Chapter 2
Logics

To learn something
one has first to realize
that one does know something
Socrates

In the present chapter we shall be concerned with *conditional propositions* or, better to say, logical relations between conditional propositions.

Let us note that by proving theorems we make use of theorems that have been proved before. Consequently, by going back we arrive at statements which are taken as true without proofs. They are called *axioms* and their place is at the foundations of every mathematical theory.

The process of reasoning from the assumptions to the proposition is called *deduction*. The process of reasoning starting from a given proposition back to the conditions under which the proposition holds is called *reduction*. The process of reasoning from particular cases to a general statement referring to all possible cases is called *induction*.

It is worth noting that reduction is a reliable method though it is more difficult than deduction for it is difficult to find a theorem which supports a conclusion which we need to prove. A drawback of the deduction method is that we draw conclusions from premises of which we are not sure whether they are true or false.

We may finally mention errors which we may arrive at in the process of reasoning. In each process of reasoning we deal with:

- a proposition,
- premises, and
- conclusions.

In the first case possible errors may be caused by a wrong formulation of the proposition or by using the same proposition which is only to be proved. The latter is called the *vicious circle*.

In the second case errors usually occur when a theorem is used without reference to the conditions under which it holds.

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In the third case a frequent source of error is when a conclusion about the consequent of a conditional statement which is false is drawn from the fact that the antecedent is false or, conversely, if from the correctness of the consequent it is concluded that the antecedent is also true.

2.1 Elementary Notions

Necessity and sufficiency
In a theory it is sometimes convenient to formulate theorems concerning properties of certain objects in terms of necessary or sufficient conditions which can be stated as follows:

Theorem 2.1 *A implies B or B follows from A, where A and B are formulas, can alternatively be expressed as: B is a necessary condition for A, or as: A is a sufficient condition for B.*

Theorem 2.2 *A is equivalent to B can be formulated as: A is a necessary and sufficient condition for B or conversely: B is a necessary and sufficient condition for A.*

A necessary and sufficient condition for an object to posses a property in one formulation of a theory can also serve as a definition of this property in an alternative formulation of the theory.

The notions of a necessary condition and a sufficient condition will be extensively used in this text.

Therefore, for a better understanding of them, they will be explained by the following examples.

Example 2.1 [4] A necessary condition for the divisibility of an integer by 4 is the divisibility of its last digit by 2. This condition, though, as it is easy to check, is not a sufficient condition.

A sufficient condition for the divisibility of an integer by 4 is that its last two digits be zeros. In other words, if the last two digits of an integer are zeros, then it is divisible by 4.

This condition, however, is not necessary because if an integer is divisible by 4, ten this does not imply that its last two digits are zeros.

The necessary and sufficient condition for the divisibility of an integer by 4 is that the number consisting of its two last digits be divisible by 4.

The necessity follows from that if an integer is divisible by 4, then the number consisting of its last digits is divisible by 4.

The sufficiency follows from that if the number consisting of the last two digits of an integer is divisible by 4, then the integer is divisible by 4.

Proof of sufficiency. An integer L of which the last two digits are a number L_2 divisible by 4 can be represented as the sum of two numbers

$$L = L_1 + L_2 \ ,$$

where L_1 is a number with its two last digits equal to zero. Thus, L_2 is divisible by 4 by assumption and the number L_1 has two zeros as its last digits and therefore is also divisible by 4 as a multiple of 100 which is divisible by 4.
Proof of necessity, by contradiction. Let L_2 be indivisible by 4. Then, sinc L_1 is divisible by 4, then the number L cannot be divisible by 4. □

The following geometric problems can be very instructive. Their solution will then be shown.

Problems 2.1 [1] 1. Give the necessary and sufficient conditions for segments a, b, c to be the sides of a triangle.

2. Prove that n lines on the plane of which no two are parallel and no three intersect at a common point cut the plane into

$$S_n = 1 + \frac{n(n+1)}{2} \quad \text{parts} \ .$$

3. Prove that for an arbitrary family of lines on the plane it is true that

$$S + N = K + 1 \ ,$$

where:

N is the number of intersection points,
K is the number of segments into which the lines cut themselves,
S is the number of parts into which the lines cut the plane.

4. Prove that if k denotes the dimension of the space, $H_n^{(k)}$ denotes the number of parts of the space into which it is cut by n subspaces of dimension $P_n^{(k-1)}$ in an arbitrary configuration, then the following recurrence relation holds

$$H_{n+1}^{(k)} = H_n^{(k)} + P_n^{(k-1)} \ .$$

In particular, for the plane divided by n lines it holds that

$$H_{n+1}^{(2)} = H_n^{(2)} + (n+1) \ .$$

and it is obvious that for $n = 1$, $H_1^{(2)} = 2$.

5. Prove the theorem (due to Euler):

Theorem 2.3 *For convex polyhedrons the following relation holds [1–3]*

$$S + N = K + 2 \ ,$$

where:

$$\begin{aligned} &S \ \textit{is the number of faces}, \\ &N \ \textit{is the number of vertices}, \\ &K \ \textit{is the number of edges}. \end{aligned}$$

6. Prove that in the three-dimensional space there are only five regular convex polyhedrons, the so called Platonic solids: the tetrahedron, cube, octahedron, dodecahedron and icosahedron.
   Moreover, how many polyhedrons, not necessarily convex, would be there?
7. Prove that in the four-dimensional space there are only six regular solids:
   (a) 5-cell with 5 vertices,
   (b) 8-cell with 16 vertices,
   (c) 16-cell with 8 vertices,
   (d) 24-cell with 24 vertices,
   (e) 120-cell with 600 vertices,
   (f) 600-cell with 120 vertices.
8. Prove that in the space of dimension $n \geq 5$ there are only tree types of regular hyper-solids:
   (a) $(n+1)$-cell with $n+1$ vertices,
   (b) 2^n-cell with 2^n vertices,
   (c) 2^n-cell with 2^n vertices.
9. Prove that there are only 7 crystallographic systems and that on the plane there can only be at most 10 crystal classes and 17 crystal groups. In the tree-dimensional space there can be 32 crystal classes and 230 crystal groups.
   And how many of them can there be in the four-dimensional space?
10. To win a prize, a son has to win two chess games in a row out of three playing in turn with his father and mother. It is known that the father is a better chess player than the mother. What strategy should the son take to have the best chance of wining the prize?
11. Prove that in the following dialog of agents (persons) A and B, B's statement is false:

A : "I have not lied more than three times in my life."
B : "Saying that you lied the fourth time."

12. The hidden treasure (a problem with a redundant information). A bottle was found and it contained the following instruction:
On the island there are only two trees and a gallows.
    (a) Starting from the gallows, walk towards one of the trees counting your steps, at the tree make a right turn in the direction that will keep you on the same side of the straight line connecting the trees and walk the same number of steps. Mark your position.
    (b) Go back to the gallows. Walk towards the other tree, again counting your steps, at the tree make a right turn in the direction that will keep you on the same side of the straight line connecting the trees and walk the same number of steps. Mark your position again. The treasure is in the middle of the distance between the marked points.

    On the island they found the trees but not a trace of the gallows. How should they find the treasure?
13. A dialog of two mathematicians (a problem with seemingly insufficient information):

    A : "How many children do you have?"
    B : "Three."
    A : "Rounding up their respective ages, how old are they, each of them?"
    B : "If you multiply their ages together you will obtain 36."
    A : "What is the sum of the numbers of their years?"
    B : "It is equal to the number of windows in the building opposite."
    A : "Er . . . I still have doubts."
    B : "My youngest son has brown eyes."
    A : "Now I know how old each of them is."

14. When observation and common sense mislead.
A solid of revolution of infinite axis cross section can have a finite volume.
15. The son wrote to his father, a computer scientist the following coded message

$$\begin{array}{r} \text{s e n d} \\ +\ \text{m o r e} \\ \hline \text{m o n e y} \end{array}$$

What is the sum in question?

2.2 Solution of the Problems

1. A necessary condition is that the inequality $a + b > c$ is satisfied.
Another necessary condition is that $|a - b| < c$ holds.

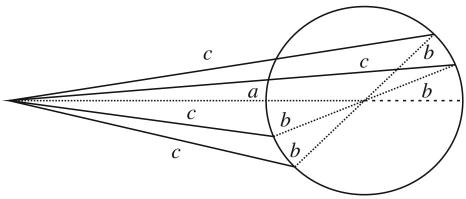


Fig. 2.1 Construction of triangles

The necessary and sufficient condition is that the conjunction of the two satisfies

$$|a-b| < c < a+b \ . \tag{2.1}$$

The inequality (2.1) can have the following wording: the necessary and sufficient condition for the three segments a, b and c to be the sides of a triangle is that the sum of each two out of the three is greater than the remaining one (Fig. 2.1).

2. For $n = 1$ the proposition holds since $1 + \dfrac{n(n+1)}{2} = 2$, i.e. a line cuts the plane into two parts.

   Assume that it holds for n lines. Then, the plane is cut into $1 + \dfrac{n(n+1)}{2}$ parts. Draw one more line. It intersects the n lines at n points and crosses the $n+1$ parts of the plane and cuts each one of them into two. Thus, $n+1$ new parts are formed. Therefore the total number of parts is now

$$1 + \frac{n(n+1)}{2} + n + 1 = 1 + \frac{(n+1)(n+2)}{2} \ .$$

   However, this expression is also obtained from the original one by substituting $n+1$ for n. Thus, the proposition has been proved by mathematical induction. See Fig. 2.2.

3. For $n = 1$ i.e. for one line we have $N = 0$, $K = 1$, $S = 2$, thus the proposition holds since $N - K + S = 1$.
   Assume that the proposition holds for n lines i.e. that $N - K + S = 1$.
   Let there be $n+1$ lines on the plane. Let $M_1, M_2, \ldots, M_k$ be the points at which the line L_{n+1} intersects the n remaining lines $L_1, L_2, \ldots, L_n$.
   Assume, for the time being, that those points are the new ones, i.e. the line

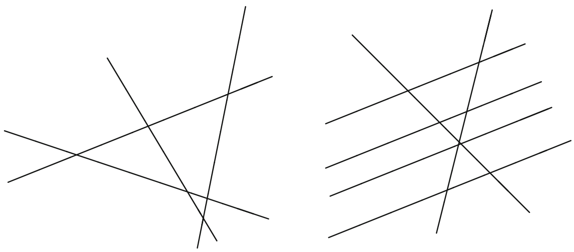


Fig. 2.2 Problem 2. An illustration

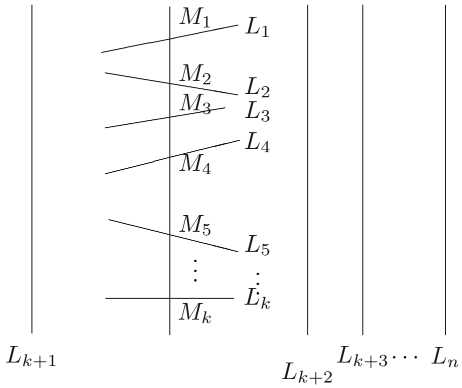


Fig. 2.3 Problem 3. An illustration

L_{n+1} does not pass through any of the points at which the n lines intersect $L_1, L_2, \ldots, L_n$ (Fig. 2.3).
Drawing the line L_{n+1} results in an increase of the number N by k and of an increase of K by $k + 1 + k = 2k + 1$ because each of the lines $L_1, L_2, \ldots, L_n$ is divided into two parts and the line L_{n+1} is divided at the points $M_1, M_2, \ldots, M_k$ into $k + 1$ parts. Now the number of the plane parts S is increased by $k + 1$, which is the number of new parts the borders of which contain, among others, those parts of the line L_{n+1} into which it is divided at the points $M_1, M_2, \ldots, M_k$. Thus we have

$$N_1 - K_1 + S_1 = (N + k) - (K + 2k + 1) + (S + k + 1) = N - K + S = 1 \ .$$

The situation does not change when among the points $M_1, M_2, \ldots, M_k$ there are old points i.e. the points of intersections of the lines $L_1, L_2, \ldots, L_n$ (Fig. 2.4) (Table 2.1).
Let the point M_1 be the point of intersection of the q lines $L_1, L_2, \ldots, L_q$. Let us denote this point by $M_{123\ldots q}$. Now the number of points N is increased up to $k - q$, the number of line parts is increased up to $k - q + 1 + k - q$ and the number of the plane parts is increased up to $k - q + 1$. Thus, again, we have

$$N_1 - K_1 + S_1 = (N + k - q) - (K + 2k - 2q + 1) + (S + k - q + 1) = 1 \ .$$

Therefore, no matter whether all the points $M_1, M - 2, \ldots, M_k$ are the new ones or there are old ones among them, and with any multiplication factor, it still holds that $N_1 - K_1 + S_1 = 1$ if it is true for n lines. Since it is true for $n = 1$, then it is true for any n.

4. The proposition has been proved for the dimension $k = 2$ with the proof of Problem 3. It is not difficult to prove it for any k by mathematical induction.
5. The proof consists in checking that the Euler formula holds for a single triangle on a sphere and showing that it also holds for a network of triangles on a sphere

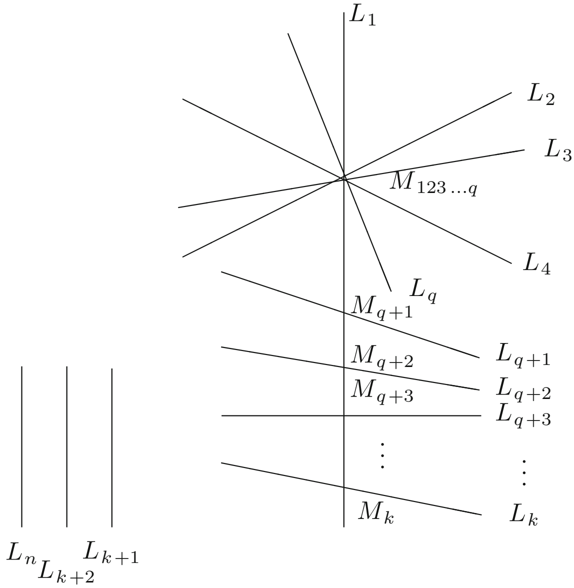


Fig. 2.4 Problem 3. An illustration

obtained through a continuous transformation of an arbitrary connected polyhedron. Let S be the number of faces, N be the number of vertices and K be the number of edges of a polyhedron. A connected polyhedron is a polyhedron which can be continuously transformed into a ball. On the sphere, which is the boundary of the ball, there are the images of the vertices, edges and faces of the polyhedron through the transformation. We still call them the vertices, edges and faces on the sphere. They form a network of the same number of vertices, edges and faces as those on the polyhedron.

Prior to the transformation, perform the following operation: divide each face of the polyhedron, which is not a triangle, into triangles by drawing diagonals. Treat the diagonals as new edges and triangles as new faces. Then, the number of new edges is equal to the number of new faces. Therefore, the difference $S - K$ remains unchanged. Also the number of vertices remains unchanged by this operation. Thus, $N + S - K$ retains its value. Now the network on the sphere as the image of the previously transformed polyhedron contains only triangles. We shall prove that such a network on the sphere satisfies the Euler formula. Choose an arbitrary triangle from the network on the sphere. Affix other triangles of the network to it. Note that it can be done in the two following ways:

(a) Add two edges, by adding one vertex and one face (Fig. 2.5); then the number $N + S - K$ remains unchanged;
(b) Add one edge, at the same time adding one face; again the number $N + S - K$ remains the same (cf. Fig. 2.6).

Table 2.1 Table divisions depending on the number of dividers

Number of dividing elements	Number of divisions		
	Subspaces by plains	Plain through lines	Lines through points
0	1	1	1
1	2	2	2
2	4	4	3
3	8	7	4
4	15	11	5
$\vdots$	V_{n-1}	P_{n-1}	n
n	V_n	P_n	$n+1$
	$(+)S$	$(-)K$	N

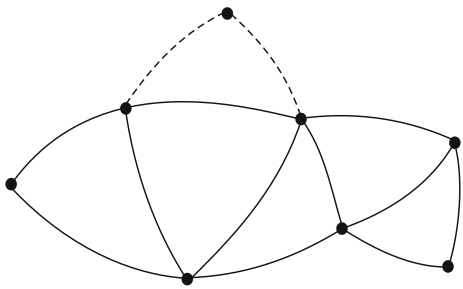


Fig. 2.5 A way of outbuilding the network

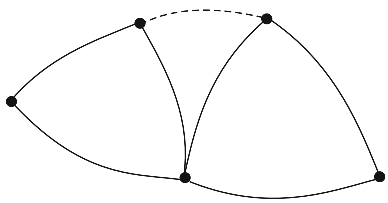


Fig. 2.6 The other way of outbuilding the network

Following this procedure reproduce the whole network.

A single triangle has three vertices, three edges and divides the sphere into two faces. Thus the equality $N + S - K = 2$ holds.

The procedure described above does not change the number $N + S - K$, so that the Euler formula holds for any connected polyhedron, not necessarily convex.

The presented argument is the proof of the Euler formula for connected polyhedrons which are relevant in geometry and topology. In modern automation its meaning is linked with its application in the theory of image recognition, for instance in the verification of the shape of three-dimensional objects approximated by polyhedrons. Moreover, the way of argumentation presented above can also be used in the analysis of networks and structures in the computer aided analysis of systems represented in the form of graphs, e.g. in operations research.

Remark 2.1 The Euler theorem has been generalized to the case of multi-dimensional spaces. Let a_k be the number of zero-dimensional objects, i.e. vertices of the n-dimensional connected polyhedron, a_1 be the number of one-dimensional objects, i.e. edges, a_2 be the number of the two-dimensional objects, and so on, and – generally – a_k be the number of k-dimensional objects constituting the n-dimensional polyhedron, then the following equality

$$a_0 - a_1 + a_2 - a_3 + \ldots + (-1)^n a_n = 1 \tag{2.2}$$

holds, i.e.

$$\sum_{k=0}^{n} (-1)^k a_k = 1 \ .$$

In the case of a three-dimensional polyhedron we have $n = 3$, $a_0 = N$, $a_1 = K$, $a_2 = S$ and $a_3 = 1$ as we deal with one polyhedron. From (2.2) we have $N - K + S - 1 = 1$, and then $N - K + S = 2$ which is the Euler equality.

6. A regular polyhedron has regular congruent polygons as its faces. Its vertices are also congruent and all their dihedral angles are equal. The equality of the flat angles follows from the congruency of faces. To specify all the possible regular polyhedrons we use the Euler theorem (Problem 5) and the observation that the sum of all flat angles of a vertex is less than 360°.

Proof 2.1 (a) Assume that the faces are equilateral triangles, then there are three possible types of vertices:

1. trihedral (the sum of flat angles is 180°),
2. tetrahedral (the sun of flat angles is 240°),
3. pentahedral (the sum of flat angles is 300°).

In the first case we have: S faces which are triangles and have $3S$ sides together but each two have an edge in common. Then the number of edges is $K = \frac{3}{2}S$; S triangles have $3S$ vertices but each polyhedron vertex is common for three faces, and then the number of polyhedron vertices is $N = S$.
So, from the Euler theorem we have

$$S + S = \frac{3}{2}S + 2 \ ,$$

and $S = 4$, $N = 4$, $K = \frac{3}{2} \cdot 4 = 6$.
Thus we have obtained the first Platonic solid: the *tetrahedron* (Fig. 2.7).
In the second case we have: S triangles as the faces and the number of edges equal to $K = \frac{3}{2}S$.

Fig. 2.7 Tetrahedron

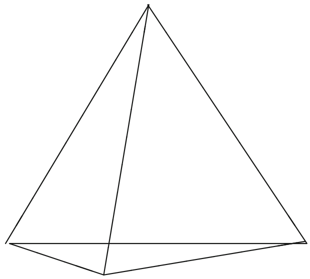
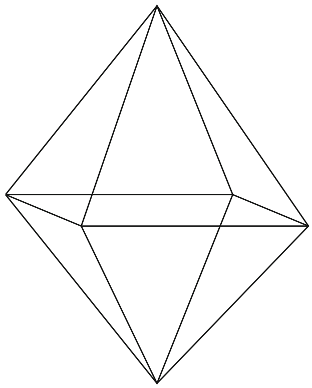


Fig. 2.8 Octahedron



Each polyhedron vertex is common for four faces, hence $N = \frac{3}{4}S$.
Therefore, we have:

$$S + \frac{3}{4}S = \frac{3}{2}S + 2 \ ,$$

then $S = 8$, $N = 6$, $K = 12$.
The second Platonic solid is the *octahedron* (Fig. 2.8).
In the third case we have: $K = \frac{3}{2}S$, and each polyhedron vertex is now common for five faces so that $N = \frac{3}{5}S$ and

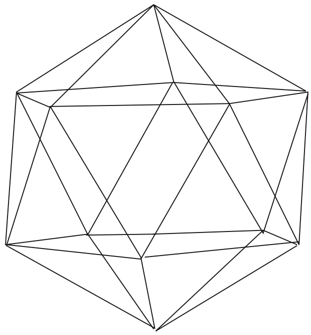
$$S + \frac{3}{5}S = \frac{3}{2}S + 2 \ .$$

Therefore, $S = 20$, $N = 12$, $K = 30$. The third Platonic solid is the *icosahedron* (Fig. 2.9).

(b) If the faces are squares, then the only possible polyhedron vertex type is trihedral and the sum of flat angles is $270°$. The number of edges is

$$K = \frac{4S}{2} = 2S \ ,$$

Fig. 2.9 Icosahedron



and the number of polyhedron vertices is $N = \frac{4}{3}S$. Therefore

$$S + \frac{4}{3}S = 2S + 2 \ ,$$

and $S = 6$, $N = 8$, $K = 12$.

Thus, we have obtained the fourth Platonic solid: the *cube* (Fig. 2.10).

(c) If the faces are pentagons, then the only possible polyhedron vertex type is trihedral. The sum of flat angles is then $3 \times 108° = 324°$. The number of edges is $K = \frac{5}{2}S$, the number of vertices is $N = \frac{5}{3}S$, and then

$$S + \frac{5}{3}S = \frac{5}{2}S + 2 \ .$$

Therefore, $S = 12$, $N = 20$, $K = 30$.

We have obtained the fifth Platonic solid: the *dodecahedron* (Fig. 2.11).

(d) The hexagon cannot be a face of a regular polyhedron since its angles are 120° each and the trihedral polyhedron vertex would have the sum of flat angles equal to $3 \times 120° = 360°$ which is not possible. Neither can a polygon of a higher number of sides than six be a face of a regular polyhedron. Then, there are only five regular polyhedrons. The octahedron is dual to the cube and the icosahedron is dual to the dodecahedron.

7. The proof is similar to that of Problem 6.
8. The proof is similar to that of Problem 6.
9. The proof is rather long and is omitted here. The interested reader is referred to [2].
10. Te son should play first with the father, then with the mother and then again with the father.

Proof 2.2 Let the probability of wining with the mother be p and that of wining with the father be q. The father is a better chess player than the mother, therefore $p > q$. There are two possible strategies for the son:

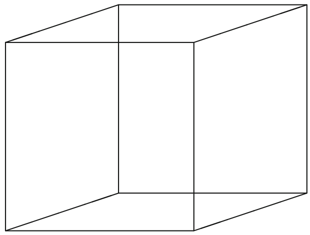


Fig. 2.10 Cube

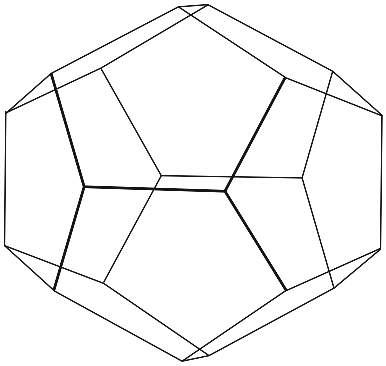


Fig. 2.11 Dodecahedron

(a) play the first game with the father, play the second game with the mother and play the last game with the father;
(b) play the first game with the mother, play the second game with the father and play the last game with the mother.

The probability of wining two games in a row in case (a) is

$$qpq + qp(1-q) + (1-q)pq = pq(2-q) \ .$$

In case (b) it is

$$pqp + pq(1-p) + (1-p)qp = pq(2-p) \ .$$

Since $p > q$ the first strategy is better.

11. If A had lied less than three times in his life or just three times, then his statement was true.
    If A had lied more than three times, then he could not lie for the fourth time. Then, B's statement was false.
12. Let us draw a plan and mark the positions of the trees and a possible position of the gallows cf. Fig. 2.12 where A is the position of one tree, B is the position of the other tree, S is an assumed position of the gallows and C is the position of the treasure, and let A, B, S, C, M, N be complex numbers on the complex plane, $j = \sqrt{-1}$.

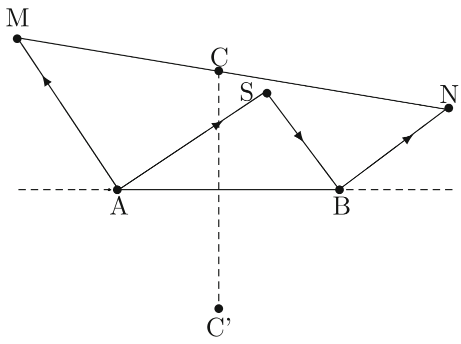


Fig. 2.12 Solution of problem 12

Then

$$M - A = j(S - A), \qquad N - B = -j(S - B) \ .$$

Adding the two equalities sidewise we have

$$M + N - (A + B) = j(B - A) \ .$$

As

$$C = \frac{1}{2}(M + N) \ ,$$

then

$$C = \frac{1}{2}(M + N) = \frac{1}{2}(A + B) + \frac{j}{2}(B - A) \ .$$

The treasure can be found in the following way: from the middle of the segment AB walk perpendicularly to it by the distance of a half of the segment AB. There are two possible places where the treasure may be found: symmetrically on both sides of the segment AB (we do not know where the gallows stood). So, the information about the gallows was redundant.

13. The factorings of the number 36 into three integers and the sums of the factors is shown below

N	Product	Sum
1	$2 \cdot 2 \cdot 9$	$2 + 2 + 9 = 13$
2	$1 \cdot 6 \cdot 6$	$1 + 6 + 6 = 13$
3	$1 \cdot 1 \cdot 36$	$1 + 1 + 36 = 38$
4	$3 \cdot 3 \cdot 4$	$3 + 3 + 4 = 10$
5	$2 \cdot 3 \cdot 6$	$2 + 3 + 6 = 11$
6	$1 \cdot 2 \cdot 18$	$1 + 2 + 18 = 21$
7	$1 \cdot 4 \cdot 9$	$1 + 4 + 9 = 14$
8	$1 \cdot 3 \cdot 12$	$1 + 3 + 12 = 16$

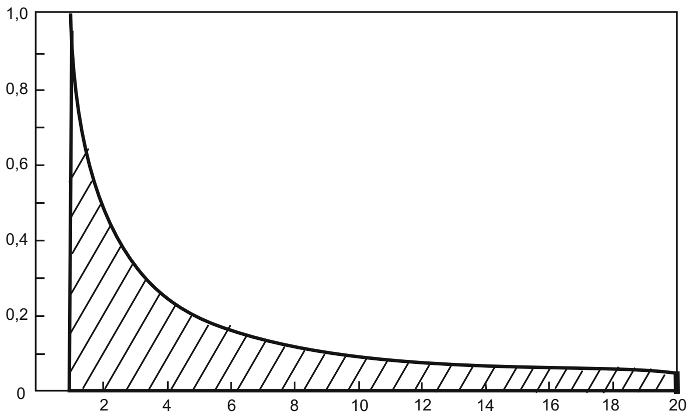


Fig. 2.13 The generating curve and the area under it

When A counted the windows, he knew the sum, but he still had doubts: there were two factorings with the same sum 13. Given the additional information that there was the youngest son (with brown eyes) he knew that the older two were twins and that the children were one and six years old.

14. Consider the volume of the solid of revolution of the infinite axis cross section area (Fig. 2.13). Let the equation of the generating curve be $y = \dfrac{1}{x}$. Calculate the area under the curve.

$$P = \lim_{\zeta\to\infty} \int_1^{\zeta} \frac{\mathrm{d}x}{x} = \lim_{\zeta\to\infty} \ln\zeta - \ln 1 = \infty \ .$$

The solid of revolution is generated by the revolution of the curve about the X-axis. Its volume is

$$V = \pi \int_1^{\infty} y^2 \mathrm{d}x = \pi \int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x = \pi \Big[-\frac{1}{x}\Big]_1^{\infty} = \pi \ ,$$

Then the volume is finite despite the cross section area is infinite.

15. We have the set of equations which follows from the assumed decimal system and the rules of addition.

$$\begin{aligned} d + e \phantom{{}+m} &= y + 10x \\ x + n + r &= e + 10p \\ p + e + o &= n + 10q \\ q + s + m &= o + 10m \end{aligned}$$

where: x, p, q may only be 0 or 1.

After a number of trials we obtain:
$m = 1, q = 0, x = 1, p = 1, r = 8, n = 6, e = 5, y = 2, d = 7.$

$$\text{Verification:}\quad \begin{array}{r} 9\,567 \\ +\quad 1\,085 \\ \hline 10\,652 \end{array}$$

The son asked for 10 652.

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