Chapter 2
Linear Feedback Control

The by far most used control method in industry is the proportional-integral-derivative or PID controller. It is currently claimed that 90 to 95% of industrial problems can be solved by this type of controller, which is easily available as an electronic module. It allies an apparent simplicity of understanding and a generally satisfactory performance. It is based on a quasi-natural principle which consists of acting on the process according to the error between the set point and the measured output. Indeed, along the chapters of this first part, it will appear that numerous variants of PID exist and that improvements can often be brought either by better tuning or by a different configuration.

2.1 Design of a Feedback Loop

2.1.1 Block Diagram of the Feedback Loop

Feedback control consists of a reinjection of the output in a loop (Fig. 2.2). The output response y or controlled variable is used to act on the control variable (or manipulated input) u in order to make the difference $(y_r - y)$ between the desired or reference set point y_r and the output y as small as possible for any value of any disturbance d. The output y is linked to the set point y_r by a system which forces the output to follow the set point (Fig. 2.1).

If a fixed value of the set point is imposed, the system is said to be regulating or in regulation mode; if the set point is variable (following a trajectory), the system is said to be tracking the set point or in tracking mode or subjected to a servomechanism. The trajectory tracking is often met, e.g. in the case of batch reactors in fine chemistry, a temperature or feed profile is imposed or, in the case of a gas chromatograph, a temperature profile is imposed on the oven temperature.

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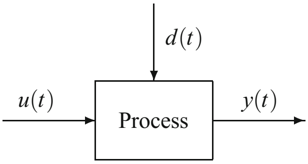


Fig. 2.1 Process representation in open loop

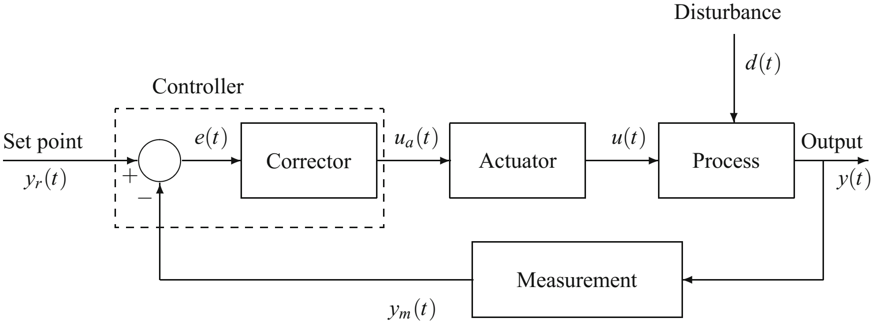


Fig. 2.2 Representation of a closed-loop process

Open-Loop System:

The vocabulary open or closed loop comes from the electricity domain. Thus, an electrical circuit must be imagined, open when it is an open loop and closed for a closed loop.

In the open loop (Fig. 2.1), the output value is not used to correct the error. The open loop can work (theoretically) only if the process model is perfect and in the absence of disturbances: in practice, many phenomena such as measurement errors, noise and disturbances are superposed so that the use of the open loop is to be proscribed. However, feedforward control (Sect. 6.6) is a special open-loop design to counterbalance the measured disturbances; in many cases, it is coupled with feedback control.

Closed-Loop System:

The process of Fig. 2.2 presents an output y, a disturbance d and a control variable u. In general, the shape of disturbance is unpredictable and the objective is to maintain the output y as close as possible to the desired set point y_r for any disturbance. A control possibility is to use a feedback realized by a closed loop (Fig. 2.2):

- The output is measured using a given measurement device; the value indicated by the sensor is y_m.
- This value is compared to the set point y_r, giving the difference [set point − measurement] to produce the error $e = y_r - y_m$.
- The value of this difference is provided to the main corrector, the function of which is to modify the value of the control variable u in order to reduce the error e. The

corrector does not operate directly, but through an actuator (valve, transducer ...) to which it gives a value u_a.

Important remark: one acts on the actuator and modifies the control variable u only after having noted the effect of the disturbance on the output. The set of the comparator and the corrector constitutes the control system and is called a controller which can perform regulation actions as well as tracking.

2.1.2 General Types of Controllers

A controller can take very different forms. In reality, it represents a control strategy, that is to say, a set of rules providing a value of the control action when the output deviates from the set point. A controller can thus be constituted by an equation or an algorithm.

In this first stage, only simple conventional controllers are considered.

2.1.2.1 Proportional (P) Controller

The operating output of the proportional controller is proportional to the error

$$u_a(t) = K_c\, e(t) + u_{ab} \tag{2.1}$$

where K_c is the proportional gain of the controller.
u_{ab} is the bias signal of the actuator (= operating signal when $e(t) = 0$), adjusted so that the output coincides with the desired output at steady state.

The proportional controller is characterized by the proportional gain K_c, sometimes by the proportional band PB defined by

$$PB = \frac{100}{K_c} \tag{2.2}$$

in the case of a dimensionless gain. In general, the proportional band takes values between 1 and 500; it represents the domain of error variation so that the operating signal covers all its domain. The higher the gain, the smaller the proportional band, and the more sensitive the controller. The sign of gain K_c can be positive or negative. K_c can be expressed with respect to the dimensions of output $u_a(t)$ and input $e(t)$ signals, or dimensionless according to the case.

A controller can saturate when its output $u_a(t)$ reaches a maximum $u_{a,\max}$ or minimum $u_{a,\min}$ value.

The controller transfer function is simply equal to the controller gain

$$G_c(s) = K_c \tag{2.3}$$

In the following, it will be noticed that the proportional controller presents the drawback to creating a deviation of the output with respect to the set point.

2.1.2.2 Proportional-Integral (PI) Controller

The operating output of the PI controller is proportional to the weighted sum of the magnitude and of the integral of the error

$$u_a(t) = K_c \left(e(t) + \frac{1}{\tau_I} \int_0^t e(x)\, dx \right) + u_{ab} \tag{2.4}$$

For chemical processes, the integral time constant is often around $0.1 \leq \tau_I \leq 60$ min.

The integral action tends to modify the controller output $u_a(t)$ as long as an error exists in the process output; thus, an integral controller can only modify small errors. The transfer function of the PI controller is equal to

$$G_c(s) = K_c \left(1 + \frac{1}{\tau_I s} \right) \tag{2.5}$$

Philosophy: the integral action takes into account (integrates) the past.

Compared to the proportional controller, the PI controller presents the advantage of eliminating the deviation between the output and the set point owing to the integral action. However, this controller can produce oscillatory responses and diminishes the closed-loop system stability. Furthermore, the integral action can become undesirable when there is saturation: the controller acts at its maximum level and nevertheless the error persists; the phenomenon is called windup. In this case, the integral term increases largely, possibly without limitation, and it is necessary to stop the integral action. An anti-windup device must be incorporated into PI controllers (see Sect. 4.6.4).

2.1.2.3 Ideal Proportional-Derivative (PD) Controller

The operating output of the ideal PD controller is proportional to the weighted sum of the magnitude and the time rate of change of the error

$$u_a(t) = K_c \left(e(t) + \tau_D \frac{de(t)}{dt} \right) + u_{ab} \tag{2.6}$$

The derivative action is intended to anticipate future errors. The transfer function of the ideal PD controller is equal to

$$G_c(s) = K_c\, (1 + \tau_D\, s) \tag{2.7}$$

This controller is theoretical because the numerator degree of the controller transfer function $G_c(s)$ is larger than the denominator degree; consequently, it is physically unrealizable. In practice, the previous derivative action is replaced by the ratio of two first-order polynomials presenting close characteristics for low and medium frequencies. Furthermore, an integral action is always added. The derivative action has a stabilizing influence on the controlled process.

Philosophy: the derivative action takes into account (anticipates) the future.

2.1.2.4 Ideal Proportional-Integral-Derivative (PID) Controller

This type of controller is the most often used, however, in a slightly different form from the ideal one which will be first presented. The operating output of the ideal PID controller is proportional to the weighted sum of the magnitude, the integral and the time rate of change of the error

$$u_a(t) = K_c \left(e(t) + \tau_D \frac{de(t)}{dt} + \frac{1}{\tau_I} \int_0^t e(x)\, dx \right) + u_{ab} \tag{2.8}$$

The transfer function of the PID controller is equal to

$$G_c(s) = K_c \left(1 + \tau_D\, s + \frac{1}{\tau_I s} \right) \tag{2.9}$$

The previous remark on the physical unrealizability of the derivative action is still valid.

Philosophy: owing to the derivative action, the PID controller takes into account (anticipates) the future, and owing to the integral action, the PID takes into account (integrates) the past.

Remark 2.1 The previous theoretical controller is, in practice, replaced by a real PID controller of the following transfer function

$$G_c(s) = K_c \left(\frac{\tau_I s + 1}{\tau_I s} \right) \left(\frac{\tau_D s + 1}{\beta\, \tau_D s + 1} \right) \tag{2.10}$$

which is physically realizable.

Remark 2.2 It is often preferred to operate the PID controller by making the derivative action act no more on the error coming from the comparator but on the measured output, under the theoretical form

$$u_a(t) = K_c \left(e(t) + \frac{1}{\tau_I} \int_0^t e(x)\, dx - \tau_D \frac{dy_m}{dt} \right) \tag{2.11}$$

or practically

$$U_a(s) = K_c \left(1 + \frac{1}{\tau_I s}\right) E(s) - \left(\frac{K_c\, \tau_D s}{\frac{\tau_D}{N} s + 1}\right) Y_m(s) \tag{2.12}$$

This method of taking into account the derivative action allows to avoid brutal changes of the controller output due to the error signal variation.

2.1.3 Sensors

This point may seem simple; indeed, in a real process, it is an essential element. Without good measurement, it is hopeless to control the process well. The sensor itself and the information transmission chain given by the sensor are concerned. The common sensors that are met on chemical processes are:

- Temperature sensors: thermocouples, platinum resistance probes, pyrometers. Temperature sensors can be modelled from the response they give to a temperature step according to a first- or second-order models, sometimes with a time delay.
- Pressure sensors: classical manometers using bellows, Bourdon tube, membrane or electronic ones using strain gauges (semiconductors whose resistance changes under strain). Diaphragm pressure sensors use detection of the diaphragm position by measurement of electrical capacitance. They are often represented by a second-order model.
- Flow rate sensors: for gas flow such as thermal mass flow meters (based on the thermal conductivity of gases, the gas flow inducing a temperature variation in a capillary tube), variable area flow meters (displacement of a float in a conical vertical tube); liquid flow such as turbine flow meters (rotation of a turbine), depression flow meters as venturi- type flow meters (the flow rate is proportional to the square root of the pressure drop), vortex flow meters (measurement of the frequency of vortex shedding due to the presence of an unstreamlined obstacle), electromagnetic flow meters (for electrically conducting fluids), sonic flow meters, Coriolis effect flow meters. Flow rate sensors have very fast dynamics and are often modelled by an equation of the form

  $$\text{flow rate} = a\sqrt{\Delta P}$$

  where the proportionality constant a is dependent on the sensor, and ΔP is the pressure drop between the section restriction point and the outlet. These signals are often noisy because of flow fluctuations and should be filtered before being used by the controller.
- Level sensors: floats (lighter than the fluid), displacement (measurement of the apparent weight of a half-submerged cylinder) through a pressure difference

measurement, conductivity probes indicating liquid presence, capacitance detectors for level variations.

- Composition sensors: potentiometers (chemical potential measurement of an ion by means of a specific electrode), conductimeters (measurement of a solution conductivity), chromatographs (separation of liquids or gases), spectrometers (visible, UV, infrared, etc.). Among these, chromatographs pose a particular and very important problem in practice: the information provided by these apparatus arrives a long time after the sampling, and thus, there exists a large time delay that must be included in the model. This time delay can be the cause of a lack of mastering or imperfect mastering of the process control.

In the absence of a measurement concerning a given variable, if a model of the process is available, it is possible to realize a state observer called a software sensor. This latter will use other available measurements to estimate the value of the unmeasured variable, and it is comparable to an indirect measurement. The linear Kalman filter or nonlinear (extended Kalman filter) is often used for this purpose (see Sect. 11.1.2.1 and 18.4.3). Chemical composition estimations are particularly concerned by this type of sensor.

The transmitter is the intermediary between the sensitive element of the sensor and the controller. It is a simple converter which is then considered as a simple gain, ratio of the difference of the output signal (often transmitted in the range 4–20 mA) over the difference of the input signal given by the sensor.

The set sensor-transmitter can be considered as a global measurement device.

2.1.4 Transmission Lines

Traditionally, transmission lines were pneumatic. Nowadays, more and more often they are electrical lines. In general, their dynamic influence on the process is neglected except in the case of very fast phenomena, which is not very common in chemical engineering.

2.1.5 Actuators

Actuators Considine (1999) constitute material elements which allow action by means of the control loop on the process. For example, as a flow rate actuator, a very common element is the pneumatic valve, operating as indicated by its name with pressurized air. It could as well be mechanically operated by a dc or a stepping motor (Fig. 2.3).

A valve is designed to be in position, either completely open (fail open, or air-to-open) or completely closed (fail closed, or air-to-close) when the air pressure is not ensured, which can happen in the case of an incident in the process. Consider the

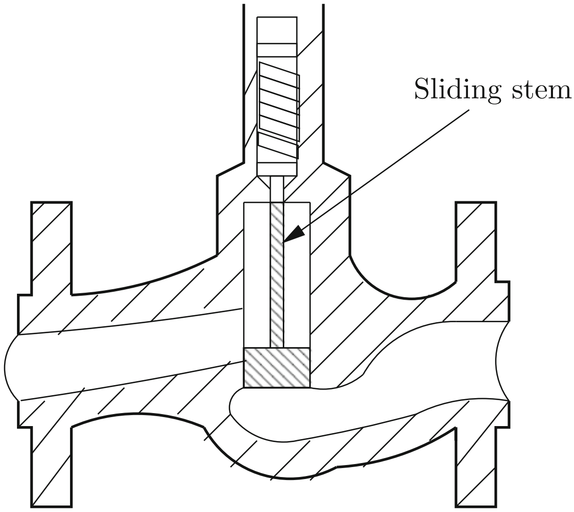


Fig. 2.3 Scheme of a typical sliding stem valve

case of a valve closed in the absence of pressure: when the air pressure increases on the diaphragm, the spring is compressed and the valve plug pulls out from its seat, thus increasing the passage section for the fluid, hence the flow rate. The inverse type of valve (open in the absence of pressure) exists where the pressure increase makes the valve plug go down (either because of the position of the air inlet with respect to the diaphragm or because of the disposal of the seat and the valve plug) and thus the cross section decreases. The choice of valves is generally made by taking into account safety rules. There also exists motorized valves: rotating valves (butterfly, ball). In general, the valve dynamics is fast. It must not be forgotten that the valve introduces a pressure drop in the pipe. With respect to control, a valve should not be operated too close to its limits, either completely open or completely closed, where its behaviour will be neither reproducible nor easily controllable. Frequently, a valve has a highly nonlinear behaviour on all its operating range, and it is necessary to linearize it piecewise and use the constructed table for the control law.

For liquids, the flow rate Q depends on the square root of the pressure drop ΔP_v caused by the valve according to

$$Q = C_v \sqrt{\frac{\Delta P}{d}} \tag{2.13}$$

where d is the fluid density (with respect to water), and C_v is a flow rate coefficient such that the ratio (C_v/D_v^2) is approximately constant for liquids for a given type of valve, D_v being the nominal valve diameter. Viscosity corrections are required for C_v in the case of viscous liquids.

For gases or vapours, when the flow is subsonic, the volume gas flow rate is

$$Q = 0.92\, C_f\, C_v\, P_{up}\, (Y - 0.148\, Y^3)\, \frac{1}{\sqrt{d_g\, T_{up}}} \tag{2.14}$$

where Q is given in $m^3.s^{-1}$ at 15 °C under 1 normal atm, P_{up} is the upstream pressure (in Pa), T_{up} is the upstream temperature, Y is the dimensionless expansion factor, and d_g is the gas density (with respect to air). C_f is a dimensionless factor which depends on the type of fittings of the valve and ranges from 0.80 to 0.98. Y is equal to

$$Y = \frac{1.63}{C_f} \sqrt{\frac{\Delta P}{P_{up}}} \tag{2.15}$$

When $Y < 1.5$, the flow is subsonic; when $Y \geq 1.5$, the flow is sonic, i.e. choked.

When the flow is sonic, the volume gas flow rate is

$$Q = 0.92\, C_f\, C_v\, P_{up}\, \frac{1}{\sqrt{d_g\, T_{up}}}. \tag{2.16}$$

The ratio q of the real flow rate Q to the maximum flow rate Q_m

$$q = \frac{Q}{Q_m} \tag{2.17}$$

can depend on the aperture degree x of the valve in several ways (Fig. 2.4). Denoting the sensibility as $\sigma = dq/dx$, the latter can be constant (linear behaviour: case 1), or increase with x (case 2), or decrease with x (case 3), or increase then decrease (case 4) Midoux (1985). Often, three main types of valves are distinguished Thomas (1999): with linear characteristics, with butterfly characteristics and with equal percentage characteristics. Denoting the position of the valve positioner by x, the equations for the valve constant are, respectively,

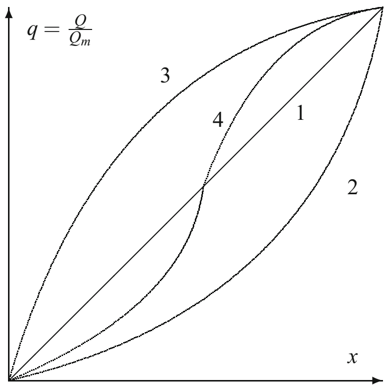


Fig. 2.4 Influence of the aperture degree of a valve on its flow rate

$$\begin{aligned} &\text{Linear:} && C_v = C_{vs}\, x \\ &\text{Butterfly:} && C_v = C_{vs} \left(1 - \cos\left(\frac{\pi}{2} x\right)\right) \\ &\text{Equal percentage:} && C_v = C_{vs}\, R_v^{x-1} \end{aligned} \tag{2.18}$$

2.2 Block Diagrams, Signal-Flow Graphs, Calculation Rules

The study of feedback control for single-input single-output processes is performed in this chapter using Laplace transfer functions. It would be possible to do the same technical realizations and their theoretical study based on state-space modelling. On the other hand, the theoretical discussion and mathematical tools would be completely different. A sketch of the state-space study nevertheless will be presented.

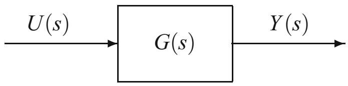
When specialized packages for solving control problems (e.g. MATLAB®) are used, it is very easy to find the state-space model equivalent to a transfer function. Furthermore, it is possible to set blocks in series or in parallel, to do feedback loops, either with transfer functions or in state space, and then to realize a complete block diagram in view of a simulation. However, an important difference exists between both approaches. When the studied system becomes complicated, the numerical solving based on transfer functions gives worse and maybe erroneous results, compared to the complete state-space solving. The reason is in the far more direct approach of the phenomena in state space and their direct numerical solving.

Given a block diagram in which each block represents a transfer function, the output of any block must be calculated with respect to the input of any other block. Beyond the blocks of transfer functions, the block diagram uses summators, which do the algebraic addition of inlet signals, and signal dividers, which separate a signal into two or several signals of same intensity. Most of the common cases are represented in Figs. 2.5, 2.6, 2.7, 2.8, 2.9, 2.10 and 2.11, and results are given in the transfer function case and in the state-space case.

Calculation Rules with Laplace Transform

Fig. 2.5 Block scheme number 2

Fig. 2.6 Block scheme number 1



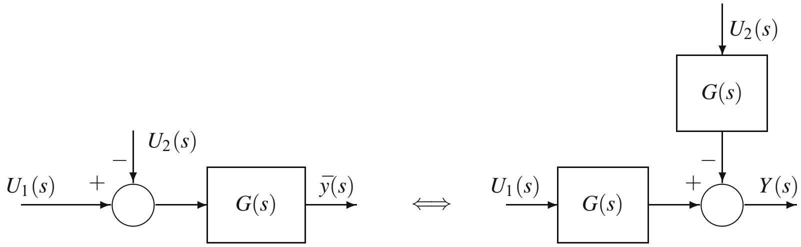


Fig. 2.7 Block scheme number 3 under two equivalent representations

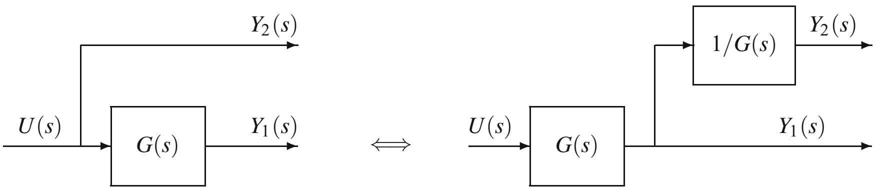


Fig. 2.8 Block scheme number 4 under two equivalent representations

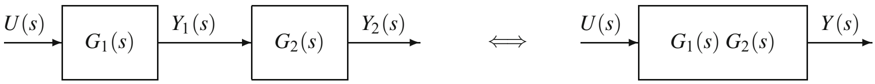


Fig. 2.9 Block scheme number 5 under two equivalent representations

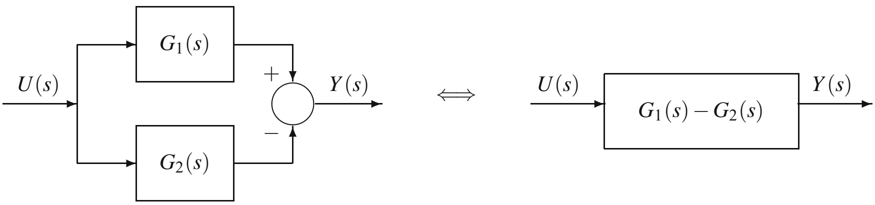


Fig. 2.10 Block scheme number 6 under two equivalent representations

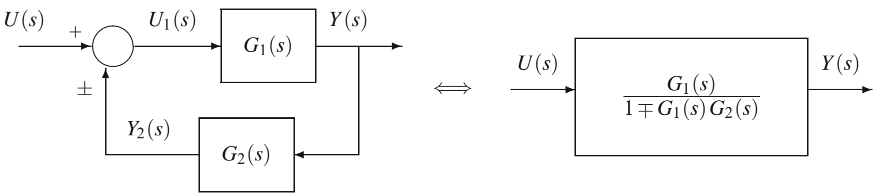


Fig. 2.11 Block scheme number 7 under two equivalent representations

For block scheme number 1 (Fig. 2.6), which contains two summators, the Laplace transform equation is

$$Y(s) = U_1(s) + U_2(s) - U_3(s) \tag{2.19}$$

For block scheme number 2 (Fig. 2.5), which contains only one transfer function, the Laplace transform equation is

$$Y(s) = GU(s) \tag{2.20}$$

For block scheme number 3 (Fig. 2.7), which contains one transfer function and a summator, the Laplace transform equation is

$$Y(s) = G\,(U_1(s) - U_2(s)) \tag{2.21}$$

For block scheme number 4 (Fig. 2.8), which contains one transfer function and a signal divider, the Laplace transform equations are

$$Y_1(s) = G\,U(s) \quad ; \quad Y_2(s) = U(s) \tag{2.22}$$

For block scheme number 5 (Fig. 2.9), which contains two transfer functions in series, the Laplace transform equation is

$$Y(s) = G_1\,G_2\,U(s) \tag{2.23}$$

For block scheme number 6 (Fig. 2.10), which contains two transfer functions in parallel and a summator, the Laplace transform equation is

$$Y(s) = (G_1 - G_2)\,U(s) \tag{2.24}$$

For block scheme number 7 (Fig. 2.11), which contains two transfer functions in a feedback loop containing a summator with the feedback of sign ε ($\varepsilon = +1$ for a positive feedback, $\varepsilon = -1$ for a negative feedback), the Laplace transform equation is

$$Y(s) = \frac{G_1}{1 + \varepsilon\,G1\,G_2}\,U(s) \tag{2.25}$$

Calculation rules in state space

In state space, signals are directly considered with respect to the time variable and each system or block number i is represented by the set of matrices (A_i, B_i, C_i, D_i). Recall that if a system can be represented by a strictly proper transfer function, the matrix D_i is zero and this is the case of most physical systems. Equations are given in the case of single-input single-output systems.
Block scheme number 1

$$y(t) = u_1(t) + u_2(t) - u_3(t) \tag{2.26}$$

Block scheme number 2

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,u(t) \\ y(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,u(t) \end{cases} \tag{2.27}$$

Block scheme number 3

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,(u_1(t) - u_2(t)) \\ y(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,(u_1(t) - u_2(t)) \end{cases} \tag{2.28}$$

Block scheme number 4

$$\begin{cases} \dot{\mathbf{x}}(t) = \;\mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,u(t) \\ y_1(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,u(t) \\ y_2(t) = u(t) \end{cases} \tag{2.29}$$

Block scheme number 5, two systems in series: equations for each block are the following

$$\begin{cases} \dot{\mathbf{x}}_1(t) = \mathbf{A}_1\,\mathbf{x}_1(t) + \mathbf{B}_1\,u(t) \\ y_1(t) = \mathbf{C}_1\,\mathbf{x}_1(t) + \mathbf{D}_1\,u(t) \\ \dot{\mathbf{x}}_2(t) = \mathbf{A}_2\,\mathbf{x}_2(t) + \mathbf{B}_2\,y_1(t) \\ y(t) = \;\mathbf{C}_2\,\mathbf{x}_2(t) + \mathbf{D}_2\,y_1(t) \end{cases} \tag{2.30}$$

Defining the global state vector, union of both state vectors:

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} \tag{2.31}$$

one obtains for two systems in series

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{\mathbf{x}}_1(t) \\ \dot{\mathbf{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & 0 \\ \mathbf{B}_2\,\mathbf{C}_1 & \mathbf{A}_2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2\,\mathbf{D}_1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \mathbf{D}_2\,\mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix} \mathbf{x}(t) + \mathbf{D}_2\,\mathbf{D}_1\,u(t) \end{cases} \tag{2.32}$$

Block scheme number 6, two systems in parallel

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{\mathbf{x}}_1(t) \\ \dot{\mathbf{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \mathbf{C}_1 & -\mathbf{C}_2 \end{bmatrix} \mathbf{x}(t) + (\mathbf{D}_1 - \mathbf{D}_2)\,u(t) \end{cases} \tag{2.33}$$

Block scheme number 7, feedback loop of sign ε:
the general case where $\mathbf{D}_1$ and $\mathbf{D}_2$ are not zero is first treated.

The basic equations are the following

$$\begin{cases} \dot{\mathbf{x}}_1(t) = \mathbf{A}_1\,\mathbf{x}_1(t) + \mathbf{B}_1\,u_1(t) \\ y(t) = \mathbf{C}_1\,\mathbf{x}_1(t) + \mathbf{D}_1\,u_1(t) \\ \dot{\mathbf{x}}_2(t) = \mathbf{A}_2\,\mathbf{x}_2(t) + \mathbf{B}_2\,y(t) \\ y_2(t) = \mathbf{C}_2\,\mathbf{x}_2(t) + \mathbf{D}_2\,y(t) \\ u_1(t) = u(t) + \varepsilon\,y_2(t) \end{cases} \tag{2.34}$$

Eliminating internal variables $u_1(t)$ and $y_2(t)$, one obtains

$$\begin{aligned} &\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{\mathbf{x}}_1(t) \\ \dot{\mathbf{x}}_2(t) \end{bmatrix} = \\ &\begin{bmatrix} \mathbf{A}_1 + \varepsilon\mathbf{B}_1\mathbf{D}_2\mathbf{C}_1\,[\mathbf{I} - \varepsilon\mathbf{D}_1\mathbf{D}_2]^{-1} & \varepsilon\mathbf{B}_1\mathbf{C}_2 + \mathbf{B}_1\mathbf{D}_2\mathbf{D}_1\mathbf{C}_2\,[\mathbf{I} - \varepsilon\mathbf{D}_1\mathbf{D}_2]^{-1} \\ \mathbf{B}_2\mathbf{C}_1\,\,[\mathbf{I} - \varepsilon\mathbf{D}_1\mathbf{D}_2]^{-1} & \mathbf{A}_2 + \varepsilon\mathbf{B}_2\mathbf{D}_1\mathbf{C}_2\,[\mathbf{I} - \varepsilon\mathbf{D}_1\mathbf{D}_2]^{-1} \end{bmatrix} \mathbf{x}(t) \\ &+ \begin{bmatrix} \mathbf{B}_1 + \varepsilon\,\mathbf{B}_1\,\mathbf{D}_2\,\mathbf{D}_1\,\,[\mathbf{I} - \varepsilon\,\mathbf{D}_1\,\mathbf{D}_2]^{-1} \\ \mathbf{B}_2\,\mathbf{D}_1\,\,[\mathbf{I} - \varepsilon\,\mathbf{D}_1\,\mathbf{D}_2]^{-1} \end{bmatrix} u(t) \\ &y(t) = \begin{bmatrix} \mathbf{C}_1\,\,[\mathbf{I} - \varepsilon\,\mathbf{D}_1\,\mathbf{D}_2]^{-1} \\ \varepsilon\,\mathbf{D}_1\,\mathbf{C}_2\,\,[\mathbf{I} - \varepsilon\,\mathbf{D}_1\,\mathbf{D}_2]^{-1} \end{bmatrix} \mathbf{x}(t) + \mathbf{D}_1\,\,[\mathbf{I} - \varepsilon\,\mathbf{D}_1\,\mathbf{D}_2]^{-1}\,u(t) \end{aligned} \tag{2.35}$$

When both transfer functions are strictly proper, and matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ zero, equations can be simplified as

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{\mathbf{x}}_1(t) \\ \dot{\mathbf{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \varepsilon\,\mathbf{B}_1\,\mathbf{C}_2 \\ \mathbf{B}_2\,\mathbf{C}_1 & \mathbf{A}_2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \mathbf{C}_1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases} \tag{2.36}$$

Mason Formula and Signal-Flow Graphs

The Mason formula allows us to quickly calculate global transfer functions for a block scheme where each block represents a transfer function. A complicated scheme such as in Fig. 2.12 is considered.

This block scheme has two external inputs y_r and d, and the transfer functions set point-output G_{yr} and disturbance-output G_{yd} must be, respectively, calculated

$$Y(s) = G_{yr}\,Y_r(s) + G_{yd}\,D(s) \tag{2.37}$$

Each block is directed from input towards output and is called unidirectional. A loop is a unidirectional path which starts and ends at a same point, and along which no point is met more than once. A loop transmittance is equal to the product of the transfer functions of the loop. When a loop includes summators, the concerned signs must be taken into account in the calculation of the loop transmittance.
Rather than directly working on the block diagram such as it is currently described, it is preferable to transform this scheme into a signal-flow graph, which contains the topological information of the set of linear equations included in the block diagram. The word signal-flow (or signal flow) means a flow of fluxes or signals.

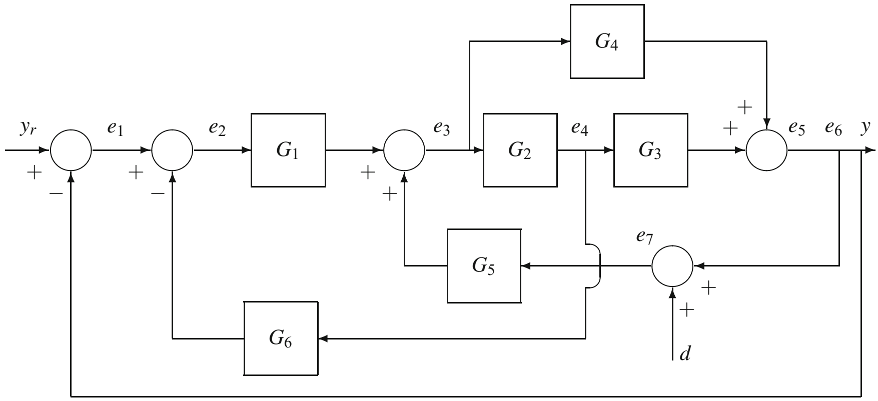


Fig. 2.12 Block scheme of a closed-loop process

To operate, some characteristic definitions of signal-flow graphs must be added:

- A signal-flow graph is made of nodes and connecting branches (a line with an arrow).
- A node is attributed to each variable which occurs in the system. The node i represents the variable y_i for example.
- For a branch beginning in i and ending in j, the transmittance a_{ij} of the branch relates variables y_i and y_j.
- A source is a node from where only branches go out.
- A sink is a node where only branches come in.
- A path is a group of connected branches having the same direction.
- A direct path comes from a source and ends in a sink; furthermore, no node should be met more than once.
- A path transmittance is the product of the transmittances associated with the branches of this path.
- A feedback loop B_i is a path coming from a node i and ending at the same node i. Along a loop, a given node cannot be met more than once.
- A transmittance of a loop B_i is the product of the transmittances associated with the branches of this loop.
- Loops B_i and B_j are nontouching when they have no node in common.

First, let us present some simple cases that allow us to understand signal-flow graphs as well as the associated equations, which are all linear.
Additions:
Graph 2.13 corresponds to the linear equation

$$y_3 = a_1\, y_1 + a_2\, y_2 \tag{2.38}$$

and graph 2.14 corresponds to the linear equation

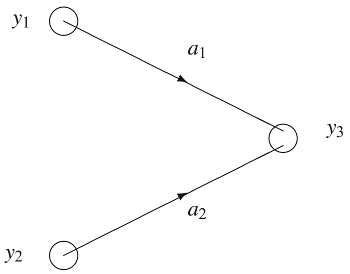


Fig. 2.13 Signal-flow graph: addition

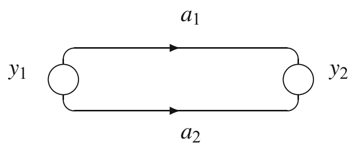


Fig. 2.14 Signal-flow graph: addition

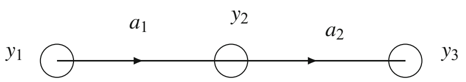


Fig. 2.15 Signal-flow graph: multiplication

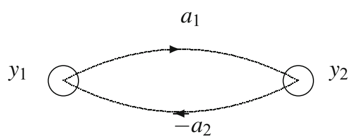


Fig. 2.16 Signal-flow graph: feedback

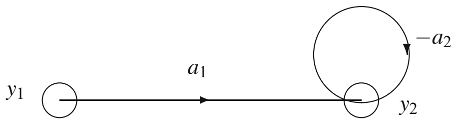


Fig. 2.17 Signal-flow graph: feedback

$$y_2 = (a_1 + a_2)\, y_1 \tag{2.39}$$

Multiplication:
Graph 2.15 corresponds to the linear equation

$$y_3 = a_2\, y_2 = a_1\, a_2\, y_1 \tag{2.40}$$

The transmittance of path from 1 to 3 is $a_1 a_2$.
Feedback:
Graph 2.16 corresponds to the linear equation

$$y_2 = a_1\, y_1 - a_2\, a_1\, y_2 \Longrightarrow y_2 = \frac{a_1}{1 + a_1\, a_2}\, y_1 \tag{2.41}$$

The transmittance of path from 1 to 2 is $\frac{a_1}{1+a_1\,a_2}$.
Graph 2.17 corresponds to the linear equation

$$y_2 = a_1\,y_1 - a_2\,y_2 \Longrightarrow y_2 = \frac{a_1}{1+a_2}\,y_1. \tag{2.42}$$

Then, the Δ characteristic function of the block scheme or determinant of the signal-flow graph is defined as

$$\begin{aligned} \Delta = 1 &- \sum \text{(transmittances of the loops)},\\ &+ \sum \text{(products of the transmittances of all nontouching loops}\\ &\qquad \text{considered two by two)},\\ &- \sum \text{(products of the transmittances of all nontouching loops}\\ &\qquad \text{considered three by three)},\\ &\ldots \end{aligned}$$

Note that Δ is independent of the input and the output.

- A direct path from an input u_i to an output y_j is any connection of directed branches and of blocks between i and j such that no point is met more than once. The input u_i and the output y_j are connected by k direct paths each having the transmittance T_{ijk}. Let Δ_{ijk} also be the determinant of each direct path calculated according to the previous formula, giving Δ by setting equal to 0 all transmittances of the loops which touch the kth direct path from i to j (suppress all the nodes and the branches of this direct path).

According to the Mason formula, the transfer function of the input u_i to the output y_j is equal to

$$G_{ij} = \frac{\sum_{k=1}^{k_{ij}} T_{ijk}\,\Delta_{ijk}}{\Delta} \tag{2.43}$$

with

$$Y_j(s) = G_{ij}\,U_i(s). \tag{2.44}$$

The signal-flow graph corresponding to the previous block diagram 2.12 is given in Fig. 2.18.

It might have been possible on this graph to merge E_6 and Y: variables have here been distinguished to make the sink Y clearly appear.

In this signal-flow graph, one wishes to calculate transfer functions from $Y_r(s)$ to $Y(s)$ and from $D(s)$ to $Y(s)$, respectively, G_{ry} and G_{dy}. In this graph, five loops exist with respective transmittances: $G_2G_3G_5$, $-G_1G_2G_3$, $-G_1G_2G_6$, $-G_1G_4$ and G_4G_5. There exist no two-by-two nontouching loops. The determinant of this graph is thus equal to

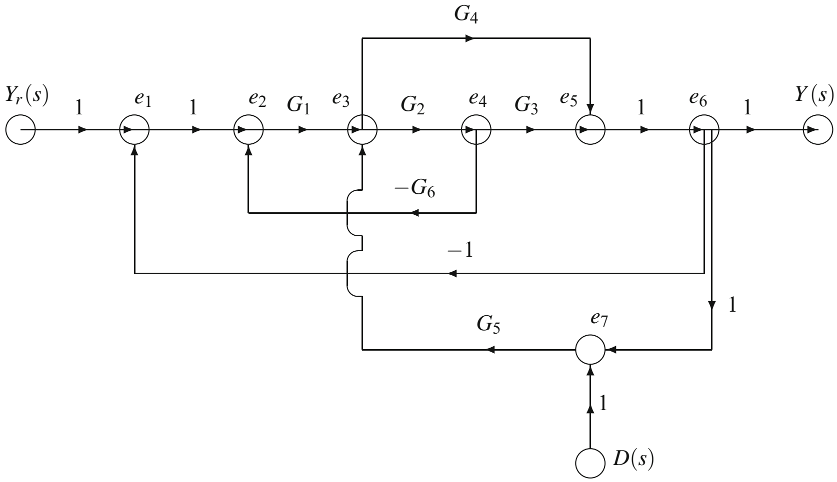


Fig. 2.18 Signal-flow graph corresponding to block diagram 2.12

$$\begin{aligned}\Delta &= 1-(G_2G_3G_5 - G_1G_2G_3 - G_1G_2G_6 - G_1G_4 + G_4G_5)\\ &= 1 + G_1G_2G_6 + (G_2G_3+G_4)(G_1-G_5).\end{aligned} \tag{2.45}$$

To find the transfer function G_{ry}, two direct paths must be noticed, one is $re_1e_2e_3e_4e_5e_6y$ with transmittance $T_{ry1} = G_1G_2G_3$ for which $\Delta_{ry1} = 1$, and the other one is $re_1e_2e_3e_5e_6y$ with transmittance $T_{ry2} = G_1G_4$ for which $\Delta_{ry2} = 1$. The transfer function G_{ry} is thus equal to

$$G_{ry} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2G_6 + (G_2G_3+G_4)(G_1-G_5)}. \tag{2.46}$$

To find the transfer function G_{dy}, there exist two direct paths: $de_7e_3e_4e_5e_6y$ with transmittance $T_{dy1} = G_2G_3G_5$ for which $\Delta_{dy1} = 1$, and the other is $de_7e_3e_5e_6y$ with transmittance $T_{dy2} = G_4G_5$ for which $\Delta_{dy2} = 1$. The transfer function G_{dy} is thus equal to

$$G_{dy} = \frac{G_2G_3G_5 + G_4G_5}{1 + G_1G_2G_6 + (G_2G_3+G_4)(G_1-G_5)}. \tag{2.47}$$

Globally, one obtains

$$Y(s) = G_{ry}\, Y_r(s) + G_{dy}\, D(s) \tag{2.48}$$

2.3 Dynamics of Feedback-Controlled Processes

The block scheme of feedback control makes use of previously studied elements with respect to their general operating principle. In the block scheme of the process and control system (Fig. 2.19), independent external inputs are on one the hand the set point $y_r(t)$ imposed by the user and on the other hand the disturbance $d(t)$ not mastered by the user; these inputs influence the output $y(t)$. Indeed, the process could be subjected to several disturbances. The process undergoes differently the action of the control variable $u(t)$ and of the disturbance $d(t)$; thus, this corresponds to distinct transfer functions denoted, respectively, by $G_d(s)$ and $G_p(s)$, so that, as a Laplace transform, the output $Y(s)$ is written as

$$Y(s) = G_p(s)U(s) + G_d(s)D(s) \tag{2.49}$$

A summator will be used for the block representation. Instead of representing the block diagram in time space, it is represented as a function of the Laplace variable s.

Other transfer functions indicate the functions of different devices:

- Measurement:

$$Y_m(s) = G_m(s)Y(s) \tag{2.50}$$

  It must be noted that the measured variable $y_m(t)$ generally does not have the same dimension as the corresponding output $y(t)$. For example, if the output is a temperature expressed in degrees Celsius, the variable measured by a thermocouple is in mV, and hence, the steady-state gain of the transfer function has the dimension of mV/Celsius. Similarly, for any transfer function, the steady-state gain has unit

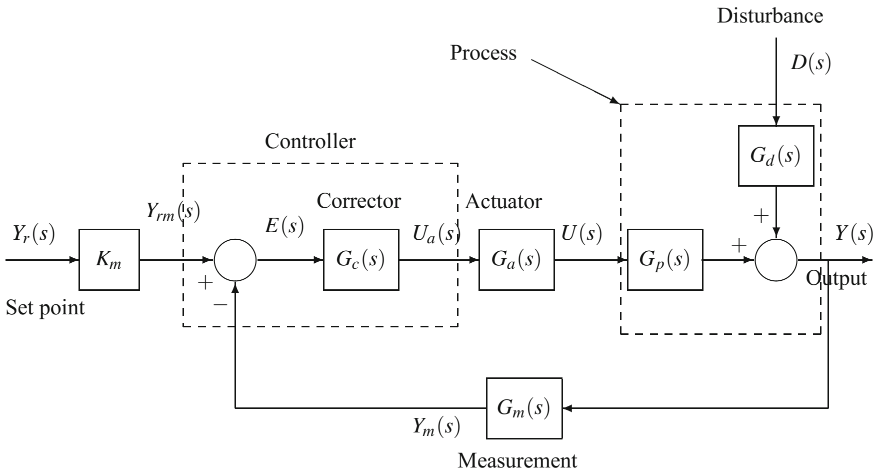


Fig. 2.19 Block scheme of the closed-loop process

dimensions. Moreover, the sensor may introduce dynamics given by the transfer function $G_m(s)$.

- Regulation:

$$U_a(s) = G_c(s)E(s) \tag{2.51}$$

with $E(s)$ being the error equal to

$$E(s) = K_m\, Y_r(s) - Y_m(s) = K_m\, Y_r(s) - G_m Y(s) \tag{2.52}$$

- Compensation of the measurement:

If the set point y_r is expressed in the same units as the output y (pressure in bar, temperature in C or K, ...), it is necessary to introduce a block to compensate the measurement (Fig. 2.19) so that the measured output y_m and the compensated set point y_{rm} have the same dimension (e.g. mA or mV), which is in general different from the output one. The gain K_m of the measurement compensation block is equal to the steady-state gain of the measurement transfer function G_m.

It is also possible to express the set point $y_r(t)$ in the same units as the measured output $y_m(t)$, and in this case, it is not necessary anymore to compensate the measurement ($K_m = 1$).

In the case of measurement compensation, this pure gain K_m is calculated by

$$K_m = \lim_{s \to 0} G_m(s) = G_m(0) \tag{2.53}$$

so that the compensated set point is equal to

$$Y_{rm}(s) = K_m\, Y_r(s) \tag{2.54}$$

- Actuator:

$$U(s) = G_a(s)U_a(s). \tag{2.55}$$

From these equations, it is interesting to express the output $Y(s)$ with respect to the set point $Y_r(s)$ and the disturbance $D(s)$. One obtains

$$Y(s) = G_p(s)\, G_a(s)\, G_c(s)\, E(s) + G_d(s)D(s) \tag{2.56}$$

or, by expressing $E(s)$,

$$Y(s) = G_p(s)\, G_a(s)\, G_c(s)\, [K_m\, Y_r(s) - G_m(s)\, Y(s)] + G_d(s)D(s) \tag{2.57}$$

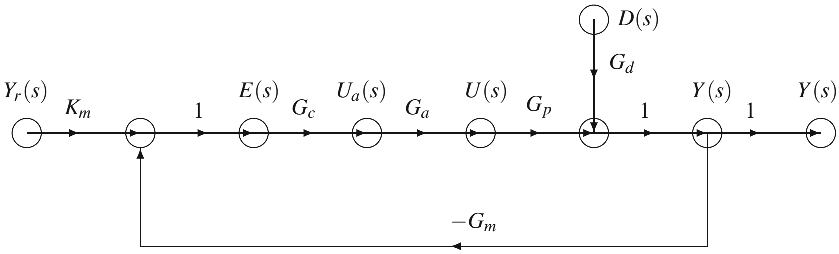


Fig. 2.20 Signal-flow graph corresponding to the block diagram of the feedback control (Fig. 2.19)

The process closed-loop response $Y(s)$ is thus equal to

$$Y(s) = \frac{G_p(s)\, G_a(s)\, G_c(s)\, K_m}{1 + G_p(s)\, G_a(s)\, G_c(s)\, G_m(s)} Y_r(s) + \frac{G_d(s)}{1 + G_p(s)\, G_a(s)\, G_c(s)\, G_m(s)} D(s) \tag{2.58}$$

The first term represents the influence of a change of the set point $Y_r(s)$ and the second term the influence of a change of disturbance $D(s)$. The closed-loop transfer function for a set point variation will be

$$G_{\text{set point}} = \frac{G_p(s)\, G_a(s)\, G_c(s)\, K_m}{1 + G_p(s)\, G_a(s)\, G_c(s)\, G_m(s)} \tag{2.59}$$

and similarly the closed-loop transfer function for a disturbance variation

$$G_{\text{disturbance}} = \frac{G_d(s)}{1 + G_p(s)\, G_a(s)\, G_c(s)\, G_m(s)} \tag{2.60}$$

The denominators of both closed-loop transfer functions are identical. These closed-loop transfer functions depend not only on the process dynamics, but also on the actuator, measurement device and the controller's own dynamics.

The application of the Mason formula would give the previous expressions. The signal-flow graph is given by Fig. 2.20.

In the present case, only one loop exists, and the graph determinant is equal to

$$\Delta = 1 - (-G_c G_a G_p G_m) \tag{2.61}$$

Between the set point and the output, only one direct path exists with transmittance: $T_1 = K_m G_c G_a G_p$ and determinant $\Delta_1 = 1$, so that the transfer function from the set point to the output is equal to

$$\frac{Y(s)}{Y_r(s)} = \frac{G_c G_a G_p K_m}{1 + G_c G_a G_p G_m} \tag{2.62}$$

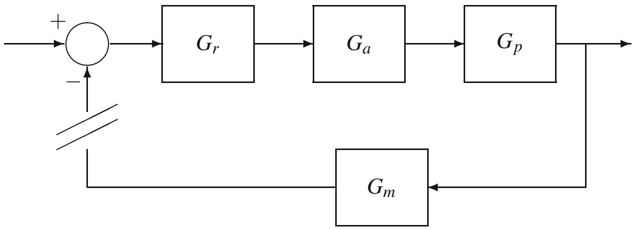


Fig. 2.21 How the "open loop" must be understood by opposition to the closed loop

Similarly between the disturbance and the output, only one direct path exists with transmittance: $T_1 = G_d$ and determinant $\Delta_1 = 1$, so that the transfer function from the disturbance to the output is equal to

$$\frac{Y(s)}{D(s)} = \frac{G_d}{1 + G_c G_a G_p G_m}. \tag{2.63}$$

The calculation of the transfer functions can be resumed in this simple case in the following manner:

The closed-loop transfer function is equal to [product of the transfer functions met on the path between an input and an output] over [1 + the product of all transfer functions met in the loop]. So, between $D(s)$ and $Y(s)$, only G_d is met, while between $Y_r(s)$ and $Y(s)$ we meet K_m, G_c, G_a, G_p. In the loop, G_c, G_a, G_p, G_m are met. The product $G_c\, G_a\, G_p\, G_m$, which appears in the denominator of the closed-loop transfer functions, is often called open-loop transfer function, as it corresponds to the transfer function of the open loop obtained by opening the loop before the comparator as can be done for an electrical circuit (Fig. 2.21). This open-loop transfer function acts as an important role in the study of the stability of the closed-loop system.
Two types of control problems will be studied in particular:

Regulation:
The set point is fixed ($Y_r(s) = 0$), and the process is subjected to disturbances. The control system reacts so as to maintain $y(t)$ at the set point value and tries to reject the disturbances: it is also called disturbance rejection.

Tracking:
It is assumed (in order to simplify the study) that the disturbance is constant ($D(s) = 0$) while the set point is now variable; the problem is to maintain $y(t)$ as close as possible to varying $y_r(t)$.

2.3.1 Study of Different Actions

To display the influence of different actions, only first- and second-order systems will be studied. Moreover, to simplify calculations, it will be assumed that the transfer

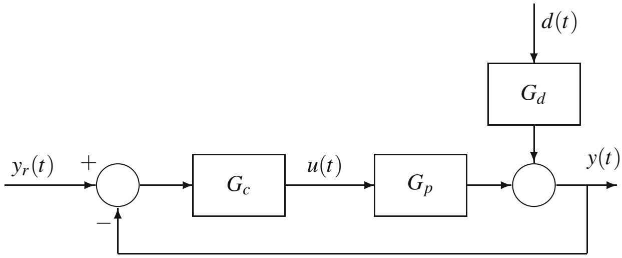


Fig. 2.22 Block diagram for the study of the action of the different controllers

functions of the actuator and of measurement are both equal to unity

$$G_a = 1\ , \ G_m = 1\ , \ K_m = 1 \tag{2.64}$$

resulting in simplified Fig. 2.22. For these first- and second-order systems with different types of controller, responses to steps of set point or disturbance are given in Figs. 2.29, 2.30, 2.31, 2.32 and commented on in the following sections.

As an aside, to simplify, it will be assumed that the order of process transfer function G_p and the order of the transfer function G_d dealing with the disturbance are equal (which is by no means compulsory), but that these transfer functions have different gains and time constants.

2.3.2 Influence of Proportional Action

As the controller is proportional, its transfer function is

$$G_c = K_c \tag{2.65}$$

2.3.2.1 First-Order Systems

A first-order system is described by a differential equation such as

$$\tau_p \frac{dy(t)}{dt} + y(t) = u(t) \tag{2.66}$$

where $y(t)$ is a deviation variable such that $y(0) = 0$ and $(dy/dt)_0 = 0$.

The process transfer function linking the output Laplace transform $Y(s)$ to the input Laplace transform $U(s)$ is equal to

$$G_p(s) = \frac{K_p}{\tau_p\, s + 1} \tag{2.67}$$

The transfer function for the disturbance is also assumed to be first-order

$$G_d(s) = \frac{K_d}{\tau_d\, s + 1} \tag{2.68}$$

The output $Y(s)$ for any set point $Y_r(s)$ and any disturbance $D(s)$ is thus equal to

$$Y(s) = \frac{K_p\, K_c}{\tau_p\, s + 1 + K_p\, K_c}\, Y_r(s) + \frac{K_d}{\tau_d\, s + 1}\, \frac{\tau_p\, s + 1}{\tau_p\, s + 1 + K_p\, K_c}\, D(s) \tag{2.69}$$

If we set $s \to 0$ (equivalent to an infinite time) in the transfer functions, the closed-loop transfer functions are, respectively, equal to the closed-loop steady-state gains (use of the final value theorem) for the set point y_r and the disturbance d

$$K'_p = \frac{K_p\, K_c}{1 + K_p\, K_c} \tag{2.70}$$

$$K'_d = \frac{K_d}{1 + K_p\, K_c} \tag{2.71}$$

The closed-loop gain K'_p is modified compared to the open-loop gain K_p; K'_p tends towards 1 when the controller gain is large. The closed-loop gain relative to the disturbance K'_d is lower than the open-loop gain K_d and tends towards 0 when the controller gain is large. The closed-loop response is still first-order with respect to set point and disturbance variations.

The open-loop time constant is τ_p; in closed loop, concerning set point variations, it is equal to

$$\tau'_p = \frac{\tau_p}{1 + K_p\, K_c} \tag{2.72}$$

thus, it has decreased; the response will be faster in closed loop than in open loop.

Consider the response to a step variation of set point (tracking) or disturbance (regulation):

Tracking study:
The set point change is a step of amplitude A

$$Y_r(s) = \frac{A}{s} \tag{2.73}$$

The disturbance is assumed constant or zero ($D(s) = 0$). The closed-loop response (Fig. 2.23) to a set point step is then equal to

$$Y(s) = \frac{K_p\, K_c}{\tau_p\, s + 1 + K_p\, K_c}\, \frac{A}{s} = \frac{K'_p}{\tau'_p\, s + 1}\, \frac{A}{s} \tag{2.74}$$

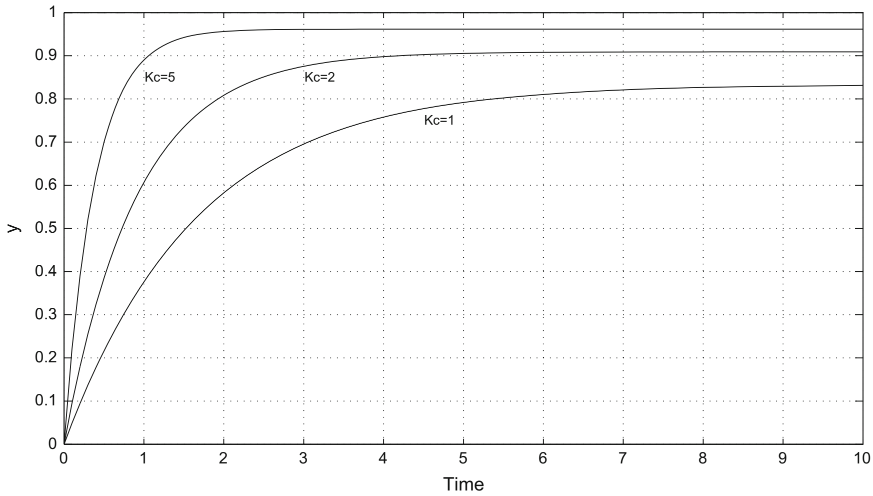


Fig. 2.23 Response of a first-order system ($K_p = 5$, $\tau_p = 10$) to a set point unit step (proportional controller with increasing gain: $K_c = 1, 2, 5$)

To get the time response $y(t)$, $Y(s)$ is decomposed into a sum of rational fractions, the first corresponding to the forced response $Y_f(s)$ and the second to the natural response $Y_n(s)$

$$Y(s) = A\frac{K'_p}{s} - A\frac{K'_p\,\tau'_p}{\tau'_p s + 1} = Y_f(s) + Y_n(s) \tag{2.75}$$

hence

$$\begin{aligned} &y(t) = AK'_p\left(1 - \exp(-t/\tau'_p)\right) = y_f(t) + y_n(t) \\ &\text{with:}\quad y_f(t) = AK'_p; \quad y_n(t) = -AK'_p\exp(-t/\tau'_p) \end{aligned} \tag{2.76}$$

Figure 2.23 was obtained for a unit step of a set point. The asymptotic value of the output presents an offset with the set point; if the controller gain K_c is increased, this offset decreases (Fig. 2.23). In practice, other transfer functions must be taken into account: actuator and measurement, so that this set may not behave exactly as a first-order process and present, e.g. a time delay, nonlinearities or neglected dynamics. The choice of too large gains for the proportional controller may render the closed-loop behaviour oscillatory or unstable. A high gain decreases the response time, imposing a more important demand to the actuator: the control variable $u(t)$ varies more strongly and more quickly so that it can reach its limits, and it is then saturated.

Regulation study:
Recall that the transfer function for the disturbance was taken to be first-order as that of the process, but with different gain $K_d \neq K_p$ and time constant $\tau_d \neq \tau_p$.

Consider a disturbance step variation of amplitude A

$$D(s) = \frac{A}{s} \tag{2.77}$$

The set point is assumed constant (regulation). The closed-loop response (Fig. 2.31) to a step disturbance is then equal to

$$Y(s) = \frac{K_d}{\tau_d s + 1} \frac{\tau_p s + 1}{\tau_p s + 1 + K_p K_c} \frac{A}{s} \tag{2.78}$$

To get the time response $y(t)$, $Y(s)$ is decomposed into

$$\begin{aligned} Y(s) &= A \left(\frac{K_d}{1 + K_p\, K_c} \frac{1}{s} + \frac{K_d\, \tau_d\, (\tau_p - \tau_d)}{\tau_d\, (1 + K_p\, K_c) - \tau_p} \frac{1}{\tau_d s + 1} \right. \\ &\quad \left. + \frac{K_d\, K_p\, K_c\, \tau_p^2}{(\tau_p - \tau_d\, (1 + K_p\, K_c))\, (1 + K_p\, K_c)} \frac{1}{\tau_p s + 1 + K_p\, K_c} \right) \\ &= A \left(\frac{c_1}{s} + \frac{c_2}{\tau_d s + 1} + \frac{c_3}{\tau_p s + 1 + K_p\, K_c} \right) \end{aligned} \tag{2.79}$$

hence the closed-loop response

$$y(t) = A \left[c_1 + c_2\, \exp(-t/\tau_d) + c_3\, \exp(-t\, (1 + K_p\, K_c)/\tau_p) \right] \tag{2.80}$$

In the absence of a controller, the process is stable and the output tends towards $A K_d$. When the proportional controller is introduced, the process remains stable; for a unit step disturbance, the output tends towards a new limit $A K_d'$ and deviates with respect to the set point from this value $A K_d'$.

When the gain of the proportional controller increases, the deviation output-set point decreases and the influence of the disturbance decreases too.

Consider now a disturbance impulse variation of amplitude A. The closed-loop response to this disturbance is then equal to

$$Y(s) = A \frac{K_d}{\tau_d s + 1} \frac{\tau_p s + 1}{\tau_p s + 1 + K_p K_c} \tag{2.81}$$

Application of the final value theorem gives

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} [s\, Y(s)] = 0 \tag{2.82}$$

Thus, impulse disturbances are rejected by a simple proportional controller.

2.3.2.2 Second-Order Systems

Again, the actuator and measurement gains and transfer functions are taken to be equal to 1.

In the case of a second-order system, the process transfer function is

$$G_p(s) = \frac{K_p}{\tau_p^2\, s^2 + 2\, \zeta_p\, \tau_p\, s + 1} \tag{2.83}$$

Assume that the transfer function for the disturbance is also second-order

$$G_d(s) = \frac{K_d}{\tau_d^2\, s^2 + 2\, \zeta_d\, \tau_d\, s + 1} \tag{2.84}$$

The output $Y(s)$ for any set point $Y_r(s)$ and any disturbance $D(s)$ is equal to

$$\begin{aligned} Y(s) = & \frac{K_p\, K_c}{\tau_p^2\, s^2 + 2\, \zeta_p\, \tau_p\, s + 1 + K_p\, K_c}\, Y_r(s) \\ & + \frac{K_d}{\tau_d^2\, s^2 + 2\, \zeta_d\, \tau_d\, s + 1}\, \frac{\tau_p^2\, s^2 + 2\, \zeta_p\, \tau_p\, s + 1}{\tau_p^2\, s^2 + 2\, \zeta_p\, \tau_p\, s + 1 + K_p\, K_c}\, D(s) \end{aligned} \tag{2.85}$$

Tracking study:
In Eq. (2.85), only the term of the set point variation is concerned. For a set point step variation of amplitude A, $Y_r(s)$ becomes A/s, and $Y(s)$ is decomposed into a sum of two fractions, the natural response of order 2 corresponding to the first factor of the previous expression and the forced response in $1/s$. The closed-loop transfer function remains second-order as in open loop. The period and the damping factor are modified

$$\tau_p' = \frac{\tau_p}{\sqrt{1 + K_p\, K_c}} \qquad \zeta_p' = \frac{\zeta_p}{\sqrt{1 + K_p\, K_c}} \tag{2.86}$$

The steady-state gain becomes

$$K_p' = \frac{K_p\, K_c}{1 + K_p\, K_c}. \tag{2.87}$$

Like for the first-order system, a deviation between the set point and the asymptotic response exists (Fig. 2.30), which is all the more important as the gain is low.

Regulation study:
As the influence of disturbance is studied, the second term of Eq. (2.85) is taken into account. The closed-loop response remains second-order as in open loop. The proportional controller is not sufficient to reject the disturbance: a deviation between the set point and the asymptotic value still exists (Fig. 2.32).

A proportional controller does not change the order of the process; the steady-state gain is modified, decreased in two cases (a/ if $K_p > 1$, or b/ if $K_c > 1/(1 - K_p)$ when $K_p < 1$): the time constants also decrease.

2.3.3 Influence of Integral Action

The study is similar to that realized in the case of the proportional controller and will be consequently less detailed.

The transfer function of a PI controller is equal to:

$$G_c(s) = K_c \left(1 + \frac{1}{\tau_I\, s}\right). \tag{2.88}$$

2.3.3.1 First-Order Process and Influence of Pure Integral Action

Though integral action is never used alone, in this section, in order to characterize its influence, we first assume that the controller is pure integral and has the following transfer function

$$G_c(s) = \frac{K_c}{\tau_I\, s} \tag{2.89}$$

In the case of a first-order process, the response $Y(s)$ to a set point or disturbance variation is equal to

$$Y(s) = \frac{\dfrac{K_p}{1+\tau_p\, s}\, \dfrac{K_c}{\tau_I\, s}}{1 + \dfrac{K_p}{1+\tau_p\, s}\, \dfrac{K_c}{\tau_I\, s}} Y_r(s) + \frac{\dfrac{K_d}{1+\tau_d\, s}}{1 + \dfrac{K_p}{1+\tau_p\, s}\, \dfrac{K_c}{\tau_I\, s}} D(s) \tag{2.90}$$

or

$$Y(s) = \frac{1}{\dfrac{\tau_p\, \tau_I}{K_p\, K_c}\, s^2 + \dfrac{\tau_I}{K_p\, K_c}\, s + 1} Y_r(s) + \frac{K_d}{\tau_d\, s + 1}\, \frac{(\tau_p\, s + 1)\, \tau_I\, s}{\tau_I \tau_p\, s^2 + \tau_I\, s + K_p\, K_c} D(s) \tag{2.91}$$

The integral controller has modified the system order: the transfer function of the closed-loop system is now of order 2, i.e. larger by one unity than the order of the open-loop system. The natural period of the closed-loop system is equal to

$$\tau'_p = \sqrt{\frac{\tau_p\, \tau_I}{K_p\, K_c}} \tag{2.92}$$

and the damping factor

$$\zeta'_p = \frac{1}{2}\sqrt{\frac{\tau_I}{\tau_p\, K_p\, K_c}} \tag{2.93}$$

As the response of a first-order process in open loop becomes second-order in closed loop, its dynamics is completely different. According to the value of the damping factor ζ', the response will be overdamped, or underdamped, possibly explosive. If the controller gain is increased, keeping constant the integral time constant, the natural period and the damping factor decrease, and thus, the response will be less sluggish, but the displacement will move progressively from overdamped responses towards oscillatory responses.

It is interesting to study the tracking, i.e. the response to a set point variation. The set point undergoes a step variation of amplitude A

$$Y_r(s) = \frac{A}{s} \tag{2.94}$$

hence

$$Y(s) = \frac{1}{\dfrac{\tau_p\,\tau_I}{K_p\,K_c}s^2 + \dfrac{\tau_I}{K_p\,K_c}s + 1}\,\frac{A}{s} \tag{2.95}$$

To find the asymptotic behaviour, the final value theorem gives

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0}[s\,Y(s)] = A \tag{2.96}$$

Thus, the limit of $y(t)$ is equal to A, the set point value. We thus find the important result that the integral action eliminates the asymptotic deviation. The value of the set point is reached faster when the gain is high, but at the expense of oscillatory responses. According to the type of controlled variable, it is preferable to rather choose an overdamped response (not going beyond the set point, e.g. for a chemical reactor which could undergo runaway above some safety temperature) or oscillatory (rapidly reach a state close to the set point).

Let us study in a similar way the regulation and thus the influence of a disturbance. Consider a disturbance step change of amplitude A that could not be rejected by the proportional controller

$$D(s) = \frac{A}{s} \tag{2.97}$$

hence

$$Y(s) = \frac{K_d}{\tau_d\,s + 1}\,\frac{(\tau_p\,s + 1)\,\tau_I\,s}{\tau_I\tau_p\,s^2 + \tau_I\,s + K_p\,K_c}\,\frac{A}{s} \tag{2.98}$$

The final value theorem gives

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0}[s\,Y(s)] = 0 \tag{2.99}$$

Thus, step-like disturbances which were not rejected by a proportional controller are perfectly rejected owing to the integral action.

Of course, impulse disturbances are also rejected by the PI controller.

2.3.3.2 First-Order Process with PI Controller

The transfer function of the PI controller is equal to

$$G_c(s) = K_c \left(1 + \frac{1}{\tau_I\, s}\right) \tag{2.100}$$

hence the general closed-loop response to set point and disturbance variations

$$Y(s) = \frac{\dfrac{K_p}{1+\tau_p\, s}\, K_c \left(1 + \dfrac{1}{\tau_I\, s}\right)}{1 + \dfrac{K_p}{1+\tau_p\, s}\, K_c \left(1 + \dfrac{1}{\tau_I\, s}\right)}\, Y_r(s) + \frac{\dfrac{K_d}{1+\tau_d\, s}}{1 + \dfrac{K_p}{1+\tau_p\, s}\, K_c \left(1 + \dfrac{1}{\tau_I\, s}\right)}\, D(s) \tag{2.101}$$

or

$$\begin{aligned} Y(s) = & \frac{\tau_I\, s + 1}{\dfrac{\tau_p\, \tau_I}{K_p\, K_c} s^2 + \tau_I\, \dfrac{K_p\, K_c + 1}{K_p\, K_c} s + 1}\, Y_r(s) + \\ & \frac{K_d}{\tau_d\, s + 1}\, \frac{\dfrac{\tau_I}{K_p\, K_c}\, s\,(1 + \tau_p\, s)}{\dfrac{\tau_p\, \tau_I}{K_p\, K_c} s^2 + \tau_I\, \dfrac{K_p\, K_c + 1}{K_p\, K_c} s + 1}\, D(s) \end{aligned} \tag{2.102}$$

Compared to the proportional action alone, the order of each transfer function output/set point or output/disturbance increases by one unit.

Previously drawn conclusions for the integral action alone remain globally true:

- In tracking, during a set point step variation (Fig. 2.29), the output tends towards the set point even for low controller gain.
- In regulation, impulse disturbances are of course rejected, but also step disturbances (Fig. 2.31).

To be convinced, it suffices to use the final value theorem.

2.3.3.3 Second-Order Process

Similarly, in the case of a second-order process in open loop, the closed-loop output with a pure integral controller would be of the immediately next order, i.e. a third-order.

The PI controller used in the case of the tracking corresponding to Fig. 2.24 leads to oscillations all the more important for the second-order system in that the integral time constant is lower and thus that the integral gain increases. The deviation with respect to the set point is cancelled by the integral action either for a set point variation (Fig. 2.30), or a disturbance variation (Fig. 2.32). With the overshoot increasing with the integral gain, it will be frequently necessary not to choose too large an integral

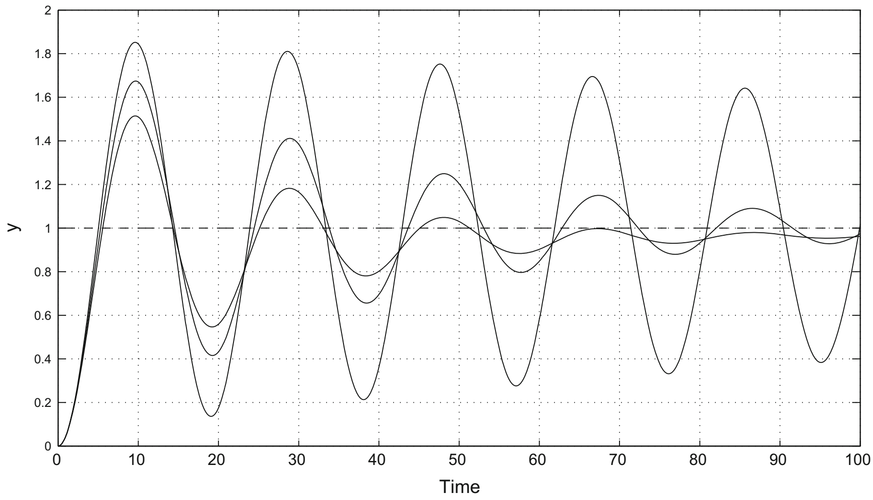


Fig. 2.24 Response of a second-order system ($K_p = 5$, $\tau_p = 10$, $\zeta_p = 0.5$) to a set point unit step with influence of the integral time constant (PI controller $K_c = 2$, $\tau_I = 10$ or 20 or 100). When τ_I increases, the oscillation amplitude decreases

gain. In those figures, the gain and the integral time constant have not been optimized, as the objective was only to display the influence of the integral action.

2.3.4 *Influence of Derivative Action*

The transfer function of a pure ideal derivative controller is equal to

$$G_c(s) = K_c\,\tau_D\,s \tag{2.103}$$

This transfer function is improper, and the ideal PID controller (Fig. 2.25) of transfer function

$$G_c(s) = K_c \left(1 + \frac{1}{\tau_I s} + \tau_D\,s\right) \tag{2.104}$$

is not realizable, as the numerator degree is larger than the denominator degree because of the ideal derivative action term. If this controller were used as such, it would amplify high-frequency noise because its amplitude ratio is unlimited at high frequency (see frequency analysis, Chap. 5).

The following study simply aims to demonstrate the characteristics of pure derivative action.

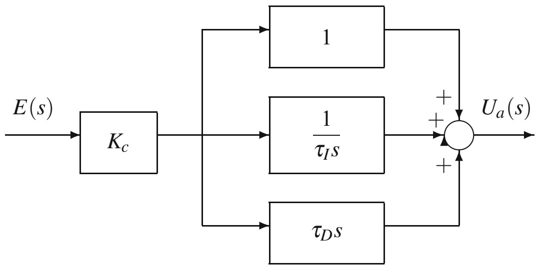


Fig. 2.25 Block diagram of the ideal PID controller

2.3.4.1 First-Order Process and Pure Derivative Action

In the case of a first-order process, if one only looks at the influence of derivative action with the controller given by Eq. (2.103), the response $Y(s)$ to a set point or disturbance variation is equal to

$$\begin{aligned} Y(s) &= \frac{\dfrac{K_p}{\tau_p\, s+1} K_c\, \tau_D\, s}{1+\dfrac{K_p}{\tau_p\, s+1} K_c\, \tau_D\, s} Y_r(s) + \frac{\dfrac{K_d}{\tau_d\, s+1}}{1+\dfrac{K_p}{\tau_p\, s+1} K_c\, \tau_D\, s} D(s) \\ &= \frac{K_p\, K_c\, \tau_D\, s}{(\tau_p + K_p\, K_c\, \tau_D)s+1} Y_r(s) + \frac{K_d}{\tau_d\, s+1}\, \frac{\tau_p\, s+1}{(\tau_p + K_p\, K_c\, \tau_D)s+1} D(s) \end{aligned} \tag{2.105}$$

Transfer functions are first-order as in open loop, and thus, the derivative action has no influence on the system order. On the other hand, the derivative action introduces a lead term in the numerator. The closed-loop time constant is equal to

$$\tau'_p = \tau_p + K_p\, K_c\, \tau_D \tag{2.106}$$

and thus is increased with respect to the open loop; the closed-loop response will be slower than the open-loop one, and this effect increases with the derivative controller gain. This will help to stabilize the process if the latter shows tendencies to oscillations in the absence of the derivative action.

2.3.4.2 First-Order Process with Real PID Controller

Compared to the PI controller previously studied, a physically realizable derivative action is introduced by the following real PID controller (Fig. 2.26) of transfer function

$$G_c(s) = K_c \left(\frac{\tau_I s + 1}{\tau_I s}\right) \left(\frac{\tau_D s + 1}{\beta\, \tau_D s + 1}\right) \tag{2.107}$$

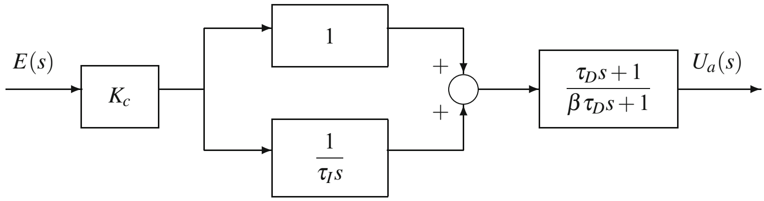


Fig. 2.26 Block diagram of the real PID controller given by Eq. (2.107)

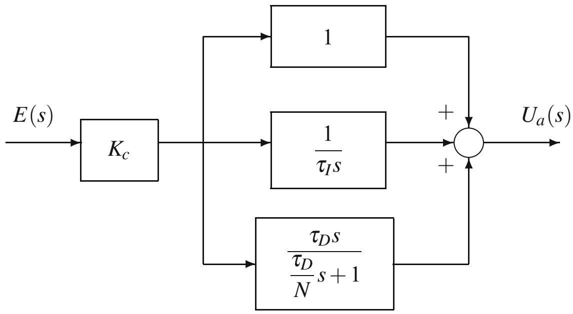


Fig. 2.27 Block diagram of the real PID controller given by Eq. (2.108)

which is physically realizable. This transfer function can be seen as the filtering of an ideal PID controller by a first-order filter. In the case of a pneumatic PID controller, β is included between 0.1 and 0.2. For the electronic PID controller, one sets $0 < \beta \ll 1$.

A real PID controller (Fig. 2.27) can also respond to the following slightly different equation, which is frequently used

$$G_c(s) = K_c \left(1 + \frac{1}{\tau_I s} + \frac{\tau_D s}{\frac{\tau_D}{N} s + 1} \right) \tag{2.108}$$

In the case of studied first-order processes, the derivative action in the PID controller does not seem to add an important effect with respect to integral action alone, as the studied process already presents a closed-loop overdamped behaviour with the PI controller. If for other parameter values the closed-loop behaviour had been underdamped, the addition of derivative action would have allowed considerable decrease of oscillations which would have become acceptable as in the following case of the second-order process (Fig. 2.28). The influence of the derivative action is clearer in response with respect to a disturbance step variation (Fig. 2.31) than in response with respect to a set point step variation (Fig. 2.29). It is shown that the overshoot is decreased. The derivative action thus brings a stabilizing influence with respect to integral action (Fig. 2.30).

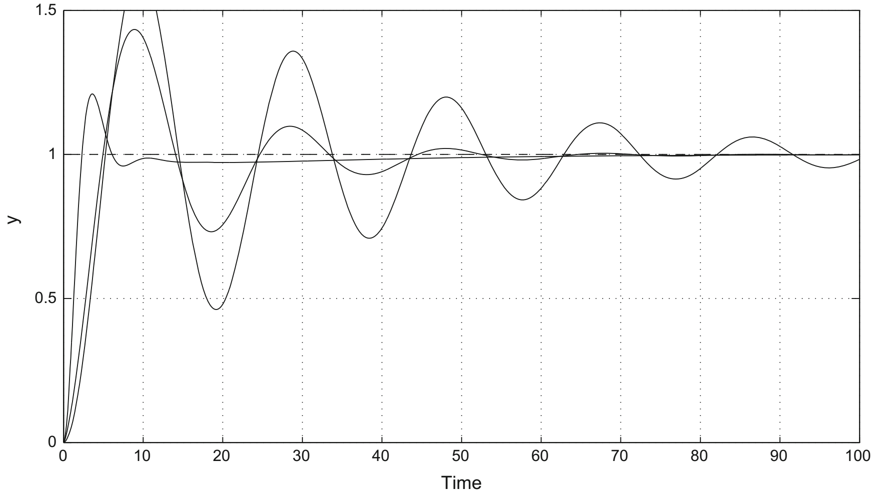


Fig. 2.28 Response of a second-order system ($K_p = 5, \tau_p = 10, \zeta_p = 0.5$) to a set point unit step (real PID controller with influence of the derivative time constant τ_D: $K_c = 2, \tau_I = 20, \tau_D = 0.1$ or 1 or 10, $\beta = 0.1$). When τ_D increases, oscillations decrease

2.3.4.3 Second-Order Process

In the case of a second-order process and pure integral action, the response $Y(s)$ to a set point variation is equal to

$$Y(s) = \frac{G_p\, K_c\, \tau_D\, s}{1 + G_p\, K_c\, \tau_D\, s} Y_r(s) = \frac{K_p\, K_c\, \tau_D\, s}{\tau^2\, s^2 + 2\,\zeta\,\tau\, s + 1 + K_p\, K_c\, \tau_D\, s} Y_r(s) \tag{2.109}$$

The closed-loop response is second-order as it was in open loop. The derivative controller does not modify the order of the response

$$Y(s) = \frac{K_p\, K_c\, \tau_D\, s}{\tau^2\, s^2 + (2\,\zeta\,\tau\; + K_p\, K_c\, \tau_D)s + 1} Y_r(s) \tag{2.110}$$

In this case, the time constant τ remains the same while the damping factor of the closed-loop response is modified with respect to the open-loop damping factor and becomes

$$\zeta_p' = \frac{2\,\zeta_p\,\tau + K_p\, K_c\, \tau_D}{2\,\tau} \tag{2.111}$$

The response is thus more damped in closed loop than in open loop, and this damping increases with the gain K_c of the derivative controller and with the derivative time constant.

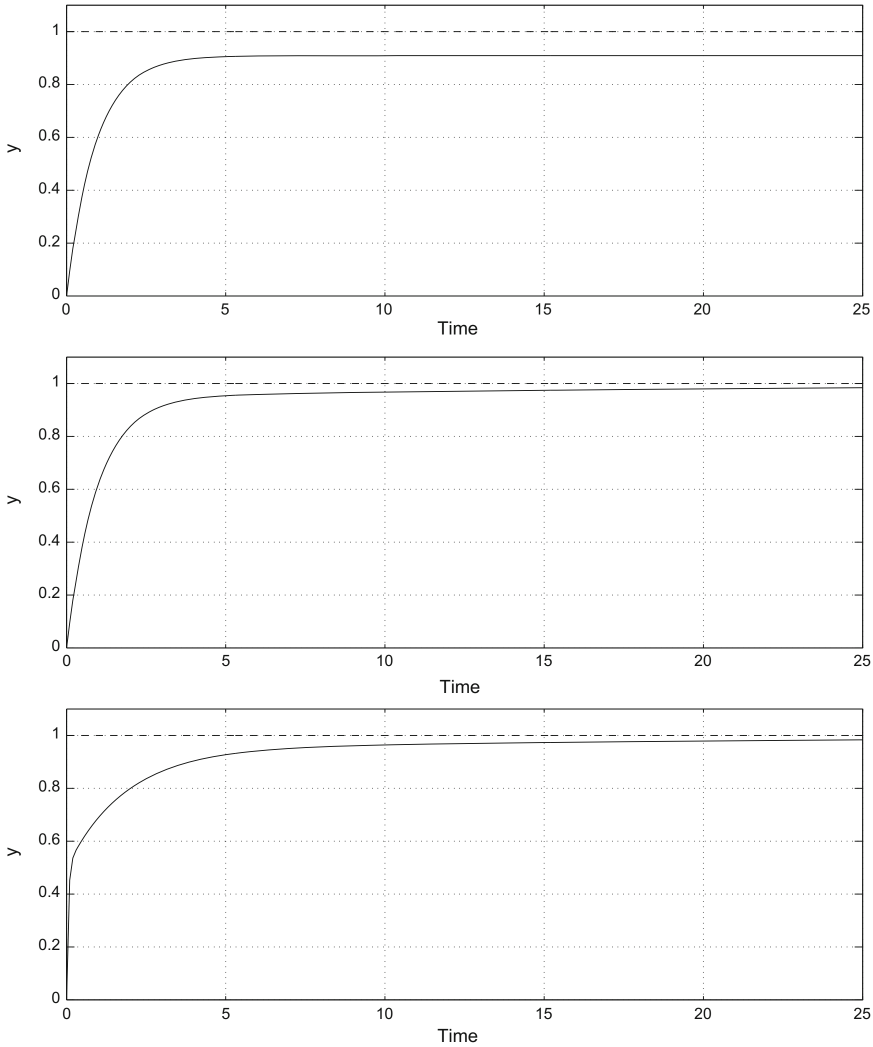


Fig. 2.29 Comparison of the influence of the controller type on the response of a first-order system ($K_p = 5$, $\tau_p = 10$) to a set point unit step. (Proportional: $K_c = 2$ (*top*). Proportional-integral: $K_c = 2$, $\tau_I = 20$ (*middle*). Real proportional-integral-derivative: $K_c = 2$, $\tau_I = 20$, $\tau_D = 1$, $\beta = 0.1$ (*bottom*))

Globally, the same effect is noticed with a real PID controller of transfer function given by Eq. (2.107).

Compared to the integral action which cancels asymptotic deviation but leads to strong oscillations, the addition of real derivative action strongly decreases oscillations which become acceptable, all the more so as the derivative time constant τ_D is higher (Fig. 2.28). However, it must be noted that the increase of derivative action

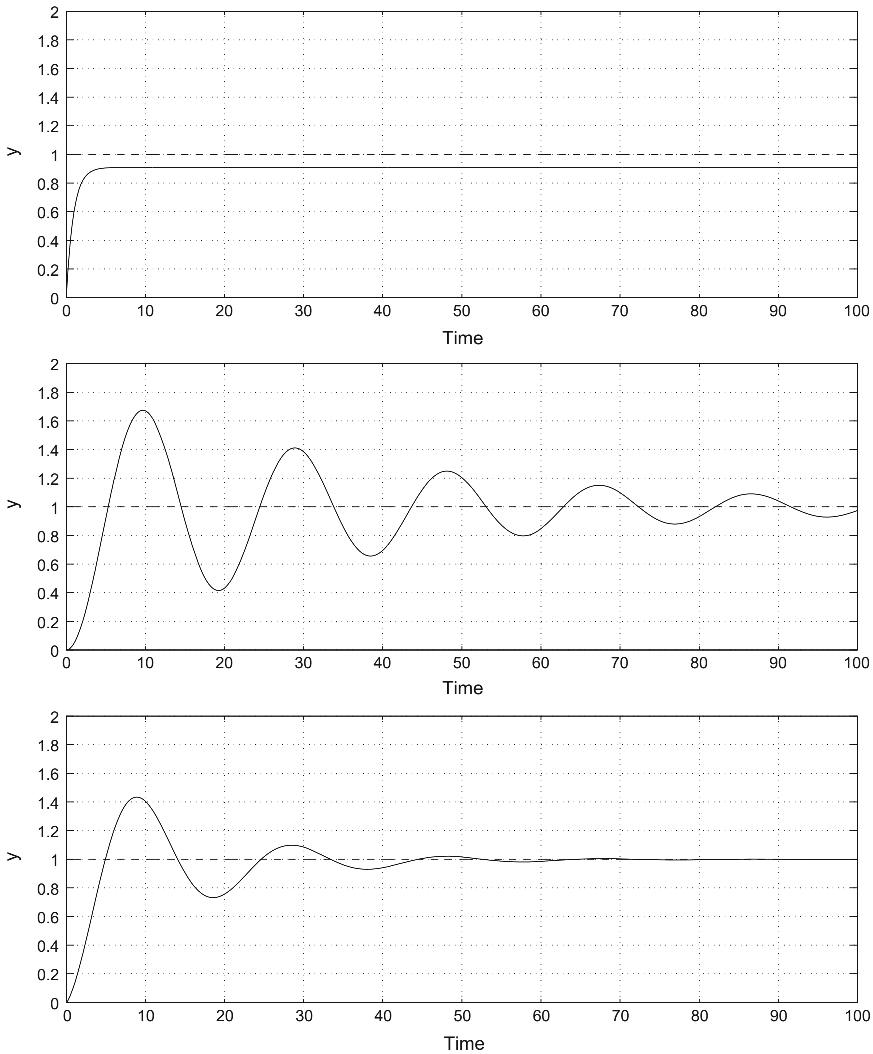


Fig. 2.30 Comparison of the influence of the controller type on the response of a second-order system ($K_p = 5$, $\tau_p = 10$, $\zeta_p = 0.5$) to a set point unit step. (Proportional: $K_c = 2$ (*top*). Proportional-integral: $K_c = 2$, $\tau_I = 20$ (*middle*). Real proportional-integral-derivative: $K_c = 2$, $\tau_I = 20$, $\tau_D = 1$, $\beta = 0.1$ (*bottom*))

tends to increase measurement noise and that this effect is not wished, so that a too large value of τ_D must be avoided in practice. The derivative action brings a stabilizing effect with respect to integral action. The overshoot is also decreased. These two effects are clear in the study of the influence of either a set point step variation (Fig. 2.30) or a disturbance step variation (Fig. 2.32).

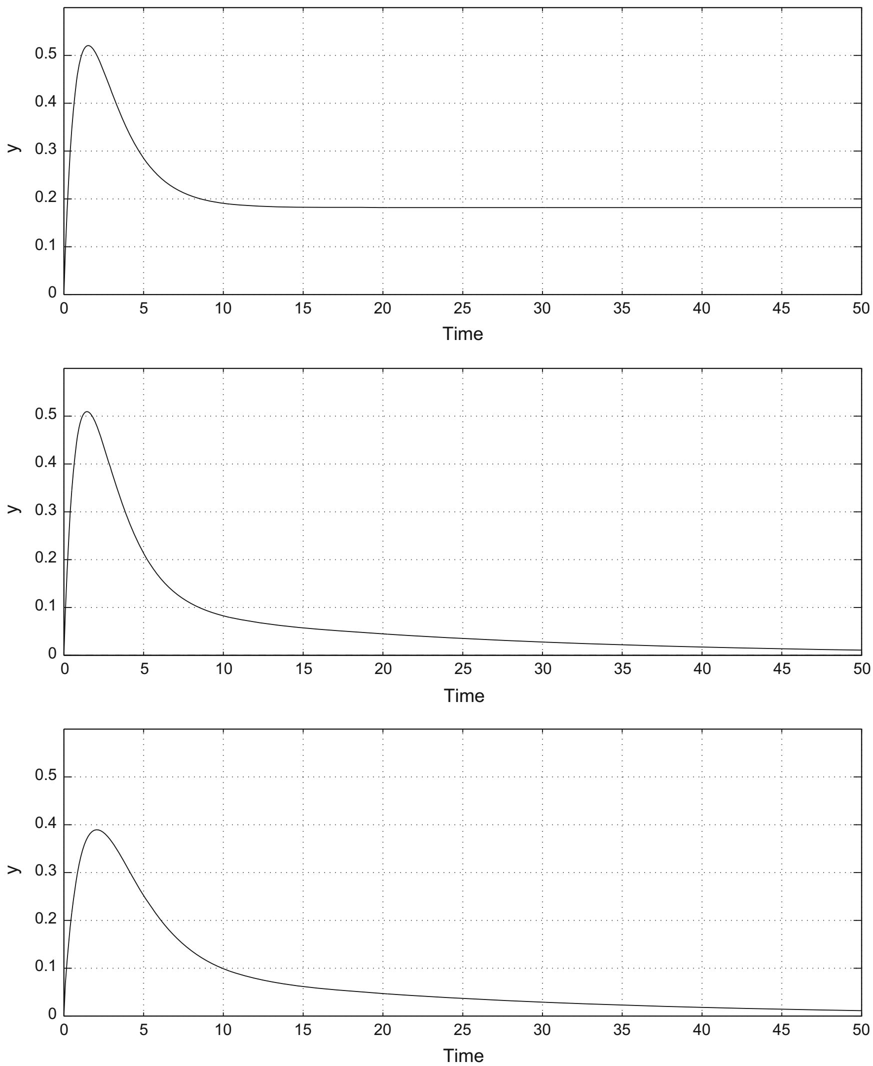


Fig. 2.31 Comparison of the influence of the controller type on the response of a first-order system ($K_p = 5, \tau_p = 10 : K_d = 2, \tau_d = 2$) to a disturbance unit step. (Proportional: $K_c = 2$ (*top*). Proportional-integral: $K_c = 2, \tau_I = 20$ (*middle*). Real proportional-integral-derivative: $K_c = 2, \tau_I = 20, \tau_D = 1, \beta = 0.1$ (*bottom*))

2.3.5 Summary of Controllers Characteristics

A proportional (P) controller contains only one tuning parameter: the controller gain. The asymptotic output presents a deviation from the set point, which can be decreased

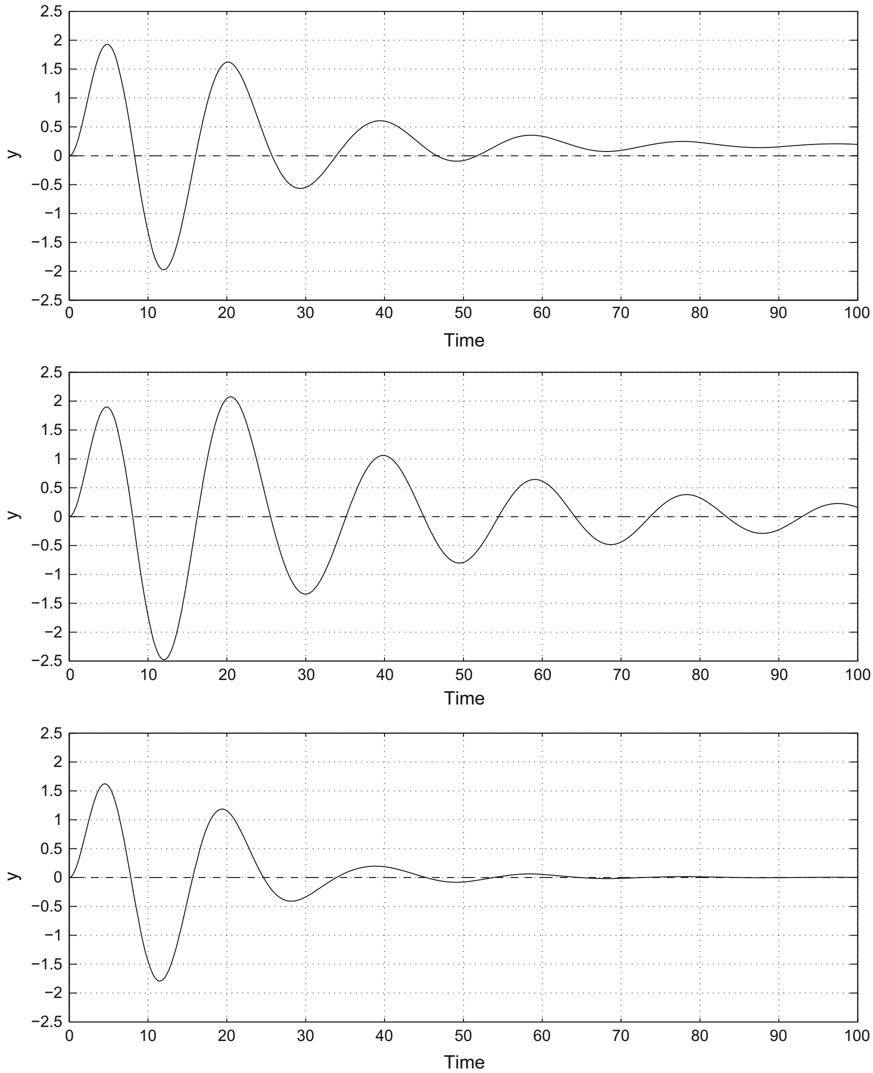


Fig. 2.32 Comparison of the influence of the controller type on the response of a second-order system ($K_p = 5$, $\tau_p = 10$, $\zeta_p = 0.5 : K_d = 2$, $\tau_d = 2$, $\zeta_d = 0.25$) to a disturbance unit step. (Proportional: $K_c = 2$ (*top*). Proportional-integral: $K_c = 2$, $\tau_I = 20$ (*middle*). Real proportional-integral-derivative: $K_c = 2$, $\tau_I = 20$, $\tau_D = 1$, $\beta = 0.1$ (*bottom*))

by increasing the controller gain. The use of too large gains can make the process unstable due to neglected dynamics or time delays.

A proportional-integral (PI) controller presents the advantage of integral action leading to the elimination of the deviation between the asymptotic state and the set point. The response is faster when the gain increases and can become oscillatory. For large values of the gain, the behaviour may even become unstable. The decreasing of the integral time constant increases the integral gain and makes the response faster. Because of the integral term, the PI controller may present a windup effect if the control variable u becomes saturated. In this case, the integral term becomes preponderant and needs time to be compensated. It is preferable to use an anti-windup system (Sect. 4.6.4).

The proportional-integral-derivative (PID) controller presents the same interest as the PI with respect to the asymptotic state. Furthermore, the derivative action allows a faster response without needing to choose too high gains as for a PI controller. This derivative action thus has a stabilizing effect.

The ideal PID controller is indeed replaced by a real PID controller, the transfer function of which given by Eq. (2.107) or (2.108) is physically realizable. In the case of a pneumatic PID controller, β is included between 0.1 and 0.2. For the PID electronic controller, one sets $0 < \beta \ll 1$.

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