

Chapter 2

Theoretical Background

2.1 Linearization of Nonlinear Equations of Motion

The equations of motion of a dynamical system in general are nonlinear. Since an analytic solution can only be found for some special cases, it is common practice to linearize these equations around a reference position, typically an equilibrium. In NEWTONIAN mechanics, nonlinear equations of motion can be written in minimal coordinates $\mathbf{q}(t) \in D \subseteq \mathbb{R}^n$ as

$$\mathbf{M}(\mathbf{q}, t)\ddot{\mathbf{q}} + \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{0}, \quad \mathbf{q}(0) = \mathbf{q}_0, \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0, \quad (2.1)$$

where $\mathbf{M} \in \mathbb{R}^{n \times n}$ is the mass matrix, $(\mathbf{q}(t), t) \in D \times [0, \infty)$ and the function $\mathbf{f} : D \rightarrow \mathbb{R}^n \times [0, \infty)$ can be interpreted as the sum of generalized (nonlinear) forces [1]. The number of DOF is equal to the dimension n of the system. Defining $\Delta \mathbf{q}(t) = \mathbf{q}(t) - \mathbf{q}_e(t)$ as a (small) deviation from an equilibrium point \mathbf{q}_e , that may be put to zero without loss of generality, a TAYLOR expansion yields the linearized equations of motion

$$\mathbf{M}(t)\ddot{\mathbf{q}} + \mathbf{B}(t)\dot{\mathbf{q}} + \mathbf{C}(t)\mathbf{q} = \mathbf{g}(t). \quad (2.2)$$

Looking for a physical meaning of $\mathbf{B}(t)$ and $\mathbf{C}(t)$, it is advantageous to decompose the matrices into their symmetric and skew-symmetric parts

$$\begin{aligned} \mathbf{D}(t) &= \frac{1}{2} (\mathbf{B}(t) + \mathbf{B}(t)^T), & \mathbf{G}(t) &= \frac{1}{2} (\mathbf{B}(t) - \mathbf{B}(t)^T), \\ \mathbf{K}(t) &= \frac{1}{2} (\mathbf{C}(t) + \mathbf{C}(t)^T), & \mathbf{N}(t) &= \frac{1}{2} (\mathbf{C}(t) - \mathbf{C}(t)^T), \end{aligned} \quad (2.3)$$

where $\mathbf{D}(t)$ is the damping matrix, $\mathbf{G}(t)$ is the gyroscopic matrix, $\mathbf{K}(t)$ is the stiffness matrix and $\mathbf{N}(t)$ is the matrix containing circulatory terms [2]. Often homogenous systems are considered, i.e. setting $\mathbf{g}(t) \equiv \mathbf{0}$ yields the MDGKN-system

$$\mathbf{M}(t)\ddot{\mathbf{q}}(t) + [\mathbf{D}(t) + \mathbf{G}(t)]\dot{\mathbf{q}}(t) + [\mathbf{K}(t) + \mathbf{N}(t)]\mathbf{q}(t) = \mathbf{0} \quad (2.4)$$

describing free linear vibrations around an equilibrium position.

2.2 Time-Invariant MDGKN-Systems

For the time being, the matrices are assumed to be time-invariant, i.e.

$$\mathbf{M}\ddot{\mathbf{q}}(t) + (\mathbf{D} + \mathbf{G})\dot{\mathbf{q}}(t) + (\mathbf{K} + \mathbf{N})\mathbf{q}(t) = \mathbf{0}. \quad (2.5)$$

Looking for a solution of the form $\mathbf{q}(t) = \hat{\mathbf{q}}e^{\lambda t}$ yields the eigenvalue problem, where $\lambda \in \mathbb{C}$ is an eigenvalue and $\hat{\mathbf{q}} \in \mathbb{C}^n$ is the corresponding (right) eigenvector. The imaginary part of λ represents the circular frequency ω , or, if divided by 2π , the frequency of the corresponding mode, i.e. $\lambda = \text{Re}(\lambda) \pm i\omega$. Nontrivial solutions only exist if

$$\det[\lambda^2\mathbf{M} + \lambda(\mathbf{D} + \mathbf{G}) + \mathbf{K} + \mathbf{N}] = 0. \quad (2.6)$$

After determining the eigenvalues λ_i ($i = 1, \dots, 2n$), the eigenvectors $\hat{\mathbf{q}}_i$ can be calculated as a solution of $[\lambda_i^2\mathbf{M} + \lambda_i(\mathbf{D} + \mathbf{G}) + \mathbf{K} + \mathbf{N}]\hat{\mathbf{q}}_i = \mathbf{0}$. Since (2.5) is linear, the general solution

$$\mathbf{q}(t) = \sum_{i=1}^{2n} C_i \hat{\mathbf{q}}_i e^{\lambda_i t} \quad (2.7)$$

can be found by superposition certainly as long as there are no multiple eigenvalues. The constants $C_i \in \mathbb{C}$ are determined by the initial conditions \mathbf{q}_0 and $\dot{\mathbf{q}}_0$ given in (2.1).

With respect to stability analysis, the real part of the eigenvalues λ_i decides whether the absolute value of (2.7) grows or decays in time. Therefore, in the linear case, the following conditions for LYAPUNOV stability can be defined [3]:

- (1) If each λ_i ($i = 1, \dots, 2n$) has negative real part, i.e. $\text{Re}(\lambda_i) < 0 \forall i \in \{1, \dots, 2n\}$, the trivial solution is *asymptotically stable*.
- (2) If at least one eigenvalue has positive real part, i.e. $\exists i \in \{1, \dots, 2n\} : \text{Re}(\lambda_i) > 0$, the trivial solution is *unstable*.

If neither condition 1 nor 2 is satisfied, a critical case occurs and the stability of the equilibrium depends on nonlinear terms.

2.3 First-Order Systems

The coordinate transformation $\mathbf{y}(t) = (\mathbf{q}(t), \dot{\mathbf{q}}(t))^T$ with $\mathbf{y} \in \mathbb{R}^{2n}$ yields a first-order formulation of (2.5), i.e.

$$\dot{\mathbf{y}}(t) = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{K} + \mathbf{N}) & -\mathbf{M}^{-1}(\mathbf{D} + \mathbf{G}) \end{bmatrix}}_{:=\mathbf{A}} \mathbf{y}(t), \quad (2.8)$$

where $\mathbf{A} \in \mathbb{R}^{2n \times 2n}$ is the square coefficient matrix and \mathbf{I} is the $n \times n$ identity matrix. The eigenvalues defined by (2.6) are identical to those of \mathbf{A} . Any set of $2n$ linearly independent solutions of (2.8) is called a *fundamental system*. The term *fundamental system* is not restricted to linear systems. But, in most cases, it can be found analytically for linear systems only. The *fundamental matrix* $\Phi(t)$ is defined as

$$\Phi(t) := (\mathbf{y}_1(t) \mid \mathbf{y}_2(t) \mid \dots \mid \mathbf{y}_{2n}(t)) \in \mathbb{R}^{2n \times 2n}. \quad (2.9)$$

It can also be determined by using the matrix exponential according to [4]. Let \mathbf{X} be an arbitrary square matrix of arbitrary dimension, the *matrix exponential* is defined as

$$e^{\mathbf{X}} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{X}^k. \quad (2.10)$$

The *fundamental matrix* of the first-order system $\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t)$ is then given by

$$\Phi(t) = \Phi(0)e^{\mathbf{A}t}. \quad (2.11)$$

Setting the initial values $\Phi(0) = \mathbf{I}$, Eq.(2.11) simplifies to

$$\Phi(t) = e^{\mathbf{A}t}. \quad (2.12)$$

2.4 Time-Periodic Systems and FLOQUET Theory

In the previous sections, two ways of analyzing and finding solutions for time-invariant MDGKN-systems are discussed. In many technical applications, e.g. automotive brakes, the assumption of time-invariance yields satisfying results. However, for time-variant linear systems, in general no solution can be found analytically. Assuming time-periodic coefficients, conclusions about the stability behavior can still be drawn. Consider the linear first-order system

$$\dot{\mathbf{y}}(t) = \mathbf{A}(t)\mathbf{y}(t), \quad \mathbf{A}(t) = \mathbf{A}(t + \frac{2\pi}{\Omega}), \quad (2.13)$$

where, similar to the time-invariant case, $\mathbf{A}(t) \in \mathbb{R}^{2n \times 2n}$. In (2.8), for example, the matrix $\mathbf{K}(t)$ may be periodic. Using FLOQUET theory the *fundamental matrix* of system (2.13) can be written as

$$\Phi(t) = \mathbf{P}_\Omega(t) \mathbf{e}^{\mathbf{B}t}, \quad (2.14)$$

where $\mathbf{P}_\Omega(t) = \mathbf{P}_\Omega(t + \frac{2\pi}{\Omega})$ is a $\frac{2\pi}{\Omega}$ -periodic matrix [5]. The index Ω denotes an angular velocity forcing the system to time-periodic parameters. Since system (2.13) repeats its properties periodically, it is sufficient to examine the stability behavior over one period Ω only. If the solution grows after one period, the system is unstable and vice versa. Hence, the main interest is not $\Phi(t)$ but $\Phi(T)$, the *fundamental matrix* at the time T after one period. The *monodromy matrix*

$$\mathcal{M} := \mathbf{e}^{T\mathbf{B}} \quad (2.15)$$

contains information about the state after one period [5, 9]. If the initial conditions are $\Phi(0) = \mathbf{I}$, the *monodromy matrix* is equal to the *fundamental matrix* evaluated after one period. The eigenvalues of \mathcal{M} are called *Floquet multipliers* and are denoted by $\mu_i \in \mathbb{C}$, $i = 1, \dots, 2n$. Similar to the time-invariant case, the following stability conditions can be defined [5]:

- (1) If the magnitude of each *Floquet multiplier* is smaller than one, i.e. $|\mu_i| < 1 \forall i \in [1, \dots, 2n]$, system (2.13) is *asymptotically stable*.
- (2) If the magnitude of at least one *Floquet multiplier* is larger than one, i.e. $\exists i \in [1, \dots, 2n] : |\mu_i| > 1$, system (2.13) is *unstable*.

The solution of time-periodic systems can be written as

$$\Phi(t) = \mathbf{e}^{\int_0^t \mathbf{A}(\tau) d\tau}, \quad (2.16)$$

where the initial conditions are set to the identity matrix. Equation (2.16) is a generalized version of (2.12), which is valid for $\mathbf{A}(t)$ being periodic or not. Since the integral of the time-dependent matrix cannot be solved analytically in general, the solution can only be computed numerically at specified time intervals. FLOQUET theory predicts $T = \frac{2\pi}{\Omega}$ as the time interval to be sufficient to study the stability behavior of (2.13). This yields another formulation of the *monodromy matrix*

$$\mathcal{M} = \mathbf{e}^{\int_0^T \mathbf{A}(\tau) d\tau}. \quad (2.17)$$

For solving the integral in (2.17) numerically, T can be divided into m subintervals, i.e. $\Delta t = \frac{T}{m}$. Discrete evaluation leads to

$$\mathcal{M} = \lim_{m \rightarrow \infty} \prod_{j=0}^m \mathbf{e}^{\mathbf{A}(j\Delta t)\Delta t}, \quad \Delta t = \frac{T}{m} = \frac{2\pi}{\Omega m}. \quad (2.18)$$

Choosing m sufficiently large, Eq. (2.18) finally gives an approximation for the *monodromy matrix* [6]. For (2.18) to be evaluated, an explicit expression of $\mathbf{A}(t)$ has to be given.

2.5 Optimization of Damping

In this study, two kinds of linear systems are considered. The decision whether the system is stable or unstable either depends on the real part of λ in the time-invariant case or on the magnitude of μ in the time-periodic case. Instability occurs if at least one eigenvalue has positive real part or if at least one eigenvalue of the *monodromy matrix* has absolute value greater than one, respectively. Thus, the optimization of the stability behavior leads to an optimization of eigenvalues $\lambda_i = \lambda_i(\mathbf{p})$ or *Floquet multipliers* $\mu_i = \mu_i(\mathbf{p})$, respectively, where $\mathbf{p} \in \mathbb{R}^s$ is a vector containing s parameters.

2.5.1 Time-Invariant Systems

As discussed in [7], the optimization of time-invariant systems can be formulated as

$$\begin{aligned} & \min_{\hat{\mathbf{p}}} \max_i \operatorname{Re}(\lambda_i) \\ & \text{s.t. } \mathbf{\Gamma} \hat{\mathbf{p}} \leq \mathbf{c}, \\ & \mathbf{\Gamma} \in \mathbb{R}^{r \times \hat{s}}, \hat{\mathbf{p}} \in \mathbb{R}^{\hat{s}} \subseteq \mathbf{p}, \mathbf{c} \in \mathbb{R}^r, \hat{s} \leq s. \end{aligned} \quad (2.19)$$

Here, $\mathbf{\Gamma} \hat{\mathbf{p}} \leq \mathbf{c}$ represents a set of r linear equations constraining $\hat{\mathbf{p}}$ in an admissible range (polyhedron). Since not all parameters are optimized in general, $\hat{\mathbf{p}}$ contains \hat{s} parameters only.

Another possibility to improve the stability is to maximize the minimum damping ratio

$$D_i = -\frac{\operatorname{Re}(\lambda_i)}{|\lambda_i|} \quad (2.20)$$

according to [8]. In this case, the first line in (2.19) changes to $\max_{\hat{\mathbf{p}}} \min_i D_i(\lambda_i)$.

2.5.2 Time-Periodic Systems

For time-periodic systems, the formulation of the optimization problem is similar, i.e.

$$\begin{aligned}
& \min_{\hat{\mathbf{p}}} \max_i |\mu_i| \\
& \text{s.t. } \mathbf{\Gamma} \hat{\mathbf{p}} \leq \mathbf{c}, \\
& \mathbf{\Gamma} \in \mathbb{R}^{r \times \hat{s}}, \hat{\mathbf{p}} \in \mathbb{R}^{\hat{s}} \subseteq \mathbf{p}, \mathbf{c} \in \mathbb{R}^r, \hat{s} \leq s.
\end{aligned} \tag{2.21}$$

In this case, the maximum magnitude of the *Floquet multipliers* has to be minimized. To define a damping ratio similar to (2.20), the relationship

$$\mu_i = e^{\lambda_i T} \tag{2.22}$$

connecting the eigenvalues of the time-invariant system with the *Floquet multipliers* is used [9]. With (2.22) it is possible to determine a formula for the real part of λ_i , i.e.

$$\text{Re}(\lambda_i) = \frac{1}{T} \ln |\mu_i|. \tag{2.23}$$

However, the imaginary part of a *Floquet multiplier* does not really represent circular frequencies, which may be strongly influenced by the function $P_\Omega(t)$. Still, to define a damping ratio for parametrically excited systems, this problem can be solved by determining numerically the dominant frequency of (2.16) with a discrete FOURIER analysis. The calculation is carried out with the fast FOURIER transformation (FFT) implemented in MATLAB. A detailed explanation of this algorithm can be found in [10]. The equivalent damping ratio becomes

$$\tilde{D}_i = -\frac{\ln |\mu_i|}{T \sqrt{(\frac{1}{T} \ln |\mu_i|)^2 + (2\pi f_i)^2}}, \tag{2.24}$$

where f_i denotes the frequency of \mathbf{q}_i resulting from the FFT. Thus, the first line in (2.21) changes to $\max_{\hat{\mathbf{p}}} \min_i \tilde{D}_i(\mu_i)$.

2.6 Linear Damping Models

Regarding optimization problems (2.19) and (2.21), the main focus of this study is to modify the properties of the damping matrix \mathbf{D} in order to stabilize or make more stable the equilibrium state subject to sensible constraints. Since most physical processes and in particular damping are inherently nonlinear, the damping models are linearized. Consider the forced one-degree-of-freedom system

$$m\ddot{q} + d_{\text{eq}}\dot{q} + kq = \hat{f}e^{i\Omega t}, \tag{2.25}$$

where d_{eq} is the equivalent viscous damping constant of the respective model. In contrast to the homogenous case, where the right-hand side of (2.25) is equal to

zero, a particular solution of (2.25) of the form

$$q = \hat{q} e^{i\Omega t + \alpha t} \quad (2.26)$$

is postulated for the definition of structural damping. The dissipated energy E_D caused by a damping force F_D over one period T is given by

$$E_D = \int_0^T F_D \dot{q} \, dq. \quad (2.27)$$

In the following, COULOMB damping (damping constant $d_{eq,c}$), viscous damping (damping constant $d_{eq,v}$) and structural damping (damping constant $d_{eq,m}$) are discussed.

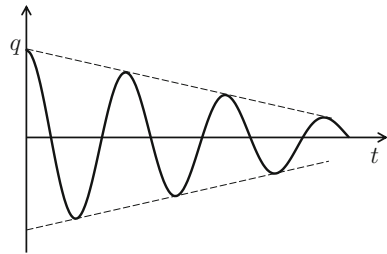
2.6.1 COULOMB Damping

The friction force $F_{D,c}$ between two bodies sliding with respect to each other is given by COULOMB's law, i.e.

$$F_{D,c} = \frac{\mu|N|}{|\dot{q}|} \dot{q}, \quad \dot{q} \neq 0, \quad (2.28)$$

where μ is the friction coefficient, N is the normal force and \dot{q} is the relative velocity between the two bodies [11]. According to COULOMB, the friction coefficient is not a function of q , \dot{q} and N and is assumed to be constant. A coefficient comparison between (2.25) and (2.28) yields the equivalent viscous damping coefficient of a system with COULOMB damping, i.e. $d_{eq,c} = \frac{\mu|N|}{|\dot{q}|}$. As can be seen in Fig. 2.1, the homogenous COULOMB damped solution of (2.25) with $\hat{f} = 0$ decays linearly in time.

Fig. 2.1 Amplitude of solution in COULOMB damped system



2.6.2 Viscous Damping

The resulting force of viscous damping $F_{D,c}$ is proportional to the relative velocity \dot{q} , i.e.

$$F_{D,v} = d\dot{q}. \quad (2.29)$$

A coefficient comparison with (2.25) yields $d_{eq,v} = d$, where d is the viscous damping coefficient. If the damping force does not only depend on the current velocity but also on the history of the velocity, Eq. (2.29) becomes the convolution integral

$$F_{D,v}(t) = \int_{-\infty}^{\infty} b(t - \tau)\dot{q}(\tau)d\tau, \quad (2.30)$$

where the function $b(t - \tau)$ relates $q(\tau)$ and $F_v(t)$. Since viscous damping cannot depend on the velocity in the future, the *causality condition*

$$b(t - \tau) \stackrel{!}{=} 0, \quad \text{if } \tau > t, \quad (2.31)$$

has to be satisfied. Inserting (2.26) in (2.29) and using (2.27) yields the dissipated energy over one period

$$E_{D,v} \propto \Omega d |\hat{q}|^2 \quad (2.32)$$

which is proportional to the square of the amplitude and proportional to the circular frequency. As can be seen in Fig. 2.2, the solution of a viscously damped homogenous system decays exponentially in time.

2.6.3 Structural Damping

Experiments on structural elements and also on complex structures indicate that the energy dissipated internally in cyclic straining is often proportional to the square of the amplitude but independent of the frequency Ω [12]. A suitable representation

Fig. 2.2 Amplitude of solution in viscously damped system

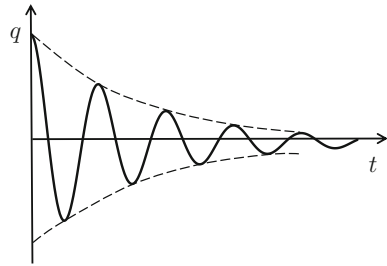
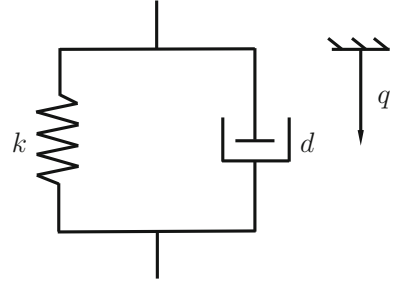


Fig. 2.3 KELVIN- VOIGT model

of material behavior is the KELVIN- VOIGT model shown in Fig. 2.3. The elastic spring (stiffness k) and the viscous damper (damping coefficient d) are connected in parallel. The resulting force $F_{D,m} = kq + d\dot{q}$ is the sum of spring and damper force [13]. Assuming harmonic oscillations according to (2.26) yields the complex stiffness notation

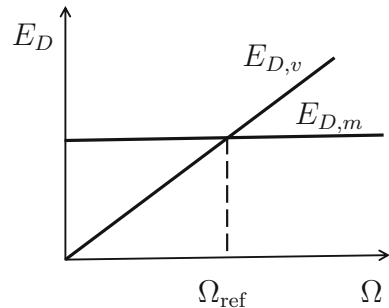
$$F_{D,m} = k(1 + ig)q, \quad (2.33)$$

where $g := \frac{d\Omega}{k}$ is the structural damping factor (loss factor) which has to be determined experimentally [14]. Detailed explanation of the experiments can be found in [15]. The dissipated energy over one period is then

$$E_{D,m} \propto gk|\hat{q}|^2. \quad (2.34)$$

Using (2.32) and (2.34) the equivalent viscous damping factor $d_{eq,m} = \frac{gk}{\Omega_{ref}}$ can be defined. As can be seen in Fig. 2.4, Ω_{ref} is the circular frequency at $E_{D,v} = E_{D,m}$. If $\Omega > \Omega_{ref}$, structural damping is overestimated and vice versa.

In general, the definition of structural damping is valid only for single-degree-of-freedom systems harmonically excited with one frequency [14]. In homogeneous systems, i.e. $\hat{f}e^{i\Omega t} \equiv 0$, Eq. (2.25) requires a solution of the form $q = \hat{q}e^{-\delta t + i(\omega_d + \gamma t)}$, where $\delta = \frac{d}{m}$ is the constant of exponential decay, ω_d is the frequency of the damped oscillation and γ is the phase shift. In this case, the resulting damping force is

Fig. 2.4 Reference circular frequency

$$F_{D,m} = k \left(\sqrt{1 - g^2} + ig \right) q. \quad (2.35)$$

In many technically relevant applications, the structural damping factor is taken to be $\mathcal{O}(h^{-2})$ or smaller [16]. With this assumption ($g \ll 1$), Eqs. (2.33) and (2.35) are approximately equal and the approach can be extended to systems with $n > 1$. Then, the complex stiffness and the equivalent viscous damping matrix read

$$\mathbf{K}^* = \mathbf{K} + i\mathbf{H}, \quad \mathbf{D}_S = \frac{1}{\Omega_{\text{ref}}} \mathbf{H}, \quad (2.36)$$

where the structural damping matrix is $\mathbf{H} := g\mathbf{K}$. A limit of this damping model is that a loss factor g independent of the circular frequency violates the *causality condition* (2.31) in the time-domain [17, 18]. However, since many problems in vibration theory are analyzed in the frequency domain and only within certain limited frequency bands, the approach remains feasible.

2.7 Modal Reduction

Mechanical systems discretized using finite elements have a large number of DOF. In order to save computing time modal reduction techniques are frequently used to reduce the number of DOF while maintaining the main dynamical characteristics of the system. In a linear combination of n eigenvectors $\hat{\mathbf{q}}_k \in \mathbb{R}^n$ the vector of minimal coordinates

$$\mathbf{q} = \sum_{k=1}^n \hat{\mathbf{q}}_k p_k = \mathbf{X} \mathbf{p} \quad (2.37)$$

can be expressed exactly, where p_k are the modal coordinates and $\mathbf{X} \in \mathbb{R}^{n \times n}$ is the modal matrix. In modal reduction, the vector of minimal coordinates is approximated by only $m \ll n$ eigenvectors, i.e.

$$\mathbf{q} \approx \sum_{k=1}^m \hat{\mathbf{q}}_k p_k = \mathbf{X}_m \mathbf{p}, \quad (2.38)$$

where $\mathbf{X}_m \in \mathbb{R}^{n \times m}$ is a rectangular matrix transforming an n -degree-of-freedom system into a subspace containing only m DOF. In order to determine the most suitable eigenvectors, different modal reduction techniques have been developed [19, 20]. The equations of motion in modal space read

$$\tilde{\mathbf{M}} \ddot{\mathbf{p}} + (\tilde{\mathbf{D}} + \tilde{\mathbf{G}}) \dot{\mathbf{p}} + (\tilde{\mathbf{K}} + \tilde{\mathbf{N}}) \mathbf{p} = \mathbf{0}, \quad (2.39)$$

where $\tilde{\mathbf{M}} = \mathbf{X}_m^T \mathbf{M} \mathbf{X}_m$, $\tilde{\mathbf{D}} = \mathbf{X}_m^T \mathbf{D} \mathbf{X}_m$, $\tilde{\mathbf{G}} = \mathbf{X}_m^T \mathbf{G} \mathbf{X}_m$, $\tilde{\mathbf{K}} = \mathbf{X}_m^T \mathbf{K} \mathbf{X}_m$, and $\tilde{\mathbf{N}} = \mathbf{X}_m^T \mathbf{N} \mathbf{X}_m$ are the respective truncated modal matrices.

2.8 Brake Squeal

A first physical explanation of brake squeal is based on the assumption of a decreasing friction coefficient, i.e. by a negative slope of the friction-velocity characteristic [21]. However, experimental tests showed the phenomenon not to be the result of oscillation components in circumferential direction [22]. Instead, it is now commonly accepted that brake squeal is initiated by an instability due to friction forces leading to self-excited vibrations even for constant coefficients of friction. These friction forces lead to an asymmetry in the matrices describing the coordinate proportional forces and to mode coupling effects which may lead to flutter-type instabilities [23, 24]. In case of nonlinear models, the disc oscillates ultimately reaching a limit cycle. Detailed explanations of the nonlinear case can be found in [11, 25–27]. Since brake squeal is due to unstable solutions, in the linear case, positive real parts of the eigenvalues λ may lead to squeal. Therefore, complex eigenvalue analysis (CEA) is the approach mostly favoured by the automotive industry [28]. However, the magnitude of the eigenvalue real part (or damping ratio) does not predict the relative amplitude, i.e. the sound pressure level. It only reveals how fast the vibration is growing and thus represents only a relative measure of squeal propensity. For example, a linear analysis can predict an unstable system, but the resulting limit cycle may be very small such that the noise generated would be inaudible.

Different techniques have been developed to avoid the occurrence of the noise phenomenon. WAGNER showed that splitting the eigenfrequencies, which is implemented using an optimization of the geometric properties of the disc, minimizes the tendency of the disc to squeal [29]. Similar results are provided by SPELSBERG-KORSPETER [30]. Another technical approach is an active control of the brake pads using piezoceramic actuators [31]. In [32], the main goal is to stabilize the disc by optimizing parameter values of the damping matrix using a minimal model of disc brake developed by VON WAGNER et al. in [33]. In another work, it is shown that damping properties, especially the damping ratio of different modes, have to be considered in detail in order to avoid mode coupling and friction induced vibrations [34].

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Wehner, J.-H.; Jekel, D.; Sampaio, R.; Hagedorn, P.
2018, VIII, 50 p. 27 illus., 23 illus. in color., Softcover
ISBN: 978-3-319-62712-0