Chapter 2
Regularity of Solutions for Nonlinear Systems

Abstract In this chapter we establish sufficient conditions for regularity of all weak solutions for nonlinear systems. We note that the respective Cauchy problems may have nonunique weak solution. In Sect. 2.1 we establish regularity of all weak solutions for parabolic feedback control problems. Section 2.2 devoted to artificial control method for nonlinear partial differential equations and inclusions. The regularity of all weak solutions is obtained. In Sect. 2.3 we consider regularity results of all weak solutions for nonlinear reaction-diffusion systems with nonlinear growth. In Sect. 2.4 we consider the following examples of applications: a parabolic feedback control problem; a model of conduction of electrical impulses in nerve axons; a climate energy balance model; FitzHugh–Nagumo System; a model of combustion in porous media.

2.1 Regularity of All Weak Solutions for a Parabolic Feedback Control Problem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be bounded and open subset with a smooth boundary $\partial\Omega$, $\underline{f}$, $\overline{f} : \mathbb{R} \to \mathbb{R}$ are some real functions. We consider the semilinear reaction-diffusion inclusion:

$$u_t - \triangle u + [\underline{f}(u), \overline{f}(u)] \ni 0 \text{ in } \Omega \times (\tau, T), \quad (-\infty < \tau < T < +\infty), \tag{2.1}$$

with boundary condition

$$u\big|_{\partial\Omega} = 0, \tag{2.2}$$

where $[a, b] = \{\alpha a + (1 - \alpha)b \mid \alpha \in [0, 1]\}, a, b \in \mathbb{R}$. We suppose that $f = [\underline{f}, \overline{f}] : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ satisfies the growth condition

$$\exists c_0 > 0 : \quad -c_0(1 + |u|) \leq \underline{f}(u) \leq \overline{f}(u) \leq c_0(1 + |u|) \quad \forall u \in \mathbb{R}. \tag{2.3}$$

Suppose also that $\underline{f}$ is lower semi-continuous, and $\overline{f}$ is upper semi-continuous.

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We shall use the following standard notations: $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, V' is the dual space of V. The function $u(\cdot) \in L^2(\tau, T; V)$ is a *weak solution* of Problem (2.1) and (2.2) on $[\tau, T]$, if there exists a measurable function $d : \Omega \times (\tau, T) \to \mathbb{R}$ such that

$$d(x,t) \in [\underline{f}(u(x,t)), \overline{f}(u(x,t))] \quad \text{for a.e. } (x,t) \in \Omega \times (\tau, T); \tag{2.4}$$

$$-\int_\tau^T \left\langle u, \frac{d\xi}{dt} \right\rangle dt + \int_\tau^T \int_\Omega (\nabla u, \nabla \xi)\, dxdt + \int_\tau^T \int_\Omega (d, \xi)\, dxdt = 0 \tag{2.5}$$

for all $\xi \in C_0^\infty(\Omega \times (\tau, T))$, where $\langle \cdot, \cdot \rangle$ denotes the pairing in the space V.

We note that Problem (2.1) and (2.2) arises in many important models for distributed parameter control problems and that large class of identification problems enter this formulation. Let us indicate a problem which is one of motivations for the study of the autonomous evolution inclusion (2.1) (cf. [37, 56] and references therein). In a subset Ω of $\mathbb{R}^3$, we consider the nonstationary heat conduction equation (Figs. 2.1 and 2.2):

$$\frac{\partial y}{\partial t} - \triangle y = f \text{ in } \Omega \times (0, +\infty)$$

with initial conditions and suitable boundary ones. Here $y = y(x,t)$ represents the temperature at the point $x \in \Omega$ and time $t > 0$. It is supposed that $f = f_1 + f_2$, where f_2 is given and f_1 is a known function of the temperature of the form

$$-f_1(x,t) \in \partial j(x, y(x,t)) \text{ a.e. } (x,t) \in \Omega \times (0, +\infty);$$

Figure 2.3 Here $\partial j(x, \xi)$ denotes generalized gradient of Clarke (see [12]) with respect to the last variable of a function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ which is assumed to be locally Lipschitz in ξ (cf. [37] and references therein). The multi-valued function $\partial j(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is generally nonmonotone and it includes the vertical jumps. In a physicist's language it means that the law is characterized by the generalized gradient of a nonsmooth potential j (cf. [39]).

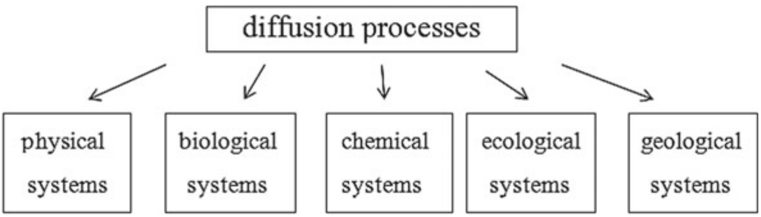

Fig. 2.1 Diffusion processes

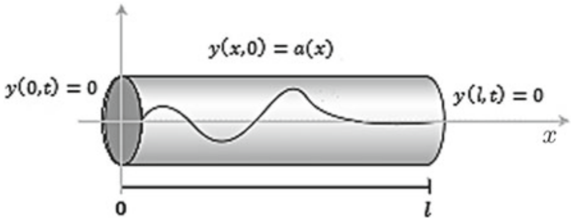

Fig. 2.2 Idealized physical setting for heat conduction in a rod with homogeneous boundary conditions

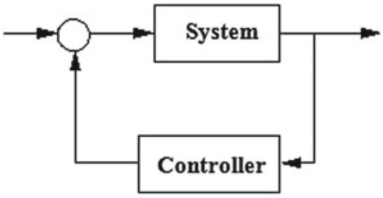

Fig. 2.3 Feedback control diagram

Another motivations connected with parabolic equations with a discontinuous nonlinearity. In [43] it is considered the case, when f is the difference of maximal monotone maps. Global attractor in phase space H for such type equations is considered there. Obtained inclusion is a particular case of an abstract differential inclusion generated by a difference of subdifferential maps of proper convex lower semicontinuous functionals [38]. Models of physical interest includes also the next (cf. [3] and references therein):

- a model of combustion in porous media;
- a model of conduction of electrical impulses in nerve axons;
- a climate energy balance model;

etc. The main purpose of this subsection is to investigate regularity properties of all globally defined weak solutions for Problem (2.1) and (2.2) with initial data $u_\tau \in H$ under listed above assumptions.

Further we need to consider the restriction of $v : [\tau, T] \to V^*$ on $[s, T]$, $s \in (\tau, T)$, $\tau < T$. To simplify conclusions denote it by the same symbol $v(\cdot)$.

Theorem 2.1 *Let $u(\cdot)$ be an arbitrary weak solution of Problem (2.1) and (2.2) on $[\tau, T]$. Then for any $\varepsilon \in (0, T - \tau)$ $u(\cdot) \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; H^2(\Omega) \cap V)$ and $u_t(\cdot) \in L^2(\tau + \varepsilon, T; H)$.*

Proof Let $u(\cdot)$ be an arbitrary weak solution of Problem (2.1) and (2.2) on $[\tau, T]$. Then there exists a measurable function $d : \Omega \times (\tau, T) \to \mathbb{R}$ such that $u(\cdot)$ and $d(\cdot)$ satisfy (2.4) and (2.5). As $u(\cdot) \in L^2(\Omega \times (\tau, T))$ and the growth condition (2.3) holds, then $d(\cdot) \in L^2(\Omega \times (\tau, T))$. The set

$$\mathscr{D} := \{s \in (\tau, T) \mid u(s) \in V\}$$

is dense in $[\tau, T]$. For any arbitrary fixed $s \in \mathscr{D}$ we note that $u(\cdot)$ is the unique weak solution on $[s, T]$ of the problem

$$\begin{cases} z_t - \triangle z = -d(x,t) \text{ in } \Omega \times (s,T), \\ z\big|_{\partial\Omega} = 0, \\ z(x,s) = u(x,s) \text{ in } \Omega. \end{cases} \tag{2.6}$$

Moreover, $u(\cdot) \in L^2(s,T;H^2(\Omega)\cap V)\cap C([s,T];V)$ and $u_t(\cdot) \in L^2(s,T;H)$, $s \in \mathscr{D}$ (cf. [40, Chap. 4.I], [42, Chap. III] and references therein). Thus for any $\varepsilon \in (0, T-\tau)$ $u(\cdot) \in C([\tau+\varepsilon, T]; V)\cap L^2(\tau+\varepsilon, T; H^2(\Omega)\cap V)$ and $u_t(\cdot) \in L^2(\tau+\varepsilon, T; H)$.

The theorem is proved.

2.2 Artificial Control Method for Nonlinear Partial Differential Equations and Inclusions: Regularity of All Weak Solutions

Let $(V; H; V^*)$ be evolution triple , where V be a real Hilbert space, such that $V \subset H$ with compact imbedding. Let $A : V \to V^*$ be a linear symmetric operator such that $\exists c > 0 : \langle Av, v\rangle_V \geq c\|v\|_V^2$, for each $v \in V$ and let $D(A) = \{u \in V : Au \in H\}$. We note that the mapping $v \to \|Av\|_H$ defines the equivalent norm on $D(A)$; Temam [42, Chap. III]. Let $B : \mathbb{R} \times V \to 2^H \setminus \{\emptyset\}$ be set-valued (in the general case) mapping such that the following assumption holds: there exists $c_1 > 0$ such that $\|y\|_H \leq c_1(1 + \|u\|_V)$, for a.e. t and each $u \in V$ and $y \in B(t,u)$.

For a set $D \subset H$ let $\overline{\text{co}}D$ be a closed convex hull of a set D. We consider the differential-operator inclusion:

$$\frac{du}{dt} + Au(t) + B(t, u(t)) \ni \bar{0} \quad (-\infty < \tau < T < +\infty). \tag{2.7}$$

The function $u(\cdot) \in L^2(\tau, T; V)$ is called a *weak solution* of Problem (2.7) on $[\tau, T]$, if there exists a Bochner-measurable function $d : (\tau, T) \to H$ such that

$$d(t) \in \overline{\text{co}}B(t, u(t)) \text{ for a.e. } t \in (\tau, T); \text{ and} \tag{2.8}$$

$$\int_\tau^T \left[-\langle u, v\rangle\, \xi'(t) + \langle Au, v\rangle\, \xi(t) + \langle d, v\rangle\, \xi(t)\right] dt = 0, \tag{2.9}$$

for all $\xi \in C_0^\infty(\tau, T)$ and for all $v \in V$, where $\langle\cdot,\cdot\rangle$ denotes the pairing in the space V.

The main regularity result of this section has the following formulation.

Theorem 2.2 *Let $-\infty < \tau < T < +\infty$ and $u_\tau \in H$. If $u(\cdot)$ is a weak solution of Problem (2.7) on $[\tau, T]$, then $u(\cdot) \in C([\tau+\varepsilon, T]; V)\cap L^2(\tau+\varepsilon, T; D(A))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau+\varepsilon, T; H)$ for each $\varepsilon \in (0, T-\tau)$.*

Proof Let $u(\cdot)$ be an arbitrary weak solution of Problem (2.7) on $[\tau, T]$. According to the definition of a weak solution of Problem (2.7) on $[\tau, T]$, there exist $d \in L^2(\tau, T; H)$ such that $u(\cdot) \in L^2(\tau, T; V)$ and $d(\cdot)$ satisfy (2.8) and (2.9). Note that the set

$$\mathscr{D} := \{s \in (\tau, T) \mid u(s) \in V\}$$

is dense in $[\tau, T]$. For an arbitrary fixed $s \in \mathscr{D}$ we remark that $u(\cdot)$ is the unique weak solution on $[\tau, T]$ of the problem

$$\begin{cases} \frac{dz}{dt} + Az(t) = -d(t) \text{ on } (s, T), \\ z(s) = u(s). \end{cases} \tag{2.10}$$

Therefore, $u(\cdot) \in L^2(s, T; D(A)) \cap C([s, T]; V)$ and $\frac{du}{dt}(\cdot) \in L^2(s, T; H)$, $s \in \mathscr{D}$ (cf. [40, Chap. 4.I], [42, Chap. III] and references therein). Thus $u(\cdot) \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; D(A))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau + \varepsilon, T; H)$ for any $\varepsilon \in (0, T - \tau)$.

The theorem is proved.

Remark 2.1 Theorem 2.2 implies that each weak solution of Problem (2.7) on $[\tau, T]$ is *regular*, that is, $u(\cdot) \in L^2(\varepsilon, T; D(A)) \cap C([\varepsilon, T]; V)$ and $\frac{du}{dt}(\cdot) \in L^2(\varepsilon, T; H)$, for each $\varepsilon \in (0, T - \tau)$.

Let $B(t, u) := \partial J_1(u) - \partial J_2(u)$ for each $u \in V$ and $t \in \mathbb{R}$, where $J_i : H \to \mathbb{R}$ be a convex, lower semi-continuous function such that the following assumptions hold: (i) (growth condition) there exists $c_1 > 0$ such that $\|y\|_H \le c_1(1 + \|u\|_H)$, for each $u \in H$ and $y \in \partial J_i(u)$ and $i = 1, 2$; (ii) (sign condition) there exist $c_2 > 0$, $\lambda \in (0, c)$ such that $(y_1 - y_2, u)_H \ge -\lambda \|u\|_H^2 - c_2$, for each $y_i \in \partial J_i(u)$, $u \in H$, where $\partial J_i(u)$ the subdifferential of $J_i(\cdot)$ at a point u. Note that $u^* \in \partial J_i(u)$ if and only if $u^*(v - u) \le J_i(v) - J_i(u)\ \forall v \in H$; $i = 1, 2$. For such B Problem (2.7) has the following formulation:

$$\frac{du}{dt} + Au(t) + \partial J_1(u(t)) - \partial J_2(u(t)) \ni \bar{0} \quad (-\infty < \tau < T < +\infty). \tag{2.11}$$

We recall that the function $u(\cdot) \in L^2(\tau, T; V)$ is called a *weak solution* of Problem (2.11) on $[\tau, T]$, if there exist Bochner measurable functions $d_i : (\tau, T) \to H$; i = 1,2, such that

$$d_i(t) \in \partial J_i(u(t)) \text{ for a.e. } t \in (\tau, T), \text{ i=1,2; and} \tag{2.12}$$

$$\int_\tau^T \left[-\langle u, v\rangle \xi'(t) + \langle Au, v\rangle \xi(t) + \langle d_1, v\rangle \xi(t) - \langle d_2, v\rangle \xi(t)\right] dt = 0, \tag{2.13}$$

for all $\xi \in C_0^\infty(\tau, T)$ and for all $v \in V$.

The following theorem provides sufficient conditions for the existence and regularity of all weak solutions for Problem (2.11).

Theorem 2.3 *Let* $-\infty < \tau < T < +\infty$ *and* $u_\tau \in H$. *If* $u(\cdot)$ *is a weak solution of Problem (2.11) on* $[\tau, T]$, *then* $u(\cdot) \in C([\tau+\varepsilon, T]; V) \cap L^2(\tau+\varepsilon, T; D(A))$ *and* $\frac{du}{dt}(\cdot) \in L^2(\tau+\varepsilon, T; H)$ *for any* $\varepsilon \in (0, T-\tau)$.

Proof The regularity of each weak solution follows from Theorem 2.2.
The theorem is proved.

2.3 Regularity of All Weak Solutions for Nonlinear Reaction-Diffusion Systems with Nonlinear Growth

In this section we establish sufficient conditions for regularity of weak solutions for both reaction-diffusion equations (Sect. 2.3.1) as well as systems of reaction-diffusion equations (Sect. 2.3.2) separately.

2.3.1 Reaction-Diffusion Equations

In a bounded domain $\Omega \subset \mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$ we consider the problem

$$\begin{cases} u_t - \Delta u + f(u) = h, \quad x \in \Omega, \ t > 0, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0, \end{cases} \tag{2.14}$$

where

$$\begin{gathered} h \in L^2(\Omega), \\ f \in C(\mathbb{R}), \\ |f(u)| \leq C_1(1 + |u|^{p-1}), \quad \forall u \in \mathbb{R}, \end{gathered} \tag{2.15}$$

with $2 \leq p \leq 3$, $C_1, C_2, \alpha > 0$.

We denote by A the operator $-\Delta$ with Dirichlet boundary conditions, so that $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. As usual, denote the eigenvalues and the eigenfunctions of A by $\lambda_i, e_i, i = 1, 2 \ldots$

Denote $F(u) = \int_0^u f(s)ds$. From (2.15) we have that $\liminf_{|u|\to\infty} \frac{f(u)}{u} = \infty$, and for some $D_1 0$,

$$|F(u)| \leq D_1(1 + |u|^p), \qquad \forall u \in \mathbb{R}. \tag{2.16}$$

In what follows we denote $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, and $\|\cdot\|$, $(\cdot,\cdot)$ will be the norm and the scalar product in $L^2(\Omega)$. We denote by $\|\cdot\|_X$ the norm in the abstract Banach space X, whereas $(\cdot,\cdot)_Y$ will be the scalar product in the abstract Hilbert space Y. Also, $P(X)$ will be the set of all non-empty subsets of X.

On the other hand, we define the usual sequence of spaces

$$V^{2\alpha} = D\left(A^{\alpha}\right) = \{u \in H : \sum_{i=1}^{\infty} \lambda_i^{2\alpha} \left|(u, e_i)\right|^2 < \infty\},$$

where $\alpha \geq 0$. We recall the following well known result, which is a particular case of [40, Lemma 37.8] for our operator $A = -\Delta$ in a three-dimensional domain.

Lemma 2.1 *$D\left(A^{\alpha}\right) \subset W^{k,q'}\left(\Omega\right)$ whenever $q' \geq 2$ and k is an integer such that*

$$k - \frac{3}{q'} < 2\alpha - \frac{3}{2}.$$

Also, it is well known that $V^s \subset H^s\left(\Omega\right)$ for all $s \geq 0$ (see [49, Chap. IV] or [34]).

A function $u \in L^2_{loc}(0, +\infty; V) \bigcap L^p_{loc}(0, +\infty; L^p(\Omega))$ is called a *weak solution* of (2.14) on $(0, +\infty)$ if for all $T > 0$, $v \in V$, $\eta \in C_0^{\infty}(0, T)$

$$-\int_0^T (u, v)\eta_t dt + \int_0^T \left((u, v)_V + (f(u), v) - (h, v)\right)\eta dt = 0.$$

It is well known [1, Theorem 2] or [9, p. 284] that for any $u_0 \in$ there exists at least one weak solution of (2.14) with $u(0) = u_0$ (and it may be non unique) and that any weak solution of (2.14) belongs to $C\left([0, +\infty); H\right)$. Moreover, the function $t \mapsto \|u(t)\|^2$ is absolutely continuous and

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|u(t)\|_V^2 + (f(u(t)), u(t)) - (h, u(t)) = 0 \text{ a.e.} \tag{2.17}$$

The function $u \in L^2_{loc}(0, +\infty; V) \bigcap L^p_{loc}(0, +\infty; L^p(\Omega))$ is called a *regular solution* of (2.14) on $(0, +\infty)$ if for all $T > 0$, $v \in V$, and $\eta \in C_0^{\infty}(0, T)$ we have

$$-\int_0^T (u, v)\eta_t dt + \int_0^T \left((u, v)_V + (f(u), v) - (h, v)\right)\eta dt = 0, \tag{2.18}$$

and

$$u \in L^{\infty}\left(\varepsilon, T; V\right), \tag{2.19}$$

$$u_t \in L^2\left(\varepsilon, T; H\right), \ \forall 0 < \varepsilon < T. \tag{2.20}$$

Any regular solution u satisfies

$$u \in L^2\left(\varepsilon, T; D\left(A\right)\right). \tag{2.21}$$

In this section we will prove that every weak solution is in fact a regular solution.

Theorem 2.4 *Assume that $2 \leq p \leq 3$ in condition (2.15). Then any weak solution $u(\cdot)$ satisfies $u \in C([\varepsilon, T]; V) \cap L^2(\varepsilon, T; D(A))$, $u_t \in L^2(\varepsilon, T; H)$ for all $\varepsilon > 0$, that is, it is a regular solution.*

Proof From

$$\int_{\Omega} |f(u(x,t))|^{\frac{p}{p-1}} dx \leq C_1 + C_2 \int_{\Omega} |u(x,t)|^p dx$$

we obtain that

$$\|f(u(t))\|^2_{L^{\frac{p}{p-1}}(\Omega)} \leq C_3 + C_4 \|u(t)\|^{2p-2}_{L^p(\Omega)}.$$

Using the Sobolev embedding $H^r(\Omega) \subset L^p(\Omega)$ if $r = \left(\frac{3}{2} - \frac{3}{p}\right) \leq \frac{1}{2}$ (as $p \leq 3$) and the Gagliardo–Nirenberg inequality

$$\|v\|_{H^r(\Omega)} \leq C_5 \|v\|_{H^{\frac{1}{2}}(\Omega)} \leq C_6 \|v\|^{\frac{1}{2}} \|v\|^{\frac{1}{2}}_{H^1(\Omega)},$$

we have

$$\begin{aligned}\|f(u(t))\|^2_{L^{\frac{p}{p-1}}(\Omega)} &\leq C_3 + C_7 \|u(t)\|^{p-1} \|u(t)\|^{p-1}_{H^1(\Omega)} \\ &\leq C_8 + C_9 \|u(t)\|^2 \|u(t)\|^2_{H^1(\Omega)}.\end{aligned}$$

Thus,

$$\|f(u)\|_{L^2\left(0,T;L^{\frac{p}{p-1}}(\Omega)\right)} \leq C_{10}\left(1 + \|u\|_{C([0,T];H)} \|u\|_{L^2(0,T;H^1(\Omega))}\right).$$

Set $d(x,t) = f(u(x,t))$ for $(x,t) \in (0,T) \times \Omega$. Then $d \in L^2\left(0,T; L^{\frac{p}{p-1}}(\Omega)\right) \subset L^2\left(0,T; H^{-r}(\Omega)\right) \subset L^2\left(0,T; V^{-r}\right)$.

We consider the problem

$$\begin{cases} v_t - \Delta v = -d(x,t) + h(x), & x \in \Omega,\ t > 0, \\ v|_{\partial\Omega} = 0, \\ v(\tau) = u(\tau). \end{cases}$$

We note that $u(\tau) \in V \subset V^r$ for a.a. $\tau > 0$. For such τ in view of [40, p. 163, Theorem 42.12] there exists a unique weak solution $v(\cdot)$ such that $v \in C([\tau, T]; V^r) \cap L^2\left(\tau, T; V^{r+1}\right)$. Hence, $u \in C([\varepsilon, T]; V^r) \cap L^2\left(\varepsilon, T; V^{r+1}\right)$ for all $\varepsilon > 0$.

We shall prove that $f(u(\cdot)) \in L^2(\varepsilon, T; H)$. As this is obvious if $p = 2$, we consider that $2 < p \leq 3$. We note that $V^r \subset H^r(\Omega) \subset L^p(\Omega)$. Also, by Lemma 2.1 with $\alpha = \frac{r+1}{2}$, $r = 3\left(\frac{1}{2} - \frac{1}{p}\right)$, $k = 1$ we obtain that $V^{r+1} \subset W^{1,q'}(\Omega)$ for any $q' < p$. On the other hand, by the Sobolev embedding theorems we have

$W^{1,q'}(\Omega) \subset L^q(\Omega)$, for $q < \frac{3p}{3-p}$. Thus, the inequality $p(p-1) < \frac{3p}{3-p}$, for all $2 \le p \le 3$, implies that $u \in C([\varepsilon, T]; L^p(\Omega)) \cap L^2\left(\varepsilon, T; L^{p(p-1)}(\Omega)\right)$. By (2.15) we have

$$\begin{aligned}\|f(u(t))\|^2 = \int_\Omega |f(u(x,t))|^2\,dx \le C_{11} + C_{12}\int_\Omega |u(x,t)|^{2(p-1)}\,dx \\ \le C_{13} + C_{14}\,\|u(t)\|^{p-1}_{L^p(\Omega)}\,\|u(t)\|^{p-1}_{L^{p(p-1)}(\Omega)}.\end{aligned}$$

Therefore, $f(u(\cdot)) \in L^2(\varepsilon, T; H)$. Then standard results imply that $u \in C([\varepsilon, T]; V) \cap L^2(\varepsilon, T; D(A))$ and $u_t \in L^2(\varepsilon, T; H)$.

The lemma is proved.

Remark 2.2 Theorem 2.4 was proved in [23].

2.3.2 Systems of Reaction-Diffusion Equations

Let us consider the following reaction-diffusion system (RD-system for short)

$$\begin{cases} u_t = a\Delta u - f(u) + h(x), & x \in \Omega,\ t > 0, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.22}$$

where $u = u(x,t) = (u^1(x,t), \ldots, u^N(x,t))$ is unknown vector-function, a is a real $N \times N$ matrix with positive symmetric part $\frac{1}{2}(a + a^*) \ge \beta I$, $\beta > 0$, $h = (h_1, \ldots, h_N)$, $f = (f_1, \ldots, f_N)$ are given functions,

$$h \in (L^2(\Omega))^N, \quad f \in C(\mathbb{R}^N; \mathbb{R}^N),$$

and for given numbers $C_1, C_2 \ge 0$, $\gamma > 0$, $p_i \ge 2$, $i = \overline{1, N}$ the following conditions hold:

$$\forall\, v \in \mathbb{R}^N \quad \sum_{i=1}^N |f_i(v)|^{q_i} \le C_1(1 + \sum_{i=1}^N |v^i|^{p_i}), \tag{2.23}$$

$$\forall\, v \in \mathbb{R}^N \quad \sum_{i=1}^N f_i(v)v^i \ge \gamma \sum_{i=1}^N |v^i|^{p_i} - C_2, \tag{2.24}$$

where $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $i = \overline{1, N}$. In further arguments we will use the standard functional spaces

$$H = (L^2(\Omega))^N \text{ with the norm } |v|^2 = \int_\Omega \sum_{i=1}^N |v^i(x)|^2 dx,$$

$$V = (H_0^1(\Omega))^N \text{ with the norm } \|v\|^2 = \int\limits_{\Omega} \sum_{i=1}^{N} |\nabla v^i(x)|^2 dx.$$

Let us denote $V' = H^{-1}(\Omega))^N$, $\mathbf{p} = (p_1, \ldots, p_N)$, $L^{\mathbf{p}}(\Omega) = L^{p_1}(\Omega) \times \cdots \times L^{p_N}(\Omega)$,

$$W = L^{\mathbf{p}}_{loc}(0, +\infty; L^{\mathbf{p}}(\Omega)) \cap L^2_{loc}(0, +\infty; V).$$

Definition 2.1 The function $u = u(x, t) \in W$ is called a (global) weak solution of Problem (2.22) on $(0, +\infty)$ if for all $T > 0$, $v \in V \cap L^{\mathbf{p}}(\Omega)$,

$$\frac{d}{dt} \int\limits_{\Omega} u(x,t)v(x)dx + \int\limits_{\Omega} \Big(a\nabla u(x,t)\nabla v(x) + f(u(x,t))v(x) - h(x)v(x)\Big)dx = 0 \tag{2.25}$$

in the sense of scalar distributions on $(0, T)$.

From (2.23) and Sobolev embedding theorem we see that every solution of (2.22) satisfies $u_t \in L^{\mathbf{q}}_{loc}(0, +\infty; H^{-\mathbf{r}}(\Omega))$, where $\mathbf{r} = (r_1, \ldots, r_N)$, $r_k = max\{1, n(\frac{1}{2} - \frac{1}{p_k})\}$. The well-known result on global resolvability of (2.22) for initial conditions from the phase space H established in [9]. Under conditions (2.23), (2.24) for every $u_0 \in H$ there exists at least one weak solution of (2.22) on $(0, +\infty)$ with $u(0) = u_0$. Every weak solution of (2.22) belongs to $C([0, +\infty); H)$, the function $t \mapsto |u(t)|^2$ is absolutely continuous and for a.a. $t \geq 0$ the following energy equality holds

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + (a\nabla u(t), \nabla u(t)) + (f(u(t)), u(t)) = (h, u(t)). \tag{2.26}$$

The function $u = u(x, t) \in W$ is called a *regular solution* of Problem (2.22) on $(0, +\infty)$ if it is weak solution on $(0, +\infty)$ and, additionally,

$$u \in L^{\infty}\left(\varepsilon, T; V \cap L^{\mathbf{p}}(\Omega)\right), \tag{2.27}$$

$$u_t \in L^2(\varepsilon, T; H) \ \forall\, 0 < \varepsilon < T. \tag{2.28}$$

Let us consider the following additional condition on vector-function f [51]:

$$\forall\, v \in \mathbb{R}^N \quad f(v) = \nabla F(v) + g(v), \tag{2.29}$$

where ∇F satisfies (2.23), (2.24), and $g \in C(\mathbb{R}^N; \mathbb{R}^N)$ is such that for some constants $C_3 \geq 0$, $C_4 \geq 0$,

$$|g(v)|^2 \leq C_3 F(v) + C_4(|v|^2 + 1), \ \forall\, v \in \mathbb{R}^N. \tag{2.30}$$

If $N = 1$ (scalar case), then (2.29), (2.30) hold with $F(v) = \int\limits_0^v f(s)ds$, $g \equiv 0$.

Conditions (2.29), (2.30) also take place if

$$f_i(v) = \alpha_i v^i |v^i|^{p_i - 2} + g_i(v),\ i = \overline{1, N},$$

where $\alpha_i > 0$, $g \in C(\mathbb{R}^N; \mathbb{R}^N)$, and $|g(v)| \leq C_4(1 + |v|)$. Another example is the FitzHugh–Nagumo system (see the example in Sect. 3.4.4 below).

Let us briefly analyze conditions (2.29), (2.30).

Using the equality

$$F(v) - F(0) = \int_0^1 \nabla F(sv) \cdot v ds = \int_0^{\frac{1}{(|v|+1)^2}} \nabla F(sv) \cdot v ds + \int_{\frac{1}{(|v|+1)^2}}^1 \nabla F(sv) \cdot v ds$$

and condition (2.24), we deduce that for some $\alpha > 0$

$$\forall\, v \in \mathbb{R}^N \quad F(v) \geq \alpha \sum_{i=1}^N |v^i|^{p_i} - C_5. \tag{2.31}$$

Again using the equality $F(v) - F(0) = \int_0^1 \nabla F(sv) v ds$, Young's inequality and condition (2.23), we obtain

$$|F(v)| \leq C_6(\sum_{i=1}^N |v^i|^{p_i} + 1). \tag{2.32}$$

Theorem 2.5 *Under conditions (2.23), (2.24), (2.29), (2.30) for every $u_0 \in H$ there exists at least one regular solution $u(\cdot)$ of (2.22) such that $u(0) = u_0$, and for some positive constants $C(g)$, $D(g)$, which depend on the function g but not on $u(\cdot)$, the following energy inequality holds for a.e. $s > 0$ and each $t \geq s$*

$$E\,(u\,(t)) + \int_s^t |u_r|^2 dr \leq E\,(u\,(s)) + C(g) \int_s^t E(u(p))dp + D(g)(t - s), \tag{2.33}$$

where $E\,(u\,(t)) = \|u\,(t)\|^2 + 2\,(F\,(u\,(t))\,, 1) - 2\,(h, u\,(t))$. *Moreover,* $C(g) = D(g) = 0$ *if in condition (2.29) we have* $g \equiv 0$.

Proof We take as in [9, p.281] the Galerkin approximations using the basis of eigenfunctions $\{w_j\,(x)\,,\ j \in \mathbb{N}\}$, of the Laplace operator with Dirichlet boundary conditions. Let $X_n = \{w_1, \ldots, w_n\}$ and let P_n be the orthogonal projector from H onto X_n. Then $u^n\,(x, t) = \sum_{j=i}^n a_{j,m}\,(t)\, w_j\,(x)$ will be a solution of the system of ordinary differential equations

$$\frac{du^n}{dt} = P_n \Delta u^n - P_n f\left(u^n\right) + P_n h,\ u^n\,(0) = P_n u_0. \tag{2.34}$$

It is proved in [9, p.281] that (2.34) is globally resolved, and for every $T > 0$ passing to a subsequence u^n converges to a weak solution u of (2.22) in $C([0,T];H)$, weakly in $L^{\mathbf{p}}(0,T;L^{\mathbf{p}}(\Omega))$ and weakly in $L^2(0,T;V)$. Also, $u_t^n \to u_t$ weakly in $L^{\mathbf{q}}\left(0,T;H^{-\mathbf{r}}(\Omega)\right)$.

Multiplying the equation in (2.34) by u_t^n we get

$$\frac{d}{dt}\left(\|u^n\|^2 + 2(F(u^n),1) - 2(h,u^n)\right) + 2|u_t^n|^2 = -2(g(u^n),u_t^n). \tag{2.35}$$

Using (2.30), we deduce from (2.35) that

$$\begin{aligned}&\frac{d}{dt}\left(\|u^n\|^2 + 2(F(u^n),1) - 2(h,u^n)\right) + |u_t^n|^2\\ &\le C_7(g)\left(\|u^n\|^2 + 2(F(u^n),1) - 2(h,u^n)\right) + C_8(g).\end{aligned} \tag{2.36}$$

In particular, u^n satisfies (2.33) $\forall\ t \ge s \ge 0$. We note that if $g \equiv 0$, then $C_7(g) = C_8(g) = 0$, so that $C(g) = D(g) = 0$ holds.

On the other hand, multiplying (2.22) by u^n and using (2.23) in a standard way we obtain

$$\frac{d}{dt}|u^n|^2 + \lambda_1|u^n|^2 + \left\|u^n\right\|^2 + \gamma\sum_{i=1}^N \left\|u_i^n\right\|_{L^{p_i}}^{p_i} \le K + |h|^2. \tag{2.37}$$

By Gronwall's lemma we obtain

$$|u^n(t)|^2 \le e^{-\lambda_1 t}|u_0^n|^2 + \frac{1}{\lambda_1}\left(K + |h|^2\right). \tag{2.38}$$

Thus integrating (2.37) over $(t, t+r)$ with $r > 0$ we have

$$\begin{aligned}&|u^n(t+r)|^2 + \int_t^{t+r}\left\|u^n\right\|^2 ds + \gamma\int_t^{t+r}\sum_{i=1}^N \left\|u_i^n(s)\right\|_{L^{p_i}}^{p_i} ds\\ &\le |u^n(t)|^2 + r\left(K + |h|^2\right)\\ &\le e^{-\lambda_1 t}|u_0^n|^2 + \left(\frac{1}{\lambda_1} + r\right)\left(K + |h|^2\right).\end{aligned} \tag{2.39}$$

Then from (2.32),

$$\int_t^{t+r}\left(\left\|u^n\right\|^2+2\left(F\left(u^n\left(s\right)\right),1\right)-2\left(h,u^n\right)\right)ds$$
$$\le\int_t^{t+r}\left\|u^n\right\|^2ds+2C_6\int_t^{t+r}\sum_{i=1}^N\left\|u_i^n(s)\right\|_{L^{p_i}}^{p_i}ds+r|h|^2+\int_t^{t+r}|u^n|^2ds+2C_6|\Omega|r$$
$$\le C_9(e^{-\lambda_1 t}|u_0^n|^2+r+1). \tag{2.40}$$

Now we can apply uniform Gronwall Lemma [46] to inequality (2.36) and obtain

$$\left\|u^n\left(t+r\right)\right\|^2+2\left(F\left(u^n\left(t+r\right)\right),1\right)-2\left(h,u^n\left(t+r\right)\right) \tag{2.41}$$
$$\le C_{10}\left(\frac{e^{-\lambda_1 t}|u_0^n|^2+1}{r}+1\right)e^r \text{ for all } 0\le t\le t+r.$$

From the last inequality and (2.31) we have

$$\left\|u^n\left(t+r\right)\right\|^2+\sum_{i=1}^N\left\|u_i^n(t+r)\right\|_{L^{p_i}}^{p_i} \tag{2.42}$$
$$\le C_{11}\left(\left(\frac{e^{-\lambda_1 t}|u_0^n|^2+1}{r}+1\right)e^r+1\right) \text{ for all } 0\le t\le t+r.$$

Therefore, the sequence $u^n(\cdot)$ is bounded in $L^\infty(r,T;V\cap L^{\mathbf{p}}(\Omega))$ for all $0<r<T$.

Integrating (2.36) over (r,T), we have

$$\int_r^T|u_t^n|^2dt\le C_7\int_r^T\left(\left\|u^n\left(s\right)\right\|^2+2\left(F\left(u^n\left(s\right)\right),1\right)-2\left(h,u^n(s)\right)\right)ds \tag{2.43}$$
$$+\left\|u^n\left(r\right)\right\|^2+2\left(F\left(u^n\left(r\right)\right),1\right)-2\left(h,u^n(r)\right)+C_8(T-r)+2C_5|\Omega|+|h|^2+|u^n(T)|^2.$$

So from (2.38), (2.40), (2.41) and the last inequality we deduce that u_t^n is bounded in $L^2(r,T;H)$ for all $0<r<T$.

Thus for the limit function u we can claim that it is regular solution of (2.22) and $u(0)=u_0$.

Let us prove that u satisfies the energy inequality (2.33). As u^n is bounded in $L^\infty(r,T;L^{\mathbf{p}}(\Omega))$, so $f(u^n)$ is bounded in $L^\infty(r,T;L^{\mathbf{q}}(\Omega))$. Therefore from [45] up to subsequence

$$u^n\to u \text{ in } L^2(r,T;V)\cap L^{\mathbf{p}}(r,T;L^{\mathbf{p}}(\Omega)). \tag{2.44}$$

and, in particular,

$$u^n(t)\to u(t) \text{ in } V \text{ for a.a. } t\in(r,T).$$

Also, it is standard to check that $u^n \to u$ in $C([r, T], H)$, for all $0 < r < T$, and that $u^n(t) \to u(t)$ weakly in V for all $0 < t \le T$.

Then by the dominated convergence theorem $F(u^n(t)) \to F(u(t))$ in $L^1(\Omega)$ for a.a. $t \in [r, T]$. Also, for any $0 < t \le T$ we have $F(u^n(x, t)) \to F(u(x, t))$ for a.a. x. Then $F(u^n(x, t)) \ge -C_5$ and Fatou's lemma imply

$$\int_\Omega F(u(x, t))dx \le \liminf \int_\Omega F(u^n(x, t))dx$$

and

$$E(u(t)) \le \liminf \; E(u^n(t)).$$

Hence, we can pass to the limit in (2.33) and obtain the required result.

The theorem is proved.

Remark 2.3 Theorem 2.5 yields only existence but not regularity of each weak solution of Problem (2.22). This theorem was proved in [24].

2.4 Examples of Applications

In this section we provide examples of applications to theorems established in Sects. 2.1–2.3. We consider a parabolic feedback control problem (Sect. 2.4.1), a model of conduction of electrical impulses in nerve axons (Sect. 2.4.2), a climate energy balance model (Sect. 2.4.3); FitzHugh–Nagumo system (Sect. 2.4.4); and a model of combustion in porous media (Sect. 2.4.5).

2.4.1 A Parabolic Feedback Control Problem

Let Ω be an open and bounded subset of $\mathbb{R}^3$. Let us consider the following nonstationary heat conduction equation

$$\frac{\partial y}{\partial t} - \triangle y = f \text{ in } \Omega \times \mathbb{R} \tag{2.45}$$

with initial condition and Dirichlet homogeneous boundary condition. Here $y = y(x, t)$ represents the temperature at the point $x \in \Omega$ and time $t > 0$.

Let $j : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function in ξ (cf. [37] and references therein) and $\partial j(x, \xi)$ denotes generalized gradient of Clarke (see [12]) with respect to the last variable. Note that the multi-valued function $\partial j(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is generally nonmonotone and it includes the vertical jumps.

We assume that $f = f_1 + f_2$, where $f_2 = f_2(x)$ is given and f_1 is a known function of the temperature of the form

$$-f_1(x,t) \in \partial j(x, y(x,t)) \text{ a.e. } (x,t) \in \Omega \times \mathbb{R}. \tag{2.46}$$

In a physicist's language it means that the law is characterized by the generalized gradient of a nonsmooth potential j (cf. [39]).

Assume also that ∂j satisfies the growth condition

$$\exists c_0 > 0: \ |p| \le c_0(1+|u|) \text{ for a.e. } x \in \Omega, \text{ and each } u \in \mathbb{R}, \text{ and } d \in \partial j(x,u);$$

and the sign condition

$$\varliminf_{u\to+\infty} \frac{\inf_{d\in\partial j(x,u)} d}{u} > -\lambda_1; \quad \varliminf_{u\to-\infty} \frac{\sup_{d\in\partial j(x,u)} d}{u} > -\lambda_1,$$

where λ_1 is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$. According to Theorem 2.2, for any $-\infty < \tau < T < +\infty$ each weak solution $u_\tau \in L^2(\Omega)$ of Problem (2.45) and (2.46) on $[\tau, T]$ belongs to $C([\tau+\varepsilon, T]; H_0^1(\Omega)) \cap L^2(\tau+\varepsilon, T; H^2(\Omega) \cap H_0^1(\Omega))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau+\varepsilon, T; L^2(\Omega))$ for each $\varepsilon \in (0, T-\tau)$.

2.4.2 A Model of Conduction of Electrical Impulses in Nerve Axons

Consider the problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u \in \lambda H(u-a), & (x,t) \in (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) = 0, & t \in \mathbb{R}, \end{cases} \tag{2.47}$$

where $a \in \left(0, \frac{1}{2}\right)$; Terman [47, 48]. Since Problem (2.47) is a particular case of Problem (2.1) and (2.2), then for each $-\infty < \tau < T < +\infty$ and a weak solution $u_\tau \in L^2((0,\pi))$ of Problem (2.47) on $[\tau, T]$ belongs to $C([\tau+\varepsilon, T]; H_0^1((0,\pi))) \cap L^2(\tau+\varepsilon, T; H^2((0,\pi)) \cap H_0^1((0,\pi)))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau+\varepsilon, T; L^2((0,\pi)))$ for each $\varepsilon \in (0, T-\tau)$; Figs. 2.4, 2.5, 2.6, and 2.7.

2.4.3 Climate Energy Balance Model

Let $(\mathscr{M}, \mathbf{g})$ be a C^∞ compact connected oriented two-dimensional Riemannian manifold without boundary (as, e.g. $\mathscr{M} = S^2$ the unit sphere of $\mathbb{R}^3$). Consider the problem:

$$\frac{\partial u}{\partial t} - \Delta u + R_e(x,u) \in QS(x)\beta(u), \quad (x,t) \in \mathscr{M} \times \mathbb{R}, \tag{2.48}$$

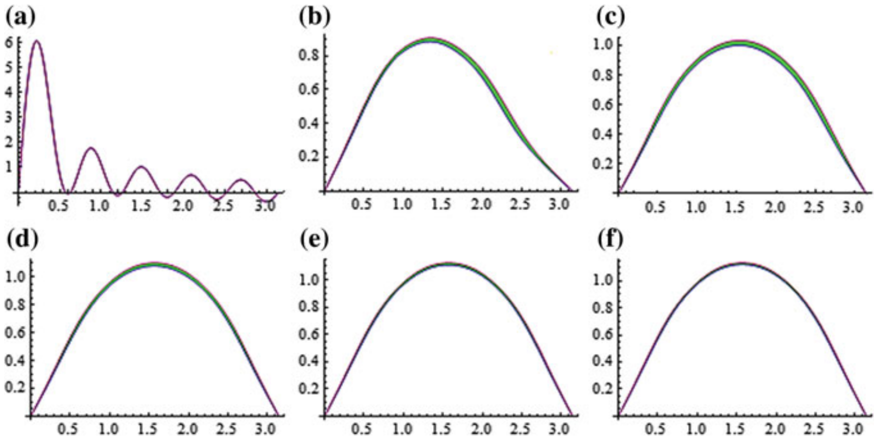

Fig. 2.4 Graphics of solutions of problem (2.47) with $a = 0.49$, $\lambda = 2$, $n = 10$, $h = 0, 001$, $N = 100$ in a moment **a** $t = 0$; **b** $t = 0.8$; **c** $t = 1.6$; **d** $t = 2.4$; **e** $t = 3.2$; **f** $t = 4$

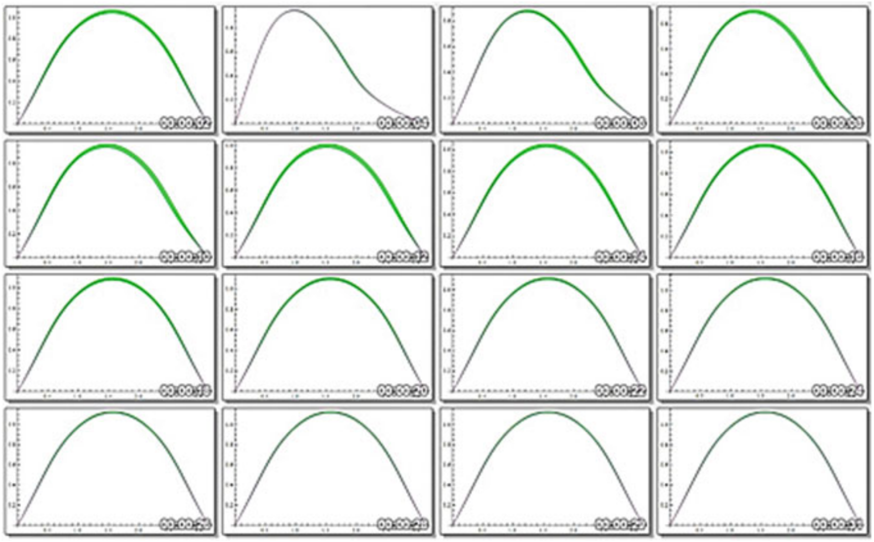

Fig. 2.5 Screenlist of animation for dynamics of solutions of problem (2.47) in 2D

where $\Delta u = \mathrm{div}_{\mathscr{M}}\left(\nabla_{\mathscr{M}} u\right)$; $\nabla_{\mathscr{M}}$ is understood in the sense of the Riemannian metric **g**. Note that (2.48) is the so-called climate energy balance model. It was proposed in Budyko [8] and Sellers [41] and examined also in Díaz et al. [13–15]. The unknown $u(x, t)$ represents the average temperature of the Earth's surface. In Budyko [8] the energy balance is expressed as

$$\text{heat variation} = R_a - R_e + D.$$

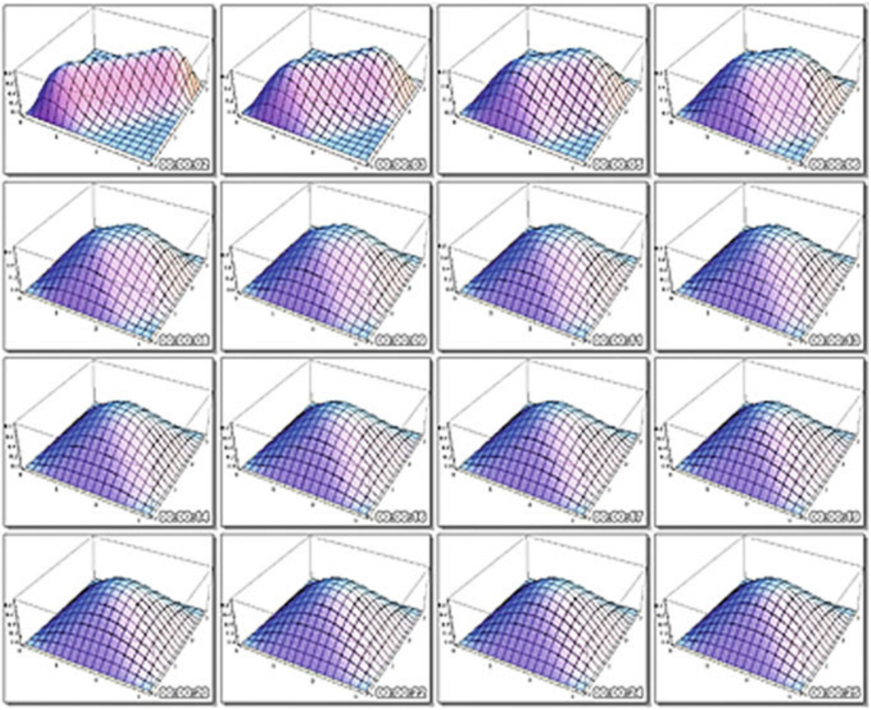

Fig. 2.6 Screenlist of animation for dynamics of solutions of problem (2.47) in 3D

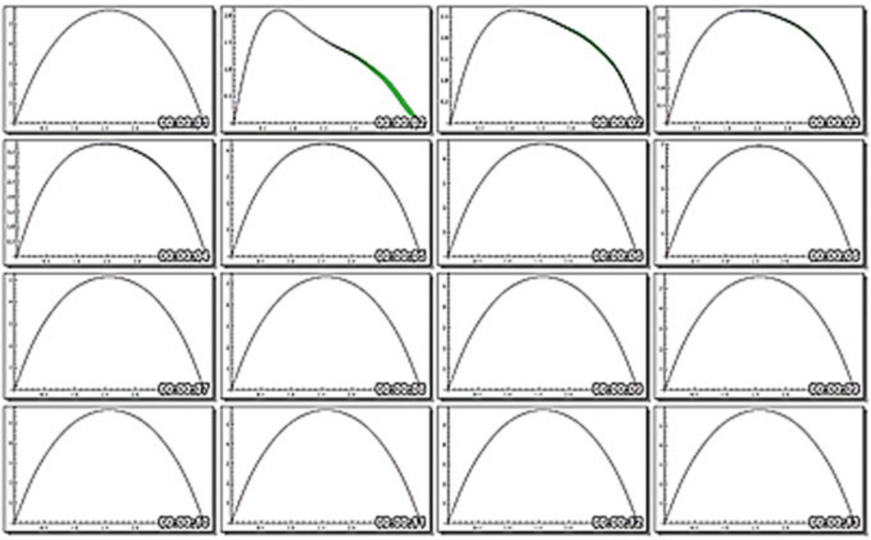

Fig. 2.7 Screenlist of animation for dynamics of solutions of problem (2.47) in section

Here $R_a = QS(x)\beta(u)$. It represents the solar energy absorbed by the Earth, $Q > 0$ is a solar constant, $S(x)$ is an insolation function, given the distribution of solar radiation falling on upper atmosphere, β represents the ratio between absorbed and incident solar energy at the point x of the Earth's surface (so-called co-albedo function). The term R_e represents the energy emitted by the Earth into space, as usual, it is assumed to be an increasing function on u. The term D is heat diffusion, we assume (for simplicity) that it is constant.

As usual, the term R_e may be chosen according to the Newton cooling law as linear function on u, $R_e = Bu + C$ (here B, C are some positive constants) [8], or according to the Stefan-Boltzman law, $R_e = \sigma u^4$ [41]. In this subsection we consider $R_e = Bu$ as in Budyko [8].

Let $S : \mathscr{M} \to \mathbb{R}$ be a function such that $S \in L^\infty(\mathscr{M})$ and there exist $S_0, S_1 > 0$ such that

$$0 < S_0 \le S(x) \le S_1.$$

Suppose also that β is a bounded maximal monotone graph of $\mathbb{R}^2$, that is there exist $m, M \in \mathbb{R}$ such that for all $s \in \mathbb{R}$ and $z \in \beta(s)$

$$m \le z \le M.$$

Let us consider real Hilbert spaces

$$H := L^2(\mathscr{M}), \quad V := \{u \in L^2(\mathscr{M}) \,:\, \nabla_{\mathscr{M}} u \in L^2(T\mathscr{M})\}$$

with respective standard norms $\|\cdot\|_H$, $\|\cdot\|_V$, and inner products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_V$, where $T\mathscr{M}$ represents the tangent bundle and the functional spaces $L^2(\mathscr{M})$ and $L^2(T\mathscr{M})$ are defined in a standard way; see, for example, Aubin [2]. According to Theorem 2.2, for any $-\infty < \tau < T < +\infty$ each weak solution $u_\tau \in L^2(\Omega)$ of Problem (2.48) on $[\tau, T]$ belongs to $C([\tau+\varepsilon, T]; H_0^1(\Omega)) \cap L^2(\tau+\varepsilon, T; H^2(\Omega) \cap H_0^1((0,\pi)))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau+\varepsilon, T; L^2(\Omega))$ for each $\varepsilon \in (0, T-\tau)$.

2.4.4 FitzHugh–Nagumo System

Let us consider generalized FitzHugh–Nagumo system [46]:

$$u_t = d_1 \Delta u - f_1(u) - v, \tag{2.49}$$

$$v_t = d_2 \Delta v + \delta u - \gamma v, \tag{2.50}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \tag{2.51}$$

where $\Omega = (0, L)$, d_1, d_2, δ, γ are positive constants, $f_1 \in C(\mathbb{R})$,

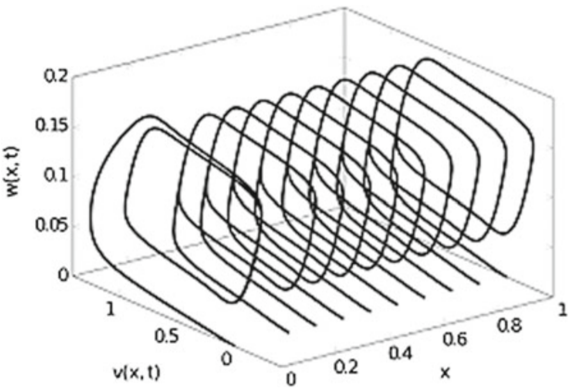

Fig. 2.8 Trajectories of FitzHugh–Nagumo system

$$|f_1(u)| \le C_1(1+|u|^3); \quad f^1(u)u \ge \alpha|u|^4 - C_2. \tag{2.52}$$

For the vector-function

$$f(u,v) = \begin{pmatrix} f_1(u)+v \\ -\delta u + \gamma v \end{pmatrix}$$

conditions (2.23), (2.24) hold with $p_1 = 4$, $p_2 = 2$. Moreover, $f = \nabla F + g$, where $F = F(u,v) = \int\limits_0^u f_1(s)ds + \frac{\gamma}{2}v^2$, $g = g(u,v) = \begin{pmatrix} v \\ -\delta u \end{pmatrix}$ and conditions (2.29), (2.30) also hold. Then all statements of Theorem 2.5 hold; Fig. 2.8.

2.4.5 A Model of Combustion in Porous Media

Let us consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - f(u) \in \lambda H(u-1), & (x,t) \in (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) = 0, \quad t \in \mathbb{R}, \end{cases} \tag{2.53}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function satisfying growth and sign assumptions, $\lambda > 0$, and $H(0) = [0,1]$, $H(s) = \mathbf{I}\{s > 0\}$, $s \ne 0$; Feireisl and Norbury [17]. Since Problem (2.53) is a particular case of Problem (2.1) and (2.2), then for any $-\infty < \tau < T < +\infty$ each weak solution $u_\tau \in L^2((0,\pi))$ of Problem (2.53) on $[\tau, T]$ belongs to $C([\tau+\varepsilon, T]; H_0^1((0,\pi))) \cap L^2(\tau+\varepsilon, T; H^2((0,\pi)) \cap H_0^1((0,\pi)))$ and $\frac{du}{dt}(\cdot) \in L^2(\tau+\varepsilon, T; L^2((0,\pi)))$ for each $\varepsilon \in (0, T-\tau)$; Fig. 2.9.

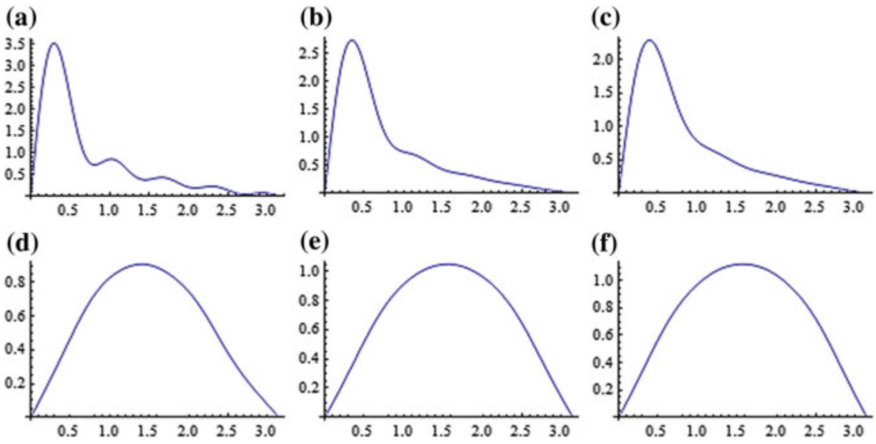

Fig. 2.9 Graphics of solutions with $f(u) = u$, $\lambda = 2$, $\varepsilon = 0.1$, $M = 100$ in moment **a** $t = 0$; **b** $t = 0.8$; **c** $t = 1.6$; **d** $t = 2.4$; **e** $t = 3.2$; **f** $t = 4$

References

1. Anguiano M., Caraballo T., Real J., Valero J.: Pullback attractors for reaction-diffusion equations in some unbounded domains with an H^{-1}-valued non-autonomous forcing term and without uniqueness of solutions. Discret. Contin. Dyn. Syst. Ser. B, **14**, 307–326 (2010)
2. Aubin, T.: Nonlinear Analysis on Manifolds. Monge-Ampére Equations. Springer, Berlin (1980)
3. Balibrea, F., Caraballo, T., Kloeden, P.E., Valero, J.: Recent developments in dynamical systems: three perspectives. Int. J. Bifurc. Chaos (2010). doi:10.1142/S0218127410027246
4. Ball, J.M.: Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations. Nonlinear Sci. **7**, 475–502 (1997). (Erratum, ibid 8:233,1998. Corrected version appears in 'Mechanics: from Theory to Computation'. pp. 447–474. Springer, Berlin (2000))
5. Ball, J.M.: Global attractors for damped semilinear wave equations. DCDS **10**, 31–52 (2004)
6. Babin, A.V., Vishik, M.I.: Attractors of Evolution Equations. Nauka, Moscow (1989)
7. Barbu, V.: Nonlinear Semigroups and Differential Equations in Banach Spaces. Editura Academiei, Bucuresti (1976)
8. Budyko, M.I.: The effects of solar radiation variations on the climate of the Earth. Tellus **21**, 611–619 (1969)
9. Chepyzhov, V.V., Vishik, M.I.: Attractors for Equations of Mathematical Physics. American Mathematical Society, Providence, Rhode Island (2002)
10. Chepyzhov, V.V., Vishik, M.I.: Evolution equations and their trajectory attractors. J. Math. Pure Appl. (1997). doi:10.1016/S0021-7824(97)89978-3
11. Chepyzhov, V.V., Vishik, M.I.: Trajectory and global attractors of three-dimensional Navier-Stokes systems. Math. Notes (2002). doi:10.1023/A:1014190629738
12. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley, Inc., New York (1983)
13. H. Díaz, J. Díaz, On a stochastic parabolic PDE arising in climatology. Rev. R. Acad. Cien. Ser. A Mat. **96**, 123–128 (2002)
14. J. Díaz, J. Hernández, L. Tello, On the multiplicity of equilibrium solutions to a nonlinear diffusion equation on a manifold arising in climatology. J. Math. Anal. Appl. **216**, 593–613 (1997)

15. Díaz, J., Hernández, J., Tello, L.: Some results about multiplicity and bifurcation of stationary solutions of a reaction diffusion climatological model. Rev. R. Acad. Cien. Ser. A. Mat. **96**(3), 357–366 (2002)
16. Díaz, J., Tello, L.: Infinitely many stationary solutions for a simple climate model via a shooting method. Math. Methods Appl. Sci. **25**, 327–334 (2002)
17. Feireisl, E., Norbury, J.: Some existence and nonuniqueness theorems for solutions of parabolic equations with discontinuous nonlinearities. Proc. R. Soc. Edinburgh A. **119**(1–2), 1–17 (1991)
18. Gajewski, H., Gröger, K., Zacharias, K.: Nichtlineare operatorgleichungen und operatordifferentialgleichungen. Akademie-Verlag, Berlin (1974)
19. Kalita, P.: Convergence of Rothe scheme for hemivariational inequalities of parabolic type. Int. J. Numer. Anal. Model. **10**, 445–465 (2013)
20. Kalita, P.: Regularity and Rothe method error estimates for parabolic hemivariational inequality. J. Math. Anal. Appl. **389**, 618–631 (2012)
21. Kapustyan, O.V., Kasyanov, P.O., Valero, J.: Pullback attractors for a class of extremal solutions of the 3D Navier-Stokes equations. J. Math. Anal. Appl. **373**(2), 535–547 (2011). doi:10.1016/j.jmaa.2010.07.040
22. Kapustyan, O.V., Valero, J.: Comparison between trajectory and global attractors for evolution systems without uniqueness of solutions. Int. J. Bifurc. Chaos (2010). doi:10.1142/S0218127410027313
23. Kapustyan, O.V., Kasyanov, P.O., Valero, J.: Structure of the global attractor for weak solutions of reaction-diffusion equation. Appl. Math. Inf. Sci. **9**(5), 2257–2264 (2015). doi:10.12785/amis/090506
24. Kapustyan, O.V., Kasyanov, P.O., Valero, J.: Regular solutions and global attractors for reaction-diffusion systems without uniqueness. Commun. Pure Appl. Anal. **13**(5), 1891–1906 (2014). doi:10.3934/cpaa.2014.13.1891
25. Kasyanov, P.O.: Multi-valued dynamics of solutions of autonomous operator differential equations with pseudomonotone nonlinearity. Math. Notes **92**(1–2), 205–218 (2012). doi:10.1134/S0001434612070231
26. Kasyanov, P.O.: Multi-valued dynamics of solutions of an autonomous differential-operator inclusion with pseudomonotone nonlinearity. Cybern. Syst. Anal. (2011). doi:10.1007/s10559-011-9359-6
27. Kas'yanov, P.O., Mel'nyk, V.S.: On properties of subdifferential mappings in Frechet spaces. Ukrainian Math. J. **57**(10), 1621–1634 (2005)
28. Kasyanov, P.O., Toscano, L., Zadoianchuk, N.V.: Long-time behaviour of solutions for autonomous evolution hemivariational inequality with multidimensional "Reaction-Displacement" law. Abstr. Appl. Anal. **2012**(450984), 21 (2012) doi:10.1155/2012/450984
29. Kasyanov, P.O., Toscano, L., Zadoianchuk, N.V.: Regularity of weak solutions and their attractors for a parabolic feedback control problem. Set-Valued Var. Anal. **21**, 271–282 (2013). doi:10.1007/s11228-013-0233-8
30. Ladyzhenskaya, O.A.: Dynamical system, generated by Navier-Stokes equations. Zap Nauch Sem LOMI. **27**, 91–115 (1972)
31. Ladyzhenskaya, O.A.: Some comments to my papers on the theory of attractors for abstract semigroups. Zap Nauchn Sem LOMI. **188**, 102–112 (1990)
32. Ladyzhenskaya, O.A.: Attractors for Semigroups and Evolution Equations. University Press, Cambridge (1991)
33. Lions, J.L.: Quelques méthodes de résolution des problémes aux limites non linéaires. Dunod, Paris (1969)
34. Lions, J.L., Magenes, E.: Problèmes aux limites non-homogènes et applications. Dunod, Paris (1968)
35. Melnik, V.S., Valero, J.: On attractors of multi-valued semiflows and differential inclusions. Set Valued Anal. (1998). doi:10.1023/A:1008608431399
36. Migórski, S.: On the existence of solutions for parabolic hemivariational inequalities. J. Comput. Appl. Math. **129**, 77–87 (2001)

37. Migórski, S., Ochal, A.: Optimal Control of Parabolic Hemivariational Inequalities. J. Glob. Optim. **17**, 285–300 (2000)
38. Otani, M., Fujita, H.: On existence of strong solutions for $\frac{du}{dt}(t) + \partial\varphi^1(u(t)) - \partial\varphi^2(u(t)) \ni f(t)$. J. Fac. Sci. Univ. Tokyo. Sect. 1 A, Math. **24**(3), 575–605 (1977)
39. Panagiotopoulos, P.D.: Inequality Problems in Mechanics and Applications. Convex and Nonconvex Energy Functions. Birkhauser, Basel (1985)
40. Sell, G.R., You, Yu.: Dynamics of Evolutionary Equations. Springer, New York (2002)
41. Sellers, W.D.: A global climatic model based on the energy balance of the Earth-atmosphere system. J. Appl. Meteorol. **8**, 392–400 (1969)
42. Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Springer, New York (1988)
43. Valero, J.: Attractors of parabolic equations without uniqueness. J. Dyn. Diff. Equ. (2001). doi:10.1023/A:1016642525800
44. Valero, J., Kapustyan, A.V.: On the connectedness and asymptotic behaviour of solutions of reaction-diffusion systems. J. Math. Anal. Appl. (2006). doi:10.1016/j.jmaa.2005.10.042
45. Vishik, M.I., Zelik, S.V., Chepyzhov, V.V.: Strong trajectory attractor of dissipative reaction-diffusion system. Doklady RAN **435**(2), 155–159 (2010)
46. Temam, R.: InfInite-dimensional Dynamical Systems in Mechanics and Physics. Springer, New York (1997)
47. Terman, D.: A free boundary problem arising from a bistable reactiondiffusion equation. Siam J. Math. Anal. **14**, 1107–1129 (1983)
48. Terman, D.: A free boundary arising from a model for nerve conduction. J. Diff. Equ. **58**(3), 345–363 (1985)
49. Vishik, M.I., Fursikov, A.V.: "Mathematical Problems of Statistical Hydromechanics. Kluwer Academic Publishers, Dordrecht (1988)
50. Vishik, M.I., Zelik, S.V., Chepyzhov, V.V.: Strong Trajectory Attractor for Dissipative Reaction-Diffusion System. Doclady Math. (2010). doi:10.1134/S1064562410060086
51. Zelik S.: The attractor for a nonlinear reaction-diffusion system with a supercritical nonlinearity and it's dimension. Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. **24**, 1–25 (2000)
52. Zgurovsky, M.Z., Kasyanov, P.O., Valero, J.: Noncoercive evolution inclusions for Sk type operators. Int. J. Bifurc. Chaos (2010). doi:10.1142/S0218127410027386
53. Zgurovsky, M.Z., Kasyanov, P.O., Zadoianchuk, N.V.: Long-time behavior of solutions for quasilinear hyperbolic hemivariational inequalities with application to piezoelectricity problem. Appl. Math. Lett. **25**(10), 1569–1574 (2012). doi:10.1016/j.aml.2012.01.016
54. Zgurovsky, M.Z., Mel'nik, V.S., Kasyanov, P.O.: Evolution Inclusions and Variation Inequalities for Earth Data Processing I. Springer, Berlin (2010). doi:10.1007/978-3-642-13837-9
55. Zgurovsky, M.Z., Mel'nik, V.S., Kasyanov, P.O.: Evolution Inclusions and Variation Inequalities for Earth Data Processing II. Springer, Berlin (2010). doi:10.1007/978-3-642-13878-2
56. Zgurovsky, M.Z., Kasyanov, P.O., Kapustyan, O.V., Valero, J., Zadoianchuk, N.V.: Evolution Inclusions and Variation Inequalities for Earth Data Processing III. Springer, Berlin (2012). doi:10.1007/978-3-642-28512-7

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