

## Chapter 2

# Advanced Concepts

**Abstract** In this chapter we introduce elements of graph theory, graphs of components, matrix formulation of Kirchhoff's laws, matrix associated spaces, and Tellegen's theorem.

### 2.1 Basic Elements of Graph Theory

A set of independent Kirchhoff's laws for a given circuit – the so-called *topological equations* – can be automatically found by relying on some concepts of *graph theory*, that is, the study of mathematical/geometrical structures, called *graphs*, used to model pairwise relations between objects. Graph theory almost certainly began when, in 1735, Leonhard Euler<sup>1</sup> solved a popular puzzle about the bridges of the East Prussian city of Königsberg (now Kaliningrad) [1]. Nowadays, graph theory is largely used in mathematics, computer science, and network science, but it can be applied in any context where many units interact in some way, such as the components in a circuit. Usually, a graph completely neglects the nature of each unit and of the interactions, just keeping information about their existence.

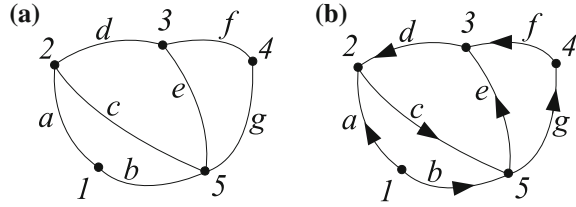
A **graph** is a finite set of  $N$  *nodes* (or vertices or points), together with a set of  $L$  *edges* (or branches or arcs or lines), each of them connecting a pair of distinct nodes.

We remark that more than one edge can connect the same pair of nodes. In this case, these edges are said to be *in parallel*. This implies that a pair of nodes can be insufficient to identify an edge univocally. Moreover, the above definition excludes the degenerate case of edges connecting one node to itself. Henceforth, we label the nodes with numbers and the edges with letters/symbols.

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<sup>1</sup>Leonhard Euler (1707–1783) was a Swiss mathematician, physicist, astronomer, logician, and engineer who made important and influential discoveries in many branches of mathematics.

**Fig. 2.1** Example of undirected (a) and directed (b) graph with  $N = 5$  and  $L = 7$



In the simplest case, the edges are not oriented: in this case we have an *undirected graph*, an example of which is shown in Fig. 2.1a. If the edges are oriented, they are called arrows (or directed edges or directed arcs or directed lines) and we have a *directed graph* or *digraph*. (See Fig. 2.1b.)

**Order** of a node: Number of edges connecting this node to other nodes.

For instance, in the figure node 1 has order 2, node 3 order 3, and node 5 order 4. The specific shape of a graph is not relevant, according to the following definition.

Two (directed) graphs  $G_1$  and  $G_2$  are **isomorphic** if it is possible to establish a bijective correspondence between:

- Each node of  $G_1$  and each node of  $G_2$
- Each edge of  $G_1$  and each edge of  $G_2$

such that corresponding edges connect (ordered) pairs of corresponding nodes.

Three examples of graphs isomorphic to the one of Fig. 2.1b are shown in Fig. 2.2. For ease of comparison, we used the same labels for nodes and edges as in Fig. 2.1b; in this case, the graph is not only isomorphic, but is essentially the same. Any change in the labels would not affect the equivalences. The graphs shown in Fig. 2.3 are in turn isomorphic to the one of Fig. 2.1b. Some of the correspondences are summarized in Table 2.1. You can check your comprehension by finding the missing correspondences.

**Planar graph:** A graph that can be embedded in the plane; that is, it can be drawn on the plane in such a way that all its edges intersect only at their endpoints.

In other words, any planar graph admits an isomorphic graph where no edges cross each other. Some examples of planar graphs are shown in Fig. 2.4.

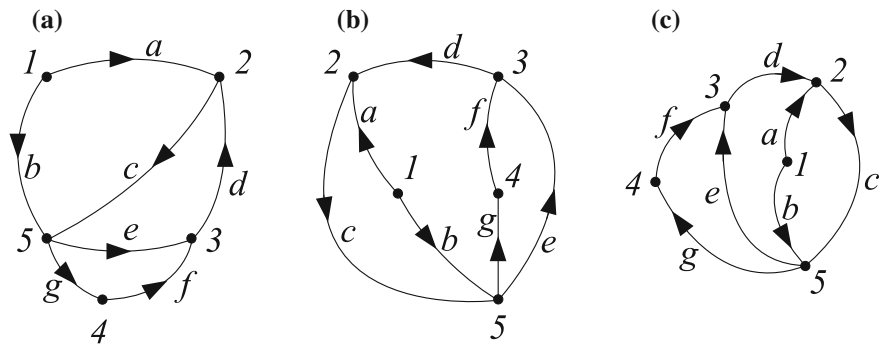


Fig. 2.2 Examples of isomorphic graphs (to be compared to Fig. 2.1b)

Fig. 2.3 Further examples of isomorphic graphs (to be compared to Fig. 2.1b)

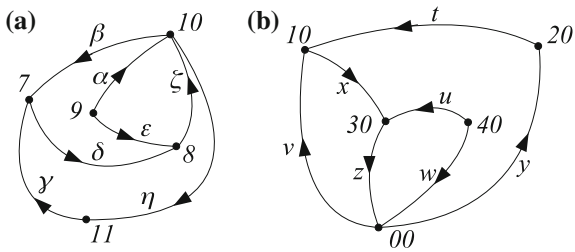
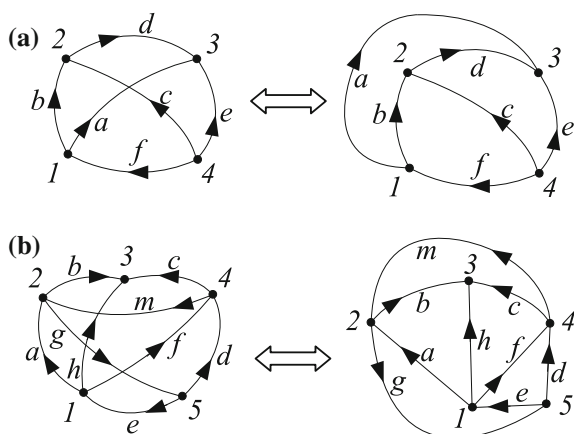


Table 2.1 Table of correspondences between elements of the isomorphic graphs of Figs. 2.1b, 2.3a, and 2.3b

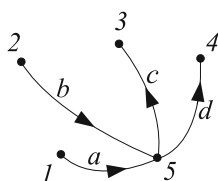
| Graph element | Graph     |           |           |
|---------------|-----------|-----------|-----------|
|               | Fig. 2.1b | Fig. 2.3a | Fig. 2.3b |
|               | a         | ε         | u         |
|               | b         | α         | w         |
|               | c         | ζ         | z         |
|               | d         | δ         | x         |
|               | 1         | 9         | 40        |
|               | 2         | 8         | 30        |
|               | 3         | 7         | 10        |
|               | 4         | 11        | 20        |
|               | 5         | 10        | 00        |

**Star graph:** A graph containing  $N - 1$  nodes of order 1 and one node of order  $N - 1$ .

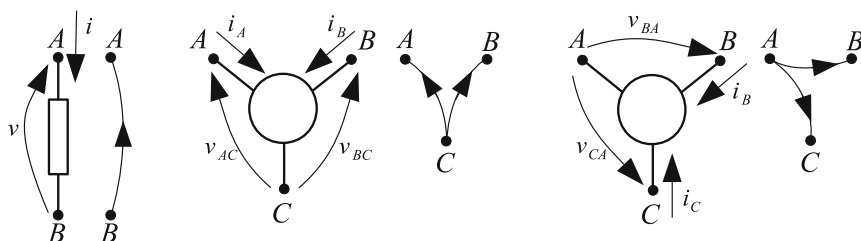
Figure 2.5 shows an example of a star graph with 5 nodes: node 5 has order 4; the other nodes have order 1.



**Fig. 2.4** Examples of planar graphs



**Fig. 2.5** Example of a star graph

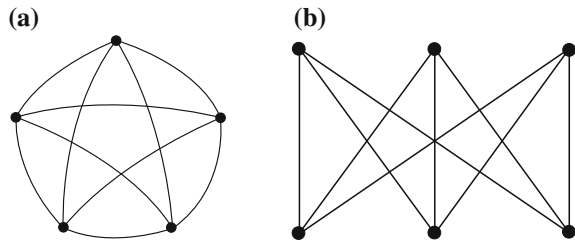


**Fig. 2.6** Examples of graphs for multiterminal components

### 2.1.1 Graphs of Components and Circuits

For a circuit, it is quite natural (even if this is not the only possible choice) to associate the circuit nodes with graph nodes and the descriptive voltages with graph arrows. By assuming the standard choice, this means that each graph arrow is associated with a voltage oriented like the arrow and to a current oriented in the opposite direction.

Figure 2.6 shows some examples of graphs for multiterminal components.

**Fig. 2.7** Kuratowski graphs

By substituting each circuit component with its graph, we obtain the circuit graph. For instance, the directed graph shown in Fig. 2.1b corresponds to the circuit of Fig. 1.8.

### 2.1.2 Subgraph, Path, Loop, and Cut-Set

In this section we define some basic graph structures.

**Subgraph:** A subset of the elements of a given graph, obtained by removing some edges and/or some nodes together with the corresponding edges.

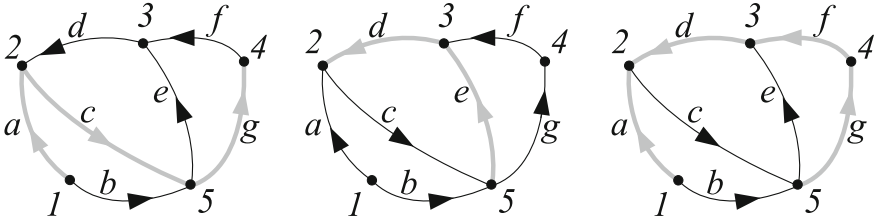
A subgraph is in turn a graph. For instance, by removing edges  $a, d, f$  from the graph of Fig. 2.1a, we obtain a subgraph which is in turn a star graph.

It has been shown [2] that a graph is nonplanar if and only if it is (or contains a subgraph) a graph isomorphic to the ones shown in Fig. 2.7, independently of the edge orientations.

**Path:** A subgraph made up of a sequence of  $k - 1$  adjacent edges (the orientation is not relevant) connecting a sequence of  $k$  nodes that, by most definitions, are all distinct from one another.

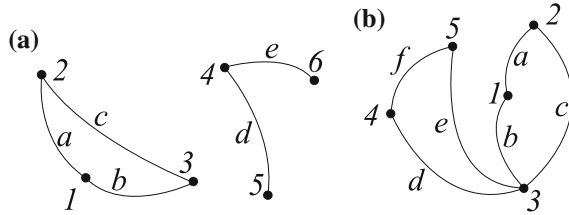
In other words, a nondegenerate path is a trail in which all nodes and all edges are distinct and then we have 2 nodes of order 1 (the first and the last) and  $k - 2$  nodes of order 2. Figure 2.8 shows some examples of paths (in grey) for the reference graph of Fig. 2.1b.

A graph is **connected** when there is a path between every pair of nodes. Otherwise it is **disconnected**.



**Fig. 2.8** Examples of paths (in grey)

**Fig. 2.9** Example of a disconnected graph (a) and hinged graph (b)



An example of a disconnected graph is shown in Fig. 2.9a.

A connected graph is **hinged** when it can be partitioned into two subgraphs connected by one node, called a *hinge*.

An example of a hinged graph is shown in Fig. 2.9b, where the hinge is node 3.

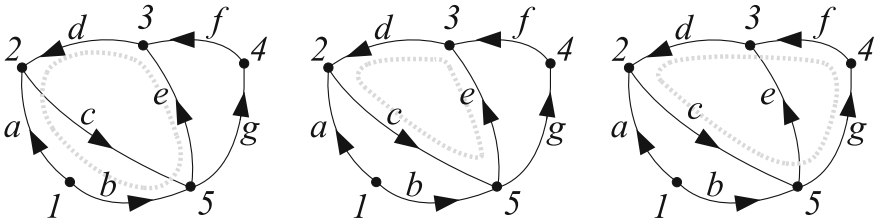
**Loop:** A subgraph containing only nodes of order 2, or a degenerate path where the first and last nodes are also of order 2, connected by an edge.

Figure 2.10 shows some examples of loops for the reference graph of Fig. 2.1b.

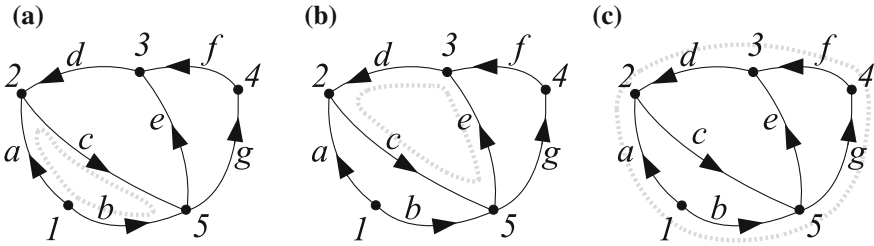
**Mesh:** A loop of a planar graph not containing any graph elements either inside (**inner loop**) or outside (**outer loop**).

Figure 2.11 shows some examples of meshes for the reference graph of Fig. 2.1b.

**Cut-set:** A set of edges of a graph which, when removed, make the graph disconnected.



**Fig. 2.10** Examples of loops (in grey)



**Fig. 2.11** Examples of meshes (in grey): inner loops (a and b) and outer loop (c)

As stated in Sect. 1.5.2, a cut-set can be easily associated with a closed path (or surface, for nonplanar graphs) crossing the cut-set edges. Actually, for each cut-set there are two possible closed paths, as shown in the examples of Fig. 2.12 for the reference graph of Fig. 2.1b.

**Nodal cut-set:** A cut-set such that one of the two disconnected parts of the resulting graph is a single node.

Figure 2.13 shows some examples of nodal cut-sets for the reference graph of Fig. 2.1b.

### 2.1.3 Tree and Cotree

We now define the two basic graph structures used to find matrix formulations of Kirchhoff's laws.

**Tree:** A subgraph containing all the  $N$  nodes and  $N - 1$  edges of a given graph and in which any two nodes are connected by exactly one path.

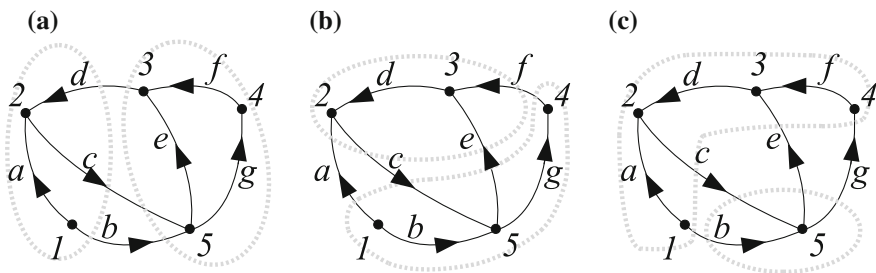


Fig. 2.12 Examples of cut-sets (corresponding to the grey dashed closed paths)

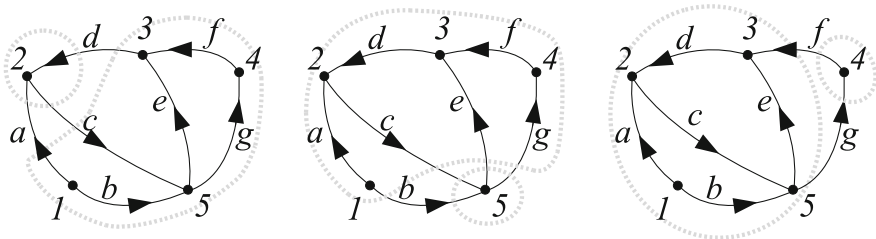


Fig. 2.13 Examples of nodal cut-sets (corresponding to the grey dashed closed paths)

Owing to this definition, a tree cannot contain any loop.

**Cotree:** A subgraph associated with a tree, containing all the  $N$  nodes and the  $L - N + 1$  edges of the graph not contained in the tree.

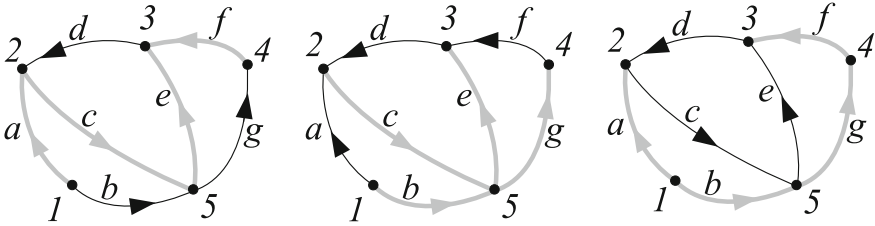
Figure 2.14 shows some examples of trees and cotrees for the reference graph of Fig. 2.1b.

## 2.2 Matrix Formulation of Kirchhoff's Laws

As stated at the beginning of Sect. 2.1, these basic elements of graph theory can be used to formulate in a compact way (i.e., in matrix form) a set of independent Kirchhoff's laws for a given circuit. The goal is to find a complete<sup>2</sup> set of independent KVLs and KCLs, which are related to corresponding sets of independent loops and cut-sets, respectively. A set of independent loops (cut-sets) is also called a *basis of fundamental loops (cut-sets)*.

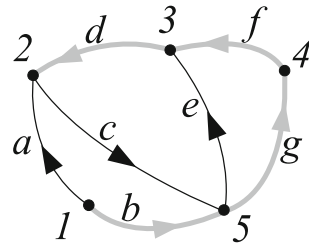
<sup>2</sup>The set is complete if any further KVL or KCL equation is linearly dependent on the equations belonging to the set.





**Fig. 2.14** Examples of trees (thick grey edges) and corresponding cotrees (thin black edges) for the reference graph of Fig. 2.1b

**Fig. 2.15** Tree (thick grey edges) and cotree (black edges) for the reference graph of Fig. 2.1b



For planar graphs, the simplest choice for these bases is the set of  $L - N + 1$  arbitrarily chosen meshes (which are independent loops) and the set of  $N - 1$  arbitrarily chosen nodal cut-sets (which are independent cut-sets).

For generic graphs, a criterion to identify these bases refers to a tree and the corresponding cotree. In the following, we use the graph, tree, and cotree shown in Fig. 2.15. Moreover, henceforth  $I_q$  denotes the identity matrix of size  $q$  (i.e., the  $q \times q$  square matrix with ones on the main diagonal and zeros elsewhere) and  $0_q$  denotes the null column vector with  $q$  elements.

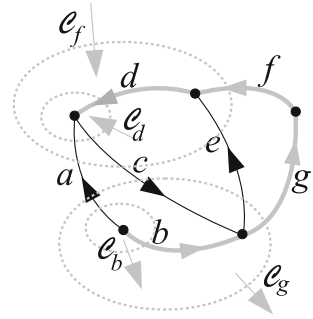
### 2.2.1 Fundamental Cut-Set Matrix

Each cut-set containing one and only one edge of the chosen tree is part of a basis of  $(N - 1)$  fundamental cut-sets. Each fundamental cut-set is oriented (inwards/outwards) like the corresponding tree edge and is labeled as  $\mathcal{C}_k$ , where  $k$  denotes the edge. Figure 2.16 shows the basis of fundamental cut-sets for the considered example and the chosen tree.

Now, we can construct a matrix (of size  $(N - 1) \times L$ ), called the *fundamental cut-set matrix*, where:

- Each row corresponds to exactly one fundamental cut-set (i.e., to the related tree edge).
- Each column corresponds to one graph edge. The columns are ordered as follows: first the cotree edges (ordered arbitrarily) and then the tree edges, in the same order

**Fig. 2.16** Basis of fundamental cut-sets for the considered example



as for the rows. In the example, we follow the alphabetical order for both edge sets.

- Each matrix entry is set to:

- 0 If the edge on the column does not belong to the fundamental cut-set on the row
- 1 If the edge on the column belongs to the fundamental cut-set on the row and has the same orientation
- −1 If the edge on the column belongs to the fundamental cut-set on the row and has the opposite orientation

In the considered example, the fundamental cut-set matrix is as follows.

$$A = \begin{matrix} & \begin{matrix} a & c & e & b & d & f & g \end{matrix} \\ \begin{matrix} b \\ d \\ f \\ g \end{matrix} & \left( \begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array} \right) = (\alpha | I_{N-1}) \end{matrix} \quad (2.1)$$

We call  $i$  the column vector of descriptive currents associated with the oriented edges of the graph and ordered exactly as are the columns of the cut-set matrix  $A$ ; that is,  $i = (i_a \ i_c \ i_e \ i_b \ i_d \ i_f \ i_g)^T$ . It is easy to check that the rows of  $A$  are linearly independent; that is, the rank of  $A$  is  $N - 1$ . This is a general property, due to the way the fundamental cut-set matrix is set up and to the fact that each row is related to one element of a basis of cut-sets.

Of course, the cut-set orientation depends on the choice of the corresponding closed path (as stated in Sect. 2.1.2), but the resulting matrix is invariant, as can be easily checked.

### Property

The system of equations

$$Ai = 0_{N-1} \quad (2.2)$$

is a set of  $N - 1$  independent KCLs for the circuit associated with the graph, corresponding to the fundamental cut-sets related to the chosen tree.

For the circuit of Fig. 1.8 and for the choice of tree of Fig. 2.15, the set of independent KCLs is:

$$Ai = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i_a \\ i_c \\ i_e \\ i_b \\ i_d \\ i_f \\ i_g \end{pmatrix} = \begin{pmatrix} i_a + i_b \\ i_a - i_c + i_d \\ i_a - i_c + i_e + i_f \\ i_a - i_c + i_e + i_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.3)$$

Each row of the submatrix  $\alpha$  contains information about the composition of the cut-set which the row refers to: for example, the nonzero elements in the row  $d$  of  $\alpha$  indicate that  $\mathcal{C}_d$  contains (in addition to  $d$ ) the edges  $a$  and  $c$ ; similarly, the row  $f$  of  $\alpha$  indicates that  $\mathcal{C}_f$  contains, in addition to  $f$ , the edges  $a, c, e$ .

We observe in passing that something similar can be observed for the columns of  $\alpha$ : for example, the nonzero elements of the column  $a$  indicate that  $b, d, f, g$  are the tree edges forming a loop with  $a$ ; similarly, the nonzero elements of the column  $c$  indicate that the tree edges  $d, f, g$  form a loop with  $c$ . Therefore  $\alpha$  also contains topological information about the loops. This fact has major consequences on the fundamental loop matrix structure, discussed soon.

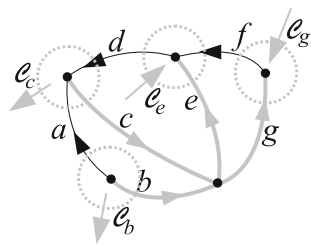
### 2.2.1.1 A Particular Case

For the specific tree choice shown in Fig. 2.17, we obtain the basis composed by nodal cut-sets only. Notice that the tree in this case is a *star subgraph*.

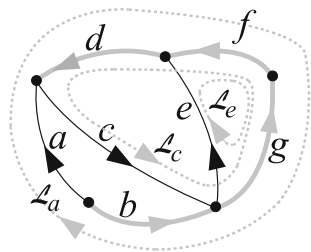
For this choice of tree, writing the cut-set matrix  $A$  according to the general rules, the set of independent KCLs is as follows.

$$Ai = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i_a \\ i_d \\ i_f \\ i_b \\ i_c \\ i_e \\ i_g \end{pmatrix} = \begin{pmatrix} i_a + i_b \\ -i_a - i_d + i_c \\ -i_d + i_f + i_e \\ -i_f + i_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.4)$$

**Fig. 2.17** Choice of tree (thick grey edges) corresponding to a basis of nodal cut-sets only (dashed lines)



**Fig. 2.18** Basis of fundamental loops for the considered example



This set of equations is completely equivalent to Eq. 2.3.

This matrix is strictly related to the so-called *incidence matrix*.

You can check your comprehension by obtaining Eq. 2.4 through linear combinations of Eq. 2.3.

### 2.2.2 Fundamental Loop Matrix

Each loop containing only one edge of a cotree is part of a basis of  $(L - N + 1)$  fundamental loops. Each fundamental loop is oriented as is the corresponding cotree edge and is labeled as  $\mathcal{L}_k$ , where  $k$  denotes the cotree edge. Figure 2.18 shows the basis of fundamental loops for the considered example and the chosen tree.

Now, we can construct a matrix (of size  $(L - N + 1) \times L$ ), called the *fundamental loop matrix*, where:

- Each row corresponds to exactly one fundamental loop (i.e., to the related cotree edge).
- Each column corresponds to one graph edge. The columns are ordered as in matrix  $A$ .
- Each matrix entry is set to:
  - 0 If the edge on the column does not belong to the fundamental loop on the row
  - 1 If the edge on the column belongs to the fundamental loop on the row and has the same orientation
  - 1 If the edge on the column belongs to the fundamental loop on the row and has the opposite orientation

In the considered example, the fundamental loop matrix is:

$$B = \begin{array}{c} a \\ c \\ e \end{array} \begin{array}{c|cccc} & a & c & e & b & d & f & g \\ \hline a & 1 & 0 & 0 & -1 & -1 & -1 & -1 \\ c & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ e & 0 & 0 & 1 & 0 & 0 & -1 & -1 \end{array} = (I_{L-N+1} | -\alpha^T) \quad (2.5)$$

We call  $v$  the column vector of descriptive voltages associated with the oriented edges of the graph and ordered exactly as are the columns of the loop matrix  $B$ ; that is,  $v = (v_a \ v_c \ v_e \ v_b \ v_d \ v_f \ v_g)^T$ . It is easy to check that the rows of  $B$  are linearly independent; this is a general property, due to the way the fundamental loop matrix is set up and to the fact that each row is related to one element of a basis of loops. For this reason, the rank of  $B$  is  $L - N + 1$ . When, as in this case, the ordering of the tree edges is the same for the matrices  $A$  and  $B$ , the elements of  $A$  and  $B$  are related very simply: the matrix part complementary to the identity submatrix is  $\alpha$  in  $A$  and  $-\alpha^T$  in  $B$ . This follows from the previously observed property concerning the columns of  $\alpha$ ; that is, for any column  $j$  of  $\alpha$ , the tree edges  $i$  with  $\alpha_{ij} \neq 0$  are the constituents of the loop  $\mathcal{L}_j$ .

### Property

The system of equations

$$Bv = 0_{L-N+1} \quad (2.6)$$

is a set of  $L - N + 1$  independent KVLs for the circuit associated with the graph, corresponding to the fundamental loops related to the chosen tree.

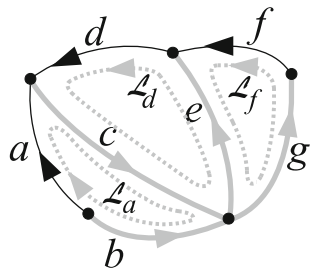
For the circuit of Fig. 1.8 and for the choice of tree of Fig. 2.15, the set of independent KVLs is as follows.

$$Bv = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} v_a \\ v_c \\ v_e \\ v_b \\ v_d \\ v_f \\ v_g \end{pmatrix} = \begin{pmatrix} v_a - v_b - v_d - v_f - v_g \\ v_c + v_d + v_f + v_g \\ v_e - v_f - v_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.7)$$

#### 2.2.2.1 A Particular Case

For the star tree shown in Fig. 2.19, we obtain the basis composed by all the inner loops.

**Fig. 2.19** Choice of tree  
(thick grey edges)  
corresponding to the basis of  
all the inner loops (dashed  
loops)



For this choice of tree, the set of independent KVLs is:

$$Bv = \begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} v_a \\ v_d \\ v_f \\ v_b \\ v_c \\ v_e \\ v_g \end{pmatrix} = \begin{pmatrix} v_a - v_b + v_c \\ v_d + v_c + v_e \\ v_f - v_e + v_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.8)$$

This set of equations is completely equivalent to Eq. 2.7. You can check your comprehension by obtaining Eq. 2.8 through linear combinations of Eq. 2.7.

### 2.2.3 Some General Concepts on Vector Spaces and Matrices

A *vector space*  $\mathcal{V}$  is a nonempty set of vectors such that, for any two vectors  $x_1$  and  $x_2$  of  $\mathcal{V}$ , any of their linear combinations  $\beta_1 x_1 + \beta_2 x_2$  ( $\beta_1, \beta_2 \in \mathbb{R}$ ) is still an element of  $\mathcal{V}$ . The null element 0 is always a vector of  $\mathcal{V}$ .

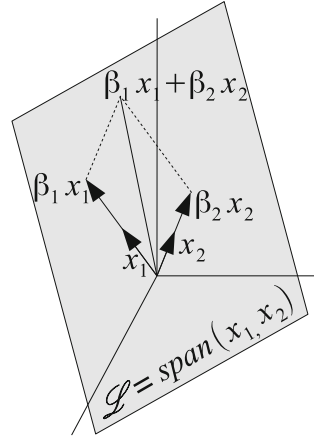
The *dimension* of  $\mathcal{V}$ , denoted as  $\dim(\mathcal{V})$ , is the maximum number of linearly independent vectors in  $\mathcal{V}$  and must not be confused with the number of components of the elements of  $\mathcal{V}$ .

A set of linearly independent vectors in  $\mathcal{V}$  consisting of  $\dim(\mathcal{V})$  vectors is called a *basis* for  $\mathcal{V}$ .

Given  $p$  vectors  $x_1, \dots, x_p$  with the same number of components, the set of all linear combinations  $\sum_{i=1}^p \beta_i x_i$  is a vector space called the *span* of these vectors. For instance, the vector space  $\mathcal{V}$  is the span of  $\dim(\mathcal{V})$  linearly independent vectors. The span of a number of linearly independent vectors lower than  $\dim(\mathcal{V})$  generates a *subspace*  $\mathcal{L}$  of  $\mathcal{V}$ . For instance, Fig. 2.20 shows an example for  $\mathcal{V} \equiv \mathbb{R}^3$ .

The vectors  $x_1$  and  $x_2$  (as well as all their linear combinations  $\beta_1 x_1 + \beta_2 x_2$ , with  $\beta_1, \beta_2 \in \mathbb{R}$ ) lie in a plane  $\mathcal{L}$ , which is a two-dimensional subspace of  $\mathbb{R}^3$  passing through the origin.

**Fig. 2.20 A**  
 two-dimensional subspace  
 $\mathcal{L}$  in  $\mathbb{R}^3$



As stated above, it is important not to confuse the dimension of the vector space (or subspace) with the number of components (the size) of its individual vectors, because they are not necessarily the same. In the considered example, for instance, the vectors  $x_1$  and  $x_2$  have three components, despite their belonging to the two-dimensional subspace  $\mathcal{L}$ .

In the following, we introduce some specific spaces and subspaces associated with a matrix [3, 4], in order to provide (in the next section) a geometrical interpretation of the matrix formulation of Kirchhoff's laws, thus settling the basis for introducing Tellegen's theorem.

Let us consider a matrix  $Q \in \mathbb{R}^{m \times n}$ . We can write  $Q$  in terms of its columns as  $Q = (q_1 \dots q_n)$ . Let  $x$  denote any vector in  $\mathbb{R}^n$ . The vector space

$$\mathcal{R}(Q) = \{y \in \mathbb{R}^m : y = Qx, x \in \mathbb{R}^n\}$$

is called the *range* of  $Q$ . We can also write, in terms of the column vectors  $q_i$ ,

$$\mathcal{R}(Q) = \text{span}(q_1, \dots, q_n).$$

In the general case, the linearly independent columns of  $Q$  can be a subset of  $\{q_1, q_2, \dots, q_n\}$ . It can be shown that the maximum number of linearly independent columns of  $Q$  and the maximum number of its linearly independent rows are equal. This common value  $r$  is the *rank* of  $Q$ . Then  $\text{rank}(Q) = \text{rank}(Q^T) = r \leq \min(m, n)$  and  $\dim(\mathcal{R}(Q)) = r$ .

The set of all solutions to the homogeneous system  $Qz = 0$ ,

$$\mathcal{N}(Q) = \{z \in \mathbb{R}^n : Qz = 0\}$$

is called the *null space* of  $Q$  (or *kernel* of  $Q$ ).

In the same way we can define the vector spaces associated with the transpose of  $Q$ :  $\mathcal{R}(Q^T)$ ,  $\mathcal{N}(Q^T)$ .

Two  $m$ -size vectors  $w_R \in \mathcal{R}(Q)$  and  $w_0 \in \mathcal{N}(Q^T)$  are always orthogonal; owing to the definition of  $\mathcal{R}(Q)$ , there must exist a vector  $\bar{x}$  such that  $w_R = Q\bar{x}$ , thus  $w_R^T w_0 = (Q\bar{x})^T w_0 = \bar{x}^T \underbrace{Q^T w_0}_0 = 0$ . An analogous result holds for two  $n$ -size

vectors  $x_R \in \mathcal{R}(Q^T)$  and  $x_0 \in \mathcal{N}(Q)$ .

These spaces are the main ingredients of two important results concerning the decomposition of vectors:

- 1 Any vector  $x \in \mathbb{R}^n$ , the domain space of  $Q$ , can be uniquely decomposed as  $x = x_R + x_0$ , where  $x_0 \in \mathcal{N}(Q)$  and  $x_R \in \mathcal{R}(Q^T)$ . Then  $\mathcal{N}(Q)$  and  $\mathcal{R}(Q^T)$  are complementary and disjoint ( $\mathcal{N}(Q) \cap \mathcal{R}(Q^T) = \emptyset$ , empty set) subspaces of  $\mathbb{R}^n$ ; that is,  $\mathbb{R}^n$  is given by the *direct sum* ( $\oplus$ ) of the two subspaces:

$$\mathbb{R}^n = \mathcal{N}(Q) \oplus \mathcal{R}(Q^T) \text{ and } n = \dim(\mathcal{N}(Q)) + r.$$

The subspace  $\mathcal{N}(Q)$  is an empty set if and only if  $r = n$ .

- 2 Any vector  $w \in \mathbb{R}^m$ , the codomain space of  $Q$ , can be uniquely decomposed as  $w = w_R + w_0$ , where  $w_0 \in \mathcal{N}(Q^T)$  and  $w_R \in \mathcal{R}(Q)$ . Then  $\mathcal{N}(Q^T)$  and  $\mathcal{R}(Q)$  are complementary and disjoint ( $\mathcal{N}(Q^T) \cap \mathcal{R}(Q) = \emptyset$ ) subspaces of  $\mathbb{R}^m$ ; that is,  $\mathbb{R}^m$  is given by the direct sum of the two subspaces

$$\mathbb{R}^m = \mathcal{N}(Q^T) \oplus \mathcal{R}(Q) \text{ and } m = \dim(\mathcal{N}(Q^T)) + r.$$

The subspace  $\mathcal{N}(Q^T)$  is an empty set if and only if  $r = m$ .

To exemplify the above concepts, let us consider the matrix

$$Q = \begin{matrix} & \begin{matrix} q_1 & q_2 \end{matrix} \\ \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{matrix}$$

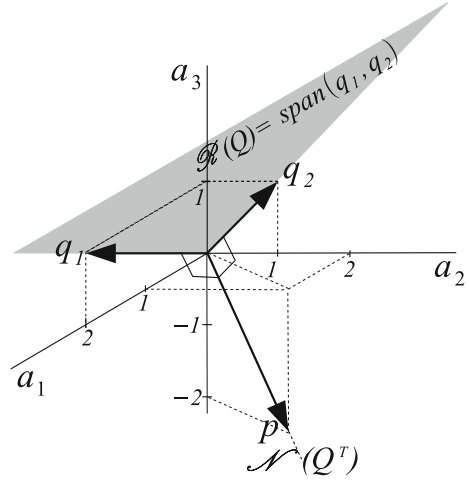
which has  $m = 3$ ,  $n = 2$  and rank  $r = 2$ . Its column vectors  $q_1$ ,  $q_2$  define the plane  $\mathcal{R}(Q)$ :

$$\mathcal{R}(Q) = \text{span}(q_1, q_2) = Q \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \beta_1 q_1 + \beta_2 q_2; \quad \beta_1, \beta_2 \in \mathbb{R}.$$

Taking as reference the orthogonal directions  $a_1, a_2, a_3$ , the vectors  $q_1$ ,  $q_2$  are shown in Fig. 2.21. The plane  $\mathcal{R}(Q)$  intersects the  $a_1 a_3$ -plane along the line of  $q_1$  and the  $a_2 a_3$ -plane along the line of  $q_2$ .



**Fig. 2.21** The plane  $\mathcal{R}(Q)$  and the line  $\mathcal{N}(Q^T)$  associated with the  $(3 \times 2)$  matrix  $Q$



Because  $\dim(\mathcal{N}(Q^T)) = m - r = 1$ , the complementary subspace  $\mathcal{N}(Q^T)$  is a straight line orthogonal to the plane  $\mathcal{R}(Q)$ . Denoting as  $p = (p_1 \ p_2 \ p_3)^T$  a vector along this line, we have

$$Q^T p = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2p_1 + p_3 = 0 \\ p_2 + p_3 = 0 \end{cases}$$

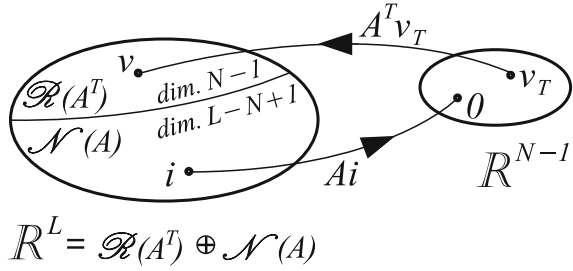
Then the components  $p_1$ ,  $p_2$  can be expressed in terms of  $p_3$ , that parameterizes the points of the subspace. The vector  $p$  plotted in the figure corresponds to  $p_3 = -2$ .

Finally, inasmuch as  $r = n$ , we have  $\dim(\mathcal{N}(Q)) = 0$  (empty subspace) and  $\mathcal{R}(Q^T) = \mathbb{R}^n$ .

### 2.2.4 The Cut-Set and Loop Matrices and Their Associate Space Vectors

Consider a directed graph with  $L$  edges and  $N$  nodes. This graph can be arbitrarily partitioned into a tree and its cotree. Such a partition leads to the definition of a cut-set matrix  $A$  and a loop matrix  $B$ , as shown in Sects. 2.2.1 and 2.2.2. A current vector  $i$  and a voltage vector  $v$ , both of size  $L$ , are said to be *compatible* with the graph if they

**Fig. 2.22** Matrix  $A$ -relationships between spaces for compatible voltage and current vectors



satisfy the KCLs and KVLs, respectively, that is, if  $Ai = 0$  and  $Bv = 0$ . Because the structure of  $A$  (where  $m = N - 1$  and  $n = L$ ) is  $(\alpha | I_{N-1})$ , from  $Ai = 0$  we have  $N - 1$  independent scalar equations, which represent as many constraints on the  $L$  elements of the vector  $i$ . Therefore, due to KCLs, the number of degrees of freedom for the current elements of a vector  $i$  compatible with the graph is  $L - N + 1$ .

In terms of vector spaces,  $Ai = 0$  means that  $i$  belongs to  $\mathcal{N}(A)$ , the null space of matrix  $A$ , whose dimension is  $L - N + 1$ .

Consider now the KVLs  $Bv = 0$ , with  $B = (I_{L-N+1} | -\alpha^T)$  (where  $m = L - N + 1$  and  $n = L$ ). The vector  $v$  can be partitioned into two subvectors  $v_C$  and  $v_T$ , which contain the  $L - N + 1$  voltages on the cotree edges and the  $N - 1$  voltages on the tree edges, respectively:

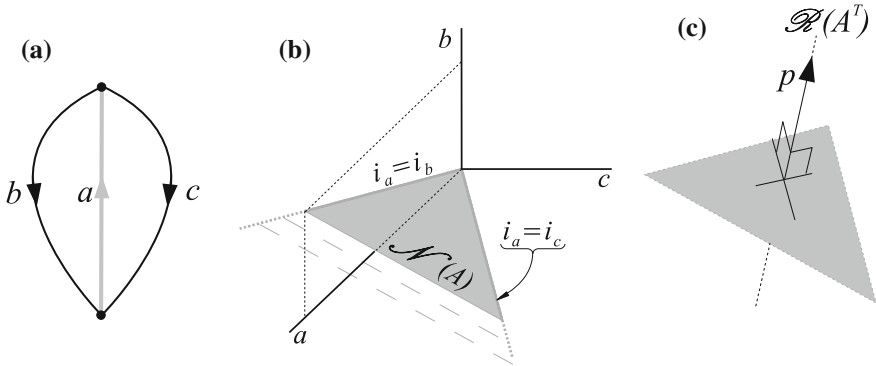
$$v = \begin{pmatrix} v_C \\ v_T \end{pmatrix} \quad (2.9)$$

Owing to this partition, the KVLs  $Bv = 0$  can be recast as  $I_{L-N+1}v_C - \alpha^T v_T = 0$ ; that is,  $v_C = \alpha^T v_T$ . Then, we directly obtain:

$$v = \begin{pmatrix} v_C \\ v_T \end{pmatrix} = \begin{pmatrix} \alpha^T \\ I_{N-1} \end{pmatrix} v_T = A^T v_T. \quad (2.10)$$

It follows that each vector  $v$  of voltages compatible with the graph can be obtained through a product  $A^T v_T$ . This means that  $v \in \mathcal{R}(A^T)$ , whose dimension is  $N - 1$ . The values of the  $N - 1$  components of the subvector  $v_T$  can be assigned independently, therefore the voltage elements of a compatible vector  $v$  can be chosen with  $N - 1$  degrees of freedom, due to KVLs, which impose  $L - N + 1$  constraints on the  $L$  components of  $v$ .

Figure 2.22 summarizes all these results and also highlights the roles of the matrices  $A$  and  $A^T$  as operators for the passage between the subspaces of  $\mathbb{R}^L$  and the space  $\mathbb{R}^{N-1}$ .



**Fig. 2.23** Case Study: **a** graph; **b, c** spaces  $\mathcal{N}(A)$  and  $\mathcal{R}(A^T)$  for compatible current and voltage vectors

### Case Study

Consider the very simple graph ( $L = 3$ ,  $N = 2$ ) shown in Fig. 2.23a. Taking the edge  $a$  as the (only) tree edge and the edges  $b, c$  as cotree edges, the fundamental cut-set matrix  $A$  is

$$A = \begin{matrix} & \begin{matrix} b & c & a \end{matrix} \\ \begin{matrix} a \end{matrix} & \begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \end{matrix} \quad (2.11)$$

Therefore, KCL reduces to a single scalar equation:

$$i = \begin{pmatrix} i_b \\ i_c \\ i_a \end{pmatrix}; \quad Ai = -i_b - i_c + i_a = 0 \quad (2.12)$$

The three components  $i_b, i_c, i_a$  of any current vector  $i$  compatible with the graph must fulfill the KCL constraint  $i_a = i_b + i_c$ , which leads to the expression for the two-dimensional subspace  $\mathcal{N}(A)$ :

$$i = \begin{pmatrix} i_b \\ i_c \\ i_b + i_c \end{pmatrix} = i_b \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + i_c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \quad (2.13)$$

In the above expression, the values of  $i_b, i_c$  play the role of span coefficients.

Denoting by  $b, c, a$  the orthogonal directions spanning the  $\mathbb{R}^3$  space as shown in Fig. 2.23b,  $\mathcal{N}(A)$  is the plane that intersects the plane  $i_c = 0$  along the straight line  $i_a = i_b$  and the plane  $i_b = 0$  along the straight line  $i_a = i_c$ . All the vectors  $i \in \mathbb{R}^3$  compatible with the graph lie on the plane  $\mathcal{N}(A)$ .

The voltage vector

$$v = \begin{pmatrix} v_b \\ v_c \\ v_a \end{pmatrix} \quad (2.14)$$

can be partitioned according to Eq. 2.9; in particular, we have  $v_T = v_a$ . With this in mind, and recalling Eq. 2.10, any vector  $v$  compatible with the graph can be obtained as

$$v = A^T v_T = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} v_a \quad (2.15)$$

or, in a more general formulation highlighting the parametric role of the term  $v_a$ , as

$$v = p\beta; \quad p = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}; \quad \beta \in \mathbb{R}. \quad (2.16)$$

Therefore, any vector  $v$  such that  $Bv = 0$  is proportional to the vector  $p$ . It is easy to verify that  $p$  is orthogonal to any vector  $i \in \mathcal{N}(A)$ , as shown in Fig. 2.23c. The way to prove it is based on the observation that, being  $i_a = i_b + i_c$ , we can write  $i$  as  $(i_b \ i_c \ (i_b + i_c))^T$  and then:

$$p^T i = (-1 \ -1 \ 1) \begin{pmatrix} i_b \\ i_c \\ i_b + i_c \end{pmatrix} = 0. \quad (2.17)$$

You can check the correspondence of these results with the general ones shown in Fig. 2.22.

In a similar fashion, denoting by  $i_C$  and  $i_T$  the subvectors containing, respectively, the  $L - N + 1$  cotree currents and the  $N - 1$  currents through the tree edges, we have

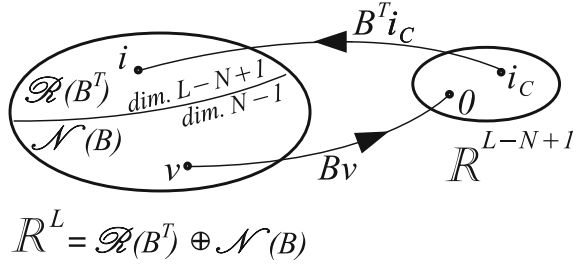
$$i = \begin{pmatrix} i_C \\ i_T \end{pmatrix} \quad (2.18)$$

which enables us to recast the KCLs  $Ai = 0$  as  $\alpha i_C + I_{N-1} i_T = 0$ ; that is,  $i_T = -\alpha i_C$ . With this in mind, we obtain

$$i = \begin{pmatrix} i_C \\ i_T \end{pmatrix} = \begin{pmatrix} I_{L-N+1} \\ -\alpha \end{pmatrix} i_C = B^T i_C. \quad (2.19)$$

Therefore, each current vector  $i$  compatible with the graph can be obtained through a product  $B^T i_C$ . This means that the current elements of any compatible vector  $i$  can be chosen with the  $L - N + 1$  degrees of freedom representing the size of the subvector

**Fig. 2.24** Matrix  $B$ -relationships between spaces for compatible voltage and current vectors



$i_C$ ; moreover,  $i \in \mathcal{R}(B^T)$ . Because  $Bv = 0$  means that  $v$  belongs to  $\mathcal{N}(B)$ , the space  $\mathbb{R}^L$  can be thought of as partitioned into the two subspaces  $\mathcal{R}(B^T)$  and  $\mathcal{N}(B)$ . This partition is shown in Fig. 2.24, which highlights the roles of the matrices  $B$  and  $B^T$  as operators for the passage between the subspaces of  $\mathbb{R}^L$  and the space  $\mathbb{R}^{L-N+1}$ .

The properties presented in this section are the basis for Tellegen's theorem, which is treated in the next section.

### 2.3 Tellegen's Theorem

**Theorem 2.1** (Tellegen's theorem) *In a directed graph, any compatible voltage vector  $v$  is orthogonal to any compatible current vector  $i$ .*

*Proof* To prove this, just consider that, thanks to the compatibility assumption, we have

$$v^T i = (A^T v_T)^T i = v_T^T \underbrace{A i}_0 = 0. \quad (2.20)$$

□

Tellegen's theorem is one of the most general theorems of circuit theory [5]. It depends only on Kirchhoff's laws and on the circuit's topology (graph), and it holds regardless of the physical nature of the circuit's components or the waveforms of voltages and currents, and so on. Therefore the voltages and currents that are used for Tellegen's theorem are not necessarily those actually present in a given circuit. By introducing specific assumptions about the physical properties of the components, waveforms and so on, Tellegen's theorem is the starting point to obtain, usually in a direct way, various specific and useful results. In the next chapters we show that for many circuit properties, the proof that can be given by relying on Tellegen's theorem is simpler than others and its range of validity is more clearly demonstrated.

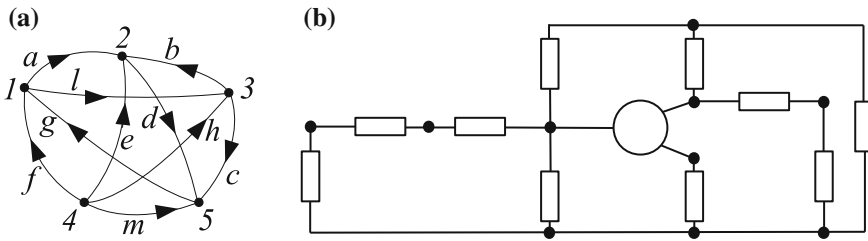


Fig. 2.25 Problems 2.1 (a) and 2.2 (b)

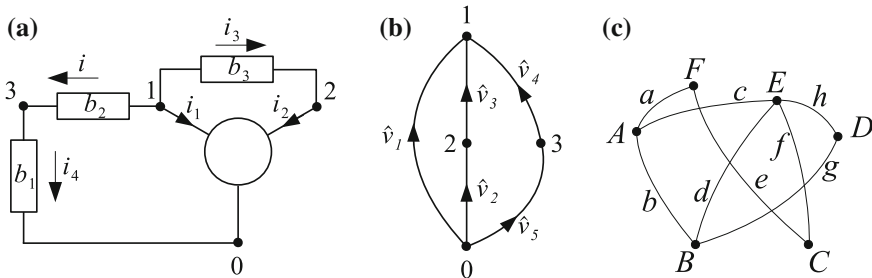


Fig. 2.26 Problems 2.3 (a, b) and 2.4 (c)

## 2.4 Problems

**2.1** Choose a tree for the nonplanar graph shown in Fig. 2.25a and find the corresponding fundamental cut-set and loop matrices.

**2.2** Determine the number of KCLs and KVLs necessary to solve the circuit shown in Fig. 2.25b. **Hint:** Consider the component connections to the lowest wire as a single node (dot).

**2.3** Assume that you can measure the voltages of another circuit whose graph is shown in Fig. 2.26b. Is it possible to determine current  $i$  in Fig. 2.26a by measuring current  $i_3$  in the same circuit? How?

**2.4** Determine the number of fundamental loops, fundamental cut-sets, and tree edges for the graph shown in Fig. 2.26c.

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