations (2.231a), (2.231b). The divergence transformation that appears in (2.232b) is allowed because, as it is shown in Appendix H, the equations of motion (2.217) are invariant under the substitution

$$L \rightarrow L + \frac{D}{D_k}\left(\delta_S \Omega_K\right) \,, \tag{2.233}$$

under the hypothesis $\partial \Omega_k / \partial \psi_k = 0$. Using (2.231a), (2.231b), (2.232a) takes the form

$$L'\left(q + \delta_S q, \psi + \delta_S \psi, \psi_k + \delta_S \psi_k\right) d(q') = L\left(q, \psi, \psi_k\right) d(q) \,, \tag{2.234}$$

and, analogously, from (2.232b) follows that

$$\begin{aligned} &L'\left(q + \delta_S q, \psi + \delta_S \psi, \psi_k + \delta_S \psi_k\right) d(q') = \\ &L\left(q + \delta_S q, \psi + \delta_S \psi, \psi_k + \delta_S \psi_k\right) d(q') + \frac{D}{Dq'_k}\left(\delta_S \Omega_k\right) d(q') \,. \end{aligned} \tag{2.235}$$

Because of (2.228a), (2.228b), and considering infinitesimal transformations of the kind (2.231a), (2.231b), at order zero the approximation

$$\frac{D}{Dq'_k}\left(\delta_S \Omega_k\right) d(q') = \frac{D}{Dq_k}\left(\delta_S \Omega_k\right) d(q)$$

holds and (2.235) yields

$$\begin{aligned} &L'\left(q + \delta_S q, \psi + \delta_S \psi, \psi_k + \delta_S \psi_k\right) d(q') = \\ &L\left(q + \delta_S q, \psi + \delta_S \psi, \psi_k + \delta_S \psi_k\right) d(q') + \frac{D}{Dq_k}\left(\delta_S \Omega_k\right) d(q) \,. \end{aligned} \tag{2.236}$$

From the comparison between (2.234) and (2.236), it follows that

$$\begin{aligned} &L\left(q, \psi, \psi_k\right) d(q) = \\ &L\left(q + \delta_S q, \psi + \delta_S \psi, \psi_k + \delta_S \psi_k\right) d(q') + \frac{D}{Dq_k}\left(\delta_S \Omega_k\right) d(q) \,, \end{aligned} \tag{2.237}$$

where, using (2.228a), (2.228b),

$$L\left(q+\delta_S q, \psi+\delta_S \psi, \psi_k+\delta_S \psi_k\right)= \left[L\left(q, \psi, \psi_k\right)-\frac{D}{D q_k}\left(\delta_S \Omega_k\right)\right]\left[1-\frac{\partial\left(\delta_S q_k\right)}{\partial q_k}\right] . \tag{2.238}$$

Equation (2.238) can be further developed using a Taylor expansion truncated at first order, and eliminating all the terms of order higher that the first, yielding

$$\left[\delta_S q \frac{\partial}{\partial q}+\delta_S \psi \frac{\partial}{\partial \psi}+\delta_S \psi_k \frac{\partial}{\partial \psi_k}+\frac{\partial\left(\delta_S q_k\right)}{\partial q_k}\right] L\left(q, \psi, \psi_k\right)=-\frac{D}{D q_k}\left(\delta_S \Omega_k\right) . \tag{2.239}$$

Equation (2.239) allows to verify the invariance of a certain Lagrangian density under a transformation of the kind (2.231a), (2.231b) and with an appropriate choice for $\Omega_i(q, \psi)$. To find the corresponding conserved quantity, (2.237) is integrated with respect to $d(q)$ over the domain R (or over R' with respect to $d(q')$), yielding

$$\int_{R'} L\left(q', \psi', \psi_k'\right) d(q')-\int_R L\left(q, \psi, \psi_k\right) d(q)+\int_R \frac{D}{D q_k}\left(\delta_S \Omega_k\right) d(q)=0 . \tag{2.240}$$

The first two terms in (2.240) are just Eq. (2.227), with δ_S in place of a generic δ. Assuming the validity of (2.217), it is possible to substitute (2.230) in (2.240), with δ_S instead of δ, yielding

$$\int_R \frac{D}{D q_k}\left(L \delta_S q_k-\frac{\partial L}{\partial \psi_k} \psi_l \delta_S q_l+\frac{\partial L}{\partial \psi_k} \delta_S \psi+\delta_S \Omega_k\right) d(q)=0 . \tag{2.241}$$

Due to the arbitrariness of the integration interval, (2.241) implies the conservation law

$$\frac{D}{D q_k}\left[\left(L \delta_{kl} q_k-\frac{\partial L}{\partial \psi_k} \psi_l\right) \delta_S q_l+\frac{\partial L}{\partial \psi_k} \delta_S \psi+\delta_S \Omega_k\right]=0 , \tag{2.242}$$

where δ_{kl} is the Kronecker delta.

Summarizing: every continuous and infinitesimal transformation (2.231a), (2.231b) that change (2.239) to an identity creates a conserved quantity with form (2.242).

2.12 Lagrangian Formulation for Fields: Lagrangian Density Dependent on Vector Functions

The previous exposition of the Lagrangian formulation for continuous systems and for the corresponding version of Noether's Theorem was based on a scalar function, i.e. the current function ψ. In this section, we will summarize the Lagrangian formalism for continuous systems, such as fluids, in which the Lagrangian is expressed as a function of a vector function. The results will be shortly presented without formal proofs, which instead follow closely, apart from technical details, the proofs for the Lagrangian formulation that makes use of the scalar function.

Introduce the notation

$$(t, x, y, z) = (x_0, x_1, x_2, x_3) \ , \tag{2.243}$$

and its abbreviation

$$(x_0, x_1, x_2, x_3) = x \ , \tag{2.244}$$

where, now, $\int_R d(x) = \int_R \prod_{n=0}^{3} dx_n$, where $R \subset \mathbb{R}^4$ is the domain of integration. The dependent variables are the components

$$q^1, \ q^2, \ q^3 \ , \tag{2.245}$$

of the position vector $\mathbf{q}$. The derivatives of each component of $\mathbf{q}$ with respect to each of the independent variable are

$$\frac{\partial q^\mu}{\partial x_n}, \ \mu = 1, 2, 3, \ n = 0, 1, 2, 3 \ . \tag{2.246}$$

The following abbreviations will also be used:

$$\frac{\partial q^\mu}{\partial x_n} = q_n^\mu, \ \frac{\partial^2 q^\nu}{\partial x_n \partial x_l} = q_{nl}^\nu \ . \tag{2.247}$$

Using this notation, the Lagrangian density is a function of the independent variables, the dependent variables and their first derivatives, and it is indicated as

$$L\left(x_n, q^\mu, q_n^\nu\right) \ . \tag{2.248}$$

The Lagrangian derivative is instead expressed as

$$\begin{aligned} \frac{D}{Dx_k} &= \frac{\partial}{\partial x_k} + \frac{\partial q^\mu}{\partial x_k}\frac{\partial}{\partial q^\mu} + \frac{\partial}{\partial x_k}\left(\frac{\partial q^\mu}{\partial x_l}\right)\frac{\partial}{\partial\left(\partial q^\mu/\partial x_l\right)} \\ &= \frac{\partial}{\partial x_k} + q_k^\mu\frac{\partial}{\partial q^\mu} + q_{kl}^\mu\frac{\partial}{\partial q_l^\mu} \ . \end{aligned} \tag{2.249}$$

It is important to notice that the sub- and superscripts should not be confused with covariant and contravariant vectors, as commonly used in physics.

As for the previous section, define the functional

$$I = \int_R L\left(x_n, q^\mu, q_n^\mu\right) d(x) \ . \tag{2.250}$$

The equations of motion can be derived from the request

$$\delta I = 0 \,. \tag{2.251}$$

To do so, label the functions $q^\mu,\ q_n^\mu$ by the label l

$$q^\mu(x,l) = q^\mu(x,0) + l\,Q^\mu(x)\,, \tag{2.252a}$$

$$q_n^\mu(x,l) = q_n^\mu(x,0) + l\,Q_n^\mu(x)\,, \tag{2.252b}$$

where $Q^\mu(x),\ Q_n^\mu(x) \in C_R^2$ are arbitrary functions satisfying $q_n = 0,\ q_n^\mu = 0$ on the boundary of the domain ∂R. Then, up to leading order in l, (2.251) corresponds to

$$\begin{aligned}\delta I &= \int_R \left[L\left(x_n, q^\mu + l\,Q^\mu, q_n^\mu + l\,Q_n^\mu\right) - L\left(x_n, q^\mu, q_n^\mu\right)\right] d(x)\\ &= \int_R \left[L\left(x_n, q^\mu, q_n^\mu\right) + \frac{\partial L}{\partial q^\mu} l\,Q^\mu + \frac{\partial L}{\partial q_n^\mu} l\,Q_n^\mu - L\left(x_n, q^\mu, q_n^\mu\right)\right] d(x)\\ &= l \int_R \left(\frac{\partial L}{\partial q^\mu} Q^\mu + \frac{\partial L}{\partial q_n^\mu} Q_n^\mu\right) d(x)\,,\end{aligned} \tag{2.253}$$

where Einstein's summation over repeated indices has been used. Equation (2.253) thus yields

$$\delta I = 0 \iff \int_R \left(\frac{\partial L}{\partial q^\mu} Q^\mu + \frac{\partial L}{\partial q_n^\mu} Q_n^\mu\right) d(x) = 0\,. \tag{2.254}$$

As in the analysis reported in the previous section, because the functions Q^μ depend only on x, it is possible to set

$$\frac{\partial Q^\mu}{\partial x_n} = \frac{D Q^\mu}{D x_n}\,, \tag{2.255}$$

so that (2.254) yields

$$\int_R \left(\frac{\partial L}{\partial q^\mu} Q^\mu + \frac{\partial L}{\partial q_n^\mu} \frac{D Q^\mu}{D x_n}\right) d(x) = 0\,. \tag{2.256}$$

The second integral in (2.256) can be rewritten as

$$\int_R \frac{\partial L}{\partial q_n^\mu} \frac{D Q^\mu}{D x_n} d(x) = \int_R \left[\frac{D}{D x_n}\left(\frac{\partial L}{\partial q_n^\mu} Q^\mu\right) - Q^\mu \frac{D}{D x_n}\left(\frac{\partial L}{\partial q_n^\mu}\right)\right] d(x)\,, \tag{2.257}$$

so that (2.256) yields

$$\int_R \left[Q^\mu \left(\frac{\partial L}{\partial q^\mu} - \frac{D}{D x_n} \frac{\partial L}{\partial q_n^\mu}\right) + \frac{D}{D x_n}\left(\frac{\partial L}{\partial q_n^\mu} Q^\mu\right)\right] d(x) = 0\,. \tag{2.258}$$

Considering the integral of the second term in the square brackets of (2.258),

$$\int_R \frac{D}{Dx_n}\left(\frac{\partial L}{\partial q_n^\mu} Q^\mu\right) d(x) \; , \tag{2.259}$$

and setting

$$P_n = \frac{\partial L}{\partial q_n^\mu} Q^\mu \; , \tag{2.260}$$

with boundary conditions

$$P_n(\partial R) = 0 \; , \tag{2.261}$$

Equation (2.259) yields

$$\int_R \frac{DP_i}{Dx_n} d(x) = \int_R \frac{DP_n}{Dx_n} dx_n \prod_{k\neq n} dx_k \; . \tag{2.262}$$

Consider now the integral

$$\int_R \frac{DP_n}{Dx_n} dx_n \; , \tag{2.263}$$

because

$$\frac{DP_n}{Dx_n} dx_n = \frac{\partial P_x}{\partial x_n} dx_n + \frac{\partial P_n}{\partial q^\mu}\frac{\partial q^\mu}{\partial x_n} dx_n + \frac{\partial P_n}{\partial q_l^\mu}\frac{\partial q_l^\mu}{\partial x_n} dx_n = dP_n \; , \tag{2.264}$$

Equation (2.263) yields

$$\int_R \frac{DP_n}{Dx_n} dx_n = \int_R dP_n = 0 \; , \tag{2.265}$$

so that (2.258) reduces to

$$\int_R \left[Q^\mu \left(\frac{\partial L}{\partial q^\mu} - \frac{D}{Dx_n}\frac{\partial L}{\partial q_n^\mu}\right)\right] d(x) = 0 \; . \tag{2.266}$$

Because the functions ϕ are arbitrary, (2.266) yields the equations of motion, i.e. the Euler–Lagrange equations

$$\frac{D}{Dx_k}\frac{\partial L}{\partial q_k^\mu} - \frac{\partial L}{\partial q^\mu} = 0 \; . \tag{2.267}$$

2.12.1 Hamilton's Equations for Vector Fields

Consider the **q** field defined in the previous section. In full analogy with point-particle dynamics and for the formulation of dynamics for scalar fields, from (2.267), we define the canonical momentum densities as

$$\pi^\mu = \frac{\partial L}{\partial \dot{q}^\mu} \,. \tag{2.268}$$

The quantities $q^\mu(x),\ p^\mu(x)$ define an infinite dimensional phase space of the field and its development in time. As was already noted in the derivation of the Hamiltonian density for scalar fields, the time variable is here separated from the spatial variables. In the following, we will thus split the domain $R = R_T \times R_V$, where $R_T \subset \mathbb{R}$ and $R_V \subset \mathbb{R}^3$ are the time and space integration domains. We will also use the contracted form $dV = dx_1 dx_2 dx_3$.

In the same way as it was done for point particles, if the Lagrangian density does not contain the field quantity q^μ explicitly, the Euler–Lagrange equation (2.267) yields the conservation law

$$\frac{D}{Dx_k}\frac{\partial L}{\partial q^\mu_k} = 0 \,, \tag{2.269}$$

i.e.

$$\frac{\partial \pi^\mu}{\partial t} + \frac{\partial}{\partial x_i}\frac{\partial L}{\partial q^\mu_i} = 0 \,. \tag{2.270}$$

Integration over R_V and use of the divergence theorem yield thus the conservation of the quantity

$$\int_{R_V} \pi^\mu dV \,. \tag{2.271}$$

Given (2.269), the Hamiltonian density can be written as

$$H = \pi^\mu \dot{q}^\mu - L \,. \tag{2.272}$$

Finally, denoting the Hamiltonian of the system with (2.223), it is possible to take the variations of the latter with respect to q^μ and π^μ, obtaining thus Hamilton's equations

$$\dot{q}^\mu = \frac{\delta \mathscr{H}}{\delta \pi^\mu} \,,\ \dot{\pi}^\mu = -\frac{\delta \mathscr{H}}{\delta q^\mu} \,. \tag{2.273}$$

2.12.2 Canonical Transformations and Generating Functionals for Vector Fields

From the previous definitions, it is possible to introduce the formalism for canonical transformations for fields depending on vector functions. Given the field functions

q^{μ} and the momentum densities π^{μ}, we can consider integral transformations into new field functions $\overline{q}^{\mu}$, $\overline{\pi}^{\mu}$. The old and new fields are related to the transformation functionals

$$\int_{R_V} \overline{q}^{\mu} dV = G^{(1,\mu)}[t, q^{\mu}, \pi^{\mu}] , \tag{2.274a}$$

$$\int_{R_V} \overline{\pi}^{\mu} dV = G^{(2,\mu)}[t, q^{\mu}, \pi^{\mu}] , \tag{2.274b}$$

where now $dV = dx_1 dx_2 dx_3$. Notice, once again, the splitting of the domain R into the time and spatial subdomains. Assuming that the integrals can be solved, we have thus

$$\int_{R_V} q^{\mu} dV = F^{(1,\mu)}[t, \overline{q}^{\mu}, \overline{\pi}^{\mu}] , \tag{2.275a}$$

$$\int_{R_V} \pi^{\mu} dV = F^{(2,\mu)}[t, \overline{q}^{\mu}, \overline{\pi}^{\mu}] . \tag{2.275b}$$

If there exists a functional

$$\overline{\mathscr{H}}[t, \overline{q}^{\mu}, \overline{\pi}^{\mu}] = \int_{R_V} \overline{H} dV , \tag{2.276}$$

preserving the form of Hamilton's equations (2.273), i.e. yielding

$$\dot{\overline{q}}^{\mu} = \frac{\delta \overline{\mathscr{H}}}{\delta \overline{\pi}^{\mu}} , \quad \dot{\overline{\pi}}^{\mu} = -\frac{\delta \overline{\mathscr{H}}}{\delta \overline{q}^{\mu}} , \tag{2.277}$$

then the transformations (2.274a), (2.274b), or equivalently (2.275a), (2.275b), are said to be canonical.

The dynamics in the original and transformed coordinates are given, respectively, by the variational principles

$$\delta \int_{R} (\pi^{\mu} \dot{q}^{\mu} - H)\, d(x) = 0 , \tag{2.278a}$$

$$\delta \int_{R} \left(\overline{\pi}^{\mu} \dot{\overline{q}}^{\mu} - \overline{H}\right) d(x) = 0 . \tag{2.278b}$$

From (2.278a) consider the functional

$$\Phi = \mathscr{L} dt = \delta \int_{R_V} (\pi^{\mu} dq^{\mu} - H dt)\, dV , \tag{2.279}$$

which takes also the name of Pfaffian functional. In (2.279), we have used the notation (2.219). It is possible to prove (see, e.g. [14]) that two Pfaffian functionals differing by a differential with respect to a parameter are equivalent. In our case, said parameter is clearly the time. This allows to set the condition for the transformation of coordinates to be canonical if

$$\int_{R_V} (\pi^\mu dq^\mu - H dt)\, dV - \int_{R_V} \left(\overline{\pi}^\mu d\overline{q}^\mu - \overline{H} dt\right) dV = d\mathscr{F} \,, \tag{2.280}$$

where

$$\mathscr{F}[t, q^\mu, \pi^\mu, \overline{q}^\mu, \overline{\pi}^\mu] = \int_{R_V} F dV \,. \tag{2.281}$$

In analogy with the case of point particles, one can define the different generating functionals

$$\mathscr{F}_1 = \mathscr{F}_1[t, q^\mu, \overline{q}^\mu] \,, \tag{2.282a}$$
$$\mathscr{F}_2 = \mathscr{F}_2[t, q^\mu, \overline{\pi}^\mu] \,, \tag{2.282b}$$
$$\mathscr{F}_3 = \mathscr{F}_3[t, \pi^\mu, \overline{q}^\mu] \,, \tag{2.282c}$$
$$\mathscr{F}_4 = \mathscr{F}_4[t, \pi^\mu, \overline{\pi}^\mu] \,, \tag{2.282d}$$

from which the transformation rules for the transformation of variables to be canonical can be derived. For example, for the particular case of the generating functional $\mathscr{F}_1$, one has

$$\pi^\mu = \frac{\delta \mathscr{F}_1}{\delta q^\mu} \,, \tag{2.283a}$$
$$\overline{\pi}^\mu = -\frac{\delta \mathscr{F}_1}{\delta \overline{q}^\mu} \,, \tag{2.283b}$$
$$\overline{H} = H + \frac{\delta \mathscr{F}_1}{\delta t} \,, \tag{2.283c}$$

which has the same form of (2.96a)–(2.96c), apart from the use of the functional derivatives instead of the partial derivatives due to the fact that $\mathscr{F}_1$ is now a functional. Analogously, the transformation rules set by the generating functionals $\mathscr{F}_2$, $\mathscr{F}_3$, $\mathscr{F}_4$ take the same form as the ones set by the generating functions introduced in Sect. 2.8.

2.13 Noether's Theorem for Fields: Lagrangian Density Dependent on Vector Functions

Having introduces the dynamics for vector fields, we can now derive Noether's Theorem for this particular case. The derivation will be similar to the one for fields depending on a scalar function.

Defining the symmetry transformations

$$x'_n = x_n + \delta_S x_n \ , \tag{2.284a}$$
$$q'^{\mu}(x') = q^{\mu}(x) + \delta_S q^{\mu}(x) \ , \tag{2.284b}$$
$$q'^{\mu}_n(x') = q^{\mu}_n(x) + \delta_S q^{\mu}_n(x) \ , \tag{2.284c}$$

the Lagrangian density transforms as

$$L'\left(x'_n, q'^{\mu}, q'^{\mu}_n\right) d(x') = L\left(x_n, q^{\mu}, q^{\mu}_n\right) d(x) \ , \tag{2.285a}$$
$$L'\left(x'_n, q'^{\mu}, q'^{\mu}_n\right) = L\left(x'_n, q'^{\mu}, q'^{\mu}_n\right) + \frac{D}{Dx'_k}\left(\delta_S \Omega_k\right) \ . \tag{2.285b}$$

Equation (2.285a) shows that the functional (2.250) is a scalar invariant under the transformations (2.284a)–(2.284c), while (2.285b) shows the invariance of the form of the Lagrangian density with respect to (2.284a)–(2.284c). The divergence transformation $D\left(\delta_S \Omega_k\right)/Dx'_k$ in (2.285b)) takes into account the invariance of (2.267) upon the substitution $L \to L + D\left(\delta_S \Omega_k\right)/Dx_k$.

Following the same way of reasoning of the previous sections, after some algebra, (2.285a) and (2.285b) yield

$$\left[\delta_S x_n \frac{\partial}{\partial x_n} + \delta_S q^{\mu} \frac{\partial}{\partial q^{\mu}_n} + \frac{\partial\left(\delta_S x_n\right)}{\partial x_n}\right] L\left(x_n, q^{\mu}, q^{\mu}_n\right) = -\frac{D}{Dx_k}\left(\delta_S \Omega_k\right) \ . \tag{2.286}$$

If (2.286) is satisfied for a certain symmetry transformation (2.284a)–(2.284c) and for a quantity $\Omega_k(x, q)$, the Lagrangian is invariant under that transformation. The corresponding conserved quantity is defined from the equation

$$\frac{D}{Dx_k}\left[\left(L\delta_{kl} - \frac{\partial L}{\partial q^{\mu}_k} q^{\mu}_l\right)\delta_S x_l + \frac{\partial L}{\partial q^{\mu}_k}\delta_S q^{\mu} + \delta_S \Omega_k\right] = 0 \ . \tag{2.287}$$

2.14 Bibliographical Note

Numerous excellent books exist on classical mechanics, such as [4]. The proofs of Noether's Theorem here exposed follow instead [7]. For a historical, critical analysis of [7], see, e.g. [10]. For a self-contained exposition of Noether's Theorem and its implications, see [15]. The same book reports an interesting discussion on why the Lagrangian is made up by the kinetic energy *minus* the potential energy based on the equipartition of energy. The original formulation of Noether's Theorem was formulated by Emmy Noether herself as *"a combination of the methods of the formal calculus of variations and of Lie's theory of groups"* [16]. A didactical proof and discussion of Noether's Theorem that makes use of Lie groups can be found for example in the book by Olver [17]. For other books see, for example [1, 3, 5, 6, 8, 9, 11, 12, 18–20].

References

1. Abraham, R., Marsden, J.: Foundations of Mechanics, vol. 36. Benjamin/Cummings Publishing Company Reading, Massachusetts (1978)
2. Arnold, V.: Mathematical Methods of Classical Mechanics. Springer, New York (1989)
3. Bühler, O.: A Brief Introduction to Classical, Statistical, and Quantum Mechanics, vol. 13. American Mathematical Soc. (2006)
4. Goldstein, H., Poole, C., Safko, J.: Classical Mechanics. Addison-Wesley, San Francisco (2001)
5. Gregory, R.D.: Classical Mechanics. Cambridge University Press, Cambridge (2006)
6. Greiner, W.: Classical Mechanics: Systems of Particles and Hamiltonian Dynamics. Springer Science & Business Media, Berlin (2009)
7. Hill, E.: Hamilton's principle and the conservation theorems of mathematical physics. Rev. Mod. Phys. **23**, 253–260 (1951)
8. Holm, D.: Geometric Mechanics. Imperial College Press, London (2008)
9. Holm, D., Schmah, T., Stoica, C.: Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions. Oxford University Press, Oxford (2009)
10. Kosmann-Schwarzbach, Y.: The Noether Theorems: Invariance and Conservation Laws in the Twentieth Century. Sources and Studies in the History of Mathematics and Physical Sciences. Springer, New York (2011)
11. Landau, L., Lifshitz, E.: Mechanics. In: Course of Theoretical Physics, vol. 1. Butterworth-Heinemann, Oxford (1976)
12. Marsden, J.: Lectures on Mechanics, vol. 174. Cambridge University Press, Cambridge (1992)
13. Marsden, J., Ratiu, T.: Introduction to Mechanics and Symmetry. Springer, New York (2003)
14. Mušicki, D.: Canonical transformations and the Hamilton-Jacobi method in the field theory. Publications de L'Institut Mathematique Beograd **2**(16), 21–34 (1962)
15. Neuenschwander, D.: Emmy Noether's Wonderful Theorem. The John Hopkins University Press, Baltimore (2011)
16. Noether, E.: Invariante Variationsprobleme. Göttinger Nachrichten, pp. 235–257 (1918)
17. Olver, P.: Applications of Lie Groups to Differential Equations, Graduate Texts in Mathematics, vol. 107. Springer, New York (1993)
18. Olver, P.: Equivalence, Invariants and Symmetry. Cambridge University Press, Cambridge (1995)
19. Taylor, J.: Classical Mechanics. University Science Books, Herndon (2005)
20. Tikhomirov, V.: Selected Works of AN Kolmogorov: Volume I: Mathematics and Mechanics, vol. 25. Springer Science & Business Media, Berlin (2012)

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