Chapter 2
Background on Gain-Scheduling

2.1 Gain-Scheduling: LPV Systems and TS Systems

After World War II, the development of advanced jet aircrafts, the advent of guided missiles and the need of stability and performance requirements for a wide set of operating conditions pushed towards a rapid adoption of *gain scheduled* autopilot systems [1]. As examples of first proposed solutions, the B-52 autopilot, developed around 1951, incorporated an airspeed-based mechanism to compensate for changes in the aero-surface effectiveness [1]. The autopilot of the Talos missile, developed in the early 1950s, adjusted the gains to compensate for changes in altitude and speed, thus exhibiting a rudimentary form of gain-scheduling [2]. Since then, gain-scheduling began to play an important role not only in military applications, but in commercial ones too. For example, in response to the dual imperatives of improved fuel economy and reduction of exhaust emissions, gain scheduling began to be used in automobile engine controllers for electronic fuel control [1], starting from [3], in which a closed-loop electronic fuel injection control with a gain influenced by measured variables was described.

The first gain scheduled controller design approach involved selecting several operating points, covering the range of the plant's working conditions, where linear time invariant (LTI) controllers were designed. Then, between these operating points, the parameters (gains) of the controller were interpolated (scheduled) [4]. However, this approach lacked in providing stability and performance guarantees for all the possible operating conditions and, moreover, it needed the assumption of slow variation in time of the parameters [5]. For this reason, the necessity for systematic analysis and design tools for gain-scheduled controllers arised. Among the most successful approaches, there are the linear parameter varying (LPV) and the Takagi-Sugeno (TS) paradigms.

LPV systems were introduced by Shamma [6] to distinguish such systems from LTI and linear time varying (LTV) ones [7]. More specifically, LPV systems are a particular class of LTV systems, where the time-varying elements depend on measurable parameters that can vary over time [8]. The LPV framework has proved to

D. Rotondo, *Advances in Gain-Scheduling and Fault Tolerant Control Techniques*,
Springer Theses, https://doi.org/10.1007/978-3-319-62902-5_2

be suitable for controlling nonlinear systems by embedding the nonlinearities in the varying parameters, that will depend on some endogenous signals, e.g. states, inputs or outputs. In this case, the system is referred to as *quasi-LPV*, to make a further distinction with respect to *pure* LPV systems, where the varying parameters only depend on exogenous signals [9].

Since the introduction of this paradigm, a lot of research has concerned the development of design techniques for LPV systems. At first, the small gain theorem was applied to LPV systems with a linear fractional transformation (LFT) form [10, 11]. However, this approach took into account complex varying parameters, that did not appear in real plants, thus introducing a strong source of conservatism [8]. For this reason, Lyapunov-based approaches were developed, allowing to take into account not only arbitrarily fast parameter variations [12], but also known bounds on the rate of parameter variation [13–15]. A unified scheme combining the small gain theorem and the Lyapunov-based approach was developed by [16].

The LPV paradigm has evolved rapidly in the last two decades and has been applied successfully to a big number of applications, e.g. active vision systems [17], airplanes [18, 19], bioreactors [20], canals [21], CD players [22], container crane load swing [23], control moment gyroscopes [24], electromagnetic actuators [25], engines [26], flexible ball screw drives [27], fuel cells [28, 29], glycemic regulation [30], induction motors [31], internet web servers [32], inverted pendula [33], ionic polymer-metal composites [34], magneto-rheological dampers [35], robots [36], unmanned aerial vehicles (UAVs) [37, 38], vehicle suspensions [39–41], wafer scanners [42], wind turbines [43] and winding machines [44]. Recently, the LPV paradigm has also been applied to time delay systems with time varying delays [45–47].

On the other hand, TS systems, introduced by [48], basically provide an effective way of representing nonlinear systems with the aid of fuzzy sets, fuzzy rules and a set of local linear models which are smoothly connected by fuzzy membership functions [49]. TS fuzzy models are universal approximators, since they can approximate any smooth nonlinear function to any degree of accuracy [50–54], such that they can represent complex nonlinear systems.

The design approaches for TS systems can be classified into six categories [49]: (i) local controller design, where feedback controllers are designed for each local model and combined to obtain the global controller, and some stability criteria is used to check stability [55, 56]; (ii) stabilization based on a nominal linear model with nonlinearities considered as uncertainties [57, 58]; (iii) stabilization based on a common quadratic Lyapunov function [59–67]; (iv) stabilization based on a piecewise quadratic Lyapunov function [57, 68–70]; (v) stabilization based on a fuzzy Lyapunov function [71, 72]; (vi) adaptive control, when the parameters of the TS fuzzy models are unknown [73–75].

Also the TS paradigm has been successfully applied in several fields, among which active suspension of vehicles [76], aircrafts [77], electromechanical systems [78], energy production systems [79], missiles [80], robotic systems [81], spark ignition engines [82], transmission systems [83] and time delay systems [84].

2.2 Modeling of LPV Systems

In this section, some basic concepts about modeling of LPV systems are recalled. Since the thesis deals with methods developed for LPV models with polytopic parameter dependence, the case of LFT parameter dependence [10, 11] will not be considered. This does not cause a loss of generality, since [8] has demonstrated that an LPV model with LFT parameter dependence can be converted into an LPV model with polytopic parameter dependence. Also, the thesis will focus on LPV state-space (SS) representations, even though LPV input-output (IO) models have been proposed too [85]. Reference [86] has suggested practically applicable approaches for the conversion of an LPV IO model in a discrete-time LPV SS representation; thus, considering SS models does not cause a loss of generality. Finally, the methods recalled hereafter provide an LPV model starting from a nonlinear model that is assumed to be available. In cases different from this, LPV models can be identified from IO data [87, 88].

An LPV system is defined as a finite-dimensional LTV system whose state space matrices are fixed functions of some varying parameters $\theta(\tau) \in \mathbb{R}^{n_\theta}$, assumed to be unknown a priori, but measured or estimated in real-time [89]:

$$\sigma.x(\tau) = A\left(\theta(\tau)\right) x(\tau) + B\left(\theta(\tau)\right) u(\tau) \tag{2.1}$$

$$y(\tau) = C\left(\theta(\tau)\right) x(\tau) + D\left(\theta(\tau)\right) u(\tau) \tag{2.2}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$ are the state, the input, and the output vector, respectively, and $A\left(\theta(\tau)\right)$, $B\left(\theta(\tau)\right)$, $C\left(\theta(\tau)\right)$ and $D\left(\theta(\tau)\right)$ are varying matrices of appropriate dimensions.

Among the available analysis/synthesis approaches, the most popular, at least taking into account the number of publications, is the polytopic approach [4]. An LPV system is called *polytopic* when it can be represented by matrices $A\left(\theta(\tau)\right)$, $B\left(\theta(\tau)\right)$, $C\left(\theta(\tau)\right)$ and $D\left(\theta(\tau)\right)$, where the parameter vector $\theta(\tau)$ ranges over a fixed polytope Θ, and the dependence of the matrices on θ is affine [12], resulting in the following representation:

$$\sigma.x(\tau) = \sum_{i=1}^{N} \mu_i\left(\theta(\tau)\right)\left(A_i x(\tau) + B_i u(\tau)\right) \tag{2.3}$$

$$y(\tau) = \sum_{i=1}^{N} \mu_i\left(\theta(\tau)\right)\left(C_i x(\tau) + D_i u(\tau)\right) \tag{2.4}$$

where the quadruples (A_i, B_i, C_i, D_i) define the so-called *vertex systems*, and μ_i are the coefficients of the polytopic decomposition, such that:

$$\sum_{i=1}^{N} \mu_i\left(\theta(\tau)\right) = 1, \quad \mu_i\left(\theta(\tau)\right) \geq 0, \ \forall i = 1, \ldots, N, \ \forall \theta \in \Theta \tag{2.5}$$

In the following, some methods for obtaining an LPV model starting from an available nonlinear SS model are recalled. For sake of simplicity, only continuous-time (CT) nonlinear systems in the form:

$$\dot{x}(t) = g\left(x(t), u(t)\right) \tag{2.6}$$

$$y(t) = h\left(x(t), u(t)\right) \tag{2.7}$$

are considered. Notice that most of the physical systems of interest for control purposes are CT, and if discrete-time (DT) LPV representations are desired for digital implementation, such models can be obtained from CT LPV models using discretization techniques, such as Euler or more sophisticated ones [90, 91].

2.2.1 Jacobian Linearization

The Jacobian linearization approach is the simplest technique that can be applied for obtaining LPV models. It assumes that the nonlinear system can be linearized around some equilibrium points of interest [9]. The basis of the method is to use a first-order Taylor-series approximation of (2.6)–(2.7), and then an interpolation of the obtained LTI models, when the system is working in operating points different from the equilibrium ones.

Despite its simplicity, the behavior of the obtained LPV model could diverge from the behavior of the nonlinear model [9]. The use of higher-order Taylor expansions could alleviate this issue, but would lead to impractical implementations [92]. Also, it is essentially impossible to capture the transient behavior of the nonlinear plant using this method [93].

Hereafter, an example of the application of the Jacobian linearization technique, taken from [94], is shown.

Consider the nonlinear system [95]:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -x_1(t) \\ x_1(t) - |x_2(t)|\, x_2(t) - 10 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \tag{2.8}$$

$$y(t) = x_2(t) \tag{2.9}$$

The set of linearized models obtained from (2.8)–(2.9) is:

$$\begin{pmatrix} \delta\dot{x}_1(t) \\ \delta\dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -2\left|x_2^{eq}(t)\right| \end{pmatrix} \begin{pmatrix} \delta x_1(t) \\ \delta x_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \delta u(t) \tag{2.10}$$

$$\delta y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \delta x_1(t) \\ \delta x_2(t) \end{pmatrix} \tag{2.11}$$

Then, by considering the scheduling parameter $\theta(t) = \left|x_2^{eq}(t)\right|$, the model (2.10)–(2.11) would appear in the form (2.1)–(2.2). The resulting system would be referred to as *quasi-LPV*, due to the dependence of $\theta(t)$ on $x_2^{eq}(t)$.

2.2.2 State Transformation

In the state transformation approach, a coordinate change is performed with the aim of removing any nonlinear term not dependent on the scheduling parameters [89]. This method assumes that the nonlinear system is in the following form:

$$\begin{pmatrix} \dot{z}(t) \\ \dot{l}(t) \end{pmatrix} = g\left(z(t)\right) + A\left(z(t)\right)\begin{pmatrix} z(t) \\ l(t) \end{pmatrix} + B\left(z(t)\right)u(t) \tag{2.12}$$

where $z(t) \in \mathbb{R}^{n_z}$ are the scheduling states, and $l(t) \in \mathbb{R}^{n_h}$ are the non-scheduling ones, with $n_z = n_u$. Under the assumptions that there exists a family of equilibrium states parameterized by $z(t)$, such that:

$$0 = g\left(z(t)\right) + A\left(z(t)\right)\begin{pmatrix} z(t) \\ l_{eq}\left(z(t)\right) \end{pmatrix} + B\left(z(t)\right)u_{eq}\left(z(t)\right) \tag{2.13}$$

with $l_{eq}\left(z(t)\right)$ and $u_{eq}\left(z(t)\right)$ continuously differentiable functions, and that $A\left(z(t)\right)$ and $B\left(z(t)\right)$ are partitioned as:

$$A\left(z(t)\right) = \begin{pmatrix} A_{11}\left(z(t)\right) & A_{12}\left(z(t)\right) \\ A_{21}\left(z(t)\right) & A_{22}\left(z(t)\right) \end{pmatrix} \tag{2.14}$$

$$B\left(z(t)\right) = \begin{pmatrix} B_1\left(z(t)\right) \\ B_2\left(z(t)\right) \end{pmatrix} \tag{2.15}$$

it is possible to rewrite the state dynamics as:

$$\begin{aligned}\begin{pmatrix} \dot{z}(t) \\ \dot{l}(t) - \dot{l}_{eq}\left(z(t)\right) \end{pmatrix} &= \begin{pmatrix} 0 & A_{12}\left(z(t)\right) \\ 0 & A_{22}\left(z(t)\right) - \left.\frac{\partial l_{eq}(z)}{\partial z}\right|_{z(t)} A_{12}\left(z(t)\right) \end{pmatrix}\begin{pmatrix} z(t) \\ l(t) - l_{eq}(t) \end{pmatrix} \\ &+ \begin{pmatrix} B_1\left(z(t)\right) \\ B_2\left(z(t)\right) - \left.\frac{\partial l_{eq}(z)}{\partial z}\right|_{z(t)} B_1\left(z(t)\right) \end{pmatrix}\left(u(t) - u_{eq}\left(z(t)\right)\right)\end{aligned} \tag{2.16}$$

thus obtaining a quasi-LPV form different from the one obtained by performing the Jacobian linearization, and exactly representing the original nonlinear system. However, the presence of an inner-loop feedback due to the term $u_{eq}\left(z(t)\right)$ can deteriorate the properties of the system by adversely exciting flexible mode dynamics [5, 6]. Hence, special care should be taken when applying this technique.

For the example (2.8)–(2.9), the quasi-LPV model:

$$\begin{pmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \end{pmatrix} = \begin{pmatrix} -1 - 2\,|\tilde{x}_2(t)| & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tilde{u}(t) \tag{2.17}$$

would be generated by changing the state coordinates as [94]:

$$\tilde{x}_1(t) = x_1(t) - x_1^{eq}\,(x_2(t)) \tag{2.18}$$

$$\tilde{x}_2(t) = x_2(t) \tag{2.19}$$

$$\tilde{u}(t) = u(t) - u_{eq}\,(x_2(t)) \tag{2.20}$$

with:

$$u_{eq}(t) = x_1^{eq}\,(x_2(t)) = |x_2(t)|\,x_2(t) + 10 \tag{2.21}$$

2.2.3 Function Substitution

An alternative approach to obtain a quasi-LPV model is the *function substitution* approach [96, 97], which consists in replacing the so-called *decomposition function* with functions that are linear with respect to the scheduling parameters. This decomposition function is formed by combining all the terms of the nonlinear system that are not both affine with respect to the non-scheduling states and control inputs, and function of the scheduling parameters alone (after a coordinate change with respect to a single equilibrium point has been performed) [9]. The decomposition is carried out through a minimization procedure, which leads to numerical optimization problems [88].

For the example (2.8)–(2.9), the nonlinear system is rewritten as [94]:

$$\begin{pmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tilde{u}(t) + \begin{pmatrix} -x_1^{eq} + u_{eq} \\ x_1^{eq} - \left|\tilde{x}_2(t) + x_2^{eq}\right| \left(\tilde{x}_2(t) + x_2^{eq}\right) - 10 \end{pmatrix} \tag{2.22}$$

where:

$$\tilde{x}_1(t) = x_1(t) - x_1^{eq} \tag{2.23}$$

$$\tilde{x}_2(t) = x_2(t) - x_2^{eq} \tag{2.24}$$

$$\tilde{u}(t) = u(t) - u_{eq} \tag{2.25}$$

with trim point $(x_1^{eq}, x_2^{eq}) = (11, 1)$. Then, by replacing the nonlinearity in (2.22) with:

$$g\left(\tilde{x}_2(t)\right) = \begin{cases} \frac{\left[\left|x_2^{eq}\right|x_2^{eq} - \left|\tilde{x}_2(t)+x_2^{eq}\right|\left(\tilde{x}_2(t)+x_2^{eq}\right)\right]}{\tilde{x}_2(t)} & \tilde{x}_2(t) \neq 0 \\ 0 & \tilde{x}_2(t) = 0 \end{cases} \tag{2.26}$$

the following quasi-LPV model is obtained:

$$\begin{pmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & g\left(\tilde{x}_2(t)\right) \end{pmatrix} \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tilde{u}(t) \tag{2.27}$$

2.2.4 *Other Approaches and Current Directions of Research*

The problem of modeling a nonlinear system as a quasi-LPV model is still a hot topic of research. For example, [98] have suggested that linearization and local controller design should be carried out not only at equilibrium states, but also in transient operating regimes.

In [99], a method for automated generation of LPV models, to be used when affine representations of polytopic models are desired, has been presented. The affine LPV representations are generated from a general nonlinear model by *hiding* the nonlinearities in the scheduling parameters. These LPV representations are not unique and different models have different properties that may facilitate, complicate, or even make impossible, the controller synthesis. For instance, two representations of the same system may differ in the number of parameters, in the property of stabilizability, or in the degree of overbounding of the admissible parameter set. Hence, [99] also proposed a heuristic measure for the quality of different LPV models.

In the case of overbounding, i.e. when the obtained quasi-LPV model displays more behaviors than the underlying nonlinear model, it is possible to use the method proposed in [100]. This method is based on parameter set mapping (PSM) [101] and leads to the generation of less conservative representations.

A SS model interpolation of local estimates (SMILE) technique has been presented in [102] for estimating LPV SS models, based on the interpolation of LTI models estimated for constant values of the scheduling parameters. The interpolation is based on the formulation of a linear least-squares problem that can be efficiently solved, yielding homogeneous polynomial LPV models that are numerically well-conditioned and therefore suitable for LPV control synthesis.

In [103], inspired by the feedback linearization theory, a systematic procedure is proposed to convert control affine nonlinear SS representation into state minimal LPV SS representations in an observable canonical form, where the scheduling parameter depends on the derivatives of the inputs and outputs of the system. In addition, if the states of the nonlinear model can be measured or estimated, then the procedure can be modified to provide LPV models scheduled by these states.

2.3 Modeling of TS Systems

In this section, some basic concepts about modeling of TS systems are recalled. Also in this case, as in the LPV modeling, the approach that constructs a TS fuzzy model using an identification procedure applied to IO data is not considered. The interested reader may find some details about this approach, suitable for plants that cannot or are too difficult to be represented by means of analytical/physical models, in [104, 105].

TS systems, as proposed by Takagi and Sugeno [48], are described by local models merged together using fuzzy IF-THEN rules [54], as follows:

$$
\begin{aligned}
&IF\ \vartheta_1(\tau)\ is\ M_{i1}\ AND\ \cdots\ AND\ \vartheta_p(\tau)\ is\ M_{ip}\\
&THEN \begin{cases} \sigma.x_i(\tau) = A_i x(\tau) + B_i u(\tau) \\ y_i(\tau) = C_i x(\tau) + D_i u(\tau) \end{cases} \quad i = 1, \ldots, N
\end{aligned} \tag{2.28}
$$

where $\vartheta_1(\tau), \ldots, \vartheta_p(\tau)$ are the *premise variables*, that can be functions of the state variables, controlled inputs, external disturbances and/or time. Each linear consequent equation represented by $A_i x(\tau) + B_i u(\tau)$ is called a *subsystem*. Given a pair $(x(\tau), u(\tau))$, the state and output of the TS system can be inferred easily as:

$$
\sigma.x(\tau) = \frac{\sum_{i=1}^{N} \varpi_i\left(\vartheta(\tau)\right)\left(A_i x(\tau) + B_i u(\tau)\right)}{\sum_{i=1}^{N} \varpi_i\left(\vartheta(\tau)\right)} = \sum_{i=1}^{N} \rho_i\left(\vartheta(\tau)\right)\left(A_i x(\tau) + B_i u(\tau)\right) \tag{2.29}
$$

$$
y(\tau) = \frac{\sum_{i=1}^{N} \varpi_i\left(\vartheta(\tau)\right)\left(C_i x(\tau) + D_i u(\tau)\right)}{\sum_{i=1}^{N} \varpi_i\left(\vartheta(\tau)\right)} = \sum_{i=1}^{N} \rho_i\left(\vartheta(\tau)\right)\left(C_i x(\tau) + D_i u(\tau)\right) \tag{2.30}
$$

where $\vartheta(\tau) = \left(\vartheta_1(\tau), \ldots, \vartheta_p(\tau)\right)^{\mathrm{T}}$ is the vector containing the premise variables, and $\varpi_i\left(\vartheta(\tau)\right)$ and $\rho_i\left(\vartheta(\tau)\right)$ are defined as follows:

$$
\varpi_i\left(\vartheta(\tau)\right) = \prod_{j=1}^{p} M_{ij}\left(\vartheta_j(\tau)\right) \tag{2.31}
$$

$$
\rho_i\left(\vartheta(\tau)\right) = \frac{\varpi_i\left(\vartheta(\tau)\right)}{\sum_{i=1}^{N} \varpi_i\left(\vartheta(\tau)\right)} \tag{2.32}
$$

where $M_{ij}\left(\vartheta_j(\tau)\right)$ is the grade of membership of $\vartheta_j(\tau)$ in M_{ij} and $\rho_i\left(\vartheta(\tau)\right)$ is such that:

$$\sum_{i=1}^{N} \rho_i\left(\vartheta(\tau)\right) = 1, \quad \rho_i\left(\vartheta(\tau)\right) \geq 0, \ \forall i = 1, \ldots, N \tag{2.33}$$

In the following, some methods for the CT TS modeling will be recalled.

2.3.1 Sector Nonlinearity

The main idea behind this method appeared for the first time in [106]. Given a nonlinear system $\dot{x}(t) = g\left(x(t)\right)$ with $g(0) = 0$, this approach aims at finding a global sector such that $\dot{x}(t) \in [a_1\ a_2]x(t)$. This approach guarantees an exact model construction [54], but in some cases it is hard to apply, and local sector nonlinearities should be considered instead.

The following example, taken from [54], shows an application of this approach. Consider the nonlinear system:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -x_1(t) + x_1(t)x_2^3(t) \\ -x_2(t) + \left(3 + x_2(t)\right) x_1^3(t) \end{pmatrix} \quad \begin{matrix} x_1(t) \in [-1, 1] \\ x_2(t) \in [-1, 1] \end{matrix} \tag{2.34}$$

Equation (2.34) can be rewritten as:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & x_1(t)x_2^2(t) \\ \left(3 + x_2(t)\right) x_1^2(t) & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{2.35}$$

By choosing the premise variables $\vartheta_1(t) = x_1(t)x_2^2(t)$ and $\vartheta_2(t) = \left(3 + x_2(t)\right) x_1^2(t)$, and calculating the minimum and maximum values of $\vartheta_1(t)$ and $\vartheta_2(t)$ over the considered intervals, i.e. $\vartheta_1(t) \in [-1, 1]$ and $\vartheta_2(t) \in [0, 4]$, the fuzzy model (2.28) is obtained, with:

$$M_{11} = M_{21} = \frac{z_1(t) + 1}{2} \tag{2.36}$$

$$M_{31} = M_{41} = \frac{1 - z_1(t)}{2} \tag{2.37}$$

$$M_{12} = M_{32} = \frac{z_2(t)}{4} \tag{2.38}$$

$$M_{22} = M_{42} = \frac{4 - z_2(t)}{4} \tag{2.39}$$

and:

$$A_1 = \begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix} \ A_2 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -1 & -1 \\ 4 & -1 \end{pmatrix} \; A_4 = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \tag{2.40}$$

It is worth mentioning that the choice of the premise variables is not unique, and different TS representations of the same nonlinear system are possible. This fact will be further investigated in the next chapter.

2.3.2 Local Approximation in Fuzzy Partition Spaces

The spirit of this approach is to approximate nonlinear terms by judiciously choosing linear terms, thus reducing the number of fuzzy rules, being this particularly important at the control system design step [54]. However, since the obtained model does not represent exactly the original nonlinear system, the designed control system could fail in guaranteeing the stability of the original nonlinear system.

The following example, taken from [54], shows an application of this approach.

Let us consider the equations of motion for an inverted pendulum [107]:

$$\dot{x}_1(t) = x_2(t) \tag{2.41}$$

$$\dot{x}_2(t) = \frac{g \sin(x_1(t)) - amlx_2^2(t)\sin(2x_1(t))/2 - a\cos(x_1(t))\,u(t)}{4l/3 - aml\cos^2(x_1(t))} \tag{2.42}$$

where $x_1(t)$ denotes the angle of the pendulum from the vertical, and $x_2(t)$ is the angular velocity; g is the gravity constant, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum, u is the force applied to the cart, and $a = 1/(m + M)$.

When $x_1(t)$ is near zero, (2.42) can be simplified as:

$$\dot{x}_2(t) = \frac{gx_1(t) - au(t)}{4l/3 - aml} \tag{2.43}$$

On the other hand, when $x_1(t)$ is near $\pm\pi/2$, (2.42) can be simplified as:

$$\dot{x}_2(t) = \frac{2gx_1(t)/\pi - a\beta u(t)}{4l/3 - aml\beta^2} \tag{2.44}$$

with $\beta = \cos(88°)$.

Then, a TS fuzzy model with two subsystems can be obtained:

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & 1 \\ \frac{g}{4l/3-aml} & 0 \end{pmatrix} \; B_1 = \begin{pmatrix} 0 \\ -\frac{a}{4l/3-aml} \end{pmatrix} \\ A_2 &= \begin{pmatrix} 0 & 1 \\ \frac{2g}{\pi(4l/3-aml\beta^2)} & 0 \end{pmatrix} \; B_2 = \begin{pmatrix} 0 \\ -\frac{a\beta}{4l/3-aml\beta^2} \end{pmatrix} \end{aligned} \tag{2.45}$$

Notice that by applying the sector nonlinearity approach described in Sect. 2.3.1, sixteen subsystems would have been obtained. Hence, the reduction of fuzzy rules is considerable.

2.4 Analysis of LPV and TS Systems

This section recalls some of the most popular approaches for analyzing an LPV or a TS system. As it will be shown in the next chapter, there are strong analogies between the two frameworks, and the tools developed for a class of system usually apply to the other one too. For this reason, the definitions and theorems recalled in this section are shown for the LPV framework only.

First of all, let us recall some definitions.

Definition 2.1 (*Poles of an LPV system* [108]) Given an autonomous LPV system:

$$\sigma.x(\tau) = A\left(\theta(\tau)\right)x(\tau) \tag{2.46}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $\theta(\tau) \in \Theta \subset \mathbb{R}^{n_\theta}$ is the varying parameter vector, $A\left(\theta(\tau)\right)$ is a varying matrix of appropriate dimensions, the poles of (2.46) are the set of all the poles of the LTI systems obtained by freezing $\theta(\tau)$ to all its possible values $\theta \in \Theta$.

Definition 2.2 (*$\mathcal{D}$-stability of an LPV system*) Given a subset $\mathcal{D}$ of the complex plane, the autonomous LPV system (2.46) is said to be $\mathcal{D}$-stable if all its poles lie in $\mathcal{D}$.

Notice that, unlike the LTI case, in general the notions of stability and $\mathcal{D}$-stability are not related. In fact, a $\mathcal{D}$-stable system could be unstable even if $\mathcal{D}$ is contained within the left-hand semiplane $Re(s) < 0$ in the CT case or the unit circle in the DT case [109]. Also, an LPV system could have some unstable poles, and yet be stable [110].

Definition 2.3 (*LMI regions* [111]) A subset $\mathcal{D}$ of the complex plane is called a linear matrix inequality (LMI) region if there exist matrices $\alpha = [\alpha_{kl}]_{k,l\in\{1,\dots,m\}} \in \mathbb{S}^{m\times m}$ and $\beta = [\beta_{k,l}]_{k,l\in\{1,\dots,m\}} \in \mathbb{R}^{m\times m}$ such that:

$$\mathcal{D} = \{\sigma \in \mathbb{C} : f_{\mathcal{D}}(\sigma) \prec O\} \tag{2.47}$$

where $f_{\mathcal{D}}(\sigma)$ is the *characteristic function* defined as:

$$f_{\mathcal{D}}(\sigma) = \alpha + \beta\sigma + \beta^{\mathrm{T}}\sigma^* = \left[\alpha_{kl} + \beta_{kl}\sigma + \beta_{lk}\sigma^*\right]_{k,l\in\{1,\dots,m\}} \tag{2.48}$$

In other words, LMI regions are subsets of the complex plane that are represented by an LMI in σ and σ^*. In [111], it was shown that LMI regions do not only include a wide variety of typical clustering regions, but also form a dense subset of the convex regions that are symmetric with respect to the real axis. Among the regions that are representable as LMI regions, there are:

- Left-hand semiplanes $Re(\sigma) < \lambda$

$$\alpha = -2\lambda \; \beta = 1$$

- Right-hand semiplanes $Re(\sigma) > \lambda$

$$\alpha = 2\lambda \; \beta = -1$$

- Disks of radius r and center $(-q, 0)$

$$\alpha = \begin{pmatrix} -r & q \\ q & -r \end{pmatrix} \beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- Horizontal strips $-\omega < Im(\sigma) < \omega$

$$\alpha = \begin{pmatrix} -2\omega & 0 \\ 0 & -2\omega \end{pmatrix} \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Definition 2.4 ($\mathcal{H}_\infty$ *norm* [112]) For a stable real-rational transfer matrix $T(\sigma)$, the $\mathcal{H}_\infty$ norm is defined as:

$$\begin{aligned} \|T(s)\|_\infty &= \sup_{\omega \in \mathbb{R}} \sigma_{\max}(T(j\omega)) && \text{CT systems} \\ \|T(z)\|_\infty &= \sup_{\omega \in [-\pi,\pi]} \sigma_{\max}\left(T(e^{j\omega})\right) && \text{DT systems} \end{aligned} \tag{2.49}$$

where $\sigma_{max}(M)$ denotes the largest singular value of the matrix M.

The $\mathcal{H}_\infty$ norm measures the system input-output gain for finite energy signals across all input/output channels.

Definition 2.5 ($\mathcal{H}_\infty$ *performance of an LPV system*) The LPV system:

$$\sigma.x(\tau) = A(\theta(\tau))\, x(\tau) + B_w(\theta(\tau))\, w(\tau) \tag{2.50}$$

$$z_\infty(\tau) = C_{z_\infty}(\theta(\tau))\, x(\tau) + D_{z_\infty w}(\theta(\tau))\, w(\tau) \tag{2.51}$$

has $\mathcal{H}_\infty$ performance γ_∞ if $\left\| T_{z_\infty w}(\sigma, \theta) \right\|_\infty < \gamma_\infty \; \forall \theta \in \Theta$, where $T_{z_\infty w}(\sigma, \theta)$ denotes the closed-loop transfer function from $w(\tau)$ to $z_\infty(\tau)$.

The $\mathcal{H}_\infty$ performance can be interpreted as a disturbance rejection performance, and is convenient to enforce robustness against model uncertainty, and to express frequency-domain specifications such as bandwidth, low-frequency gain, and roll-off [113].

Definition 2.6 ($\mathcal{H}_2$ *norm* [113]) For a stable real-rational transfer matrix $T(\sigma)$, the $\mathcal{H}_2$ norm is defined as:

$$\begin{aligned} \|T(s)\|_2 &= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} Tr\left(T(j\omega)^H T(j\omega)\right) d\omega} \text{ CT systems} \\ \|T(z)\|_2 &= \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} Tr\left(T(e^{j\omega})^H T(e^{j\omega})\right) d\omega} \text{ DT systems} \end{aligned} \tag{2.52}$$

where $Tr(M)$ denotes the trace of the matrix M.

The $\mathcal{H}_2$ norm is equal to the root-mean-square of the impulse response of the system. It measures the steady-state covariance (or power) of the output response $z_2 = T(\sigma)w$ to unit white noise inputs w.

Definition 2.7 ($\mathcal{H}_2$ *performance of an LPV system*) The LPV system (2.50) and:

$$z_2(\tau) = C_{z_2}(\theta(\tau))\, x(\tau) \tag{2.53}$$

has $\mathcal{H}_2$ performance γ_2 if $\left\|T_{z_2 w}(\sigma, \theta)\right\|_2 < \gamma_2\ \forall \theta \in \Theta$, where $T_{z_2 w}(\sigma, \theta)$ denotes the closed-loop transfer function from $w(\tau)$ to $z_2(\tau)$.

The $\mathcal{H}_2$ performance is useful to handle stochastic aspects such as measurement noise and random disturbances [113].

Definition 2.8 (*Finite time stability* [114, 115]) The autonomous LPV system (2.46) is said to be *finite time stable* (FTS) with respect to (c_1, c_2, T, R) with $c_2 > c_1 > 0$ and $R \succ 0$ if:

$$x(0)^{\mathrm{T}} R x(0) \le c_1 \;\Rightarrow\; x(\tau)^{\mathrm{T}} R x(\tau) < c_2 \quad \begin{array}{ll} \forall t \in [0, T] & \text{CT systems} \\ \forall k \in \{1, \dots, T\} & \text{DT systems} \end{array} \tag{2.54}$$

The idea of finite time stability, originally formulated by [116], concerns the boundedness of the state of a system over a finite time interval for given initial conditions. Notice that this definition of finite time stability is different from the one provided in other works, e.g. [117], where the property of a given system to be driven to the equilibrium point in finite time is considered instead.

Definition 2.9 (*Finite time boundedness* [114, 115]) The CT LPV system:

$$\dot{x}(t) = A(\theta(t))\, x(t) + B_w(\theta(t))\, w(t) \tag{2.55}$$

and the DT LPV system:

$$\begin{cases} x(k+1) = A(\theta(k))\, x(k) + B_w(\theta(k))\, w(k) \\ w(k+1) = W(\theta(k))\, w(k) \end{cases} \tag{2.56}$$

are said to be *finite time bounded* (FTB) with respect to (c_1, c_2, T, R, d), with $c_2 > c_1 > 0$, $R \succ 0$ and $d > 0$ if:

$$\begin{cases} x(0)^{\mathrm{T}} R x(0) \leq c_1 \\ w(t)^{\mathrm{T}} w(t) \leq d \end{cases} \Rightarrow \; x(\tau)^{\mathrm{T}} R x(\tau) < c_2 \quad \begin{array}{ll} \forall t \in [0, T] & \text{CT systems} \\ \forall k \in \{1, \ldots, T\} & \text{DT systems} \end{array} \tag{2.57}$$

The idea of state boundedness is more general and concerns the behavior of the state in presence of external disturbances. Notice that FTS can be recovered as a special case of FTB when $w = 0$.

2.4.1 Analysis Based on a Common Quadratic Lyapunov Function

The simplest approach for the analysis of LPV/TS systems is the one based on a common quadratic Lyapunov function. In this case, the Lyapunov candidate function used to assess the chosen specification is:

$$V(x(\tau)) = x(\tau)^{\mathrm{T}} P x(\tau) \tag{2.58}$$

where $P \succ O$.

Theorem 2.1 (Quadratic stability of CT LPV systems) *The autonomous LPV system (2.46) with $t = \tau$ is quadratically stable:*

1. *if there exists $P \succ O$ such that [118]:*

$$A(\theta)^{\mathrm{T}} P + P A(\theta) \prec O \qquad \forall \theta \in \Theta \tag{2.59}$$

2. *if there exists $Q \succ 0$ such that [119]:*

$$Q A(\theta)^{\mathrm{T}} + A(\theta) Q \prec O \qquad \forall \theta \in \Theta \tag{2.60}$$

Proof It is straightforward to obtain (2.59) by calculating $\dot{V}(x(t))$, replacing $\dot{x}(t)$ with (2.46), and imposing the condition $\dot{V}(x(t)) < 0$. Then, (2.60) can be obtained from (2.59) with $Q = P^{-1}$ [119]. A relevant consequence is that the stability of the dual system:

$$\dot{x}(t) = A(\theta(t))^{\mathrm{T}} x(t) \tag{2.61}$$

is also characterized by (2.59)–(2.60). ■

Theorem 2.2 (Quadratic stability of DT LPV systems) *The autonomous LPV system (2.46) with $\tau = k$ is quadratically stable:*

1. *if there exists $P \succ O$ such that [120]:*

$$A(\theta)^{\mathrm{T}} P A(\theta) - P \prec O \qquad \forall \theta \in \Theta \tag{2.62}$$

2. *if there exists $P \succ O$ such that:*

$$\begin{pmatrix} -P & PA(\theta) \\ A(\theta)^{\mathrm{T}}P & -P \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{2.63}$$

3. *if there exists $Q \succ O$ such that [49]:*

$$\begin{pmatrix} -Q & A(\theta)Q \\ QA(\theta)^{\mathrm{T}} & -Q \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{2.64}$$

Proof It is straightforward to obtain (2.62) by calculating $\Delta V(x(k))$, replacing $x(k+1)$ with (2.46) and imposing the condition $\Delta V(x(k)) < 0$. Then, (2.63) can be easily obtained from (2.62) by using the Schur complement. Finally, (2.64) can be obtained from (2.63) with $P = Q^{-1}$. ■

Theorem 2.3 (Quadratic $\mathcal{D}$-stability of LPV systems) *Given an LMI region $\mathcal{D}$ defined as in (2.47), the autonomous LPV system (2.46) is quadratically $\mathcal{D}$-stable:*

1. *if there exists $P \succ O$ such that [121]:*

$$\begin{aligned} &\alpha \otimes P + \beta \otimes PA(\theta) + \beta^{\mathrm{T}} \otimes A(\theta)^{\mathrm{T}}P \\ &\quad = \left[\alpha_{kl}P + \beta_{kl}PA(\theta) + \beta_{lk}A(\theta)^{\mathrm{T}}P\right]_{k,l\in\{1,\ldots,m\}} \prec O \qquad \forall\theta \in \Theta \end{aligned} \tag{2.65}$$

2. *if there exists $Q \succ 0$ such that:*

$$\begin{aligned} &\alpha \otimes Q + \beta \otimes A(\theta)Q + \beta^{\mathrm{T}} \otimes QA(\theta)^{\mathrm{T}} \\ &\quad = \left[\alpha_{kl}Q + \beta_{kl}A(\theta)Q + \beta_{lk}QA(\theta)^{\mathrm{T}}\right]_{k,l\in\{1,\ldots,m\}} \prec O \qquad \forall\theta \in \Theta \end{aligned} \tag{2.66}$$

Proof The proof follows from the reasoning provided in [111], and (2.66) can be obtained as the dual matrix inequality of (2.65) [119]. ■

For an LPV system quadratically $\mathcal{D}$-stable, it is assured that its poles are in $\mathcal{D}$. As shown by [108], the quadratic $\mathcal{D}$-stability also affects the dynamical behavior of the system, justifying from the engineering point of view the definition of LPV poles given in Definition 2.1. It should be highlighted that in the case of CT systems, it can be proved that some transient properties, usually defined in terms of pole location in the case of LTI systems, hold for the LPV case too. This fact has been shown by [121], taking into account the reasoning in [122].

Corollary 2.1 (Exponential decay/growth of LPV systems) *Let $V(x(t))$ be defined as in (2.58), and let the autonomous LPV system (2.46) be quadratically $\mathcal{D}$ stable, i.e. (2.65) holds. Then, the Lyapunov function $V(x(t))$ satisfies, for all $x(t) \neq 0$:*

$$\frac{1}{2}\frac{\dot{V}(x(t))}{V(x(t))} \in \mathcal{D} \cap \mathbb{R} \tag{2.67}$$

Proof Pre-multiplying (2.65) by $I \otimes x(t)^{\mathrm{T}}$, and post-multiplying it by $I \otimes x(t)$, respectively, the following is obtained for all $x(t) \neq 0$:

$$\alpha \otimes x(t)^{\mathrm{T}} P x(t) + \beta \otimes x(t)^{\mathrm{T}} P A(\theta(t))\, x(t) + \beta^{\mathrm{T}} \otimes x(t)^{\mathrm{T}} A(\theta(t))^{\mathrm{T}} P x(t) \prec O \tag{2.68}$$

Recalling that:

$$\frac{1}{2}\dot{V}(x(t)) = x(t)^{\mathrm{T}} P A(\theta(t))\, x(t) = x(t)^{\mathrm{T}} A(\theta(t))^{\mathrm{T}} P x(t) \tag{2.69}$$

and dividing (2.68) by $V(x(t))$, this process leads to:

$$\alpha \otimes 1 + \beta \otimes \frac{1}{2}\frac{\dot{V}(x(t))}{V(x(t))} + \beta^{\mathrm{T}} \otimes \frac{1}{2}\frac{\dot{V}(x(t))}{V(x(t))} \prec O \tag{2.70}$$

which implies (2.67). ■

As a consequence of Corollary 2.1, the system's decay/growth rate lies within the LMI region $\mathcal{D}$. Reference [121] have shown that the concept of $\mathcal{D}$-stability can also be used for imposing limits on the energy of the rate of state change, thus imposing a limit on the system's oscillatory behaviors.

Theorem 2.4 (Quadratic $\mathcal{H}_\infty$ performance of CT LPV systems) *The LPV system (2.50)–(2.51) with $\tau = t$ has quadratic $\mathcal{H}_\infty$ performance γ_∞ [111]:*

1. *if there exists $P \succ O$ such that:*

$$\begin{pmatrix} A(\theta)^{\mathrm{T}} P + P A(\theta) & P B_w(\theta) & C_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}} P & -I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ C_{z_\infty}(\theta) & D_{z_\infty w}(\theta) & -\gamma_\infty^2 I \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.71}$$

2. *if there exists $Q \succ O$ such that:*

$$\begin{pmatrix} A(\theta) Q + Q A(\theta)^{\mathrm{T}} & B_w(\theta) & Q C_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}} & -I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ C_{z_\infty}(\theta) Q & D_{z_\infty w}(\theta) & -\gamma_\infty^2 I \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.72}$$

Proof See [123]. ■

It is worth recalling that (2.71)–(2.72) can be replaced with [12]:

$$\begin{pmatrix} A(\theta)^{\mathrm{T}} P + P A(\theta) & P B_w(\theta) & C_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}} P & -\gamma_\infty I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ C_{z_\infty}(\theta) & D_{z_\infty w}(\theta) & -\gamma_\infty I \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.73}$$

$$\begin{pmatrix} A(\theta) Q + Q A(\theta)^{\mathrm{T}} & B_w(\theta) & Q C_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}} & -\gamma_\infty I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ C_{z_\infty}(\theta) Q & D_{z_\infty w}(\theta) & -\gamma_\infty I \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.74}$$

Theorem 2.5 (Quadratic $\mathcal{H}_\infty$ performance of DT LPV systems) *The LPV system (2.50)–(2.51) with $\tau = k$ has quadratic $\mathcal{H}_\infty$ performance γ_∞ [124]:*

1. if there exists $P \succ O$ such that:

$$\begin{pmatrix} P & PA(\theta) & PB_w(\theta) & O \\ A(\theta)^{\mathrm{T}}P & P & O & C_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}}P & O & I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ O & C_{z_\infty}(\theta) & D_{z_\infty w}(\theta) & \gamma_\infty^2 I \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.75}$$

2. if there exists $Q \succ O$ such that:

$$\begin{pmatrix} Q & A(\theta)Q & B_w(\theta) & O \\ QA(\theta)^{\mathrm{T}} & Q & O & QC_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}} & O & I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ O & C_{z_\infty}(\theta)Q & D_{z_\infty w}(\theta) & \gamma_\infty^2 I \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.76}$$

Proof See [123]. ■

Also in this case, (2.75)–(2.76) can be rewritten as [12]:

$$\begin{pmatrix} P & PA(\theta) & PB_w(\theta) & O \\ A(\theta)^{\mathrm{T}}P & P & O & C_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}}P & O & \gamma_\infty I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ O & C_{z_\infty}(\theta) & D_{z_\infty w}(\theta) & \gamma_\infty I \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.77}$$

$$\begin{pmatrix} Q & A(\theta)Q & B_w(\theta) & O \\ QA(\theta)^{\mathrm{T}} & Q & O & QC_{z_\infty}(\theta)^{\mathrm{T}} \\ B_w(\theta)^{\mathrm{T}} & O & \gamma_\infty I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ O & C_{z_\infty}(\theta)Q & D_{z_\infty w}(\theta) & \gamma_\infty I \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.78}$$

The results provided in Theorems 2.4 and 2.5 are also known as the *bounded real lemma* (BRL). Several results developed throughout this thesis based on (2.71)–(2.72) and (2.75)–(2.76) can be easily extended to the alternative formulations given by (2.73)–(2.74) and (2.77)–(2.78).

Theorem 2.6 (Quadratic $\mathcal{H}_2$ performance of CT LPV systems) *The LPV system (2.50) and (2.53) with $\tau = t$ has quadratic $\mathcal{H}_2$ performance γ_2 [111]:*

1. if there exist $P \succ O$ and $Y(\theta) \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ such that $Tr\,(Y(\theta)) < \gamma_2^2\ \forall\theta \in \Theta$ and:

$$\begin{pmatrix} A(\theta)^{\mathrm{T}}P + PA(\theta) & B_w(\theta) \\ B_w(\theta)^{\mathrm{T}} & -I \end{pmatrix} \prec O \quad \forall\theta \in \Theta \tag{2.79}$$

$$\begin{pmatrix} Y(\theta) & C_{z_2}(\theta) \\ C_{z_2}(\theta)^{\mathrm{T}} & P \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.80}$$

2. *if there exist* $Q \succ O$ *and* $Y(\theta) \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ *such that* $Tr\,(Y(\theta)) < \gamma_2^2\ \forall \theta \in \Theta$ *and:*

$$\begin{pmatrix} A(\theta)Q + QA(\theta)^{\mathrm{T}} & B_w(\theta) \\ B_w(\theta)^{\mathrm{T}} & -I \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.81}$$

$$\begin{pmatrix} Y(\theta) & C_{z_2}(\theta)Q \\ QC_{z_2}(\theta)^{\mathrm{T}} & Q \end{pmatrix} \succ O \quad \forall \theta \in \Theta \tag{2.82}$$

Proof See [111]. ■

Theorem 2.7 (Quadratic $\mathcal{H}_2$ performance of DT LPV systems) *The LPV system (2.50) and (2.53) with* $\tau = k$ *has quadratic* $\mathcal{H}_2$ *performance* γ_2 *[124]:*

1. *if there exist* $P \succ O$ *and* $Y(\theta) \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ *such that* $Tr\,(Y(\theta)) < \gamma_2^2\ \forall \theta \in \Theta$ *and:*

$$\begin{pmatrix} P & PA(\theta) & PB_w(\theta) \\ A(\theta)^{\mathrm{T}}P & P & O \\ B_w(\theta)^{\mathrm{T}}P & O & I \end{pmatrix} \succ O \quad \forall \theta \in \Theta \tag{2.83}$$

$$\begin{pmatrix} Y(\theta) & C_{z_2}(\theta) \\ C_{z_2}(\theta)^{\mathrm{T}} & P \end{pmatrix} \succ O \quad \forall \theta \in \Theta \tag{2.84}$$

2. *if there exist* $Q \succ O$ *and* $Y(\theta) \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ *such that* $Tr\,(Y(\theta)) < \gamma_2^2\ \forall \theta \in \Theta$ *and:*

$$\begin{pmatrix} Q & A(\theta)Q & B_w(\theta) \\ QA(\theta)^{\mathrm{T}} & Q & O \\ B_w(\theta)^{\mathrm{T}} & O & I \end{pmatrix} \succ O \quad \forall \theta \in \Theta \tag{2.85}$$

$$\begin{pmatrix} Y(\theta) & C_{z_2}(\theta)Q \\ QC_{z_2}(\theta)^{\mathrm{T}} & Q \end{pmatrix} \succ O \quad \forall \theta \in \Theta \tag{2.86}$$

Proof See [124]. ■

Notice that, according to Schur's complements [125], in case a multiobjective specification is considered, some of the provided conditions are redundant. For example, the stability conditions provided in Theorems 2.1–2.2 can be found in the upper-left parts of (2.71)–(2.79), (2.81), (2.83) and (2.85). Also, if both $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performances are considered at the same time, (2.79), (2.81), (2.83) and (2.85) are not needed, since they are already included in (2.71)–(2.76).

Theorem 2.8 (Quadratic FTB of CT LPV systems) *The LPV system (2.55) is quadratically FTB with respect to* (c_1, c_2, T, R, d) *if, letting* $\tilde{Q}_1 = R^{-1/2}Q_1R^{-1/2}$, *there exist positive scalars* a, λ_1, λ_2, λ_3 *and two positive definite matrices* $Q_1 \in \mathbb{S}^{n_x \times n_x}$ *and* $Q_2 \in \mathbb{S}^{n_w \times n_w}$ *such that:*

$$\begin{pmatrix} A(\theta)\tilde{Q}_1 + \tilde{Q}_1A(\theta)^{\mathrm{T}} - a\tilde{Q}_1 & B_w(\theta)Q_2 \\ Q_2B_w(\theta)^{\mathrm{T}} & -aQ_2 \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.87}$$

$$\lambda_1 I \prec Q_1 \prec I \tag{2.88}$$

$$\lambda_2 I \prec Q_2 \prec \lambda_3 I \tag{2.89}$$

$$\begin{pmatrix} c_2 e^{-aT} & \sqrt{c_1} & \sqrt{d} \\ \sqrt{c_1} & \lambda_1 & 0 \\ \sqrt{d} & 0 & \lambda_2 \end{pmatrix} \succ O \tag{2.90}$$

Proof It is obtained straightforwardly, taking into account that the conditions presented in Lemma 6 of [114] should hold for every possible value of θ. ■

Theorem 2.9 (Quadratic FTB of DT LPV systems) *The discrete-time LPV system (2.56) is quadratically FTB with respect to* (c_1, c_2, T, R, d) *if there exist positive scalars* a, λ_1, λ_2 *with* $a \geq 1$ *and two positive definite matrices* $Q_1 \in \mathbb{S}^{n_x \times n_x}$ *and* $Q_2 \in \mathbb{S}^{n_w \times n_w}$ *such that:*

$$\begin{pmatrix} -aQ_1 & Q_1 A(\theta)^{\mathrm{T}} & O & O \\ A(\theta) Q_1 & -Q_1 & B_w(\theta) & O \\ O & B_w(\theta)^{\mathrm{T}} & -aQ_2 & W(\theta)^{\mathrm{T}} Q_2 \\ O & O & Q_2 W(\theta) & -Q_2 \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.91}$$

$$\lambda_1 R^{-1} \prec Q_1 \prec R^{-1} \tag{2.92}$$

$$O \prec Q_2 \prec \lambda_2 I \tag{2.93}$$

$$\begin{pmatrix} \frac{c_2}{a^T} - \lambda_2 d & \sqrt{c_1} \\ \sqrt{c_1} & \lambda_1 \end{pmatrix} \succ O \tag{2.94}$$

Proof It is obtained straightforwardly, taking into account that the conditions presented in Lemma 1 of [115] should hold for every possible value of θ. ■

Theorem 2.10 (Quadratic FTS of CT LPV systems) *The autonomous LPV system (2.46) with* $\tau = t$ *is quadratically FTS with respect to* (c_1, c_2, T, R) *if, letting* $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$, *there exist positive scalars* a, λ_1 *and a positive definite matrix* $Q_1 \in \mathbb{S}^{n_x \times n_x}$ *such that (2.88) and:*

$$A(\theta) \tilde{Q}_1 + \tilde{Q}_1 A(\theta)^{\mathrm{T}} - a\tilde{Q}_1 \prec O \quad \forall \theta \in \Theta \tag{2.95}$$

$$\begin{pmatrix} c_2 e^{-aT} & \sqrt{c_1} \\ \sqrt{c_1} & \lambda_1 \end{pmatrix} \succ O \tag{2.96}$$

hold.

Proof It is a direct consequence of Theorem 2.8, when $B_w(\theta(t)) = O$ and $d = 0$. ■

Theorem 2.11 (Quadratic FTS of DT LPV systems) *The autonomous LPV system (2.46) with $\tau = k$ is quadratically FTS with respect to (c_1, c_2, T, R) if there exist positive scalars a, λ_1 with $a \geq 1$ and a positive definite matrix $Q_1 \in \mathbb{S}^{n_x \times n_x}$ such that:*

$$\begin{pmatrix} -a\mathrm{Q}_1 & \mathrm{Q}_1 A(\theta)^{\mathrm{T}} \\ A(\theta)\, Q_1 & -Q_1 \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.97}$$

$$\begin{pmatrix} \frac{c_2}{a^T} & \sqrt{c_1} \\ \sqrt{c_1} & \lambda_1 \end{pmatrix} \succ O \tag{2.98}$$

$$\lambda_1 R^{-1} \prec Q_1 \prec R^{-1} \tag{2.99}$$

Proof It is a direct consequence of Theorem 2.9, when $W(\theta(k)) = B_w(\theta(k)) = O$ and $d = 0$. ∎

The problem with the conditions provided in Theorems 2.1–2.11 is that they rely on the satisfaction of infinite constraints. However, this difficulty can be overcome by considering the polytopic approach, as recalled in Sect. 2.2. In the following, for each theorem, an appropriate corollary is obtained. A mathematical proof is provided for Corollary 2.2 only, while it is omitted for the other corollaries, due to the similarity of the reasoning behind their proofs with the provided one.

Corollary 2.2 (Quadratic stability of CT LPV systems, polytopic version) *The autonomous polytopic CT LPV system:*

$$\dot{x}(t) = \sum_{i=1}^{N} \mu_i(\theta(t)) A_i x(t) \tag{2.100}$$

with coefficients μ_i such that (2.5) holds, is quadratically stable:

1. *if there exists $P \succ O$ such that:*

$$A_i^{\mathrm{T}} P + P A_i \prec O \qquad \forall i = 1, \ldots, N \tag{2.101}$$

2. *if there exists $Q \succ O$ such that:*

$$Q A_i^{\mathrm{T}} + A_i Q \prec O \qquad \forall i = 1, \ldots, N \tag{2.102}$$

Proof Due to a basic property of matrices [126], any linear combination of (2.101) and (2.102) with non-negative coefficients, of which at least one different from zero, is negative definite. Hence, using the coefficients $\mu_i(\theta(t))$, and taking into account (2.5), (2.59) and (2.60) are obtained. ∎

Corollary 2.3 (Quadratic stability of DT LPV systems, polytopic version) *The autonomous polytopic DT LPV system:*

$$x(k+1) = \sum_{i=1}^{N} \mu_i\,(\theta(k)) A_i x(k) \tag{2.103}$$

with coefficients μ_i such that (2.5) holds, is quadratically stable:

1. *if there exists $P \succ O$ such that:*

$$\begin{pmatrix} -P & PA_i \\ A_i^{\mathrm{T}} P & -P \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.104}$$

2. *if there exists $Q \succ O$ such that:*

$$\begin{pmatrix} -Q & A_i Q \\ Q A_i^{\mathrm{T}} & -Q \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.105}$$

Proof Similar to that of Corollary 2.2, thus omitted. ∎

Corollary 2.4 (Quadratic $\mathcal{D}$-stability of LPV systems, polytopic version) *Given an LMI region $\mathcal{D}$ defined as in (2.47), the autonomous polytopic LPV system:*

$$\sigma.x(\tau) = \sum_{i=1}^{N} \mu_i\,(\theta(\tau)) A_i x(\tau) \tag{2.106}$$

with coefficients μ_i such that (2.5) holds, is quadratically $\mathcal{D}$-stable:

1. *if there exists $P \succ O$ such that:*

$$\begin{aligned} &\alpha \otimes P + \beta \otimes PA_i + \beta^{\mathrm{T}} \otimes A_i^{\mathrm{T}} P \\ &\quad = \left[\alpha_{kl} P + \beta_{kl} PA_i + \beta_{lk} A_i^{\mathrm{T}} P\right]_{k,l \in \{1,\ldots,m\}} \prec O \quad \forall i = 1, \ldots, N \end{aligned} \tag{2.107}$$

2. *if there exists $Q \succ O$ such that:*

$$\begin{aligned} &\alpha \otimes Q + \beta \otimes A_i Q + \beta^{\mathrm{T}} \otimes Q A_i^{\mathrm{T}} \\ &\quad = \left[\alpha_{kl} Q + \beta_{kl} A_i Q + \beta_{lk} Q A_i^{\mathrm{T}}\right]_{k,l \in \{1,\ldots,m\}} \prec O \quad \forall i = 1, \ldots, N \end{aligned} \tag{2.108}$$

Proof Similar to that of Corollary 2.2, thus omitted. ∎

Corollary 2.5 (Quadratic $\mathcal{H}_\infty$ performance of CT LPV systems, polytopic version) *The polytopic CT LPV system:*

$$\dot{x}(t) = \sum_{i=1}^{N} \mu_i\,(\theta(t)) \left[A_i x(t) + B_{w,i} w(t)\right] \tag{2.109}$$

$$z_\infty(t) = \sum_{i=1}^{N} \mu_i\left(\theta(t)\right)\left[C_{z_\infty,i}x(t) + D_{z_\infty w,i}w(t)\right] \tag{2.110}$$

with coefficients μ_i such that (2.5) holds, has quadratic $\mathcal{H}_\infty$ performance γ_∞:

1. *if there exists $P \succ O$ such that:*

$$\begin{pmatrix} A_i^{\mathrm{T}}P + PA_i & PB_{w,i} & C_{z_\infty,i}^{\mathrm{T}} \\ B_{w,i}^{\mathrm{T}}P & -I & D_{z_\infty w,i}^{\mathrm{T}} \\ C_{z_\infty,i} & D_{z_\infty w,i} & -\gamma_\infty^2 I \end{pmatrix} \prec O \qquad \forall i = 1,\ldots,N \tag{2.111}$$

2. *if there exists $Q \succ O$ such that:*

$$\begin{pmatrix} A_iQ + QA_i^{\mathrm{T}} & B_{w,i} & QC_{z_\infty,i}^{\mathrm{T}} \\ B_{w,i}^{\mathrm{T}} & -I & D_{z_\infty w,i}^{\mathrm{T}} \\ C_{z_\infty,i}Q & D_{z_\infty w,i} & -\gamma_\infty^2 I \end{pmatrix} \prec O \qquad \forall i = 1,\ldots,N \tag{2.112}$$

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.6 (Quadratic $\mathcal{H}_\infty$ performance of DT LPV systems, polytopic version) *The polytopic DT LPV system:*

$$x(k+1) = \sum_{i=1}^{N} \mu_i\left(\theta(k)\right)\left[A_ix(k) + B_{w,i}w(k)\right] \tag{2.113}$$

$$z_\infty(k) = \sum_{i=1}^{N} \mu_i\left(\theta(k)\right)\left[C_{z_\infty,i}x(k) + D_{z_\infty w,i}w(k)\right] \tag{2.114}$$

with coefficients μ_i such that (2.5) holds, has quadratic $\mathcal{H}_\infty$ performance γ_∞:

1. *if there exists $P \succ O$ such that:*

$$\begin{pmatrix} P & PA_i & PB_{w,i} & O \\ A_i^{\mathrm{T}}P & P & O & C_{z_\infty,i}^{\mathrm{T}} \\ B_{w,i}^{\mathrm{T}}P & O & I & D_{z_\infty w,i}^{\mathrm{T}} \\ O & C_{z_\infty,i} & D_{z_\infty w,i} & \gamma_\infty^2 I \end{pmatrix} \succ O \qquad \forall i = 1,\ldots,N \tag{2.115}$$

2. *if there exists $Q \succ O$ such that:*

$$\begin{pmatrix} Q & A_iQ & B_{w,i} & O \\ QA_i^{\mathrm{T}} & Q & O & QC_{z_\infty,i}^{\mathrm{T}} \\ B_{w,i}^{\mathrm{T}} & O & I & D_{z_\infty w,i}^{\mathrm{T}} \\ O & C_{z_\infty,i}Q & D_{z_\infty w,i} & \gamma_\infty^2 I \end{pmatrix} \succ O \qquad \forall i = 1,\ldots,N \tag{2.116}$$

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.7 (Quadratic $\mathcal{H}_2$ performance of CT LPV systems, polytopic version) *The polytopic CT LPV system (2.109) and:*

$$z_2(t) = \sum_{i=1}^{N} \mu_i\left(\theta(t)\right) C_{z_2,i}\,x(t) \tag{2.117}$$

with coefficients μ_i such that (2.5) holds, has quadratic $\mathcal{H}_2$ performance γ_2:

1. *if there exist $P \succ O$ and N matrices $Y_i \in \mathbb{S}^{n_{z_2}\times n_{z_2}}$ such that $Tr(Y_i) < \gamma_2^2$ $\forall i = 1,\ldots,N$ and:*

$$\begin{pmatrix} A_i^{\mathrm{T}} P + P A_i & B_{w,i} \\ B_{w,i}^{\mathrm{T}} & -I \end{pmatrix} \prec O \qquad \forall i = 1,\ldots,N \tag{2.118}$$

$$\begin{pmatrix} Y_i & C_{z_2,i} \\ C_{z_2,i}^{\mathrm{T}} & P \end{pmatrix} \succ O \qquad \forall i = 1,\ldots,N \tag{2.119}$$

2. *if there exist $Q \succ O$ and N matrices $Y_i \in \mathbb{S}^{n_{z_2}\times n_{z_2}}$ such that $Tr(Y_i) < \gamma_2^2$ $\forall i = 1,\ldots,N$ and:*

$$\begin{pmatrix} A_i Q + Q A_i^{\mathrm{T}} & B_{w,i} \\ B_{w,i}^{\mathrm{T}} & -I \end{pmatrix} \prec O \qquad \forall i = 1,\ldots,N \tag{2.120}$$

$$\begin{pmatrix} Y_i & C_{z_2,i} Q \\ Q C_{z_2,i}^{\mathrm{T}} & Q \end{pmatrix} \succ O \qquad \forall i = 1,\ldots,N \tag{2.121}$$

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.8 (Quadratic $\mathcal{H}_2$ performance of DT LPV systems, polytopic version) *The polytopic DT LPV system (2.113) and:*

$$z_2(k) = \sum_{i=1}^{N} \mu_i\left(\theta(k)\right) C_{z_2,i}\,x(k) \tag{2.122}$$

with coefficients μ_i such that (2.5) holds, has quadratic $\mathcal{H}_2$ performance γ_2:

1. *if there exist $P \succ O$ and N matrices $Y_i \in \mathbb{S}^{n_{z_2}\times n_{z_2}}$ such that $Tr(Y_i) < \gamma_2^2$ $\forall i = 1,\ldots,N$ and:*

$$\begin{pmatrix} P & P A_i & P B_{w,i} \\ A_i^{\mathrm{T}} P & P & O \\ B_{w,i}^{\mathrm{T}} P & O & I \end{pmatrix} \succ O \qquad \forall i = 1,\ldots,N \tag{2.123}$$

$$\begin{pmatrix} Y_i & C_{z_2,i} \\ C_{z_2,i}^{\mathrm{T}} & P \end{pmatrix} \succ O \qquad \forall i = 1, \ldots, N \tag{2.124}$$

2. *if there exist $Q \succ O$ and N matrices $Y_i \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ such that $Tr(Y_i) < \gamma_2^2$ $\forall i = 1, \ldots, N$ and:*

$$\begin{pmatrix} Q & A_i Q & B_{w,i} \\ Q A_i^{\mathrm{T}} & Q & O \\ B_{w,i}^{\mathrm{T}} & O & I \end{pmatrix} \succ O \qquad \forall i = 1, \ldots, N \tag{2.125}$$

$$\begin{pmatrix} Y_i & C_{z_2,i} Q \\ Q C_{z_2,i}^{\mathrm{T}} & Q \end{pmatrix} \succ O \qquad \forall i = 1, \ldots, N \tag{2.126}$$

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.9 (Quadratic FTB of CT LPV systems, polytopic version) *The polytopic CT LPV system (2.109), with coefficients μ_i such that (2.5) holds, is quadratically FTB with respect to (c_1, c_2, T, R, d) if, letting $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$, there exist positive scalars a, λ_1, λ_2, λ_3 and two positive definite matrices $Q_1 \in \mathbb{S}^{n_x \times n_x}$ and $Q_2 \in \mathbb{S}^{n_w \times n_w}$ such that:*

$$\begin{pmatrix} A_i \tilde{Q}_1 + \tilde{Q}_1 A_i^{\mathrm{T}} - a \tilde{Q}_1 & B_{w,i} Q_2 \\ Q_2 B_{w,i}^{\mathrm{T}} & -a Q_2 \end{pmatrix} \prec O \quad \forall i = 1, \ldots, N \tag{2.127}$$

and (2.88)–(2.90) hold.

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.10 (Quadratic FTB of DT LPV systems, polytopic version) *The polytopic DT LPV system (2.113) and:*

$$w(k+1) = \sum_{i=1}^{N} \mu_i\left(\theta(k)\right) W_i w(k) \tag{2.128}$$

with coefficients μ_i such that (2.5) holds, is quadratically FTB with respect to (c_1, c_2, T, R, d) if there exist positive scalars a, λ_1, λ_2, with $a \geq 1$ and two positive definite matrices $Q_1 \in \mathbb{S}^{n_x \times n_x}$ and $Q_2 \in \mathbb{S}^{n_w \times n_w}$ such that:

$$\begin{pmatrix} -a Q_1 & Q_1 A_i^{\mathrm{T}} & O & O \\ A_i Q_1 & -Q_1 & B_{w,i} & O \\ O & B_{w,i}^{\mathrm{T}} & -a Q_2 & W_i^{\mathrm{T}} Q_2 \\ O & O & Q_2 W_i & -Q_2 \end{pmatrix} \prec O \quad \forall i = 1, \ldots, N \tag{2.129}$$

and (2.92)–(2.94) hold.

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.11 (Quadratic FTS of CT LPV systems, polytopic version) *The autonomous polytopic CT LPV system (2.100), with coefficients μ_i such that (2.5) holds, is quadratically FTS with respect to (c_1, c_2, T, R) if, letting $\tilde{Q}_1 = R^{-1/2} Q_1 R^{-1/2}$, there exist positive scalars a, λ_1 and a positive definite matrix $Q_1 \in \mathbb{S}^{n_x \times n_x}$ such that:*

$$A_i \tilde{Q}_1 + \tilde{Q}_1 A_i^{\mathrm{T}} - a\tilde{Q}_1 \prec O \quad \forall i = 1, \ldots, N \tag{2.130}$$

(2.88) and (2.96) hold.

Proof Similar to that of Corollary 2.2, thus omitted. ■

Corollary 2.12 (Quadratic FTS of DT LPV systems, polytopic version) *The autonomous polytopic DT LPV system (2.103), with coefficients μ_i such that (2.5) holds, is quadratically FTS with respect to (c_1, c_2, T, R) if there exist positive scalars a, λ_1 with $a \geq 1$ and a positive definite matrix $Q_1 \in \mathbb{S}^{n_x \times n_x}$ such that:*

$$\begin{pmatrix} -a\mathrm{Q}_1 & \mathrm{Q}_1 A_i^{\mathrm{T}} \\ A_i Q_1 & -Q_1 \end{pmatrix} \prec O \quad \forall \theta \in \Theta \tag{2.131}$$

and (2.98)–(2.99) hold.

Proof Similar to that of Corollary 2.2, thus omitted. ■

2.4.2 Analysis Based on Other Lyapunov Functions

In some situations, using a common quadratic Lyapunov function, as shown in Sect. 2.4.1, could not be enough, due to the introduction of conservativeness of these functions. In these cases, other types of Lyapunov functions could be used, even though at the expense of increasing the complexity of the analysis. This section reviews some of the results in this field. Mathematical details will not be provided, but the interested reader could find easily further informations in the references provided throughout this section.

The main weakness of quadratic stability is that it considers arbitrarily fast parameter variations. As a consequence, the analysis performed using the conditions presented in Sect. 2.4.1 can be very conservative for constant or slowly-varying parameters. In order to reduce the conservatism, [127] proposed extending the class of Lyapunov functions to include parameter-dependent Lyapunov functions:

$$V\left(x(\tau)\right) = x(\tau)^{\mathrm{T}} P\left(\theta(\tau)\right) x(\tau) \tag{2.132}$$

Also, [128] showed that robust stability of a time-varying system is equivalent to the existence of a parameter-dependent Lyapunov function (2.132) for some augmented system. However, the approaches proposed in [127, 128] are non-convex, and thus hardly tractable from a computational point of view. For this reason, [13] proposed a way to convexify the problem by imposing additional constraints on the parameter-dependent Lyapunov functions, obtaining a numerically tractable LMI feasibility problem. Since the bounds on the derivatives of the scheduling parameters are explicitly taken into account, the approach proposed in [13] provides a smooth transition between time invariant parameters and arbitrarily fast parameter variations. Further development of this approach can be found in [15], where $\mathcal{H}_\infty$ control synthesis was considered, in [129], where $\mathcal{H}_2$ control synthesis was considered, and in [130], where an extended characterization of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms was provided, allowing to further decrease the conservatism when using parameter-dependent Lyapunov functions. Homogeneous polynomially parameter-dependent quadratic Lyapunov functions were proposed by [131], demonstrating their effectiveness with respect to linearly parameter-dependent Lyapunov functions. A systematic procedure for constructing a family of LMI conditions of increasing precision is given in [132]. At each step, a set of LMIs provides sufficient conditions for the existence of an affine parameter-dependent Lyapunov function. Necessity is asymptotically attained through a relaxation based on a generalization of Pólya's theorem. A robust stability approach based on a Lyapunov function which depends quadratically both on the system state and the varying parameters (biquadratic stability) has been proposed by [133]. References [134–136] have shown that, by employing Lyapunov functions associated with higher-order time-derivatives of the state, simpler inequalities in a higher-dimensional space can be obtained, leading to not only simple and tractable, but also less conservative LMI conditions.

Notice that the use of parameter-dependent Lyapunov functions in the case of LPV systems is akin to the use of fuzzy Lyapunov functions in TS systems, as proposed in [72, 137].

It is worth recalling an additional line of research, that tries to enhance the concept of LMI region provided in Definition 2.3. For example, [138] have introduced $\mathcal{D}_R$ regions, obtained modifying the characteristic function (2.48), as follows:

$$f_{D_R}(\sigma) = \alpha + \beta\sigma + \beta^{\mathrm{T}}\sigma^* + \chi\sigma\sigma^* = \left[\alpha_{kl} + \beta_{kl}\sigma + \beta_{lk}\sigma^* + \chi_{kl}\sigma\sigma^*\right]_{k,l\in\{1,\ldots,m\}} \tag{2.133}$$

with $\chi = [\chi_{k,l}]_{k,l\in\{1,\ldots,m\}} \in \mathbb{S}^{m\times m}$. Without any assumption on the matrix χ, $\mathcal{D}_R$ regions are not convex, but when χ is positive semidefinite, $\mathcal{D}_R$ are only a slight modification of LMI regions [138], that allow applying parameter-dependent Lyapunov functions for assessing the pole clustering property. On the other hand, [139] have developed an approach that allows specifying not only a simple convex region, but also a non-convex region, defined as a number of convex subregions. The introduction of extra variables and the use of additional LMIs have been considered by [140], requiring greater computational effort, but providing sufficient conditions that are much more close to necessity. A Kalman-Yakubovich-Popov (KYP) lemma for LMI regions, to guarantee the satisfaction of a frequency domain inequality, has been discussed in [141].

2.5 Control of LPV and TS Systems

Taking into account the analysis conditions presented in Sect. 2.4, the problem of designing a control law such that the resulting closed-loop system has some desired properties will be analyzed hereafter.

For the sake of simplicity, only the case of a state-feedback control law of the form:

$$u(\tau) = K(\theta(\tau))\, x(\tau) \tag{2.134}$$

will be considered. Even though in many situations the state is not available, in most of them the system is observable, thus it is possible to add a state observer to the control loop. Then, the state observer would provide an estimation of the state to be fed back to the controller [142]. In cases where this would not be possible, other approaches may be viable, e.g. output-feedback controller synthesis [143–145] or IO controller synthesis [85, 146, 147].

The following theorems can be easily obtained taking into account the results presented in the previous section.

Theorem 2.12 (Quadratic stabilization of CT LPV systems) *The LPV system (2.1) with control law (2.134) and $\tau = t$ is quadratically stabilizable if there exist $Q \succ O$ and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:*

$$He\{A(\theta)Q + B(\theta)K(\theta)Q\} \prec O \qquad \forall\theta \in \Theta \tag{2.135}$$

Proof It is obtained straightforwardly from Theorem 2.1, by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the autonomous state matrix $A(\theta)$. ■

Theorem 2.13 (Quadratic stabilization of DT LPV systems) *The LPV system (2.1) with control law (2.134) and $\tau = k$ is quadratically stabilizable if there exist $Q \succ O$ and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:*

$$\begin{pmatrix} -Q & A(\theta)Q + B(\theta)K(\theta)Q \\ * & -Q \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{2.136}$$

Proof It is obtained straightforwardly from Theorem 2.2, by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the autonomous state matrix $A(\theta)$. ■

Theorem 2.14 (Quadratic $\mathcal{D}$-stabilizability of LPV systems) *Given an LMI region $\mathcal{D}$ defined as in (2.47), the LPV system (2.1) with control law (2.134) is quadratically $\mathcal{D}$-stabilizable if there exist $Q \succ O$ and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:*

$$\alpha \otimes Q + He\{\beta \otimes [A(\theta)Q + B(\theta)K(\theta)Q]\} \prec O \qquad \forall\theta \in \Theta \tag{2.137}$$

Proof It is obtained straightforwardly from Theorem 2.3, by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the autonomous state matrix $A(\theta)$. ■

Theorem 2.15 (Quadratic $\mathcal{H}_\infty$ state-feedback for CT LPV systems) *The CT LPV system:*

$$\dot{x}(t) = A\left(\theta(t)\right)x(t) + B\left(\theta(t)\right)u(t) + B_w\left(\theta(t)\right)w(t) \tag{2.138}$$

$$z_\infty(t) = C_{z_\infty}\left(\theta(t)\right)x(t) + D_{z_\infty u}\left(\theta(t)\right)u(t) + D_{z_\infty w}\left(\theta(t)\right)w(t) \tag{2.139}$$

with control law (2.134) has quadratic $\mathcal{H}_\infty$ performance γ_∞ if there exist $Q \succ O$ and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:

$$\begin{pmatrix} He\left\{A(\theta)Q + B(\theta)K(\theta)Q\right\} & * & * \\ B_w(\theta)^{\mathrm{T}} & -I & * \\ C_{z_\infty}(\theta)Q + D_{z_\infty u}(\theta)K(\theta)Q & D_{z_\infty w}(\theta) & -\gamma_\infty^2 I \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{2.140}$$

Proof It is obtained straightforwardly from Theorem 2.4 by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the state matrix $A(\theta)$, and the closed-loop z_∞ output matrix $C_{z_\infty}(\theta) + D_{z_\infty u}(\theta)K(\theta)$ instead of the z_∞ output matrix $C_{z_\infty}(\theta)$. ■

Theorem 2.16 (Quadratic $\mathcal{H}_\infty$ state-feedback for DT LPV systems) *The DT LPV system:*

$$x(k+1) = A\left(\theta(k)\right)x(k) + B\left(\theta(k)\right)u(k) + B_w\left(\theta(k)\right)w(k) \tag{2.141}$$
$$z_\infty(k) = C_{z_\infty}\left(\theta(k)\right)x(k) + D_{z_\infty u}\left(\theta(k)\right)u(k) + D_{z_\infty w}\left(\theta(k)\right)w(k) \tag{2.142}$$

with control law (2.134) has quadratic $\mathcal{H}_\infty$ performance γ_∞ if there exist $Q \succ O$ and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:

$$\begin{pmatrix} Q & A(\theta)Q + B(\theta)K(\theta)Q & B_w(\theta) & O \\ * & Q & O & QC_{z_\infty}(\theta)^{\mathrm{T}} + QK(\theta)^{\mathrm{T}}D_{z_\infty u}(\theta)^{\mathrm{T}} \\ * & * & I & D_{z_\infty w}(\theta)^{\mathrm{T}} \\ * & * & * & \gamma_\infty^2 I \end{pmatrix} \succ O \quad \forall\theta \in \Theta \tag{2.143}$$

Proof It is obtained straightforwardly from Theorem 2.5 by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the state matrix $A(\theta)$, and the closed-loop z_∞ output matrix $C_{z_\infty}(\theta) + D_{z_\infty u}(\theta)K(\theta)$ instead of the z_∞ output matrix $C_{z_\infty}(\theta)$. ■

Theorem 2.17 (Quadratic $\mathcal{H}_2$ state-feedback for CT LPV systems) *The CT LPV system (2.138) and:*

$$z_2(t) = C_{z_2}\left(\theta(t)\right)x(t) + D_{z_2 u}\left(\theta(t)\right)u(t) \tag{2.144}$$

with control law (2.134) has quadratic $\mathcal{H}_2$ performance γ_2 if there exist $Q \succ O$ and matrix functions $K(\theta) \in \mathbb{R}^{n_u \times n_x}$, $Y(\theta) \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ such that $Tr\,(Y(\theta)) < \gamma_2^2\ \forall\theta \in \Theta$ and:

$$\begin{pmatrix} He\,\{A(\theta)Q + B(\theta)K(\theta)Q\} & B_w(\theta) \\ * & -I \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{2.145}$$

$$\begin{pmatrix} Y(\theta) & C_{z_2}(\theta)Q + D_{z_2u}(\theta)K(\theta)Q \\ * & Q \end{pmatrix} \succ O \qquad \forall\theta \in \Theta \tag{2.146}$$

Proof It is obtained straightforwardly from Theorem 2.6 by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the state matrix $A(\theta)$, and the closed-loop z_2 output matrix $C_{z_2}(\theta) + D_{z_2u}(\theta)K(\theta)$ instead of the z_2 output matrix $C_{z_2}(\theta)$. ■

Theorem 2.18 (Quadratic $\mathcal{H}_2$ state-feedback for DT LPV systems) *The DT LPV system (2.141) and:*

$$z_2(k+1) = C_{z_2}\,(\theta(k))\,x(k) + D_{z_2u}\,(\theta(k))\,u(k) \tag{2.147}$$

with control law (2.134) has quadratic $\mathcal{H}_2$ performance γ_2 if there exist $Q \succ O$ and matrix functions $K(\theta) \in \mathbb{R}^{n_u \times n_x}$, $Y(\theta) \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$ such that $Tr\,(Y(\theta)) < \gamma_2^2\ \forall\theta \in \Theta$ and:

$$\begin{pmatrix} Q & A(\theta)Q + B(\theta)K(\theta)Q & B_w(\theta) \\ * & Q & O \\ * & * & I \end{pmatrix} \succ O \qquad \forall\theta \in \Theta \tag{2.148}$$

$$\begin{pmatrix} Y(\theta) & C_{z_2}(\theta)Q + D_{z_2u}(\theta)K(\theta)Q \\ * & Q \end{pmatrix} \succ O \qquad \forall\theta \in \Theta \tag{2.149}$$

Proof It is obtained straightforwardly from Theorem 2.7 by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the state matrix $A(\theta)$, and the closed-loop z_2 output matrix $C_{z_2}(\theta) + D_{z_2u}(\theta)K(\theta)$ instead of the z_2 output matrix $C_{z_2}(\theta)$. ■

Theorem 2.19 (Quadratic FTB state-feedback for CT LPV systems) *The CT LPV system (2.138) with control law (2.134) is quadratically FTB with respect to (c_1, c_2, T, R, d) if, letting $\tilde{Q}_1 = R^{-1/2}Q_1R^{-1/2}$, there exist positive scalars a, λ_1, λ_2, λ_3, two positive definite matrices $Q_1 \in \mathbb{S}^{n_x \times n_x}$ and $Q_2 \in \mathbb{S}^{n_w \times n_w}$, and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:*

$$\begin{pmatrix} He\left\{A(\theta)\tilde{Q}_1 + B(\theta)K(\theta)\tilde{Q}_1\right\} - a\tilde{Q}_1 & B_w(\theta)Q_2 \\ * & -aQ_2 \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{2.150}$$

and (2.88)–(2.90) hold.

Proof It is obtained straightforwardly from Theorem 2.8 by considering the closed-loop state matrix $A(\theta) + B(\theta)K(\theta)$ instead of the state matrix $A(\theta)$. ■

Theorem 2.20 (Quadratic FTB state-feedback for DT LPV systems) *The DT LPV system (2.141) and:*

$$w(k+1) = W(\theta(k))\, w(k) \tag{2.151}$$

with control law (2.134) is quadratically FTB with respect to (c_1, c_2, T, R, d) if there exist positive scalars a, λ_1, λ_2 with $a \geq 1$, two positive definite matrices $Q_1 \in \mathbb{S}^{n_x \times n_x}$ and $Q_2 \in \mathbb{S}^{n_w \times n_w}$ and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:

$$\begin{pmatrix} -aQ_1 & * & * & * \\ A(\theta)Q_1 + B(\theta)K(\theta)Q_1 & -Q_1 & * & * \\ O & B_w(\theta)^{\mathrm{T}} & -aQ_2 & * \\ O & O & Q_2W(\theta) & -Q_2 \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{2.152}$$

and (2.92)–(2.94) hold.

Proof It is obtained straightforwardly from Theorem 2.9 by considering $A(\theta) + B(\theta)K(\theta)$ instead of $A(\theta)$. ■

Theorem 2.21 (Quadratic finite time stabilization of CT LPV systems) *The LPV system (2.1) with control law (2.134) and $\tau = t$ is quadratically finite time stabilizable with respect to (c_1, c_2, T, R) if, letting $\tilde{Q}_1 = R^{-1/2}Q_1R^{-1/2}$, there exist positive scalars a, λ_1, a positive definite matrix $Q_1 \in \mathbb{S}^{n_x \times n_x}$ and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:*

$$He\left\{A(\theta)\tilde{Q}_1 + B(\theta)K(\theta)\tilde{Q}_1\right\} - a\tilde{Q}_1 \prec O \qquad \forall\theta \in \Theta \tag{2.153}$$

(2.88) and (2.96) hold.

Proof It is obtained straightforwardly from Theorem 2.10 by considering $A(\theta) + B(\theta)K(\theta)$ instead of $A(\theta)$. ■

Theorem 2.22 (Quadratic finite time stabilization of DT LPV systems) *The LPV system (2.1) with control law (2.134) and $\tau = k$ is quadratically finite time stabilizable with respect to (c_1, c_2, T, R) if there exist positive scalars a, λ_1 with $a \geq 1$, a positive definite matrix $Q_1 \in \mathbb{S}^{n_x \times n_x}$ and a matrix function $K(\theta) \in \mathbb{R}^{n_u \times n_x}$ such that:*

$$\begin{pmatrix} -\alpha Q_1 & * \\ A(\theta)Q_1 + B(\theta)K(\theta)Q_1 & -Q_1 \end{pmatrix} \prec O \qquad \forall\theta \in \Theta \tag{2.154}$$

and (2.98)–(2.99) hold.

Proof It is obtained straightforwardly from Theorem 2.11 by considering $A(\theta) + B(\theta)K(\theta)$ instead of $A(\theta)$. ■

However, similarly to the previous section, Theorems 2.12–2.22 imply infinite constraints to be checked, that can be reduced to a finite number using the polytopic approach described in Sect. 2.2. In this case, the matrices $A(\theta(\tau))$, $B_w(\theta(\tau))$, $C_{z_\infty}(\theta(\tau))$, $D_{z_\infty w}(\theta(\tau))$, $C_{z_2}(\theta(\tau))$, $W(\theta(k))$ are assumed to be polytopic, as follows:

$$\begin{pmatrix} A(\theta(\tau)) \\ B_w(\theta(\tau)) \\ C_{z_\infty}(\theta(\tau)) \\ D_{z_\infty w}(\theta(\tau)) \\ C_{z_2}(\theta(\tau)) \\ W(\theta(k)) \end{pmatrix} = \sum_{i=1}^{N} \mu_i(\theta(\tau)) \begin{pmatrix} A_i \\ B_{w,i} \\ C_{z_\infty,i} \\ D_{z_\infty w,i} \\ C_{z_2,i} \\ W_i \end{pmatrix} \tag{2.155}$$

where the coefficients μ_i satisfy the property (2.5). On the other hand, the matrices B, $D_{z_\infty u}$ and $D_{z_2 u}$ are assumed to be constant. This assumption is not restrictive, since in the case of varying matrices $B(\theta(\tau))$, $D_{z_\infty u}(\theta(\tau))$ and $D_{z_2 u}(\theta(\tau))$, a prefiltering of the input $u(\tau)$ would lead to obtain a new system with constant matrices $\tilde{B}$, $\tilde{D}_{z_\infty u}$ and $\tilde{D}_{z_2 u}$ [12]. More specifically, for the system:

$$\sigma.x(\tau) = A(\theta(\tau))\,x(\tau) + B(\theta(\tau))\,u(\tau) + B_w(\theta(\tau))\,w(\tau) \tag{2.156}$$

$$z_\infty(\tau) = C_{z_\infty}(\theta(\tau))\,x(\tau) + D_{z_\infty u}(\theta(\tau))\,u(\tau) + D_{z_\infty w}(\theta(\tau))\,w(\tau) \tag{2.157}$$

$$z_2(\tau) = C_{z_2}(\theta(\tau))\,x(\tau) + D_{z_2 u}(\theta(\tau))\,u(\tau) \tag{2.158}$$

let us define a new control input $\tilde{u}(\tau)$ such that:

$$\sigma.x_u(\tau) = A_u(\theta(\tau))\,x_u(\tau) + B_u\tilde{u}(\tau) \tag{2.159}$$

$$u(\tau) = C_u x_u(\tau) \tag{2.160}$$

with $A_u(\theta(\tau))$ stable. Then, the resulting LPV system would be:

$$\begin{pmatrix} \sigma.x(\tau) \\ \sigma.x_u(\tau) \end{pmatrix} = \begin{pmatrix} A(\theta(\tau)) & B(\theta(\tau))\,C_u \\ O & A_u(\theta(\tau)) \end{pmatrix} \begin{pmatrix} x(\tau) \\ x_u(\tau) \end{pmatrix} + \begin{pmatrix} O \\ B_u \end{pmatrix} \tilde{u}(\tau) + \begin{pmatrix} B_w(\theta(\tau)) \\ O \end{pmatrix} w(\tau) \tag{2.161}$$

$$z_\infty(\tau) = \begin{pmatrix} C_{z_\infty}(\theta(\tau)) & D_{z_\infty u}(\theta(\tau))\,C_u \end{pmatrix} \begin{pmatrix} x(\tau) \\ x_u(\tau) \end{pmatrix} + D_{z_\infty w}(\theta(\tau))\,w(\tau) \tag{2.162}$$

$$z_2(\tau) = \begin{pmatrix} C_{z_2}(\theta(\tau)) & D_{z_2 u}(\theta(\tau))\,C_u \end{pmatrix} \begin{pmatrix} x(\tau) \\ x_u(\tau) \end{pmatrix} \tag{2.163}$$

that are in the desired form.

It is worth recalling that some recent research has developed design conditions that would work in the case where the matrices $B(\theta(\tau))$, $D_{z_\infty u}(\theta(\tau))$ and $D_{z_2 u}(\theta(\tau))$

are varying, without the need of resorting to the input prefiltering [148]. Since these conditions are in some way conservative, many works try to reduce their pessimism. Among these works, [149] is recognized to lead to a good compromise between complexity and conservatism.

The following corollaries are obtained from Theorems 2.12–2.22, and consider a polytopic state-feedback control law (2.134), as follows:

$$u(\tau) = \sum_{i=1}^{N} \mu_i \left(\theta(\tau)\right) K_i x(\tau) \tag{2.164}$$

The mathematical proof is provided only for Corollary 2.13, since the proofs of the remaining ones can be presented by a similar reasoning.

Corollary 2.13 (Design of a quadratically stabilizing polytopic state-feedback controller for CT LPV systems) *Let $Q \succ O$ and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$ be such that:*

$$He\left\{A_i Q + B\Gamma_i\right\} \prec O \qquad \forall i = 1, \ldots, N \tag{2.165}$$

Then, the closed-loop system made up by the LPV system (2.1), with $\tau = t$, $B\left(\theta(t)\right) = B$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q^{-1}$, $i = 1, \ldots, N$, is quadratically stable.

Proof By considering that $K_i = \Gamma_i Q^{-1}$ is equivalent to $\Gamma_i = K_i Q$, (2.165) can be rewritten as:

$$He\left\{A_i Q + BK_i Q\right\} \prec O \qquad \forall i = 1, \ldots, N \tag{2.166}$$

Then, taking into account the basic property of matrices [126] that any linear combination of (2.166) with non-negative coefficients, of which at least one different from zero, is negative definite, using the coefficients $\mu_i\left(\theta(\tau)\right)$, and taking into account (2.155) and (2.164), (2.135) is obtained. ■

Corollary 2.14 (Design of a quadratically stabilizing polytopic state-feedback controller for DT LPV systems) *Let $Q \succ O$ and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, be such that:*

$$\begin{pmatrix} -Q & A_i Q + B\Gamma_i \\ * & -Q \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.167}$$

Then, the closed-loop system made up by the LPV system (2.1), with $\tau = k$, $B\left(\theta(k)\right) = B$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q^{-1}$, $i = 1, \ldots, N$, is quadratically stable.

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.15 (Design of a quadratically $\mathcal{D}$-stabilizing polytopic state-feedback controller for LPV systems) *Given an LMI region $\mathcal{D}$ defined as in (2.47), let $Q \succ O$ and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, be such that:*

$$\alpha \otimes Q + He\{\beta \otimes [A_i Q + B\Gamma_i]\} \prec O \qquad \forall i = 1, \ldots, N \tag{2.168}$$

Then, the closed-loop system made up by the LPV system (2.1), with $B(\theta(\tau)) = B$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q^{-1}$, $i = 1, \ldots, N$, is quadratically $\mathcal{D}$-stable.

Proof Similar to that of Corollary 2.13, thus omitted. ∎

Corollary 2.16 (Design of a quadratic $\mathcal{H}_\infty$ polytopic state-feedback controller for CT LPV systems) *Let $Q \succ O$ and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, be such that:*

$$\begin{pmatrix} He\{A_i Q + B\Gamma_i\} & * & * \\ B_{w,i}^{\mathrm{T}} & -I & * \\ C_{z_\infty,i} Q + D_{z_\infty u}\Gamma_i & D_{z_\infty w,i} & -\gamma_\infty^2 I \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.169}$$

Then, the closed-loop system made up by the LPV system (2.138)–(2.139), with $B(\theta(t)) = B$, $D_{z_\infty u}(\theta(t)) = D_{z_\infty u}$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q^{-1}$, $i = 1, \ldots, N$, has quadratic $\mathcal{H}_\infty$ performance γ_∞.

Proof Similar to that of Corollary 2.13, thus omitted. ∎

Corollary 2.17 (Design of a quadratic $\mathcal{H}_\infty$ polytopic state-feedback controller for DT LPV systems) *Let $Q \succ O$ and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, be such that:*

$$\begin{pmatrix} Q & A_i Q + B\Gamma_i & B_{w,i} & O \\ * & Q & O & QC_{z_\infty,i}^{\mathrm{T}} \\ * & * & I & D_{z_\infty w,i}^{\mathrm{T}} \\ * & * & * & \gamma_\infty^2 \end{pmatrix} \succ O \qquad \forall i = 1, \ldots, N \tag{2.170}$$

Then, the closed-loop system made up by the LPV system (2.141)–(2.142), with $B(\theta(k)) = B$, $D_{z_\infty u}(\theta(k)) = D_{z_\infty u}$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q^{-1}$, $i = 1, \ldots, N$, has quadratic $\mathcal{H}_\infty$ performance γ_∞.

Proof Similar to that of Corollary 2.13, thus omitted. ∎

Corollary 2.18 (Design of a quadratic $\mathcal{H}_2$ polytopic state-feedback controller for CT LPV systems) *Let $Q \succ O$, $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$ and $Y_i \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$, $i = 1, \ldots, N$, be such that:*

$$Tr(Y_i) < \gamma_2^2 \qquad \forall i = 1, \ldots, N \tag{2.171}$$

$$\begin{pmatrix} He\{A_i Q + B\Gamma_i\} & B_{w,i} \\ * & -I \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.172}$$

$$\begin{pmatrix} Y_i & C_{z_2,i}Q + D_{z_2u}\Gamma_i \\ * & Q \end{pmatrix} \succ O \qquad \forall i = 1, \ldots, N \tag{2.173}$$

Then, the closed-loop system made up by the CT LPV system (2.138) and (2.144), with $B(\theta(t)) = B$, $D_{z_2u}(\theta(t)) = D_{z_2u}$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q^{-1}$, $i = 1, \ldots, N$, has quadratic $\mathcal{H}_2$ performance γ_2.

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.19 (Design of a quadratic $\mathcal{H}_2$ polytopic state-feedback controller for DT LPV systems) *Let $Q \succ O$, $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$ and $Y_i \in \mathbb{S}^{n_{z_2} \times n_{z_2}}$, $i = 1, \ldots, N$, be such that:*

$$Tr(Y_i) < \gamma_2^2 \qquad \forall i = 1, \ldots, N \tag{2.174}$$

$$\begin{pmatrix} Q & A_iQ + B\Gamma_i & B_{w,i} \\ * & Q & O \\ * & * & I \end{pmatrix} \succ O \qquad \forall i = 1, \ldots, N \tag{2.175}$$

$$\begin{pmatrix} Y_i & C_{z_2,i}Q + D_{z_2u}\Gamma_i \\ * & Q \end{pmatrix} \succ O \qquad \forall i = 1, \ldots, N \tag{2.176}$$

Then, the closed-loop system made up by the DT LPV system (2.141) and (2.147), with $B(\theta(k)) = B$, $D_{z_2u}(\theta(k)) = D_{z_2u}$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i Q^{-1}$, $i = 1, \ldots, N$, has quadratic $\mathcal{H}_2$ performance γ_2.

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.20 (Design of a quadratic FTB polytopic state-feedback controller for CT LPV systems) *Fix $a > O$, and let $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $Q_1 \succ O$, $Q_2 \succ O$, and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, be such that:*

$$\begin{pmatrix} He\left\{A_i\tilde{Q}_1 + B\Gamma_i\right\} - a\tilde{Q}_1 & B_{w,i}Q_2 \\ * & -aQ_2 \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.177}$$

and (2.88)–(2.90) hold, where $\tilde{Q}_1 = R^{-1/2}Q_1R^{-1/2}$. Then, the closed-loop system made up by the CT LPV system (2.138), with $B(\theta(t)) = B$, and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as $K_i = \Gamma_i\tilde{Q}_1^{-1}$, $i = 1, \ldots, N$, is quadratically FTB with respect to (c_1, c_2, T, R, d).

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.21 (Design of a quadratic FTB polytopic state-feedback controller for DT LPV systems) *Fix $a \geq 1$, and let $\lambda_1 > 0$, $\lambda_2 > 0$, $Q_1 \succ O$, $Q_2 \succ O$ and $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, be such that:*

$$\begin{pmatrix} -aQ_1 & * & * & * \\ A_iQ_1 + B\Gamma_i & -Q_1 & * & * \\ O & B_{w,i}^{\mathrm{T}} & -aQ_2 & * \\ O & O & Q_2W_i & -Q_2 \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.178}$$

and (2.92)–(2.94) hold. Then, the closed-loop system made up by the DT LPV system (2.141) and (2.151), with $B(\theta(k)) = B$, *and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as* $K_i = \Gamma_i Q_1^{-1}$, $i = 1, \ldots, N$, *is quadratically FTB with respect to* (c_1, c_2, T, R, d).

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.22 (Design of a quadratically finite time stabilizing polytopic state-feedback controller for CT LPV systems) *Fix* $a > 0$, *and let* $\lambda_1 > 0$, $Q_1 \succ O$ *and* $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, *be such that:*

$$He\left\{A_i\tilde{Q}_1 + B\Gamma_i\right\} - a\tilde{Q}_1 \prec O \qquad \forall i = 1, \ldots, N \tag{2.179}$$

Equation (2.88) and (2.96) hold, where $\tilde{Q}_1 = R^{-1/2}Q_1R^{-1/2}$. *Then, the closed-loop system made up by the CT LPV system (2.1), with* $\tau = t$, $B(\theta(t)) = B$, *and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as* $K_i = \Gamma_i\tilde{Q}_1^{-1}$, $i = 1, \ldots, N$, *is quadratically FTS with respect to* (c_1, c_2, T, R).

Proof Similar to that of Corollary 2.13, thus omitted. ■

Corollary 2.23 (Design of a quadratically finite time stabilizing polytopic state-feedback controller for DT LPV systems) *Fix* $a \geq 1$, *and let* $Q_1 \succ O$ *and* $\Gamma_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \ldots, N$, *be such that:*

$$\begin{pmatrix} -aQ_1 & * \\ A_iQ_1 + B\Gamma_i & -Q_1 \end{pmatrix} \prec O \qquad \forall i = 1, \ldots, N \tag{2.180}$$

and (2.98)–(2.99) hold. Then, the closed-loop system made up by the DT LPV system (2.1), with $\tau = k$, $B(\theta(k)) = B$, *and polytopic matrices as in (2.155), and the polytopic state-feedback control law (2.164) with gains calculated as* $K_i = \Gamma_i Q_1^{-1}$, $i = 1, \ldots, N$, *is quadratically FTS with respect to* (c_1, c_2, T, R).

Proof Similar to that of Corollary 2.13, thus omitted. ■

2.6 Conclusions

This chapter has presented some background on gain-scheduling. Some basic concepts about modeling of LPV and TS systems have been recalled, and different methods for obtaining such models starting from an available nonlinear state-space model have been illustrated using some examples. For LPV systems, the following methods have been recalled: (a) the Jacobian linearization approach, based on the interpolation of LTI models obtained as first-order Taylor-series approximations of the nonlinear systems around some equilibrium points of interest; (b) the state transformation approach, where a coordinate change is performed with the aim of removing any nonlinear term not dependent on the scheduling parameters; and (c) the function substitution approach, that replaces a *decomposition function* with functions that are linear with respect to the scheduling parameters. For TS systems, the following methods have been recalled: (d) the sector nonlinearity approach, that aims at finding global sectors through which an exact model representation is guaranteed; and (e) the local approximation in fuzzy partition spaces, where nonlinear terms are approximated by judiciously choosing linear terms, with the effect of reducing the number of fuzzy rules.

Afterwards, the problem of analyzing whether or not some properties hold for a given LPV system has been considered. The definitions in the case of LPV systems of poles, LMI regions, $\mathcal{H}_\infty$ norm, $\mathcal{H}_\infty$ performance, $\mathcal{H}_2$ norm, $\mathcal{H}_2$ performance, finite time stability and finite time boundedness have been provided. Detailed conditions to perform the analysis based on a common quadratic Lyapunov function have been listed, and it has been shown that a finite number of LMIs can be obtained by considering the polytopic approach.

Finally, it has been shown how the analysis conditions can be taken into account for designing a state-feedback control law such that the resulting closed-loop system has some desired properties.

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http://www.springer.com/978-3-319-62901-8

Advances in Gain-Scheduling and Fault Tolerant Control Techniques
Rotondo, D.
2018, XXIII, 255 p. 63 illus., 34 illus. in color., Hardcover
ISBN: 978-3-319-62901-8