Chapter 2
Convex Hull Representations

2.1 Introduction

The problem of estimating the domain of attraction for control systems with actuator saturation, both in the continuous-time and discrete-time settings, has been extensively studied during the past decades. A large number of results have emerged in the literature. In particular, the contractively invariant sets, from which all trajectories remain inside them and converge to the origin, have been widely used as estimates of the domain of attraction of a saturated system. The determination of the contractive invariance of a set involves the treatment of the saturation function. Researchers in the control community have paid considerable attention to handling saturation functions and have developed different treatments of the saturation function. Based on these treatments, various conditions have been established under which an ellipsoid is contractively invariant.

Consider the following linear system under a saturated linear feedback,

$$\dot{x} = Ax + B\text{sat}(Fx), \quad x \in \mathbf{R}^n, \ F \in \mathbf{R}^{m\times n}, \tag{2.1}$$

where x is the state, F is the state feedback gain, and $\text{sat} : \mathbf{R}^m \to \mathbf{R}^m$ is the vector-valued standard saturation function, defined as,

$$\begin{aligned} \text{sat}(u) &= \left[\text{sat}(u_1) \ \ \text{sat}(u_2) \ \ \cdots \ \ \text{sat}(u_m)\right], \\ \text{sat}(u_i) &= \text{sgn}(u_i)\min\{1, |u_i|\}, \end{aligned}$$

with $u = [u_1 \ u_2 \ \cdots \ u_m]^{\mathrm{T}}$. Here we have again slightly abused the notation by using sat to denote both the vector valued and scalar saturation functions. Also, recall that non-unity saturation levels can be accommodated by scaling the matrices B and F.

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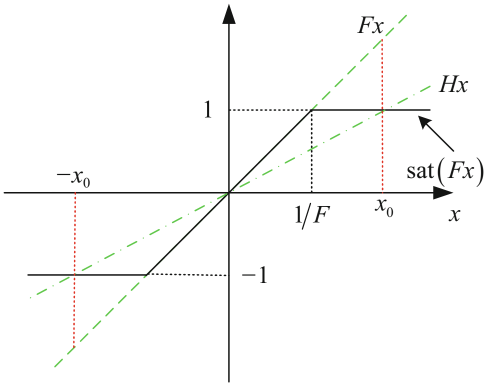

Fig. 2.1 An illustration of the regional sector condition.

One of the popular treatments of the saturation function is the regional sector condition. In this treatment, the saturated linear feedback $\mathrm{sat}(Fx)$ can be placed into a regional linear sector. Given a matrix $H \in \mathbf{R}^{m\times n}$, let

$$\mathcal{L}(H) = \left\{x \in \mathbf{R}^n : \ |Hx|_\infty \le 1\right\}.$$

Clearly, $\mathcal{L}(H)$ represents the region where Hx does not saturate. The regional sector condition is described in the following lemma.

Lemma 2.1.1 ([30]) *Let $F, H \in \mathbf{R}^{n\times m}$. For any $x \in \mathcal{L}(H)$ and any diagonal matrix $S \in \mathbf{R}^{n\times n}$ satisfying $S > 0$, the following inequality holds,*

$$(Fx - \mathrm{sat}(Fx))^{\mathrm{T}} S(\mathrm{sat}(Fx) - Hx) \ge 0. \tag{2.2}$$

Lemma 2.1.1 is illustrated in Figure 2.1 for the case of $m = 1$ and $n = 1$. Without loss of generality, we assume that $H > 0$. Note that

$$\begin{aligned}\mathcal{L}(H) &= [-x_0,\ x_0]\\ &= \left[-\frac{1}{H},\ \frac{1}{H}\right].\end{aligned}$$

It is clear that, for any $x \in \mathcal{L}(H)$, the saturated linear feedback $\mathrm{sat}(Fx)$ resides in the sector formed by Fx and Hx. Note that inequality (2.2) is satisfied only for $x \in \mathcal{L}(H)$. If we set $H = 0$, the regional sector condition (2.2) will become the global sector condition, that is,

$$(Fx - \mathrm{sat}(Fx))^{\mathrm{T}} S\mathrm{sat}(Fx) \ge 0, \quad \forall\ x \in \mathbf{R}^{n\times n}.$$

The regional sector condition (2.2) is usually applied to determine a contractively invariant set in such a way that the term $(Fx - \mathrm{sat}(Fx))^{\mathrm{T}} S(\mathrm{sat}(Fx) - Hx)$ is added to the time derivative of a quadratic Lyapunov function such that a quadratic function of an augmented state vector containing the system state and the saturation or deadzone function can be formed. The negative definiteness of the associated matrix of the resulting quadratic function then ensures the contractive invariance of a level set of the Lyapunov function.

Consider a quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$ and its level set $\mathcal{E}(P) := \{x \in \mathbf{R}^n : x^{\mathrm{T}}Px \leq 1\}$. The ellipsoid $\mathcal{E}(P)$ is contractively invariant if $\mathcal{E}(P) \subseteq \mathcal{L}(H)$ and

$$\begin{aligned}\dot{V}(x) &= 2x^{\mathrm{T}}P\dot{x} \\ &\leq 2x^{\mathrm{T}}P(Ax + B\mathrm{sat}(Fx)) + 2(Fx - \mathrm{sat}(Fx))^{\mathrm{T}}S(\mathrm{sat}(Fx) - Hx) \\ &\leq \begin{bmatrix} x \\ \mathrm{sat}(Fx) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathrm{He}(PA - F^{\mathrm{T}}SH) & PB + F^{\mathrm{T}}S + H^{\mathrm{T}}S \\ \star & -2S \end{bmatrix} \begin{bmatrix} x \\ \mathrm{sat}(Fx) \end{bmatrix} \\ &< 0, \quad \forall x \in \mathcal{E}(P) \setminus \{0\}.\end{aligned}$$

Based on the analysis above, we establish in the following conditions for the contractive invariance of $\mathcal{E}(P)$.

Theorem 2.1.1 *Let $P \in \mathbf{R}^{n\times n}$ be a positive definite matrix and $Q = P^{-1}$. If there exist a positive definite diagonal matrix $W \in \mathbf{R}^{m\times n}$ and matrix $Z \in \mathbf{R}^{m\times n}$ such that*

$$\begin{bmatrix} Q(A+BF)^{\mathrm{T}} + (A+BF)Q & -BW + QF^{\mathrm{T}} - Z^{\mathrm{T}} \\ \star & -2W \end{bmatrix} < 0, \tag{2.3}$$

and

$$\begin{bmatrix} 1 & z_j \\ z_j^{\mathrm{T}} & Q \end{bmatrix} \geq 0, \quad j \in I[1, m], \tag{2.4}$$

where z_j is the jth row of matrix $Z = HQ$, then the ellipsoid $\mathcal{E}(P)$ is a contractively invariant set of system (2.1).

Proof Matrix inequalities (2.4) are equivalent to

$$\begin{bmatrix} 1 & h_j \\ h_j^{\mathrm{T}} & P \end{bmatrix} \geq 0, \quad j \in I[1, m],$$

or,

$$P \geq h_j^{\mathrm{T}}h_j, \quad j \in I[1, m],$$

where h_j is the jth row of matrix H. For any $x \in \mathcal{E}(P)$, $x^{\mathrm{T}}Px \leq 1$, and $x^{\mathrm{T}}Px \geq x^{\mathrm{T}}h_j^{\mathrm{T}}h_jx$. Thus, $|h_jx| \leq 1$ for all $x \in \mathcal{E}(P)$. This implies that $\mathcal{E}(P) \subseteq \mathcal{L}(H)$.

Multiplying

$$\begin{bmatrix} P & 0 \\ 0 & S \end{bmatrix},$$

where $S = W^{-1}$, to the left and the right of both sides of Inequality (2.3), we have

$$\begin{bmatrix} (A+BF)^{\mathrm{T}}P + P(A+BF) & -PB + (F-H)^{\mathrm{T}}S \\ \star & -2S \end{bmatrix} < 0.$$

Noting the non-singularity of matrix

$$\begin{bmatrix} I_n & 0 \\ F & -I_m \end{bmatrix},$$

we have

$$\begin{aligned} & \begin{bmatrix} I_n & 0 \\ F & -I_m \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} (A+BF)^{\mathrm{T}}P + P(A+BF) & -PB + (F-H)^{\mathrm{T}}S \\ \star & -2S \end{bmatrix} \begin{bmatrix} I_n & 0 \\ F & -I_m \end{bmatrix} \\ = & \begin{bmatrix} \mathrm{He}(PA - F^{\mathrm{T}}SH) & PB + F^{\mathrm{T}}S + H^{\mathrm{T}}S \\ \star & -2S \end{bmatrix} \\ < & \ 0. \end{aligned}$$

Since $\mathcal{E}(P) \subseteq \mathcal{L}(H)$, for any $x \in \mathcal{E}(P)$,

$$(Fx - \mathrm{sat}(Fx))^{\mathrm{T}}S(\mathrm{sat}(Fx) - Hx) \geq 0.$$

Then, the time derivative of $V(x) = x^{\mathrm{T}}Px$ along the trajectory of system (2.1) can be evaluated as,

$$\begin{aligned} \dot{V}(x) \leq & \begin{bmatrix} x \\ \mathrm{sat}(Fx) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathrm{He}(PA - F^{\mathrm{T}}SH) & PB + F^{\mathrm{T}}S + H^{\mathrm{T}}S \\ \star & -2S \end{bmatrix} \begin{bmatrix} x \\ \mathrm{sat}(Fx) \end{bmatrix} \\ < & \ 0, \quad \forall x \in \mathcal{E}(P) \setminus \{0\}. \end{aligned}$$

This implies that the ellipsoid $\mathcal{E}(P)$ is a contractively invariant set of system (2.1). □

Remark 2.1.1 *The results of Theorem 2.1.1 can be arrived at in a different way. By the relationship between* $\mathrm{sat}(Fx)$ *and* $\mathrm{dz}(Fx)$*, where the deadzone function* $\mathrm{dz}(Fx)$ *is defined as*

$$\mathrm{dz}(Fx) = Fx - \mathrm{sat}(Fx),$$

system (2.1) can be rewritten as

$$\dot{x} = (A + BF)x - B\mathrm{dz}(Fx), \tag{2.5}$$

and the regional sector condition (2.2) can be rewritten as

$$\mathrm{dz}^{\mathrm{T}}(Fx)S((F - H)x - \mathrm{dz}(Fx)) \geq 0, \quad \forall\, x \in \mathcal{L}(H). \tag{2.6}$$

It is easy to see that

$$\begin{aligned} \dot{V}(x) &\leq 2x^{\mathrm{T}}P((A + BF)x - B\mathrm{dz}(Fx)) + 2\mathrm{dz}^{\mathrm{T}}(Fx)S((F - H)x - \mathrm{dz}(Fx)) \\ &= \begin{bmatrix} x \\ \mathrm{dz}(Fx) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} (A + BF)^{\mathrm{T}}P + P(A + BF) & -PB + (F - H)^{\mathrm{T}}S \\ \star & -2S \end{bmatrix} \begin{bmatrix} x \\ \mathrm{dz}(Fx) \end{bmatrix}. \end{aligned}$$

Then the same conditions as those in Theorem 2.1.1, which guarantee the contractive invariance of $\mathcal{E}(P)$, can be obtained.

Let

$$\Pi = \begin{bmatrix} (A + BF)^{\mathrm{T}}P + P(A + BF) & -PB + (F - H)^{\mathrm{T}}S \\ \star & -2S \end{bmatrix}.$$

The negative definiteness of matrix $\Pi \in \mathbf{R}^{(n+m)\times(n+m)}$ implies the contractive invariance of $\mathcal{E}(P)$, as seen in the proof of Theorem 2.1.1. However, because of the dependence of $\dot{V}(x)$ on x and $\mathrm{sat}(Fx)$ (or $\mathrm{dz}(Fx)$), the requirement of the negative definiteness of Π is conservative. This is partially due to the fact that the regional sector condition (2.2) contains the term $\mathrm{sat}(Fx)$. Hence, eliminating $\mathrm{sat}(Fx)$ in the treatment of $\mathrm{sat}(Fx)$ can be expected to reduce this conservativeness.

Considering the relation between $\mathrm{sat}(Fx)$ and Fx, we can rewrite $\mathrm{sat}(Fx)$ as

$$\mathrm{sat}(Fx) = D(\alpha(x))Fx,$$

where $D(\alpha(x)) = \mathrm{diag}\{\alpha_1(x), \ \alpha_2(x), \cdots, \alpha_m(x)\} \in \mathbf{R}^{m\times m}$, and each diagonal element $\alpha_j(x), j \in I[1, m]$, is defined as

$$\alpha_j(x) = \begin{cases} \frac{1}{f_j x}, & \text{if } f_j x > 1, \\ 1, & \text{if } -1 \leq f_j x \leq 1, \\ -\frac{1}{f_j x}, & \text{if } f_j x < -1, \end{cases}$$

with $f_j \in \mathbf{R}^{1\times n}$ being the jth row of matrix F. It is clear that $0 < \alpha_j(x) \leq 1$ for all $j \in I[1, m]$. Moreover, the larger $|f_j x|$ is, the smaller $\alpha_j(x)$ is. Let $\beta = [\beta_1, \beta_2, \cdots, \beta_m]^{\mathrm{T}} \in \mathbf{R}^m$, where $\beta_j \geq 1$. Define a state region

$$\mathcal{S}(F, \beta) = \Big\{x \in \mathbf{R}^n : \ |f_j x| \leq \beta_j, \ j \in I[1, m]\Big\}.$$

We can easily verify that

$$\begin{aligned}\frac{1}{\beta_j} &= \min_{x\in\mathcal{S}(F,\beta)} \alpha_j(x)\\ &\le 1.\end{aligned}$$

Thus, $\frac{1}{\beta_j} \le \mathrm{sat}(f_j x) \le 1$ for each $x \in \mathcal{S}(F,\beta)$, that is, $\mathrm{sat}(f_j x) \in \mathrm{co}\{\frac{1}{\beta_j}, 1\}$ for each $x \in \mathcal{S}(F,\beta)$. Define a set of 2^m diagonal matrices, D_i^β, in such a way that the jth diagonal element of each matrix is either $\frac{1}{\beta_j}$ or 1. For example, for $m = 2$,

$$D_1^\beta = \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix},\ \ D_2^\beta = \begin{bmatrix} \frac{1}{\beta_1} & 0\\ 0 & 1\end{bmatrix},\ \ D_3^\beta = \begin{bmatrix} 1 & 0\\ 0 & \frac{1}{\beta_2}\end{bmatrix},\ \ D_4^\beta = \begin{bmatrix} \frac{1}{\beta_1} & 0\\ 0 & \frac{1}{\beta_2}\end{bmatrix}.$$

From the fact that $\mathrm{sat}(f_j x) \in \mathrm{co}\left\{\frac{1}{\beta_j}, 1\right\}, j \in I[1,m]$, the following lemma is clear.

Lemma 2.1.2 ([29]) *Given a set $\mathcal{S}(F,\beta)$. If $x \in \mathcal{S}(F,\beta)$, then*

$$\mathrm{sat}(Fx) \in \mathrm{co}\left\{D_i^\beta Fx,\ i \in I[1,2^m]\right\}. \tag{2.7}$$

Note that, unlike the sector condition (2.2), each vertex of the polytope (2.7) into which the saturated linear feedback can be placed does not contain the term $\mathrm{sat}(Fx)$. Based on this treatment of the saturated linear feedback, the following theorem establishes a set of conditions under which an ellipsoid is contractively invariant for system (2.1).

Theorem 2.1.2 *Let $P \in \mathbf{R}^{n\times n}$ be a positive definite matrix. If there exists a vector $\beta = [\beta_1\ \beta_2\ \cdots\ \beta_m]^{\mathrm{T}} \in \mathbf{R}^m$ satisfying $\beta_j \in [1,+\infty)$, $j \in I[1,m]$, such that*

$$(A + BD_i^\beta F)^{\mathrm{T}} P + P(A + BD_i^\beta F) < 0,\ \ i \in I[1,2^m], \tag{2.8}$$

$$\begin{bmatrix} P & \frac{1}{\beta_j} f_j^{\mathrm{T}}\\ \frac{1}{\beta_j} f_j & 1\end{bmatrix} \ge 0,\ \ j \in I[1,m], \tag{2.9}$$

then the ellipsoid $\mathcal{E}(P)$ is a contractively invariant set of system (2.1).

Proof From (2.9), we have $P \ge \frac{1}{\beta_j^2} f_j^{\mathrm{T}} f_j$, $j \in I[1,m]$. Thus, for any $x \in \mathcal{E}(P)$, then

$$1 \ge x^{\mathrm{T}} P x \ge \frac{1}{\beta_j^2} x^{\mathrm{T}} f_j^{\mathrm{T}} f_j x,\ \ \forall\, j \in I[1,m].$$

By the definition of the set $\mathcal{S}(F,\beta)$, we have $\mathcal{E}(P) \subseteq \mathcal{S}(F,\beta)$, from which and Lemma 2.1.2 the saturated linear feedback sat(Fx) can be expressed as

$$\mathrm{sat}(Fx) \in \mathrm{co}\left\{D_i^{\beta}Fx : \ i \in I[1,2^m]\right\}.$$

Thus, there exist a set of scalars $\lambda_i \in [0,1]$, $i \in I[1,2^m]$, satisfying $\sum_{i=1}^{2^m}\lambda_i = 1$, such that

$$\mathrm{sat}(Fx) = \sum_{i=1}^{2^m}\lambda_i D_i^{\beta}Fx.$$

Then for any $x \in \mathcal{E}(P)$, the time derivative of the quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$ along the trajectory of system (2.1) can be evaluated as

$$\begin{aligned}\dot{V}(x) &= 2x^{\mathrm{T}}P(Ax + B\mathrm{sat}(Fx)) \\ &= \sum_{i=1}^{2^m}\lambda_i x^{\mathrm{T}}\left((A + BD_i^{\beta}F)^{\mathrm{T}}P + P(A + BD_i^{\beta}F)\right)x.\end{aligned}$$

From matrix inequalities (2.8), we have $\dot{V}(x) < 0$ for all $x \in \mathcal{E}(P)\setminus\{0\}$. This implies that the ellipsoid $\mathcal{E}(P)$ is a contractively invariant set of system (2.1). □

Theorem 2.1.2 presents a set of sufficient conditions for the determination of the contractive invariance of $\mathcal{E}(P)$ for system (2.1) by using the polytope representation (2.7). However, each vertex of this polytope depends on all elements of feedback gain F. As a result, the conservativeness still exists and can be further reduced. Indeed, a generalization of this polytope representation has been proposed in [36, 45, 46]. The generalized polytope representation, referred to as the convex hull representation, introduces an auxiliary feedback matrix such that each vertex is not required to depend on all the information of matrix F. This convex hull representation of sat(Fx) has been a popular treatment used in the estimation of the domain of attraction of a saturated system. Here and throughout the book, by the domain of attraction of a system we mean the domain of attraction of an equilibrium of the system, which is usually the origin of the state space.

In the remaining sections of this chapter, we first recall the convex hull representation of a saturated linear feedback with the single auxiliary matrix, proposed in [36, 45, 46], and then present an improved convex hull representation containing multiple auxiliary matrices. The multiple auxiliary matrix method can be further applied to treat the nestedly saturated linear feedbacks and linear feedbacks subject to piecewise linear functions with multiple bends. These treatments provide necessary preliminaries for the estimation of the domain of attraction of a saturated system. Based on these treatments of the saturation function, the regional stability conditions with respect to a quadratic Lyapunov function are established to guarantee the contractive invariance of the level sets of a quadratic Lyapunov function.

2.2 Single and Multiple Auxiliary Matrices

2.2.1 Single-Layer Saturated Linear Feedbacks

In contrast with the nestedly saturated feedbacks to be considered in Section 2.2.2, we refer to the saturated feedbacks of the form $\text{sat}(Fx)$ as single-layer saturated linear feedbacks, which are also simply referred to as saturated linear feedbacks. We first review the convex hull representation of saturated linear feedbacks in [36, 45, 46]. Let $\mathcal{D}$ be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. In the case of $m = 2$,

$$\mathcal{D} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

It is clear that there are 2^m elements in $\mathcal{D}$. We label the elements of $\mathcal{D}$ as D_i, $i = 1, 2, \cdots, 2^m$. In particular, we specify $D_1 = I_m$ and $D_{2^m} = 0_{m\times m}$ throughout this book. Denote $D_i^- = I_m - D_i$. Clearly, D_i^- is also an element of $\mathcal{D}$ if $D_i \in \mathcal{D}$. Given two matrices $F,\ H \in \mathbf{R}^{m\times n}$, $\{D_iF + D_i^-H : \ i \in I[1, 2^m]\}$ is the set of matrices formed by some corresponding rows of F and the rest of H. Note that for each diagonal matrix D_i, except for $D_1 = I_m$, the matrix $D_iF + D_i^-H$ only contains partial information of matrix F.

Lemma 2.2.1 **([36, 46])** *Let $F, H \in \mathbf{R}^{m\times n}$. Then, for any $x \in \mathcal{L}(H)$,*

$$\text{sat}(Fx) \in \text{co}\Big\{D_iFx + D_i^-Hx,\ i \in I[1, 2^m]\Big\}. \tag{2.10}$$

Lemma 2.2.1 indicates that the saturated linear feedback $\text{sat}(Fx)$ belongs to a convex hull whose vertices are a group of auxiliary linear feedbacks. The associated feedback gains are all possible mixtures of the rows of F and the auxiliary matrix H.

Remark 2.2.1 *When $m = 1$, $\text{sat}(Fx) \in \text{co}\{Fx, Hx\}$, that is, $\text{sat}(Fx)$ is located in the sector formed by Fx and Hx when $x \in \mathcal{L}(H)$. Clearly, in this case, the convex hull representation (2.10) is geometrically equivalent to the regional sector condition.*

Remark 2.2.2 *Let $D^\beta = \text{diag}\left\{\frac{1}{\beta_1}\ \frac{1}{\beta_2}\ \cdots\ \frac{1}{\beta_m}\right\} \in \mathbf{R}^{m\times m}$. By the definition of matrices D_i^β, $i \in I[1, 2^m]$, there always exist a pair of diagonal matrices (D_i, D_i^-) such that*

$$D_i^\beta F = D_iF + D_i^- D^\beta F.$$

Thus, the convex hull representation (2.7) can be rewritten as

$$\text{sat}(Fx) \in \text{co}\Big\{(D_iF + D_i^- D^\beta F)x :\ i \in I[1, 2^m]\Big\}.$$

If we set $H = D^\beta F$, then $\mathcal{L}(H) = \mathcal{S}(F, \beta)$, and Lemma 2.2.1 reduces to Lemma 2.1.2.

By using the convex hull representation (2.10) to deal with the saturated linear feedback, the following theorem establishes a set of sufficient conditions under which an ellipsoid is an estimate of the domain of attraction of system (2.1).

Theorem 2.2.1 *Let $P \in \mathbf{R}^{n\times n}$ be a positive definite and $Q = P^{-1}$. If there exists a matrix $Z \in \mathbf{R}^{m\times n}$ such that*

$$\mathrm{He}\left((A + BD_iF)Q + BD_i^- Z\right) < 0, \quad i \in I[1, 2^m], \tag{2.11}$$

$$\begin{bmatrix} 1 & z_j \\ z_j^\mathrm{T} & Q \end{bmatrix} \geq 0, \quad j \in I[1, m], \tag{2.12}$$

then, the ellipsoid $\mathcal{E}(P)$ is a contractively invariant set of system (2.1).

Proof Let $H = ZQ^{-1}$. From the proof of Theorem 2.1.1, matrix inequalities (2.12) imply that $\mathcal{E}(P) \subseteq \mathcal{L}(H)$. For any $x \in \mathcal{E}(P)$, by the convex hull representation (2.10), the saturated linear feedback can be expressed as

$$\mathrm{sat}(Fx) = \sum_{i=1}^{2^m} \lambda_i (D_iF + D_i^- H)x,$$

for some scalars $\lambda_i \in [0, 1]$ such that $\sum_{i=1}^{2^m} \lambda_i = 1$. By matrix inequalities (2.11), we have

$$\begin{aligned} \dot{V}(x) &= \sum_{i=1}^{2^m} \lambda_i x^\mathrm{T} \left(\mathrm{He}(P(A + BD_iF + BD_i^- H))\right) x \\ &< 0, \quad \forall\, x \in \mathcal{E}(P) \setminus \{0\}, \end{aligned}$$

which implies that $\mathcal{E}(P)$ is a contractively invariant set of system (2.1). □

In order to intuitively understand the convex hull representation (2.10), we illustrate Lemma 2.2.1 in Figure 2.2 for the case of $m = 2$, where f_j and h_j, $j = 1, 2$, are the jth rows of matrices F and H, respectively. Note that

$$\begin{bmatrix} f_1x \\ f_2x \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} F + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} H\right) x,$$

$$\begin{bmatrix} h_1x \\ f_2x \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} F + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} H\right) x,$$

$$\begin{bmatrix} f_1x \\ h_2x \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} F + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} H\right) x,$$

$$\begin{bmatrix} h_1x \\ h_2x \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} F + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} H\right) x.$$

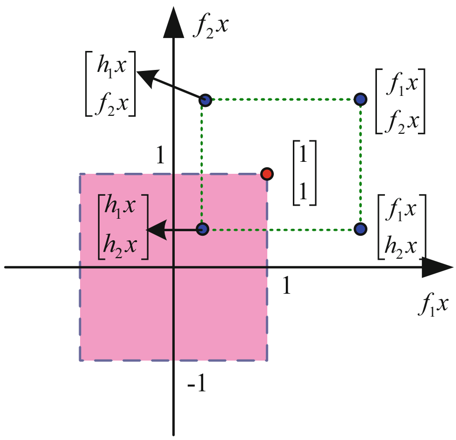

Fig. 2.2 A geometric illustration of the convex hull representation (2.10).

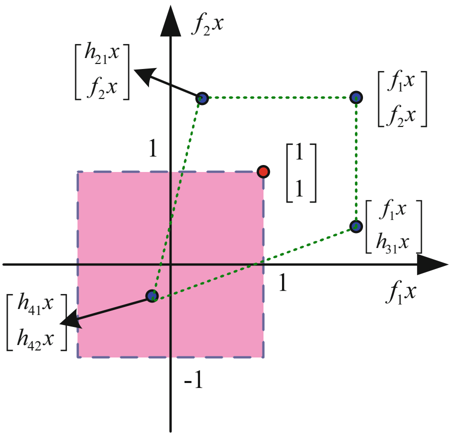

Fig. 2.3 A geometric illustration of the convex hull representation (2.13).

In Figure 2.2, the shaded zone represents the region $\mathcal{L}(H)$ in the input space. Note that all vertices of the convex hull in (2.10) share a common auxiliary feedback matrix H. This results in a rectangular convex hull, enclosed by the dotted lines. It is clear that such a constraint in the shape leads to the conservativeness of the convex hull representation (2.10). From the geometric illustration of sat(Fx) in Figure 2.2 we see that sat(Fx) still resides in the resulting convex hull if the vertex associated with Hx is moved to other points in the shaded area. The resulting convex hull forms a general quadrangle that is not necessarily a rectangular, as shown in Figure 2.3, where h_{ij} is the jth row of matrix $H_i \in \mathbf{R}^{2\times 2}$, $j = 1, 2$, $i = 1, 2, 3, 4$, and

$$\begin{bmatrix} f_1 x \\ f_2 x \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} F + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} H_1 \right) x,$$

$$\begin{bmatrix} h_{21} x \\ f_2 x \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} F + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} H_2 \right) x,$$

$$\begin{bmatrix} f_1 x \\ h_{32} x \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} F + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} H_3 \right) x,$$

$$\begin{bmatrix} h_{41} x \\ h_{42} x \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} F + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} H_4 \right) x.$$

Each vertex of the quadrangle is associated with an independent auxiliary matrix H_i. Hence, we have the following lemma, which presents an improved convex hull representation of $\mathrm{sat}(Fx)$.

Lemma 2.2.2 *Let $F, H_i \in \mathbf{R}^{m\times n}$, $i \in I[1, 2^m]$. Then, for any $x \in \bigcap_{i=1}^{2^m} \mathcal{L}(H_i)$,*

$$\mathrm{sat}(Fx) \in \mathrm{co}\Big\{D_i Fx + D_i^- H_i x,\ i \in I[1, 2^m]\Big\}. \tag{2.13}$$

Lemma 2.2.2 recovers the result of Lemma 2.2.1 if we set $H_i = H$, $i \in I[1, 2^m]$. The introduction of multiple auxiliary matrices increases the degree of freedom in dealing with $\mathrm{sat}(Fx)$, and results in a less conservative approach to expressing $\mathrm{sat}(Fx)$. The proof of Lemma 2.2.2 follows directly from the following lemma.

Lemma 2.2.3 *Let $u = [u_1\ u_2\ \cdots\ u_m]^{\mathrm{T}} \in \mathbf{R}^m$, and $v_i = [v_{i1}\ v_{i2}\ \cdots\ v_{im}]^{\mathrm{T}} \in \mathbf{R}^m$, $i \in I[1, 2^m]$. Suppose that $|v_{ij}| \le 1$ for all $i \in I[1, 2^m], j \in I[1, m]$, then*

$$\mathrm{sat}(u) \in \mathrm{co}\Big\{D_i u + D_i^- v_i : i \in I[1, 2^m]\Big\}. \tag{2.14}$$

To prove Lemma 2.2.3, we need the following lemma as a preliminary result.

Lemma 2.2.4 *Let u, $u_1^1, u_1^2, \cdots, u_1^p, u_2^1, u_2^2, \cdots, u_2^p, \cdots, u_q^1, u_q^2, \cdots, u_q^p \in \mathbf{R}^{m_1}$, and v, $v_1^1, v_1^2, \cdots, v_1^q, v_2^1, v_2^2, \cdots, v_2^q, \cdots, v_p^1, v_p^2, \cdots, v_p^q \in \mathbf{R}^{m_2}$. If $u \in \mathrm{co}\{u_j^i : i \in I[1, p]\}$ for all $j \in I[1, q]$, and $v \in \mathrm{co}\{v_i^j : j \in I[1, q]\}$ for all $i \in I[1, p]$, satisfying $v \in \mathrm{co}\{\sum_{i=1}^p \alpha_j^i v_i^j : j \in I[1, q]\}$ for all $\alpha_j^i \in [0, 1]$ and $\sum_{i=1}^p \alpha_j^i = 1$, then*

$$\begin{bmatrix} u \\ v \end{bmatrix} \in \mathrm{co}\left\{ \begin{bmatrix} u_j^i \\ v_i^j \end{bmatrix} : i \in I[1, p], j \in I[1, q] \right\}.$$

Proof Since $u \in \mathrm{co}\{u_j^i : i \in I[1, p]\}$, for all $j \in I[1, q]$, there exist a set of α_j^i, $i \in I[1, p]$, satisfying $\alpha_j^i \in [0, 1]$ and $\sum_{i=1}^p \alpha_j^i = 1$, such that

$$u = \sum_{i=1}^{p} \alpha_j^i u_j^i, \quad \forall j \in I[1, q].$$

Similarly, there exist a set of β_j, $j \in I[1, q]$, satisfying $\beta_j \in [0, 1]$ and $\sum_{j=1}^q \beta_j = 1$, such that

$$\begin{aligned} v &= \sum_{j=1}^{q} \beta_j \sum_{i=1}^{p} \alpha_j^i v_i^j \\ &= \sum_{j=1}^{q} \sum_{i=1}^{p} \beta_j \alpha_j^i v_i^j. \end{aligned}$$

Hence, we have

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} \sum_{j=1}^{q} \beta_j \sum_{i=1}^{p} \alpha_j^i u_j^i \\ \sum_{j=1}^{q} \sum_{i=1}^{p} \beta_j \alpha_j^i v_i^j \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=1}^{q} \sum_{i=1}^{p} \beta_j \alpha_j^i u_j^i \\ \sum_{j=1}^{q} \sum_{i=1}^{p} \beta_j \alpha_j^i v_i^j \end{bmatrix} \\ &= \sum_{j=1}^{q} \sum_{i=1}^{p} \beta_j \alpha_j^i \begin{bmatrix} u_j^i \\ v_i^j \end{bmatrix}. \end{aligned}$$

Noting that

$$\begin{aligned} \sum_{j=1}^{q} \sum_{i=1}^{p} \beta_j \alpha_j^i &= \sum_{j=1}^{q} \beta_j \sum_{i=1}^{p} \alpha_j^i \\ &= \sum_{j=1}^{q} \beta_j \\ &= 1, \end{aligned}$$

we have

$$\begin{bmatrix} u \\ v \end{bmatrix} \in \mathrm{co} \left\{ \begin{bmatrix} u_j^i \\ v_i^j \end{bmatrix} : i \in I[1,p], j \in I[1,q] \right\} .$$

$\square$

We next prove Lemma 2.2.3.

Proof of Lemma 2.2.3 Since $|v_{ij}| \leq 1$, we have $\mathrm{sat}(u_j) \in \mathrm{co}\{u_j, v_{ij}\}$, for all $i \in I[1, 2^m]$ and $j \in I[1, m]$.

For $m = 1$, that is, $u = u_1$, let $v_1 = v_{11}$ and $v_2 = v_{21}$. Hence,

$$\begin{aligned} \mathrm{sat}(u) &= \mathrm{sat}(u_1) \\ &\in \mathrm{co}\{u_1, v_{21}\} \\ &= \mathrm{co}\{u, v_2\} \\ &= \mathrm{co}\Big\{D_i u + D_i^- v_i, i = 1, 2\Big\}, \end{aligned}$$

where $D_1 = 1$ and $D_2 = 0$.

For $m = 2$, $\text{sat}(u_1) \in \text{co}\{u_1, v_{21}\}$, $\text{sat}(u_1) \in \text{co}\{u_1, v_{41}\}$, $\text{sat}(u_2) \in \text{co}\{u_2, v_{32}\}$, and $\text{sat}(u_2) \in \text{co}\{u_2, v_{42}\}$. For any α_1 and α_2 satisfying $\alpha_1, \alpha_2 \in [0, 1]$ and $\alpha_1+\alpha_2 = 1$, we have $\text{sat}(u_2) \in \text{co}\{\alpha_1 u_2 + \alpha_2 u_2, \alpha_1 v_{32} + \alpha_2 v_{42}\}$, since $|\alpha_1 v_{32} + \alpha_2 v_{42}| \le \alpha_1 |v_{32}| + \alpha_2 |v_{42}| \le \alpha_1 + \alpha_2 = 1$. By Lemma 2.2.4, we have

$$
\begin{aligned}
\text{sat}(u) &= \text{sat}\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) \\
&\in \text{co}\left\{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_{21} \\ u_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ v_{32} \end{bmatrix}, \begin{bmatrix} v_{41} \\ v_{42} \end{bmatrix}\right\} \\
&= \text{co}\Big\{D_i u + D_i^- v_i, i = 1, 2, 3, 4\Big\},
\end{aligned}
$$

where $D_1 = \text{diag}\{1, 1\}$, $D_2 = \text{diag}\{0, 1\}$, $D_3 = \text{diag}\{1, 0\}$ and $D_4 = \text{diag}\{0, 0\}$.

For $m = 3$,

$$
\begin{aligned}
\text{sat}(u_3) &\in \text{co}\{u_3, v_{53}\}, \\
\text{sat}(u_3) &\in \text{co}\{u_3, v_{63}\}, \\
\text{sat}(u_3) &\in \text{co}\{u_3, v_{73}\}, \\
\text{sat}(u_3) &\in \text{co}\{u_3, v_{83}\}.
\end{aligned}
$$

Thus, for any $\alpha_1, \alpha_2, \alpha_3$ and α_4, satisfying $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0, 1]$ and $\sum_{i=1}^{4} \alpha_i = 1$, we have

$$
\text{sat}(u_3) \in \text{co}\{u_3, \alpha_1 v_{53} + \alpha_2 v_{63} + \alpha_3 v_{73} + \alpha_4 v_{83}\}.
$$

By Lemma 2.2.4, we have

$$
\begin{aligned}
\text{sat}(u) &= \text{sat}\left(\begin{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ u_3 \end{bmatrix}\right) \\
&\in \text{co}\left\{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} v_{21} \\ u_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} u_1 \\ v_{32} \\ u_3 \end{bmatrix}, \begin{bmatrix} v_{41} \\ v_{42} \\ u_3 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \\ v_{53} \end{bmatrix}, \begin{bmatrix} v_{61} \\ u_2 \\ v_{63} \end{bmatrix}, \begin{bmatrix} u_1 \\ v_{72} \\ v_{73} \end{bmatrix}, \begin{bmatrix} v_{81} \\ v_{82} \\ v_{83} \end{bmatrix}\right\} \\
&= \text{co}\Big\{D_i u + D_i^- v_i, i = 1, 2, \cdots, 8\Big\},
\end{aligned}
$$

where $D_1 = \text{diag}\{1, 1, 1\}$, $D_2 = \text{diag}\{0, 1, 1\}$, $D_3 = \text{diag}\{1, 0, 1\}$, $D_4 = \text{diag}\{0, 0, 1\}$, $D_5 = \text{diag}\{1, 1, 0\}$, $D_6 = \text{diag}\{0, 1, 0\}$, $D_7 = \text{diag}\{1, 0, 0\}$ and $D_8 = \text{diag}\{0, 0, 0\}$.

For $m \geq 4$, by recursively considering $u_4, u_5, \cdots, u_m$ as we consider u_3, u_2 and u_1 above, we can deduce that

$$\mathrm{sat}(u) \in \mathrm{co}\Big\{D_i u + D_i^- v_i, i = 1, 2, \cdots, 2^m\Big\}.$$

This completes the proof. □

Note that some matrices D_i^- contain zero elements in their diagonal, the corresponding rows of matrices H_i do not appear in the improved convex hull representation (2.13), and are thus irrelevant, since some D_i^- contain zero columns. Denote $\mathcal{D}_i^-$, $i \in I[2, 2^m]$, to be the D_i^- with all its zero columns removed, and we have $\mathcal{D}_i^- \in \mathbf{R}^{m \times p_i}$, where p_i is the number of all nonzero columns of D_i^-. For example, if

$$D_i^- = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$\mathcal{D}_i^- = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and $p_i = 2$. Note that $p_{2^m} = m$ since $D_{2^m}^- = I_m$. Noting that the irrelevant rows of H_i can be replaced with zero elements, we denote $\overline{H}_i$, $i \in I[2, 2^m]$, to be H_i whose irrelevant columns are set to be zeros. Let $\mathcal{H}_i \in \mathbf{R}^{p_i \times n}$, $i \in I[2, 2^m]$, be $\overline{H}_i$ whose invalid elements are removed. Then, we have $\mathcal{L}(\mathcal{H}_i) = \mathcal{L}(\overline{H}_i)$, $i \in I[2, 2^m]$. Then, the improved convex hull representation (2.13) can be equivalently written as

$$\mathrm{sat}(Fx) \in \mathrm{co}\Big\{Fx, D_i F x + \mathcal{D}_i^- \mathcal{H}_i x,\ i \in I[2, 2^m]\Big\}, \quad \forall\, x \in \bigcap_{i=2}^{2^m} \mathcal{L}(\mathcal{H}_i). \tag{2.15}$$

All elements of $\mathcal{H}_i$'s are necessary to construct such a more compact form. By simple calculation, we can obtain that there are $mn2^{m-1}$ elements of all the auxiliary matrices in (2.15).

The following theorem provides a set of sufficient conditions with less conservativeness than Theorem 2.2.1 for the determination of the contractive invariance of an ellipsoid.

Theorem 2.2.2 *Let $P \in \mathbf{R}^{n \times n}$ be a positive definite matrix and $Q = P^{-1}$. If there exist matrices $\mathcal{Z}_i \in \mathbf{R}^{p_i \times n}$, $i \in I[2, 2^m]$, such that $Q(A + BF)^{\mathrm{T}} + (A + BF)Q < 0$,*

$$\mathrm{He}\left((A + BD_i F)Q + B\mathcal{D}_i^- \mathcal{Z}_i\right) < 0, \quad i \in I[2, 2^m], \tag{2.16}$$

$$\begin{bmatrix} 1 & z_{ij} \\ z_{ij}^{\mathrm{T}} & Q \end{bmatrix} \geq 0, \quad j \in I[1, p_i],\ i \in I[2, 2^m], \tag{2.17}$$

where p_i is the number of all nonzero columns of D_i^-, $i \in I[2, 2^m]$, and $z_j \in \mathbf{R}^{1\times n}$ is the jth row of $\mathcal{Z}_i$, then the ellipsoid $\mathcal{E}(P)$ is a contractively invariant set of system (2.1).

Proof Matrix inequalities (2.17) imply that $\mathcal{E}(P) \subseteq \bigcap_{i=2}^{2^m} \mathcal{L}(\mathcal{H}_i)$, where $\mathcal{H}_i = \mathcal{Z}_i Q^{-1}$. The rest of the proof is the same as that of Theorem 2.2.1. □

Remark 2.2.3 *A treatment of saturated linear feedbacks with multiple auxiliary matrices that is equivalent to the compact form of convex hull representation (2.13) or (2.15) was earlier proposed by [2]. Other equivalent results can be found in [21, 92, 105, 107].*

2.2.2 Nestedly Saturated Linear Feedbacks

Dynamical systems subject to nested saturation in their input have attracted significant attention from control system researchers due to their frequent occurrence in various engineering applications. As an example, control systems subject to simultaneous actuator magnitude and rate saturation in the input (see, *e.g.*, [6, 7, 11, 83, 98, 101]) can be modeled with nested saturation functions. In this subsection, we consider a linear system with a nestedly saturated linear feedback,

$$\dot{x} = Ax + B_1 \mathrm{sat}(F_1 x + B_2 \mathrm{sat}(F_2 x + B_3 \mathrm{sat}(F_3 x + \cdots + B_q \mathrm{sat}(F_q x)))), \tag{2.18}$$

where $F_k \in \mathbf{R}^{m_k \times n}$, $k \in I[1, q]$, $B_k \in \mathbf{R}^{m_{k-1} \times m_k}$, $k \in I[1, q]$, and $m_0 = n$. We number the saturation functions from the outermost layer inward, with the outermost layer as the first layer saturation function.

The nestedly saturated linear system (2.18) with B_k being diagonal matrices was originally considered in [7], and the general case was later studied (see, *e.g.*, [21, 93, 107]). Here we consider the regional stability of the general case of (2.18).

The regional sector condition (2.6) with respect to deadzone function, which was used to handle the single layer saturated linear feedback in Section 2.2.1, can also be adopted to treat the nestedly saturated linear feedback, as has been done in [93].

Let

$$\begin{aligned}
\mathcal{A} &= A + \sum_{j=1}^{q} \left(\prod_{l=1}^{j} B_l \right) F_j, \\
\mathcal{F}_j &= F_j + \sum_{l=j+1}^{q} \left(\prod_{k=j+1}^{l} B_k \right) F_l, \quad j \in I[2, q-1], \\
\mathcal{F}_q &= F_q.
\end{aligned}$$

Then system (2.18) can be rewritten as

$$\dot{x} = \mathcal{A}x - B_1\mathrm{dz}(\psi_1) - B_1B_2\mathrm{dz}(\psi_2) - \cdots - B_1B_2\cdots B_q\mathrm{dz}(\psi_q), \tag{2.19}$$

where

$$\begin{aligned} \psi_q &= F_q x, \\ \psi_j &= \mathcal{F}_j x - \sum_{l=j+1}^{q}\left(\prod_{k=1}^{l} B_k\right)\mathrm{dz}(\psi_l),\ j \in I[q-1,1]. \end{aligned}$$

Recall the regional sector condition of the deadzone function $\mathrm{dz}(v) \in \mathbf{R}^m$,

$$\mathrm{dz}^{\mathrm{T}}(v)T(\mathrm{dz}(v) - v + w) \le 0, \quad \forall\, w \in \mathbf{R}^m,\ |w|_\infty \le 1,$$

where T is a diagonal and positive definite matrix. It then follows that

$$\mathrm{dz}^{\mathrm{T}}(\psi_j)T_j(\mathrm{dz}(\psi_j) - \psi_j + E_j\Psi_j) \le 0, \quad |E_j\Psi_j|_\infty \le 1,\ j \in I[1,q], \tag{2.20}$$

where T_j's are diagonal and positive definite matrices and

$$\begin{aligned} E_q &= E_{0q}, \\ E_j &= [E_{0j}\ \ E_{qj}\ \ E_{(q-1)j}\ \ \cdots\ \ E_{(j+1)j}],\ \ j \in I[q-1,1], \\ &\qquad E_{0j} \in \mathbf{R}^{m_j\times n},\ E_{kj} \in \mathbf{R}^{m_j\times m_k},\ \ k \in I[q,j], \\ \Psi_q &= x, \\ \Psi_j &= [x^{\mathrm{T}}\ \ \mathrm{dz}^{\mathrm{T}}(\psi_q)\ \ \mathrm{dz}^{\mathrm{T}}(\psi_{q-1})\ \ \cdots\ \ \mathrm{dz}^{\mathrm{T}}(\psi_j)]^{\mathrm{T}},\ \ j \in I[q-1,1]. \end{aligned}$$

In view of the expression of ψ_j, Inequalities (2.20) can be rewritten as

$$\begin{aligned} &\mathrm{dz}^{\mathrm{T}}(\psi_q)T_q(\mathrm{dz}(\psi_q) + (E_{0q} - \mathcal{F}_q)x) \le 0, \quad |E_{0q}x|_\infty \le 1, \\ &\mathrm{dz}^{\mathrm{T}}(\psi_j)T_j\left(\mathrm{dz}(\psi_j) + (E_{0j} - \mathcal{F}_j)x + \sum_{l=j+1}^{q}\left(E_{lj} + \prod_{k=1}^{l} B_k\right)\mathrm{dz}(\psi_l)\right) \le 0, \\ &|E_j\Psi_j|_\infty \le 1,\ j \in I[q-1,1]. \end{aligned}$$

Denote

$$\begin{aligned} \Omega_q &= \left\{x \in \mathbf{R}^n :\ |E_{0q}x|_\infty \le 1\right\}, \\ \Omega_j &= \left\{x \in \mathbf{R}^n :\ |E_j\Psi_j|_\infty \le 1\right\},\ \ j \in I[q-1,1]. \end{aligned}$$

In order to apply the regional sector conditions to determine if an ellipsoid $\mathcal{E}(P)$ is a contractively invariant set of system (2.18), we need the condition $\mathcal{E}(P) \subseteq \bigcap_{j=1}^{q} \Omega_j$. The condition $\mathcal{E}(P) \subseteq \Omega_q$ is implied by

$$\begin{bmatrix} 1 & e_{0qk} \\ \star & P \end{bmatrix} \geq 0, \quad k \in I[1, m_q], \tag{2.21}$$

where e_{0qk} is the kth row of matrix E_{0q}. For the ellipsoid $\mathcal{E}(P)$ contained in Ω_q, the condition $\mathcal{E}(P) \subseteq \Omega_{q-1}$ holds if

$$\begin{aligned}
& x^{\mathrm{T}} P x - \Psi_{q-1}^{\mathrm{T}} e_{(q-1)k}^{\mathrm{T}} e_{(q-1)k} \Psi_{q-1} + 2\mathrm{dz}^{\mathrm{T}}(\psi_q) T_q (\mathrm{dz}(\psi_q) + (E_{0q} - \mathcal{F}_q)x) \\
&= \Psi_{q-1}^{\mathrm{T}} \begin{bmatrix} P & (E_{0q} - \mathcal{F}_q)^{\mathrm{T}} T_q \\ \star & 2T_q \end{bmatrix} \Psi_{q-1} - \Psi_{q-1}^{\mathrm{T}} e_{(q-1)k}^{\mathrm{T}} e_{(q-1)k} \Psi_{q-1} \\
&=: \Psi_{q-1}^{\mathrm{T}} (\Xi_{q-1} - e_{(q-1)k}^{\mathrm{T}} e_{(q-1)k}) \Psi_{q-1} \\
&\geq 0,
\end{aligned}$$

which is implied by

$$\begin{bmatrix} 1 & e_{(q-1)k} \\ \star & \Xi_{q-1} \end{bmatrix} \geq 0, \quad k \in I[1, m_{q-1}], \tag{2.22}$$

where $e_{(q-1)k}$ is the kth row of E_{q-1}. Moreover, for the ellipsoid $\mathcal{E}(P)$ contained in $\bigcap_{l=j+1}^{q} \Omega_l$, the condition $\mathcal{E}(P) \subseteq \Omega_j$ holds if

$$\begin{aligned}
& x^{\mathrm{T}} P x - \Psi_j^{\mathrm{T}} e_{jk}^{\mathrm{T}} e_{jk} \Psi_j + 2\mathrm{dz}^{\mathrm{T}}(\psi_q) T_q (\mathrm{dz}(\psi_q) + (E_{0q} - \mathcal{F}_q)x) \\
&\quad + 2 \sum_{g=j+1}^{q-1} \left(\mathrm{dz}^{\mathrm{T}}(\psi_g) T_g \left(\mathrm{dz}(\psi_g) + (E_{0g} - \mathcal{F}_g)x + \sum_{l=g+1}^{q} \left(E_{lg} + \prod_{k=1}^{l} B_k \right) \mathrm{dz}(\psi_l) \right) \right) \\
&= \Psi_j^{\mathrm{T}} (\Xi_j - e_{jk}^{\mathrm{T}} e_{jk}) \Psi_j \\
&\geq 0,
\end{aligned}$$

where e_{jk} is the kth row of matrix E_j and

$$\Xi_j = \begin{bmatrix} P & (E_{0q} - \mathcal{F}_q)^{\mathrm{T}} T_q & (E_{0(q-1)} - \mathcal{F}_{q-1})^{\mathrm{T}} T_{q-1} & \cdots & (E_{0(j+1)} - \mathcal{F}_{j+1})^{\mathrm{T}} T_{j+1} \\ \star & 2T_q & (E_{qj} + B_q) T_{q-1} & \cdots & (E_{q(j+1)} + \prod_{k=1}^{q} B_k) T_{j+1} \\ \star & \star & 2T_{q-1} & \cdots & (E_{(q-1)(j+1)} + \prod_{k=1}^{q-1} B_k) T_{j+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \cdots & 2T_{j+1} \end{bmatrix}.$$

Thus, the satisfaction of

$$\begin{bmatrix} 1 & e_{jk} \\ \star & \Xi_j \end{bmatrix} \geq 0, \quad j \in I[1, q-1], \tag{2.23}$$

guarantees that $\mathcal{E}(P) \subseteq \bigcap_{l=j}^{q} \Omega_l$.

Based on the above analysis, we can establish the following theorem, which presents a set of sufficient conditions under which an ellipsoid is contractively invariant with respect to system (2.18).

Theorem 2.2.3 *Let $P \in \mathbf{R}^{n\times n}$ be a positive definite matrix. If there exist diagonal positive definite matrices $T_j \in \mathbf{R}^{m_j\times m_j}$, $j \in I[1, q]$, and matrices $E_{0j} \in \mathbf{R}^{m_j\times n}$, $E_{kj} \in \mathbf{R}^{m_j\times m_k}$, $k \in I[j, q]$, $j \in I[1, q]$, such that*

$$\mathcal{M} < 0, \tag{2.24}$$

(2.21) and (2.23) hold, where

$$\mathcal{M} = \begin{bmatrix} \mathcal{A}^{\mathrm{T}}P + P\mathcal{A} & -PB_1 - (E_{01} - \mathcal{F}_1)^{\mathrm{T}}T_1 & -PB_1B_2 - (E_{02} - \mathcal{F}_2)^{\mathrm{T}}T_2 & \cdots & M_1 \\ \star & -2T_1 & -T_1(E_{21} + B_1B_2) & \cdots & M_2 \\ \star & \star & -2T_2 & \cdots & M_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \cdots & -2T_q \end{bmatrix}$$

with $M_1 = -P\prod_{k=1}^{q} B_k - (E_{0q} - \mathcal{F}_q)^{\mathrm{T}}T_q$, $M_2 = -T_q(E_{2q} + \prod_{k=1}^{q} B_k)$ and $M_3 = -T_q(E_{3q} + \prod_{k=1}^{q} B_k)$, then the ellipsoid $\mathcal{E}(P)$ is a contractively invariant set of system (2.18).

Proof Because of Inequalities (2.21) and (2.23), the regional sector conditions (2.20) are satisfied for $x \in \mathcal{E}(P)$. Consider the quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$. Its time derivative along the trajectory of system (2.20) can be evaluated as

$$\dot{V}(x) = 2x^{\mathrm{T}}P\left(\mathcal{A}x - \sum_{j=1}^{q}\left(\prod_{k=1}^{j} B_k\right)\mathrm{dz}(\psi_j)\right).$$

Thus, for any $x \in \mathcal{E}(P)$, we have

$$\begin{aligned} \dot{V}(x) &\leq \dot{V}(x) + 2\mathrm{dz}^{\mathrm{T}}(\psi_q)T_q(\mathrm{dz}(\psi_q) + (E_{0q} - \mathcal{F}_q)x) \\ &\quad + 2\sum_{g=j+1}^{q-1}\left(\mathrm{dz}^{\mathrm{T}}(\psi_g)T_g\left(\mathrm{dz}(\psi_g) + (E_{0g} - \mathcal{F}_g)x + \sum_{l=g+1}^{q}\left(E_{lg} + \prod_{k=1}^{l} B_k\right)\mathrm{dz}(\psi_l)\right)\right) \\ &= \phi^{\mathrm{T}}\mathcal{M}\phi, \end{aligned}$$

where $\phi = [x^{\mathrm{T}}\ \mathrm{dz}^{\mathrm{T}}(\psi_1)\ \mathrm{dz}^{\mathrm{T}}(\psi_2)\ \cdots\ \mathrm{dz}^{\mathrm{T}}(\psi_q)]^{\mathrm{T}}$. Since $\mathcal{M} < 0$, we have $\dot{V}(x) < 0$ for all $x \in \mathcal{E}(P) \setminus \{0\}$. Hence, $\mathcal{E}(P)$ is a contractively invariant set of system (2.18). $\square$

In what follows, we will review the convex hull representation of the nested saturation. Such a treatment of nested saturation functions was first proposed in [7], where the case that B_k's are diagonal was considered. Here, we deal with the general case. Let $H_k(i_k) \in \mathbf{R}^{m_k \times n}$, $i_k \in I[1, 2^{m_k}]$, $k \in I[1, q]$. For each k, the set $\{H_k(i_k) :\ i_k \in I[1, 2^{m_k}]\}$ has 2^{m_k} elements.

Lemma 2.2.5 *For an $x \in \mathbf{R}^n$, if $x \in \mathcal{L}(H_k(i_k))$, $i_k \in I[1, 2^{m_k}]$, $k \in I[1, q]$, then*

$$\begin{aligned}
&\mathrm{sat}(F_1x + B_2\mathrm{sat}(F_2x + B_3\mathrm{sat}(F_3x + \ldots + B_q\mathrm{sat}(F_qx)))) \\
&\in \mathrm{co}\left\{\sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} D_{i_l}B_{l+1}\right) D_{i_k}F_kx \right.\\
&\qquad \left. + \sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} D_{i_l}B_{l+1}\right) D_{i_k}^{-}H_k(i_k)x :\ \ i_k \in I[1, 2^{m_k}], k \in I[1, q]\right\},
\end{aligned} \tag{2.25}$$

where we have defined $\prod_{l=1}^{0} D_{i_l}B_{l+1} = I$.

Proof Let

$$\begin{aligned}
v_1 &= F_1x + B_2\mathrm{sat}(F_2x + B_3\mathrm{sat}(F_3x + \cdots + B_q\mathrm{sat}(F_qx))),\\
v_2 &= F_2x + B_3\mathrm{sat}(F_3x + \cdots + B_q\mathrm{sat}(F_qx)),\\
&\ \ \vdots\\
v_q &= F_qx.
\end{aligned}$$

We first consider the first layer saturation function, that is, the outmost layer saturation function. For any $x \in \mathcal{L}(H_k(i_k))$, $i_k \in I[1, 2^{m_k}]$, $k \in I[1, q]$, by Lemma 2.2.2, we have

$$\begin{aligned}
\mathrm{sat}(v_1) &\in \mathrm{co}\left\{D_{i_1}v_1 + D_{i_1}^{-}H_1(i_1)x :\ i_1 \in I[1, 2^{m_1}]\right\}\\
&= \mathrm{co}\left\{D_{i_1}F_1x + D_{i_1}B_2\mathrm{sat}(v_2) + D_{i_1}^{-}H_1(i_1)x :\ i_1 \in I[1, 2^{m_1}]\right\}.
\end{aligned}$$

For each function $D_{i_1}F_1x + D_{i_1}B_2\mathrm{sat}(v_2) + D_{i_1}^{-}H_1(i_1)x$, $i_1 \in I[1, 2^{m_1}]$, by applying again Lemma 2.2.2, we have

$$\begin{aligned}
&D_{i_1}F_1x + D_{i_1}B_2\mathrm{sat}(v_2) + D_{i_1}^{-}H_1(i_1)\\
&\in \mathrm{co}\left\{D_{i_1}F_1x + D_{i_1}B_2D_{i_2}F_2x + D_{i_1}B_2D_{i_2}B_3\mathrm{sat}(v_3)\right.\\
&\qquad \left. + D_{i_1}B_2D_{i_2}^{-}H_2(i_2)x + D_{i_1}^{-}H_1(i_1)x :\ i_2 \in I[1, 2^{m_2}]\right\}.
\end{aligned}$$

Thus, we can get

$$\begin{aligned}\mathrm{sat}(v_1) \in \mathrm{co}\Big\{& D_{i_1}F_1x + D_{i_1}B_2D_{i_2}F_2x + D_{i_1}B_2D_{i_2}B_3\mathrm{sat}(v_3)\\ &+ D_{i_1}B_2D_{i_2}^-H_2(i_2)x + D_{i_1}^-H_1(i_1)x:\ i_1 \in I[1,2^{m_1}], i_2 \in I[1,2^{m_2}]\Big\}.\end{aligned}$$

By repeating the above procedure for $v_3, v_4, \cdots, v_q$, we can finally obtain that

$$\begin{aligned}\mathrm{sat}(v_1) \in \mathrm{co}\Bigg\{&\sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} D_{i_l}B_{l+1}\right)D_{i_k}F_kx\\ &+\sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} D_{i_l}B_{l+1}\right)D_{i_k}^-H_k(i_k)x:\quad i_k \in I[1,2^{m_k}], k \in I[1,q]\Bigg\}.\end{aligned}$$

□

From Lemma 2.2.5 and its proof, we can observe that some vertices of the convex hull (2.25) share a set of common auxiliary matrices, which clearly results in conservativeness of this treatment of nestedly saturated linear feedback. In order to reduce the conservativeness of the convex hull representation (2.25), we introduce more auxiliary feedback matrices, defined as $H_k(i_1, i_2, \cdots, i_k) \in \mathbf{R}^{m_k\times n}$, $i_k \in I[1,2^{m_k}]$, $k \in I[1,q]$, and obtain the following lemma, which is a generalization of Lemma 2.2.5.

Lemma 2.2.6 *For an $x \in \mathbf{R}^n$, if $x \in \mathcal{L}(H_k(i_1, i_2, \cdots, i_k))$, $i_k \in I[1,2^{m_k}]$, $k \in I[1,q]$, then*

$$\begin{aligned}&\mathrm{sat}(F_1x + B_2\mathrm{sat}(F_2x + B_3\mathrm{sat}(F_3x + \cdots + B_q\mathrm{sat}(F_qx))))\\ &\in \mathrm{co}\Bigg\{\sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} D_{i_l}B_{l+1}\right)D_{i_k}F_kx\\ &\quad+\sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} D_{i_l}B_{l+1}\right)D_{i_k}^-H_k(i_1, i_2, \cdots, i_k)x:\quad i_k \in I[1,2^{m_k}], k \in I[1,q]\Bigg\},\end{aligned} \tag{2.26}$$

where we have defined $\prod_{l=1}^{0} D_{i_l}B_{l+1} = I$.

Proof Let $v_1, v_2, \cdots, v_q$ be defined as in the proof of Lemma 2.2.5. For any $x \in \mathcal{L}(H_k(i_1, i_2, \cdots, i_k))$, $i_k \in I[1,2^{m_k}]$, $k \in I[1,q]$, by applying Lemma 2.2.2, we have

$$\begin{aligned}\mathrm{sat}(v_1) &\in \mathrm{co}\Big\{D_{i_1}v_1 + D_{i_1}^-H_1(i_1)x:\ i_1 \in I[1,2^{m_1}]\Big\}\\ &= \mathrm{co}\Big\{D_{i_1}F_1x + D_{i_1}B_2\mathrm{sat}(v_2) + D_{i_1}^-H_1(i_1)x:\ i_1 \in I[1,2^{m_1}]\Big\}.\end{aligned}$$

For any given $i_1 \in I[1, 2^{m_1}]$, $D_{i_1}F_1x + D_{i_1}B_2\mathrm{sat}(v_2) + D_{i_1}^-H_1(i_1)x$, by Lemma 2.2.2, can be expressed as,

$$\begin{aligned}
&D_{i_1}F_1x + D_{i_1}B_2\mathrm{sat}(v_2) + D_{i_1}^-H_1(i_1)\\
&\in \mathrm{co}\Big\{D_{i_1}F_1x + D_{i_1}B_2D_{i_2}F_2x + D_{i_1}B_2D_{i_2}B_3\mathrm{sat}(v_3)\\
&\qquad + D_{i_1}B_2D_{i_2}^-H_2(i_1, i_2)x + D_{i_1}^-H_1(i_1)x : \ i_2 \in I[1, 2^{m_2}]\Big\}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mathrm{sat}(v_1) \in \mathrm{co}\Big\{&D_{i_1}F_1x + D_{i_1}B_2D_{i_2}F_2x + D_{i_1}B_2D_{i_2}\mathrm{sat}(v_3)\\
&+ D_{i_1}B_2D_{i_2}^-H_2(i_1, i_2)x + D_{i_1}^-H_1(i_1)x : \ i_1 \in I[1, 2^{m_1}], i_2 \in I[1, 2^{m_2}]\Big\}.
\end{aligned} \tag{2.27}$$

Similarly, for any given $i_1 \in I[1, 2^{m_1}]$ and $i_2 \in I[1, 2^{m_2}]$,

$$\begin{aligned}
&D_{i_1}F_1x + D_{i_1}B_2D_{i_2}F_2x + D_{i_1}B_2D_{i_2}B_3\mathrm{sat}(v_3) + D_{i_1}B_2D_{i_2}^-H_2(i_1, i_2)x + D_{i_1}^-H_1(i_1)x\\
&\in \mathrm{co}\Big\{D_{i_1}F_1x + D_{i_1}B_2D_{i_2}F_2x + D_{i_1}B_2D_{i_2}B_3D_{i_3}F_3x\\
&\qquad + D_{i_1}B_2D_{i_2}B_3D_{i_3}B_4\mathrm{sat}(v_4) + D_{i_1}B_2D_{i_2}B_3D_{i_3}^-H_3(i_1, i_2, i_3)x\\
&\qquad + D_{i_1}B_2D_{i_2}^-H_2(i_1, i_2)x + D_{i_1}^-H_1(i_1)x : \ i_3 \in I[1, 2^{m_3}]\Big\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathrm{sat}(v_1) \in \mathrm{co}\Bigg\{&\sum_{k=1}^{3}\left(\prod_{l=1}^{k-1} D_{i_l}B_{l+1}\right)D_{i_k}F_kx + \prod_{k=1}^{3}(D_{i_k}B_{k+1})\,\mathrm{sat}(v_4)\\
&+ \sum_{k=1}^{3}\left(\prod_{l=1}^{k-1} D_{i_l}B_{l+1}\right)D_{i_k}^-H_k(i_k)x : \quad i_k \in I[1, 2^{m_k}], k \in I[1, 3]\Bigg\}.
\end{aligned}$$

Further considering $v_4, \ v_5, \ \cdots, \ v_q$ in a similar way, we can finally obtain the convex hull representation (2.26). □

Remark 2.2.4 *In [21], a two-layer nestedly saturated linear feedback is treated by a convex hull representation, with a different format and notation but otherwise equivalent to (2.27). Lemma 2.2.6 is a generalization of the result in [21] for multiple layer saturation functions.*

Although the convex hull representation (2.26) contains more auxiliary matrices than (2.25), there still exist some common auxiliary matrices, such as the auxiliary matrices associated with the outmost layer saturation $H_1(i_1)$, $i_1 \in I[1, 2^{m_1}]$. Introducing as many auxiliary matrices as possible is an obvious and tractable approach to further reduction of the conservativeness of the convex hull representations (2.25) and (2.26). Define a set of auxiliary matrices $H_k(i_1, i_2, \cdots, i_q) \in \mathbf{R}^{m_k \times n}$, $(i_1, i_2, \cdots, i_q) \in \Pi$, $k \in I[1, q]$, where $\Pi = I[1, 2^{m_1}] \times I[1, 2^{m_2}] \times \cdots \times I[1, 2^{m_q}]$, for the kth layer saturation function. For a given k, there are $2^{\sum_{r=1}^{q} m_r}$ such auxiliary matrices. Following the approach to expressing the saturated linear feedback on the convex hull of a group of auxiliary linear feedbacks, as described in Lemma 2.2.2, we can establish the following lemma that provides a similar treatment for the nested saturation function found in (2.18).

Lemma 2.2.7 *For an* $x \in \mathbf{R}^n$*, if* $x \in \mathcal{L}(H_k(i_1, i_2, \cdots, i_q))$*,* $k \in I[1, q]$*,* $(i_1, i_2, \cdots, i_q) \in \Pi$*, then*

$$\begin{aligned} &\mathrm{sat}(F_1 x + B_2 \mathrm{sat}(F_2 x + B_3 \mathrm{sat}(F_3 x + \ldots + B_q \mathrm{sat}(F_q x)))) \\ &\in \mathrm{co}\left\{ \sum_{k=1}^{q} \left(\prod_{l=1}^{k-1} D_{i_l} B_{l+1} \right) D_{i_k} F_k x \right. \\ &\qquad \left. + \sum_{k=1}^{q} \left(\prod_{l=1}^{k-1} D_{i_l} B_{l+1} \right) D_{i_k}^{-} H_k(i_1, i_2, \ldots, i_q) x : (i_1, i_2, \ldots, i_q) \in \Pi \right\}, \end{aligned} \tag{2.28}$$

where we have defined $\prod_{l=1}^{0} D_{i_l} B_{l+1} = I$*.*

Proof Let $v_1, v_2, \cdots, v_q$ be as defined in the proof of Lemma 2.2.5. Since $x \in \mathcal{L}(H_k(i_1, i_2, \cdots, i_q))$, $(i_1, i_2, \cdots, i_q) \in \Pi$, $k \in I[1, q]$, for any nonnegative α_{i_r}, $i_r \in I[1, 2^{m_r}]$, $r \in I[1, q]$, satisfying $\sum_{i_r=1}^{2^{m_r}} \alpha_{i_r} = 1$, $r \in I[1, q]$, we have $x \in \mathcal{L}(\mathcal{G}_k)$, where

$$\begin{aligned} \mathcal{G}_k &= \sum_{i_{k+1}=1}^{2^{m_{k+1}}} \alpha_{i_{k+1}} \cdots \sum_{i_q=1}^{2^{m_q}} \alpha_{i_q} H_k(i_1, i_2, \cdots, i_q), \ k \in I[1, q-1], \\ \mathcal{G}_q &= H_q(i_1, i_2, \cdots, i_q). \end{aligned}$$

By Lemma 2.2.2, for each $k \in I[1, q]$, there exist a set of nonnegative α_{i_k}, $i_k \in I[1, 2^{m_k}]$, satisfying $\sum_{i_k=1}^{2^{m_k}} \alpha_{i_k} = 1$, such that

$$\mathrm{sat}(v_k) = \sum_{i_k=1}^{2^{m_k}} \alpha_{i_k} \left(D_{i_k} v_k + D_{i_k}^{-} \mathcal{G}_k x \right), \quad k \in I[1, q].$$

Then we have

$$\begin{aligned} \mathrm{sat}(v_1) = &\sum_{i_1=1}^{2^{m_1}} \alpha_{i_1} D_{i_1} F_1 x + \sum_{i_1=1}^{2^{m_1}} \alpha_{i_1} D_{i_1} B_2 \sum_{i_2=1}^{2^{m_2}} \alpha_{i_2} D_{i_2} F_2 x + \cdots \\ &+ \sum_{i_1=1}^{2^{m_1}} \alpha_{i_1} D_{i_1} B_2 \sum_{i_2=1}^{2^{m_2}} \alpha_{i_2} D_{i_2} \cdots B_q \sum_{i_q=1}^{2^{m_q}} \alpha_{i_q} D_{i_q} F_q x \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i_1=1}^{2^{m_1}}\alpha_{i_1}D_{i_1}B_2\sum_{i_2=1}^{2^{m_2}}\alpha_{i_2}D_{i_2}\cdots B_q\sum_{i_q=1}^{2^{m_q}}\alpha_{i_q}D_{i_q}^{-}H_q(i_1,i_2,\cdots,i_q)x\\
&+\sum_{i_1=1}^{2^{m_1}}\alpha_{i_1}D_{i_1}B_2\sum_{i_2=1}^{2^{m_2}}\alpha_{i_2}D_{i_2}\cdots B_q\sum_{i_{q-1}=1}^{2^{m_{q-1}}}\alpha_{i_{q-1}}D_{i_{q-1}}^{-}\\
&\times\sum_{i_q=1}^{2^{m_q}}\alpha_{i_q}H_{q-1}(i_1,i_2,\cdots,i_q)x+\cdots\\
&+\sum_{i_1=1}^{2^{m_1}}\alpha_{i_1}D_{i_1}^{-}\sum_{i_2=1}^{2^{m_2}}\alpha_{i_2}\cdots\sum_{i_q=1}^{2^{m_q}}\alpha_{i_q}H_1(i_1,i_2,\cdots,i_q)x\\
=&\sum_{i_1=1}^{2^{m_1}}\alpha_{i_1}\sum_{i_2=1}^{2^{m_2}}\alpha_{i_2}\cdots\sum_{i_q=1}^{2^{m_q}}\alpha_{i_q}\left(\sum_{k=1}^{q}\left(\prod_{l=1}^{k-1}D_{i_l}B_{l+1}\right)D_{i_k}F_kx\right.\\
&\left.+\sum_{k=1}^{q}\left(\prod_{l=1}^{k-1}D_{i_l}B_{l+1}\right)D_{i_k}^{-}H_k(i_1,i_2,\cdots,i_q)x\right),\\
=&\sum_{i_1=1}^{2^{m_1}}\sum_{i_2=1}^{2^{m_2}}\cdots\sum_{i_q=1}^{2^{m_q}}\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_q}\left(\sum_{k=1}^{q}\left(\prod_{l=1}^{k-1}D_{i_l}B_{l+1}\right)D_{i_k}F_kx\right.\\
&\left.+\sum_{k=1}^{q}\left(\prod_{l=1}^{k-1}D_{i_l}B_{l+1}\right)D_{i_k}^{-}H_k(i_1,i_2,\cdots,i_q)x\right),
\end{aligned}
$$

which is equivalent to (2.28), since for all $i_k \in I[1, 2^{m_k}]$, $k \in I[1, q]$, we have $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_q} \in [0, 1]$ and

$$\sum_{i_1=1}^{2^{m_1}}\sum_{i_2=1}^{2^{m_2}}\cdots\sum_{i_q=1}^{2^{m_q}}\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_q} = 1.$$

□

In this generalized convex hull representation of a nested saturated linear feedback, as given in Lemma 2.2.7, each vertex of the convex hull is associated with a separate set of auxiliary linear feedbacks $H_k(i_1, i_2, \cdots, i_q)$. On the other hand, a group of vertices of the convex hull proposed by [21, 105, 107] share one common auxiliary linear feedback. If we set $H_k(i_1, i_2, \cdots, i_q) = H_k(i_1, i_2, \cdots, i_k)$, Lemma 2.2.7 will be equivalent to Theorem 2 in [21].

Note that each of the convex hulls in (2.25)–(2.28) has $2^{\sum_{k=1}^{q} m_k}$ vertices. However, some vertices expressed in different auxiliary linear feedbacks are actually the same since auxiliary matrices $H_s(i_1, i_2, \cdots, i_q)$, $s \in I[k+1, q]$ will not appear in the expression if the associated D_{i_k}'s are zero matrices. As with the improved convex hull representation (2.13) of the single-layer saturated linear feedback, the convex hull representations (2.25)–(2.28) also contain some redundant elements of auxiliary feedback matrices $H_k(i_1, i_2, \cdots, i_q)$'s. Here we consider (2.28). Let $\mathcal{D}_{i_k}^-$ be the $D_{i_k}^-$ with its zero columns removed, and we have $\mathcal{D}_{i_k}^- \in \mathbf{R}^{n \times p_{i_k}}$, where p_{i_k} is the number of nonzero columns of $\mathcal{D}_{i_k}^-$. Let $\mathcal{H}_k(i_1, i_2, \cdots, i_q) \in \mathbf{R}^{p_{i_k} \times n}$, $k \in I[1, q]$, be the $H_i(i_1, i_2, \cdots, i_q)$ with its columns that are associated with the zero rows of $D_{i_k}^-$ removed. Moreover, we define

$$
\begin{aligned}
\Xi_1 &= \mathcal{H}_1, \\
\Xi_k &= \left\{ \sum_{l=1}^{k-1} \left(\prod_{j=1}^{l-1} D_{i_j} B_{j+1} \right) \left(D_{i_l} F_l + D_{i_l}^- \mathcal{H}_l(i_1, i_2, \cdots, i_{k-1}) \right) \right. \\
&\quad \left. + \prod_{l=1}^{k-1} (D_{i_l} B_{l+1})\, \mathcal{H}_l(i_1, i_2, \cdots, i_{k-1}) : \ i_l \in I[1, 2^{m_l} - 1],\ l \in I[1, k-1] \right\}, \\
&\quad k = 2, 3, \cdots, q,
\end{aligned}
$$

where $\mathcal{H}_1 \in \mathbf{R}^{m_1 \times n}$, $\mathcal{H}_l(i_1, i_2, \cdots, i_{k-1}) \in \mathbf{R}^{p_{i_l} \times n}$. Let $\Pi_\circ = I[2, 2^{m_1}] \times I[2, 2^{m_2}] \times \cdots \times I[2, 2^{m_q}]$. A simplified convex hull representation of the nestedly saturated linear feedback can be written as

$$
\begin{aligned}
& \operatorname{sat}(F_1 x + B_2 \operatorname{sat}(F_2 x + B_3 \operatorname{sat}(F_3 x + \ldots + B_q \operatorname{sat}(F_q x)))) \\
& \in \operatorname{co} \left\{ \Xi_k x,\ \sum_{k=1}^{q} \left(\prod_{l=1}^{k-1} D_{i_l} B_{l+1} \right) D_{i_k} F_k x + \sum_{k=1}^{q} \left(\prod_{l=1}^{k-1} D_{i_l} B_{l+1} \right) \right. \\
& \qquad \left. \times \mathcal{D}_{i_k}^- \mathcal{H}_k(i_1, i_2, \ldots, i_q) x : \ (i_1, i_2, \ldots, i_q) \in \Pi_\circ, k \in I[1, q] \right\}. \qquad (2.29)
\end{aligned}
$$

This resulting convex hull does not contain any repetitive vertices as in the convex hull (2.28). Moreover, the convex hull in (2.29) has M_1 vertices and nM_2 elements of the auxiliary feedback matrices, where

$$
\begin{aligned}
M_1 &= \prod_{l=1}^{q} (2^{m_l} - 1) + \sum_{k=1}^{q} \prod_{l=0}^{k-1} (2^{m_l} - 1), \\
M_2 &= m_1 + \sum_{k=2}^{q} \prod_{l=1}^{k-1} (2^{m_l} - 1) \left(m_k + \sum_{l=1}^{k-1} (2^{m_l} - m_l) \right) + \prod_{k=1}^{q} (2^{m_k} - 1).
\end{aligned}
$$

In what follows, we will consider the problem of estimating the domain of attraction of system (2.18) by using the improved convex hull representation (2.28) of the nested saturated linear feedback, in which each vertex of the convex hull has been assigned as large a set of independent auxiliary feedback matrices as possible. The following theorem establishes a set of sufficient conditions with less conservativeness for the contractive invariance of an ellipsoid.

Theorem 2.2.4 *Let $P \in \mathbf{R}^{n\times n}$ be a positive definite matrix and $Q = P^{-1}$. If there exist matrices $Z_k(i_1, i_2, \cdots, i_q) \in \mathbf{R}^{m_k\times n}$, $k \in I[1, q]$, $(i_1, i_2, \cdots, i_q) \in \Pi$, such that*

$$\begin{aligned}\mathrm{He}\Bigg(\Bigg(A + \sum_{k=1}^{q}\left(\prod_{l=1}^{k} B_l D_{i_l}\right) F_k Q \\ + \sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} B_l D_{i_l}\right) B_k D_{i_k}^{-} Z_k(i_1, i_2, \cdots, i_q)\Bigg)\Bigg) < 0, \quad (i_1, i_2, \cdots, i_q) \in \Pi,\end{aligned} \tag{2.30}$$

and

$$\begin{bmatrix} 1 & z_{j_k}(i_1, i_2, \cdots, i_q) \\ \star & Q \end{bmatrix} \geq 0, \quad j_k \in I[1, m_k], k \in I[1, q], (i_1, i_2, \cdots, i_q) \in \Pi, \tag{2.31}$$

are satisfied, where $z_{j_k}(i_1, i_2, \cdots, i_q)$ is the j_kth row of matrix $Z_k(i_1, i_2, \cdots, i_q)$, then the origin of system (2.18) is exponentially stable, and $\mathcal{E}(P)$ is a contractively invariant set of system (2.18).

Proof Let $H_k(i_1, i_2, \cdots, i_q) = Z_k(i_1, i_2, \cdots, i_q)Q^{-1}, k \in I[1, q], (i_1, i_2, \cdots, i_q) \in \Pi$. From the proof of Theorem 2.1.1, one can verify that Inequalities (2.31) imply that $\mathcal{E}(P) \subseteq \mathcal{L}(H_k(i_1, i_2, \cdots, i_q))$, $k \in I[1, q]$, $(i_1, i_2, \cdots, i_q) \in \Pi$. Choose a quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$. By Lemma 2.2.7, the time derivative of $V(x)$ along the trajectory of system (2.18) within $\mathcal{E}(P)$, is given by

$$\begin{aligned}\dot{V}(x) &= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + 2x^{\mathrm{T}}PB_1\mathrm{sat}(F_1x + B_2\mathrm{sat}(F_2x + B_3\mathrm{sat}(F_3x + \cdots + B_q\mathrm{sat}(F_qx)))) \\ &\leq x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + 2\max_{(i_1,i_2,\cdots,i_q)\in\Pi} \\ &\quad \left\{x^{\mathrm{T}}PB_1\left(\sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} D_{i_l}B_{l+1}\right) D_{i_k}F_kx + \sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} D_{i_l}B_{l+1}\right) D_{i_k}^{-}H_k(i_1, i_2, \cdots, i_q)x\right)\right\} \\ &= \max_{(i_1,i_2,\cdots,i_q)\in\Pi}\Bigg\{x^{\mathrm{T}}\mathrm{He}\Bigg(P\Bigg(A + B_1\sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} D_{i_l}B_{l+1}\right) D_{i_k}F_k \\ &\qquad + B_1\sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} D_{i_l}B_{l+1}\right) D_{i_k}^{-}H_k(i_1, i_2, \cdots, i_q)\Bigg)\Bigg)x\Bigg\} \\ &= \max_{(i_1,i_2,\cdots,i_q)\in\Pi} x^{\mathrm{T}}\mathrm{He}\Bigg(P\Bigg(A + \sum_{k=1}^{q}\left(\prod_{l=1}^{k} B_l D_{i_l}\right) F_k \\ &\qquad + \sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} B_l D_{i_l}\right) B_k D_{i_k}^{-} H_k(i_1, i_2, \cdots, i_q)\Bigg)\Bigg)x.\end{aligned}$$

Note that matrix inequalities (2.30) are equivalent to the negative definiteness of matrices $\mathrm{He}(P(A + \sum_{k=1}^{q}(\prod_{l=1}^{k} B_l D_{i_l})F_k + \sum_{k=1}^{q}(\prod_{l=1}^{k-1} B_l D_{i_l})B_k D_{i_k}^{-} H_k(i_1, i_2, \cdots, i_q)))$, $(i_1, i_2, \cdots, i_q) \in \Pi$. Therefore, if (2.30) hold, there must exist a scalar $\beta > 0$ such that $\dot{V}(x) < -\beta x^{\mathrm{T}} P x$ for any $x \in \mathcal{E}(P) \setminus \{0\}$, which implies that $V(x(t)) < e^{-\beta t} V(x(0))$. Thus, system (2.18) is exponentially stable at the origin with $\mathcal{E}(P)$ contained in the domain of attraction. □

2.2.3 Piecewise Linear Functions with Multiple Bends

In this subsection, we will present a convex hull based approach to handling an odd symmetric piecewise linear function with multiple bends, which is a generalization of the standard saturation function. A linear feedback subject to such a nonlinearity is obviously more complex than a saturated linear feedback. A one-dimensional piecewise linear function with one bend,

$$\psi(u) = \begin{cases} k_0 u, & \text{if } u \in [0, b_1], \\ k_1 u + c_1, & \text{if } u \in (b_1, +\infty), \end{cases}$$

whose values for $u < 0$ can be determined by its odd symmetry, can be equivalently rewritten in terms of a standard saturation as,

$$\psi(u) = k_1 u + c_1 \mathrm{sat}\left(\frac{k_0 - k_1}{c_1} u\right).$$

In what follows, we will consider a one-dimensional piecewise linear function with N bends, defined by

$$\psi(u) = \begin{cases} k_0 u, & \text{if } u \in [0, b_1], \\ k_1 u + c_1, & \text{if } u \in (b_1, b_2], \\ \quad\vdots & \quad\quad\vdots \\ k_N u + c_N, & \text{if } u \in (b_N, +\infty), \end{cases} \tag{2.32}$$

where $k_0 < k_1 < \cdots < k_N$. An illustration of $\psi(u)$ can be found in Figure 2.4.

Define a set of piecewise linear functions with one bend

$$\psi^l(u) = \begin{cases} k_0 u, & \text{if } u \in [0, g_l], \\ k_l u + c_l, & \text{if } u \in (g_l, +\infty), \end{cases} \quad l = 1, 2, \cdots, N,$$

as shown in Figure 2.4, where $g_1 = b_1$, and

$$g_l = \frac{c_l}{k_0 - k_l}, \quad l = 2, 3, \cdots, N. \tag{2.33}$$

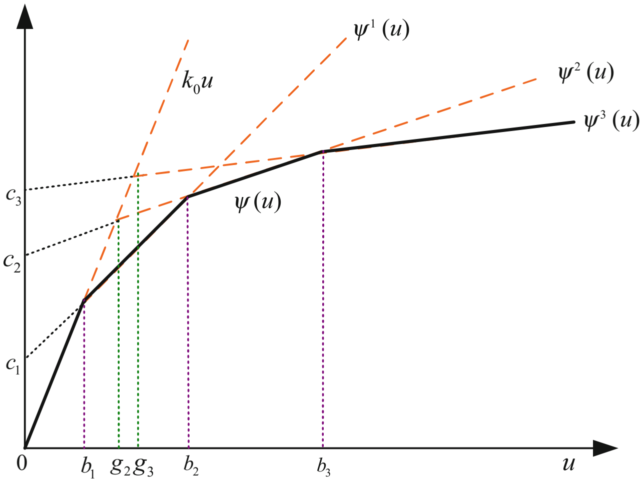

Fig. 2.4 A geometric illustration of a piecewise linear function with two bends.

It is clear in Figure 2.4 that

$$\psi(u) = \begin{cases} \psi^1(u), & \text{if } u \in [0, b_2], \\ \psi^2(u), & \text{if } u \in (b_2, b_3], \\ \quad\vdots & \quad\vdots \\ \psi^N(u), & \text{if } u \in (b_N, +\infty). \end{cases}$$

Then we have

$$\begin{aligned} \psi(u) &\in \mathrm{co}\left\{\psi^l(u) : \ l \in I[1,N]\right\} \\ &= \mathrm{co}\left\{k_l u + c_l \mathrm{sat}\left(\frac{k_0 - k_l}{c_l} u\right) : \ l \in I[1,N]\right\}. \end{aligned} \tag{2.34}$$

The convex hull in (2.34) actually forms a generalized sector whose boundaries are a group of linear and saturation functions [41].

We next consider the one-dimensional feedback $\psi(Fx)$, where $F \in \mathbf{R}^{1\times n}$. The following lemma establishes a convex hull representation of $\psi(Fx)$.

Lemma 2.2.8 ([41]) *Let $F, H_l \in \mathbf{R}^{1\times n}$, $l \in I[1,N]$. For an $x \in \mathbf{R}^n$, if $x \in \bigcap_{l=1}^N \mathcal{L}(H_l)$, then*

$$\psi(Fx) \in \mathrm{co}\left\{k_0 Fx, (k_l F + c_l H_l)x : \ l \in I[1,N]\right\}. \tag{2.35}$$

We next consider the multiple-dimensional piecewise linear function $\psi(u)$, where $u = [u_1 \;\; u_2 \;\; \cdots \;\; u_m]^{\mathrm{T}} \in \mathbf{R}^m$, and

$$\psi(u) = [\psi_1(u_1) \;\; \psi_2(u_2) \;\; \cdots \;\; \psi_m(u_m)]^{\mathrm{T}}, \quad \psi_j(u_j) \in \mathbf{R},\ j \in I[1,m]. \tag{2.36}$$

Each scalar function $\psi_j(u_j)$ is continuous, odd symmetric, and is defined for $u_j \geq 0$ as

$$\psi_j(u_j) = \begin{cases} k_{j0}u_j, & \text{if } u_j \in [0, b_{j1}], \\ k_{j1}u_j + c_{j1}, & \text{if } u_j \in (b_{j1}, b_{j2}], \\ \quad\vdots & \quad\vdots \\ k_{jN_j}u_j + c_{jN_j}, & \text{if } u_j \in (b_{jN_j}, +\infty), \end{cases} \tag{2.37}$$

where $k_{j0} < k_{j1} < \cdots < k_{jN_j}, j \in I[1,m]$.

Denote

$$\psi_j^{l_j}(u_j) = k_{jl_j}u_j + c_{l_j}\mathrm{sat}\left(\frac{k_{j0} - k_{jl_j}}{c_{l_j}}u_j\right).$$

From the analysis above, we have

$$\psi_j(u_j) \in \mathrm{co}\left\{\psi_j^{l_j}(u_j) :\ l_j \in I[1,N_j]\right\}.$$

Let $K = N_1N_2\cdots N_m$ and let each $q \in I[1,K]$ be associated with a different set of integers $(l_1, l_2, \cdots, l_m)$, $l_j \in I[1,N_j]$, $j \in I[1,m]$. Denote $\psi_q(u) = \left[\psi_1^{l_1}(u_1) \;\; \psi_2^{l_2}(u_2) \;\; \cdots \;\; \psi_m^{l_m}(u_m)\right]^{\mathrm{T}}$. Thus, we have

$$\begin{aligned} \psi(u) &\in \mathrm{co}\left\{\psi_q(u) :\ q \in I[1,K]\right\} \\ &= \left\{\Pi_q u + \Omega_q \mathrm{sat}(\Gamma_q u) :\ q \in I[1,K]\right\}, \end{aligned}$$

where

$$\begin{aligned} \Pi_q &= \mathrm{diag}\left\{k_{1l_1}u_1, k_{2l_2}u_2, \cdots, k_{ml_m}u_m\right\}, \\ \Omega_q &= \mathrm{diag}\left\{c_{l_1}, c_{l_2}, \cdots, c_{l_N}\right\}, \\ \Gamma_q &= \mathrm{diag}\left\{\frac{k_{10} - k_{1l_1}}{c_{l_1}}, \frac{k_{20} - k_{2l_2}}{c_{l_2}}, \cdots, \frac{k_{m0} - k_{ml_m}}{c_{l_m}}\right\}. \end{aligned}$$

Clearly, the convex hull representation of the piecewise linear feedback $\psi(Fx)$ can be stated in the following lemma.

Lemma 2.2.9 ([43]) *Let $H_q \in \mathbf{R}^{m\times n}$, $q \in I[1, K]$. For any $x \in \bigcap_{q=1}^{K} \mathcal{L}(H_q)$, we have*

$$\psi(Fx) \in \mathrm{co}\Big\{D_i(\Pi_q + \Omega_q)Fx + D_i^-(\Pi_q F + H_q)x : \ q \in I[1, K], \ i \in I[1, 2^m]\Big\}. \tag{2.38}$$

As with the convex hull representation (2.10), each H_q in (2.38) is shared by a group of vertices. This will lead to conservativeness in treating piecewise linear functions. In order to reduce such conservativeness, we assign an independent auxiliary matrix for each vertex of the convex hull in (2.38). Then we have the following lemma.

Lemma 2.2.10 *Let $H_{qi} \in \mathbf{R}^{m\times n}$, $q \in I[1, K]$, $i \in I[1, 2^m]$. For any $x \in \bigcap_{i=1}^{2^m} \bigcap_{q=1}^{K} \mathcal{L}(H_{qi})$, we have*

$$\psi(Fx) \in \mathrm{co}\Big\{D_i(\Pi_q + \Omega_q)Fx + D_i^-(\Pi_q F + H_{qi})x : \ q \in I[1, K], \ i \in I[1, 2^m]\Big\}. \tag{2.39}$$

Consider the following linear system subject to a piecewise linear function in the input,

$$\dot{x} = Ax + B\psi(Fx), \tag{2.40}$$

where $x \in \mathbf{R}^n$, $F \in \mathbf{R}^{m\times n}$ and $\psi(\cdot)$ is as defined in (2.36). Here, we focus on the problem of estimating the domain of attraction of system (2.40). Regarding the convex hull representation (2.39) of $\psi(Fx)$, similarly as Theorem 2.2.2, we can establish a set of sufficient conditions under which the ellipsoid $\mathcal{E}(P)$ is an estimate of the domain of attraction of system (2.40). These conditions are summarized in the following theorem.

Theorem 2.2.5 *If there exist a positive definite matrix $P \in \mathbf{R}^{n\times n}$, and matrices $H_{qi} \in \mathbf{R}^{m\times n}$, $q \in I[1, K]$, $i \in I[1, 2^m]$, such that*

$$\mathrm{He}\big(P(A + BD_i(\Pi_q + \Omega_q)F + BD_i^-(\Pi_q F + H_{qi}))\big) < 0, \quad q \in I[1, K], \ i \in I[1, 2^m], \tag{2.41}$$

and

$$\mathcal{E}(P) \subseteq \bigcap_{i=1}^{2^m} \bigcap_{q=1}^{K} \mathcal{L}(H_{qi}), \tag{2.42}$$

then the ellipsoid $\mathcal{E}(P)$ is contractively invariant and can be used as an estimate of the domain of attraction of system (2.40).

Proof The proof is similar to that of Theorem 2.2.2 and is omitted here. □

We next consider a special case of system (2.40) where the function ψ is a one-dimensional piecewise linear function with N bends. By using the convex hull representation (2.35) for one-dimensional piecewise linear feedback, we have the following theorem which is special case of Theorem 2.2.5.

Theorem 2.2.6 *Let $\psi(\cdot)$ be as defined in (2.32). An ellipsoid $\mathcal{E}(P)$ is contractively invariant for system (2.40) if*

$$\text{He}\,(P(A + k_0 BF)) < 0, \tag{2.43}$$

and there exist matrices $H_l \in \mathbf{R}^{m\times n}$, $l \in I[1,N]$, such that

$$\text{He}\,(P(A + k_l BF + c_l BH_l)) < 0, \quad l \in I[1,N], \tag{2.44}$$

and

$$\mathcal{E}(P) \subseteq \bigcap_{l=1}^{N} \mathcal{L}(H_l). \tag{2.45}$$

Remark 2.2.5 *For system (2.40) with a one-dimensional N-bend piecewise linear function ψ as defined in (2.32), Theorem 2.2.6 presents a set of sufficient conditions under which the ellipsoid $\mathcal{E}(P)$ is contractively invariant. It has been proven in [41] that these conditions are also necessary.*

2.3 Optimization Problems

2.3.1 Single-Layer Saturated Linear Feedback

Theorems 2.1.1–2.2.2 present several sets of conditions, by employing different treatments of the saturated linear feedback, under which an ellipsoid $\mathcal{E}(P)$ is contractively invariant and can be used as an estimate of the domain of attraction of system (2.1). Among all the ellipsoids satisfying the contractively invariant conditions, we are interested in the one with the "largest" size. To find the "largest" invariant ellipsoid, we can formulate the following optimization problem,

$$\begin{aligned} &\max \quad s(\mathcal{E}(P)) \\ &\text{s.t. a) Inequalities (2.3) – (2.4), or} \\ &\qquad \text{a) Inequalities (2.8) – (2.9), or} \\ &\qquad \text{a) Inequalities (2.11) – (2.12), or} \\ &\qquad \text{a) Inequalities (2.16) – (2.17),} \end{aligned} \tag{2.46}$$

where the optimization criterion, the function $s(\mathcal{E}(P))$, is a measure of the size of the ellipsoid $\mathcal{E}(P)$, such as the length of one of its axes and the relative size with respect to a shape reference set.

- Each eigenvalue of matrix P corresponds to the length of one ellipsoid axis. The minimization of $\mathrm{tr}(P)$, the sum of all eigenvalues of P, represents the maximization of $\mathcal{E}(P)$ [92]. A smaller $\mathrm{tr}(P)$ implies a larger ellipsoid. Thus, we can set $s(\mathcal{E}(P)) = -\mathrm{tr}(P)$ as the optimization criterion of (2.46).
- We can also set $s(\mathcal{E}(P)) = \alpha$, for the largest α such that $\alpha\mathcal{X}_{\mathrm{R}} \subseteq \mathcal{E}(P)$, where $\mathcal{X}_{\mathrm{R}}$ is a given set, referred to as the shape reference set. If $\mathcal{X}_{\mathrm{R}}$ is a polyhedron, that is, $\mathcal{X}_{\mathrm{R}} = \mathrm{co}\{x_1, x_2, \cdots, x_p\}$, where $x_l \in \mathbf{R}^n$, $l \in I[1,p]$, then the maximization of α implies enlarging $\mathcal{E}(P)$ along the directions of the vectors $x_1, x_2, \cdots, x_p$. Another convenient choice of $\mathcal{X}_{\mathrm{R}}$ is an ellipsoid.

In this chapter, we will adopt the largest α such that $\alpha\mathcal{X}_{\mathrm{R}} \subseteq \mathcal{E}(P)$ with $\mathcal{X}_{\mathrm{R}} = \mathrm{co}\{x_1, x_2, \cdots, x_p\}$ as the optimization criterion of (2.46). Note that $\alpha\mathcal{X}_{\mathrm{R}} \subseteq \mathcal{E}(P)$ is equivalent to

$$x_l^{\mathrm{T}} P x_l \leq \frac{1}{\alpha^2}, \quad l \in I[1,p].$$

from which, by Schur complement, it follows that

$$\begin{bmatrix} \frac{1}{\alpha^2} & x_l^{\mathrm{T}} \\ x_l & Q \end{bmatrix} \geq 0, \quad l \in I[1,p],$$

where $Q = P^{-1}$. If $\mathcal{X}_{\mathrm{R}} = \mathcal{E}(P_0)$ for some $P_0 > 0$, then, $\alpha\mathcal{X}_{\mathrm{R}} \subseteq \mathcal{E}(P)$ is equivalent to $\alpha^2 P \leq P_0$, which, by the Schur complement, is further equivalent to

$$\begin{bmatrix} \frac{1}{\alpha^2}P_0 & I \\ I & Q \end{bmatrix} \geq 0.$$

In what follows, we will formulate four optimization problems, each of which corresponds to one of the four treatments (2.2), (2.7), (2.10), and (2.13), presented in Sections 2.1 and 2.2.

- If we use the regional sector condition (2.2) to deal with the saturated linear feedback, the optimization problem (2.46) can be rewritten as,

$$\begin{aligned} &\min_{Q>0,\ W>0,\ Z} \quad \gamma \\ &\text{s.t. a)} \begin{bmatrix} \gamma & x_l^{\mathrm{T}} \\ x_l & Q \end{bmatrix} \geq 0, \quad l \in I[1,p], \\ &\quad\ \ \text{b)} \begin{bmatrix} \mathrm{He}((A+BF)Q) & -BW + QF^{\mathrm{T}} - Z^{\mathrm{T}} \\ \star & -2W \end{bmatrix} \leq 0, \\ &\quad\ \ \text{c)} \begin{bmatrix} 1 & z_j \\ z_j^{\mathrm{T}} & Q \end{bmatrix} \geq 0, \quad j \in I[1,m], \end{aligned} \tag{2.47}$$

where $\gamma = \frac{1}{\alpha^2}$, matrix W is a diagonal matrix and z_j is the jth of matrix Z. Let the optimal solution of (2.47) be $(\gamma^{\mathrm{R}}, Q^{\mathrm{R}}, W^{\mathrm{R}}, Z^{\mathrm{R}})$. Then we have the optimal ellipsoid $\mathcal{E}(P^{\mathrm{R}})$, $P^{\mathrm{R}} = (Q^{\mathrm{R}})^{-1}$, and $\alpha^{\mathrm{R}} = \frac{1}{\sqrt{\gamma^{\mathrm{R}}}}$. The associated auxiliary matrix is $H^{\mathrm{R}} = Z^{\mathrm{R}}(Q^{\mathrm{R}})^{-1}$.

- If the saturated linear feedback $\mathrm{sat}(Fx)$ is expressed as

$$\mathrm{sat}(Fx) \in \mathrm{co}\left\{D_i^{\beta} Fx, \quad i \in I[1, 2^m]\right\},$$

shown in Lemma 2.1.2, the optimization problem (2.46) is given as,

$$\min_{P>0,\ \beta\in\mathbf{R}^m} \quad \gamma \tag{2.48}$$
$$\text{s.t. a)}\ x_l^{\mathrm{T}} P x_l \geq \gamma, \quad l \in I[1, p],$$
$$\text{b)}\ \mathrm{He}\left(P(A + BD_i^{\beta} F)\right) \leq 0, \quad i \in I[1, 2^m],$$
$$\text{c)}\ \begin{bmatrix} P & \frac{1}{\beta_j} f_j^{\mathrm{T}} \\ \frac{1}{\beta_j} f_j & 1 \end{bmatrix} \geq 0, \quad j \in I[1, m],$$

where $\gamma = \frac{1}{\alpha^2}$.

- If we use the convex hull (2.10) with a single auxiliary matrix to represent $\mathrm{sat}(Fx)$, the optimization problem (2.46) is given by

$$\min_{Q>0,\ Z} \quad \gamma \tag{2.49}$$
$$\text{s.t. a)}\ \begin{bmatrix} \gamma & x_l^{\mathrm{T}} \\ x_l & Q \end{bmatrix} \geq 0, \quad l \in I[1, p],$$
$$\text{b)}\ \mathrm{He}\left((A + BD_i F)Q + BD_i^{-} Z\right) \leq 0, \quad i \in I[1, 2^m],$$
$$\text{c)}\ \begin{bmatrix} 1 & z_j \\ z_j^{\mathrm{T}} & Q \end{bmatrix} \geq 0, \quad j \in I[1, m],$$

where $\gamma = \frac{1}{\alpha^2}$. Denote the optimal solution as $(\gamma^{\mathrm{C}}, Q^{\mathrm{C}}, Z^{\mathrm{C}})$. Then the optimal ellipsoidal estimate is $\mathcal{E}(P^{\mathrm{C}})$, where $P^{\mathrm{C}} = (Q^{\mathrm{C}})^{-1}$, and the optimal value of α is $\alpha^{\mathrm{C}} = \frac{1}{\sqrt{\gamma^{\mathrm{C}}}}$. The associated auxiliary matrix can be computed as $H^{\mathrm{C}} = Z^{\mathrm{C}}(Q^{\mathrm{C}})^{-1}$.

- If the improved convex hull representation (2.13) with multiple auxiliary matrices is employed, the optimization problem (2.46) can be rewritten as,

$$\min_{Q>0,\ Z_i, i\in I[1,2^m]} \quad \gamma \tag{2.50}$$

$$\text{s.t. a)} \begin{bmatrix} \gamma & x_l^{\mathrm{T}} \\ x_l & Q \end{bmatrix} \geq 0, \quad l \in I[1,p],$$

$$\text{b) } \mathrm{He}\left((A + BD_iF + BD_i^- Z_i)Q\right) \leq 0, \quad i \in I[1,2^m],$$

$$\text{c)} \begin{bmatrix} 1 & z_{ij} \\ z_{ij}^{\mathrm{T}} & Q \end{bmatrix} \geq 0, \quad j \in I[1,m],\ i \in I[1,2^m],$$

where z_{ij} is the jth row of matrix Z_i. Let the optimal solution be $(\gamma^{\mathrm{I}}, Q^{\mathrm{I}}, Z_i^{\mathrm{I}})$. Then the optimal ellipsoid is $\mathcal{E}(P^{\mathrm{I}})$, where $P^{\mathrm{I}} = (Q^{\mathrm{I}})^{-1}$, and the optimal criterion is $\alpha^{\mathrm{I}} = \frac{1}{\sqrt{\gamma^{\mathrm{I}}}}$. The associated auxiliary matrices can be computed as $H_i^{\mathrm{I}} = Z_i^{\mathrm{I}}(Q^{\mathrm{I}})^{-1}$, $i \in I[1,2^m]$.

Remark 2.3.1 *Note that the matrix inequalities a)-c) of optimization problems (2.47), (2.49) and (2.50) are LMIs in Q, γ, W, and Z (or Z_i's). One can easily obtain their global optimal solutions by using the standard computation softwares. However, the optimization problem (2.48) is bilinear, that is, it is a BMI problem, since its constraints contain the product terms of two unknown matrices, such as $PBD_i^{\beta}F$. A commonly adopted approach to dealing with BMI problems is to develop LMI-based iterative algorithms, which however may require a large amount of computation. Note that the optimization problem (2.48) will reduce to an LMI-based problem if P or β is fixed. This observation has led to the following LMI-based iterative algorithm, originally proposed in [92], to solve the problem (2.48).*

Algorithm 2.3.1

- *Step 1. Initialize β.*
- *Step 2. Solve the optimization problem (2.48) with the fixed β, and denote the optimal solution as (γ_0, P^S).*
- *Step 3. Set $P = P^S$. Solve the optimization problem (2.48) with the fixed P. Denote the optimal solution as (γ_1, β^S).*
- *Step 4. If $|\gamma_0 - \gamma_1| < \delta$, where δ is a pre-determined tolerance, then, (γ_1, P^S, β^S) is a feasible solution of (2.48) and let $\alpha^S = \frac{1}{\sqrt{\gamma_1}}$, Stop. Else, let $\beta = \beta^S$, and go to Step 2.*

Since it has been proven that the treatment (2.7) is a special case of the both convex hull representations (2.10) and (2.13), the optimal value α^S obtained from Algorithm 2.3.1 must be smaller than or equal to α^C and α^I, that is, the estimate $\mathcal{E}(P^S)$ will be contained in the other two ellipsoidal estimates $\mathcal{E}(P^C)$ and $\mathcal{E}(P^I)$.

In what follows, we will discuss the computational complexity of the optimization problems (2.47)–(2.50). Since the optimization problem (2.48) is a BMI problem, it has the highest computational complexity when we use Algorithm 2.3.1 to solve it. We next focus on the other three optimization problems whose constraints are all LMIs. The numbers of constraints and decision variables determine the

Table 2.1 Numbers of lines and decision variables of LMIs in different optimization problems

Optimization problems	Number of lines	Number of decision variables
(2.47)	$n+m+(p+m)(n+1)$	$m+\frac{1}{2}n(n+1)+mn$
(2.49)	$n2^m+(p+m)(n+1)$	$\frac{1}{2}n(n+1)+mn$
(2.50)	$n2^m+(p+m2^m)(n+1)$	$\frac{1}{2}n(n+1)+mn2^{m-1}$

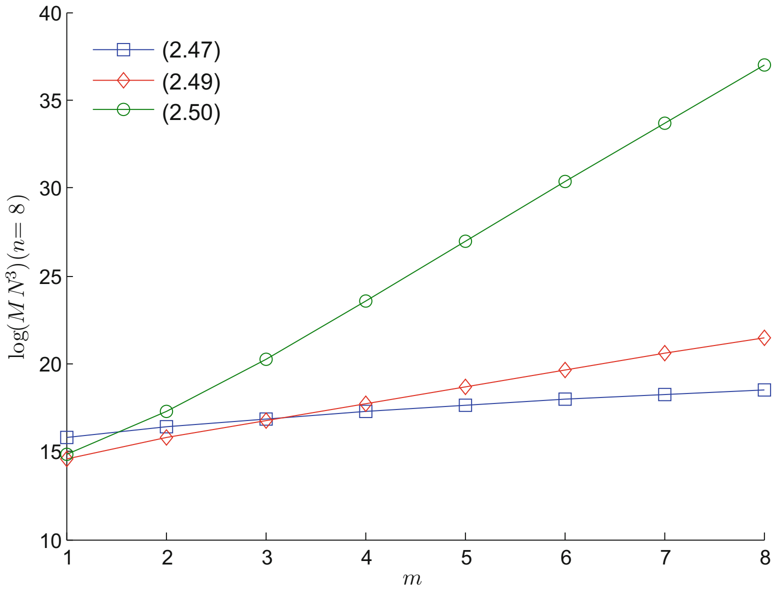

Fig. 2.5 A comparison of the computational complexity among different optimization problems ($n=8$).

computational complexity. The comparison results are shown in Table 2.1, where for the optimization problem (2.50) the irrelevant decision variables in matrices Z_i's have been excluded.

As shown in [93, 107], $\log(MN^3)$ can be used to measure the computational complexity of an LMI-based optimization problem, where M is the total number of the lines of LMIs and N is the total number of decisions variables of LMIs. For an easy comparison between the three optimization problems, we set $n = 8$ and $m = 4$, and observe, as shown in Figures 2.5 and 2.6, that the function $\log(MN^3)$ that characterizes the computational complexity increases as n and m become larger. Moreover, the optimization problem (2.50) has the highest computational complexity as its associated treatment of saturated linear feedback introduces multiple auxiliary matrices such that the LMIs in (2.50) contain the most decision variables that result in the least conservativeness among all the convex hull representations (2.7), (2.10) and (2.13). This implies that less conservativeness is achieved at the cost of higher computational complexity. Hence, a tradeoff should be considered

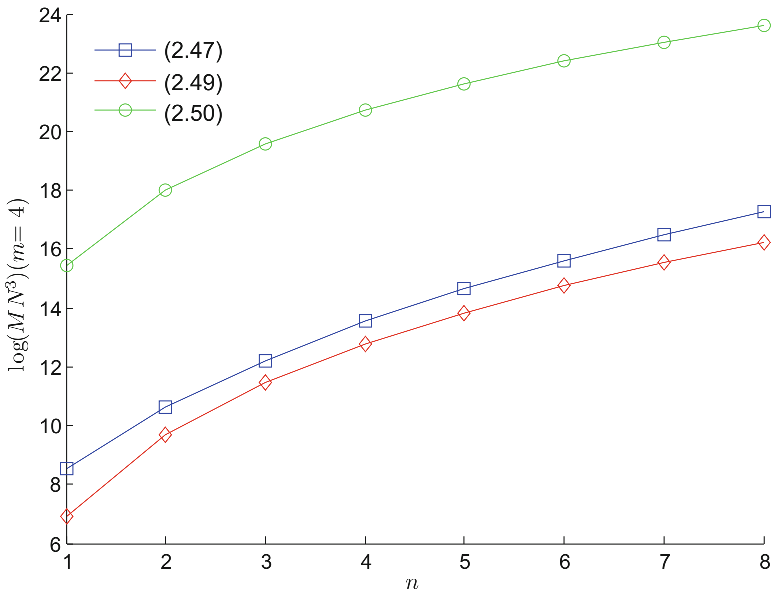

Fig. 2.6 A comparison of the computational complexity among different optimization problems ($m = 4$).

between the conservativeness of the results and computational complexity. For system (2.1) with lower dimensions of the states and inputs, we can solve (2.50) for a larger estimate of the domain of attraction. If the numbers of the states and inputs are large, we can choose to solve (2.47) or (2.49) to avoid excessive computation.

We next discuss the stabilization problem of designing a state feedback gain $F \in \mathbf{R}^{m\times n}$ such that the estimate of the domain of attraction of the closed-loop system (2.1) is as large as possible. To solve this stabilization problem, we treat F as an additional variable in the above optimization problems to obtain the optimal controller gain that maximizes the estimate of the domain of attraction. If we use the regional sector conditions to deal with the saturated linear feedback, the optimization problem for stabilization can be formulated, by setting $G = FQ$, as follows,

$$\min_{Q>0,\ W>0,\ G,\ Z} \quad \gamma \tag{2.51}$$

$$\text{s.t. a)} \quad \begin{bmatrix} \gamma & x_l^{\mathrm{T}} \\ x_l & Q \end{bmatrix} \geq 0, \quad l \in I[1,p],$$

$$\text{b)} \quad \begin{bmatrix} \mathrm{He}(AQ+BG) & -BW+G^{\mathrm{T}}-Z^{\mathrm{T}} \\ \star & -2W \end{bmatrix} \leq 0,$$

$$\text{c)} \quad \begin{bmatrix} 1 & z_j \\ z_j^{\mathrm{T}} & Q \end{bmatrix} \geq 0, \quad j \in I[1, m].$$

Let $(\gamma^{\mathrm{S}}, Q^{\mathrm{S}}, G^{\mathrm{S}}, Z^{\mathrm{S}})$ denote the optimal solution of (2.51). Then, the optimal controller gain $F^{\mathrm{S}} = G^{\mathrm{S}}(Q^{\mathrm{S}})^{-1}$ and the associated ellipsoidal estimate is $\mathcal{E}(P^{\mathrm{S}})$ with $P^{\mathrm{S}} = (Q^{\mathrm{S}})^{-1}$.

If we use the improved convex hull representation (2.13) and set $G = FQ$ as additional decision variables, the optimization problem is given by,

$$\min_{Q>0,\ G,\ Z_i, i \in I[1,2^m]} \quad \gamma \tag{2.52}$$

$$\text{s.t. a)} \quad \begin{bmatrix} \gamma & x_l^{\mathrm{T}} \\ x_l & Q \end{bmatrix} \geq 0, \quad l \in I[1, p],$$

$$\text{b)} \quad \mathrm{He}\left(AQ + BD_iG + BD_i^- Z_i\right) \leq 0, \quad i \in I[1, 2^m],$$

$$\text{c)} \quad \begin{bmatrix} 1 & z_{ij} \\ z_{ij}^{\mathrm{T}} & Q \end{bmatrix} \geq 0, \quad j \in I[1, m],\ i \in I[1, 2^m].$$

Let $(\gamma^{\mathrm{I}}, Q^{\mathrm{I}}, G^{\mathrm{I}}, Z_i^{\mathrm{I}})$ be the optimal solution of (2.52). If the optimization problem (2.52) is further constrained by $G = Z_i$, $i \in I[1, 2^m]$, it will reduce to the following problem,

$$\min_{Q>0,\ G} \quad \gamma \tag{2.53}$$

$$\text{s.t. a)} \quad \begin{bmatrix} \gamma & x_l^{\mathrm{T}} \\ x_l & Q \end{bmatrix} \geq 0, \quad l \in I[1, p],$$

$$\text{b)} \quad \mathrm{He}\,(AQ + BG) \leq 0,$$

$$\text{c)} \quad \begin{bmatrix} 1 & g_j \\ g_j^{\mathrm{T}} & Q \end{bmatrix} \geq 0, \quad j \in I[1, m],$$

where g_j is the jth row of matrix G. Denote the optimal solution as γ°. It is clear that $\gamma^{\circ} \leq \gamma^{\mathrm{I}}$. On the other hand, noting that Constraints b) and c) in (2.53) are included in Constraints b) and c) of (2.52), we can obtain from this fact that $\gamma^{\circ} \geq \gamma^{\mathrm{I}}$. Thus, we have $\gamma^{\circ} = \gamma^{\mathrm{I}}$.

If we adopt the convex hull representation (2.10), the optimization problem for stabilization also reduces to (2.53). This fact implies that, as far as the stabilization problem is concerned, the convex hull with multiple auxiliary matrices does not improve the resulting estimate of the domain of attraction.

2.3.2 Nestedly Saturated Linear Feedback

In this subsection, we will consider the optimization problem of obtaining an ellipsoidal estimate of the domain of attraction of system (2.18). By using the regional sector condition of saturation/deadzone functions, Theorem 2.2.3 presents a set of conditions under which an ellipsoid is a contractively invariant set of system (2.18). Viewing these conditions as constraints, we can formulate an LMI-based optimization problem to maximize the invariant ellipsoid $\mathcal{E}(P)$. Let $Q = P^{-1}$, $S_j = T_j^{-1}, j \in I[1,q]$, $\mathcal{P}_j = \mathrm{diag}\{Q, S_q, S_{q-1}, \cdots, S_{j+1}\}$, $Z_{0q} = E_{0q}Q$, $Z_j = E_j\mathcal{P}_j$, $W_{lj} = E_{lj}S_l$, $l \in I[j+1,q], j \in I[1,q-1]$. The optimization problem is given as,

$$\min_{Q>0,\ Z_{0q},\ Z_j,\ W_j\, j\in I[1,q-1], S_j,\, j\in I[1,q]} \quad \gamma \tag{2.54}$$

$$\text{s.t. a)} \begin{bmatrix} \gamma & x_l^{\mathrm{T}} \\ x_l & Q \end{bmatrix} \geq 0, \quad l \in I[1,p],$$

$$\text{b)} \begin{bmatrix} 1 & z_{0qk} \\ \star & Q \end{bmatrix} \geq 1, \quad k \in I[1,m_q],$$

$$\text{c)} \begin{bmatrix} 1 & z_{jk} \\ \star & \bar{\Xi}_j \end{bmatrix} \geq 0, \quad k \in I[1,m_j],\ j \in I[1,q-1],$$

$$\text{d)} \ \mathcal{N} \leq 0,$$

where

$$\bar{\Xi}_j = \begin{bmatrix} Q & Z_{0q}^{\mathrm{T}} - Q\mathcal{F}_q^{\mathrm{T}} & Z_{0(q-1)}^{\mathrm{T}} - Q\mathcal{F}_{q-1}^{\mathrm{T}} & \cdots & Z_{0(j+1)}^{\mathrm{T}} - Q\mathcal{F}_{j+1}^{\mathrm{T}} \\ \star & 2S_q & W_{qj}^{\mathrm{T}} + S_q B_q^{\mathrm{T}} & \cdots & W_{q(j+1)}^{\mathrm{T}} + S_q(\prod_{k=1}^{q} B_k))^{\mathrm{T}} \\ \star & \star & 2S_{q-1} & \cdots & W_{(q-1)(j+1)}^{\mathrm{T}} + S_q(\prod_{k=1}^{q-1} B_k)^{\mathrm{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \cdots & 2S_{j+1} \end{bmatrix},$$

$$\mathcal{N} = \begin{bmatrix} Q\mathcal{A}^{\mathrm{T}} + \mathcal{A}Q & -B_1S_1 - Z_{01}^{\mathrm{T}} + Q\mathcal{F}_1^{\mathrm{T}} & -B_1B_2S_2 - Z_{02}^{\mathrm{T}} + Q\mathcal{F}_2^{\mathrm{T}} & \cdots & N_1 \\ \star & -2S_1 & -W_{21} - B_1B_2S_2 & \cdots & N_2 \\ \star & \star & -2S_2 & \cdots & N_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \cdots & -2S_q \end{bmatrix},$$

$N_1 = -\prod_{k=1}^{q} B_k S_q - Z_{0q}^{\mathrm{T}} + Q\mathcal{F}_q^{\mathrm{T}}$, $N_2 = -W_{q1} - \prod_{k=1}^{q} B_k S_q$, $N_3 = -W_{q2} - \prod_{k=1}^{q} B_k S_q$, and z_{0qk} and z_{jk} are the kth rows of matrices $Z_{0q} = E_{0q}Q$ and $Z_j = E_j\mathcal{P}_j$, respectively. Note that $\bar{\Xi}_j = \mathcal{P}_j^{\mathrm{T}}\Xi_j\mathcal{P}_j$ and $\mathcal{N} = \mathcal{P}^{\mathrm{T}}\mathcal{M}\mathcal{P}$, where $\mathcal{P} = [Q \ \ S_1 \ \ S_2 \ \ \cdots \ \ S_q]$.

To solve the stabilization problem, that is, to determine feedback gains F_j's, $j \in I[1,q]$, such that the ellipsoid $\mathcal{E}(P)$ is as large an estimate of the domain of attraction of system (2.18) as possible, we can set $Y_j = F_jQ, j \in I[1,q]$, as additional decision variables to result in the following LMI-based optimization problem,

$$\min_{Q>0,\ Z_{0q},\ Z_j,\ W_j\ j\in I[1,q-1], Y_j, S_j,\ j\in I[1,q]} \gamma \tag{2.55}$$

$$\text{s.t. a) Inequalities a) and b) in (2.54),}$$

$$\text{b)} \begin{bmatrix} 1 & z_{jk} \\ \star & \hat{\Xi}_j \end{bmatrix} \geq 0, \quad k \in I[1,m_j],\ j \in I[1,q-1],$$

$$\text{c)}\ \hat{\mathcal{N}} \leq 0,$$

where

$$\hat{\Xi}_j = \begin{bmatrix} Q & Z_{0q}^{\mathrm{T}} - Y_q^{\mathrm{T}} & Z_{0(q-1)}^{\mathrm{T}} - Y_{q-1}^{\mathrm{T}} - Y_q^{\mathrm{T}} B_q^{\mathrm{T}} & \cdots & Z_{0(j+1)}^{\mathrm{T}} - Y_{j+1}^{\mathrm{T}} - \sum_{l=j+2}^{q} (\prod_{l=j+2}^{l} B_k) Y_l \\ \star & 2S_q & W_{qj}^{\mathrm{T}} + S_q B_q^{\mathrm{T}} & \cdots & W_{q(j+1)}^{\mathrm{T}} + S_q (\prod_{k=1}^{q} B_k))^{\mathrm{T}} \\ \star & \star & 2S_{q-1} & \cdots & W_{(q-1)(j+1)}^{\mathrm{T}} + S_q (\prod_{k=1}^{q-1} B_k)^{\mathrm{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \cdots & 2S_{j+1} \end{bmatrix},$$

$$\hat{\mathcal{N}} = \begin{bmatrix} N_0 & -B_1 S_1 - Z_{01}^{\mathrm{T}} + N_4^{\mathrm{T}} & -B_1 B_2 S_2 - Z_{02}^{\mathrm{T}} + N_5^{\mathrm{T}} & \cdots & N_1 \\ \star & -2S_1 & -W_{21} - B_1 B_2 S_2 & \cdots & N_2 \\ \star & \star & -2S_2 & \cdots & N_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \cdots & -2S_q \end{bmatrix},$$

$N_0 = \mathrm{He}(AQ + \sum_{j=1}^{q} (\prod_{l=1}^{q} B_l) Y_j)$, $N_1 = -\prod_{k=1}^{q} B_k S_q - Z_{0q}^{\mathrm{T}} + Y_q^{\mathrm{T}}$, $N_4 = Y_1 + \sum_{l=2}^{q} (\prod_{k=2}^{l}) Y_l$, $N_5 = Y_1 + \sum_{l=3}^{q} (\prod_{k=3}^{l}) Y_l$, and N_2 and N_3 are as defined in (2.54).

Based on the conditions of Theorem 2.2.4, which characterizes the contractive invariance of an ellipsoid by using the convex hull representation (2.28) without using common auxiliary matrices to treat the nested saturation function, we can formulate the following LMI-based optimization problem,

$$\min_{Q>0,\ Z_k(i_1,i_2,\cdots,i_q)} \gamma \tag{2.56}$$

$$\text{s.t. a)} \begin{bmatrix} \gamma & x_l^{\mathrm{T}} \\ x_l & Q \end{bmatrix} \geq 0, \quad l \in I[1,p],$$

$$\text{b)} \quad \begin{cases} \text{He}\left(\left(A + \sum_{k=1}^{q}\left(\prod_{l=1}^{k} B_l D_{i_l}\right) F_k Q \right.\right. \\ \qquad \left.\left. + \sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} B_l D_{i_l}\right) B_k D_{i_k}^{-} Z_k(i_1, i_2, \cdots, i_q)\right)\right) < 0, \quad (i_1, i_2, \cdots, i_q) \in \Pi, \end{cases}$$

$$\text{c)} \quad \begin{bmatrix} 1 & z_{j_k}(i_1, i_2, \cdots, i_q) \\ \star & Q \end{bmatrix} \geq 0, \quad j_k \in I[1, m_k], k \in I[1, q], (i_1, i_2, \cdots, i_q) \in \Pi.$$

If we use the convex hull (2.25) or (2.26) to handle the nestedly saturated linear feedback, Constraints b) and c) of the LMI-based optimization problem (2.55) should be modified as,

$$\text{b)} \quad \begin{cases} \text{He}\left(\left(A + \sum_{k=1}^{q}\left(\prod_{l=1}^{k} B_l D_{i_l}\right) F_k Q \right.\right. \\ \qquad \left.\left. + \sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} B_l D_{i_l}\right) B_k D_{i_k}^{-} Z_k(i_k)\right)\right) < 0, \quad i_k \in I[1, 2^{m_k}], \ k \in I[1, q], \end{cases}$$

$$\text{c)} \quad \begin{bmatrix} 1 & z_{j_k}(i_k) \\ \star & Q \end{bmatrix} \geq 0, \quad j_k \in I[1, m_k], i_k \in I[1, 2^{m_k}], \ k \in I[1, q],$$

and

$$\text{b)} \quad \begin{cases} \text{He}\left(\left(A + \sum_{k=1}^{q}\left(\prod_{l=1}^{k} B_l D_{i_l}\right) F_k Q \right.\right. \\ \qquad \left.\left. + \sum_{k=1}^{q}\left(\prod_{l=1}^{k-1} B_l D_{i_l}\right) B_k D_{i_k}^{-} Z_k(i_1, i_2, \cdots, i_k)\right)\right) < 0, \\ \qquad i_k \in I[1, 2^{m_k}], \ k \in I[1, q], \end{cases}$$

$$\text{c)} \quad \begin{bmatrix} 1 & z_{j_k}(i_1, i_2, \cdots, i_k) \\ \star & Q \end{bmatrix} \geq 0, \quad j_k \in I[1, m_k], i_k \in I[1, 2^{m_k}], \ k \in I[1, q],$$

respectively.

We next discuss the computational complexity of optimization problems that involve different approaches to dealing with the nested saturation function. For brevity, we assume, without loss of significant generality, that $m_1 = m_2 = \cdots = m_q = m$. Similarly as for the single-layer saturated linear feedback, we list the number of lines and decision variables of constraints in these optimization problems involving different treatments of the nested saturation function in Table. 2.2, where

Table 2.2 Numbers of lines and decision variables of LMIs in different optimization problems

Treatments	Number of lines	Number of decision variables
Regional sector	$(p+mq)(n+1)+\frac{m^2}{2}q(q-1)$	$mq(n+1)+\frac{m^2}{2}q(q-1)+\frac{n}{2}(n+1)$
Convex hull (2.25)	$nG+(n+1)(p+q2^{m-1})$	$\frac{1}{2}n(n+1)+nq2^{m-1}$
Convex hull (2.26)	$nG+(n+1)(p+F)$	$\frac{1}{2}n(n+1)+nF$
Convex hull (2.28)	$nG+(n+1)(p+T)$	$\frac{1}{2}n(n+1)+nT$

$$G=\begin{cases} q+1, & m=1,\\ (2^m-1)^q+\frac{(2^m-1)^q-1}{2^m-2}, & m>1, \end{cases}$$

$$F=\begin{cases} q+1 & m=1,\\ 2^{m-1}\frac{(2^m-1)^q-2^m+1}{2^m-2}, & m>1, \end{cases}$$

$$T=\begin{cases} q+1, & m=1,\\ m+(2^m-1)^q+\frac{(2^m-m)(2^m-1)}{2^m-2}\left(\frac{(2^m-1)^q-2^m+1}{2^m-2}-q+1\right), & m>1. \end{cases}$$

As shown in Table. 2.2, the LMIs in (2.52) contain the most lines and decision variables among all the optimization problems as the treatment (2.28) involves the largest number of auxiliary feedback matrices. The fact that each of these auxiliary matrices is independent from each other leads to the least conservativeness under the convex hull framework for treating nestedly saturated linear feedbacks. To illustrate the computational complexity of different optimization problems, we plot in Figure 2.7 the function $\log(MN^3)$ that characterizes the computational complexity of LMI optimization problems in the case of $q=3$ and $n=8$. It is clear that the optimization problems based on the convex hull representations have significantly higher computational complexity than the one based on the regional sector condition.

For solving the problem of designing linear feedback gains $F_k\in\mathbf{R}^{m_k\times n}$, $k\in I[1,q]$, such that the domain of attraction of system (2.18) is as large as possible, we can formulate the following LMI-based optimization problem,

$$\min_{Q>0,\ G_k}\quad \gamma \tag{2.57}$$

$$\text{s.t. a)}\ \begin{bmatrix}\gamma & x_l^{\mathrm{T}}\\ x_l & Q\end{bmatrix}\geq 0,\quad l\in I[1,p],$$

$$\text{b)}\ \operatorname{He}\left(AQ+\sum_{k=1}^{q}\left(\prod_{l=1}^{k-1}B_lD_{i_l}\right)B_kG_k\right)<0,\quad i_k\in I[1,2_k^m],\ k\in I[1,q],$$

$$\text{c)}\ \begin{bmatrix}1 & g_{j_k}\\ \star & Q\end{bmatrix}\geq 0,\quad j_k\in I[1,m_k],k\in I[1,q],$$

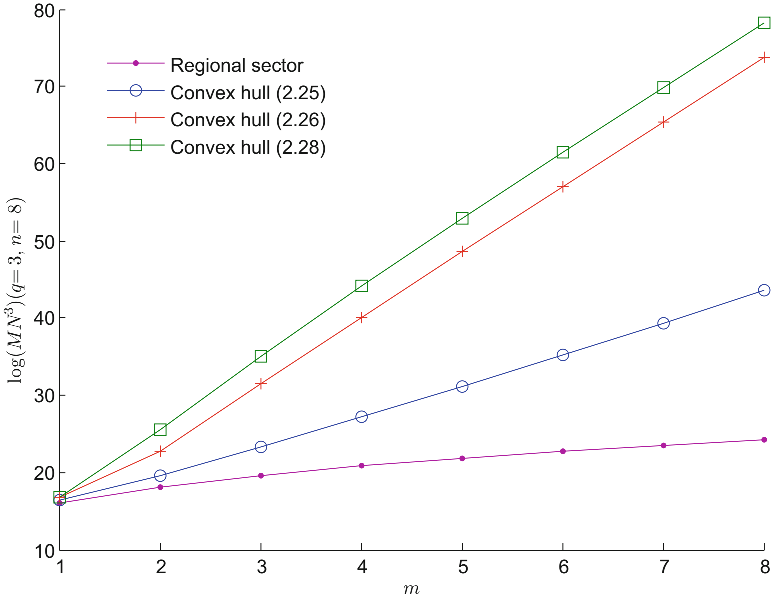

Fig. 2.7 A comparison of the computational complexity among different optimization problems ($q = 3$ and $n = 8$).

where g_{j_k} is the j_kth row of G_k, and $\prod_{l=1}^{0} B_l D_{i_l} = I$. Denote the optimal solution of (2.57) as $(\gamma^{\mathrm{N}}, Q^{\mathrm{N}}, G_k^{\mathrm{N}})$. Then we have the optimal controller gains $F_k^{\mathrm{N}} = G_k^{\mathrm{N}} P^{\mathrm{N}}$ where $P^{\mathrm{N}} = (Q^{\mathrm{N}})^{-1}$ and the optimal ellipsoidal estimate $\mathcal{E}(P^{\mathrm{N}})$. Actually, the optimization problem (2.57) results from (2.56) with F_k's set as additional decision variables. The verification of this fact is similar to that of (2.53). Note that no matter which of the convex hull representations is employed to express the nestedly saturated linear feedback, the optimization problem for the largest contractively invariant ellipsoid by designing linear feedback gains will become (2.57).

2.3.3 Piecewise Linear Functions

In this subsection, we focus on the problem of estimating the domain of attraction of system (2.40) whose linear feedback is subject to a piecewise linear function. Based on the conditions of Theorem 2.2.5, which characterizes the contractive invariance of an ellipsoid by introducing multiple auxiliary matrices to express the piecewise linear feedback, the following LMI-based optimization problem can be formulated to obtain the largest ellipsoidal estimate of the domain of attraction of system (2.40),

$$\min_{Q>0,\ Z_{qi}} \quad \gamma \tag{2.58}$$

$$\text{s.t. a)} \begin{bmatrix} \gamma & x_l^{\mathrm{T}} \\ x_l & Q \end{bmatrix} \geq 0, \quad l \in I[1,p],$$

$$\text{b)} \begin{cases} \mathrm{He}\left(AQ + B\Pi_q FQ + BD_i\Omega_q FQ + BD_i^- Z_{qi}\right) < 0, \\ \qquad q \in I[1,K],\ i \in I[1,2^m], \end{cases}$$

$$\text{c)} \begin{bmatrix} 1 & z_{jqi} \\ \star & Q \end{bmatrix} \geq 0, \quad j \in I[1,m],\ q \in I[1,K],\ i \in I[1,2^m],$$

where $\gamma = \frac{1}{\alpha^2}$ and z_{jqi} is the jth row of Z_{qi}. Note that inequalities in Constraints c) are equivalent to Condition (2.42). Let $(Q^{\mathrm{P}}, Z_{qi}^{\mathrm{P}})$ be the optimal solution of (2.58). Then the ellipsoid $\mathcal{E}(P^{\mathrm{P}})$ with $P^{\mathrm{P}} = (Q^{\mathrm{P}})^{-1}$ is the largest ellipsoidal estimate of the domain of attraction.

2.4 Discrete-Time Systems

In Section 2.2, we presented treatments of saturated linear feedbacks via multiple auxiliary matrices, and by using these treatments, established sets of sufficient conditions under which an ellipsoid is a contractively invariant set and thus is an estimate of the domain of attraction for a continuous-time system with saturated linear feedbacks. Moreover, in Section 2.3, based on these conditions we formulated and solved the optimization problems for obtaining the largest such contractively invariant set.

In this section, we consider the following discrete-time linear system subject to saturated linear feedback,

$$x^+ = Ax + B\mathrm{sat}(Fx), \quad x \in \mathbf{R}^n,\ F \in \mathbf{R}^{m\times n}, \tag{2.59}$$

where x^+ is the successor of x.

Based on the convex hull representation (2.13), which contains multiple auxiliary feedback matrices, as the treatment of saturated linear feedback, the following theorem establishes conditions under which an ellipsoid is contractively invariant and can be used as an estimate of the domain of attraction of system (2.59).

Theorem 2.4.1 *Consider system (2.59). For a given positive definite $P \in \mathbf{R}^{n\times n}$, if there exist matrices $H_i \in \mathbf{R}^{m\times n}$, $i \in I[1,2^m]$, such that*

$$(A + BD_iF + BD_i^- H_i)^{\mathrm{T}} P (A + BD_iF + BD_i^- H_i) - P < 0, \quad i \in I[1,2^m], \tag{2.60}$$

and $\mathcal{E}(P) \subseteq \bigcap_{i=1}^{2^m} \mathcal{L}(H_i)$, then the ellipsoid $\mathcal{E}(P)$ is contractively invariant.

Proof By Lemma 2.2.2, condition $\mathcal{E}(P) \subseteq \bigcap_{i=1}^{2^m} \mathcal{L}(H_i)$ implies that $\text{sat}(Fx)$ can be expressed as

$$\text{sat}(Fx) \in \text{co}\left\{D_i Fx + D_i^- H_i x :\ i \in I[1, 2^m]\right\}.$$

By the definition of the convex hull, there are a set of nonnegative scalars α_i satisfying $\sum_{i=1}^{2^m} \alpha_i = 1$ such that

$$\text{sat}(Fx) = \sum_{i=1}^{2^m} \alpha_i (D_i F + D_i^- H_i) x. \tag{2.61}$$

By the Schur complement, (2.60) is equivalent to

$$\begin{bmatrix} P & \star \\ A + BD_i F + BD_i^- H_i & P^{-1} \end{bmatrix} > 0, \quad i \in I[1, 2^m].$$

Multiplying Ξ and Ξ^{T} respectively to the left and the right sides of the above matrix, where

$$\Xi = \begin{bmatrix} x & 0_{n\times n} \\ 0_{n\times 1} & I_n \end{bmatrix},$$

and $x \in \mathbf{R}^n$ is any nonzero vector, we have

$$\begin{bmatrix} x^{\mathrm{T}} Px & \star \\ (A + BD_i F + BD_i^- H_i)x & P^{-1} \end{bmatrix} > 0, \quad i \in I[1, 2^m].$$

In view of (2.61) and the nonnegativeness of α_i's, we have

$$\begin{aligned} &\begin{bmatrix} x^{\mathrm{T}} Px & \star \\ \sum_{i=1}^{2^m} \alpha_i (A + BD_i F + BD_i^- H_i)x & P^{-1} \end{bmatrix} \\ &= \begin{bmatrix} x^{\mathrm{T}} Px & \star \\ A + B\text{sat}(Fx) & P^{-1} \end{bmatrix} \\ &> 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &(A + B\text{sat}(Fx))^{\mathrm{T}} P (A + B\text{sat}(Fx)) - x^{\mathrm{T}} Px \\ &= (x^+)^{\mathrm{T}} Px^+ - x^{\mathrm{T}} Px \\ &= V(x^+) - V(x) \\ &= \Delta V(x) \\ &< 0, \end{aligned}$$

for all nonzero $x \in \mathcal{E}(P)$. Hence, we can conclude that $\mathcal{E}(P)$ is a contractively invariant set of system (2.59). $\square$

The ellipsoids which satisfy Inequalities (2.60) and $\mathcal{E}(P) \subseteq \bigcap_{i=1}^{2^m} \mathcal{L}(H_i)$ can be used as the estimates of the domain of attraction of system (2.59). To choose the largest among these ellipsoids, we formulate the following LMI-based optimization problem,

$$\begin{aligned} &\min_{Q>0,\ Z_i} \quad \gamma \\ &\text{s.t. a)} \begin{bmatrix} \gamma & x_l^{\mathrm{T}} \\ x_l & Q \end{bmatrix} \geq 0, \quad l \in I[1,p], \\ &\qquad \text{b)} \ \operatorname{He}\big((A + BD_iF)Q + BD_i^- Z_i\big) < 0, \quad i \in I[1,2^m], \\ &\qquad \text{c)} \begin{bmatrix} 1 & z_{ji} \\ \star & Q \end{bmatrix} \geq 0, \ \ j \in I[1,m], \ \ i \in I[1,2^m]. \end{aligned} \tag{2.62}$$

Denote the optimal solution as $(\gamma^{\mathrm{D}}, Q^{\mathrm{D}}, Z_i^{\mathrm{D}})$. Let $P^{\mathrm{D}} = (Q^{\mathrm{D}})^{-1}$. Then the ellipsoid $\mathcal{E}(P^{\mathrm{D}})$ is the largest estimate obtained from (2.62). Optimization problems can also be formulated if we use the other treatments of saturated linear feedback. It is clear that the optimal values of γ for these optimization problems are no less than γ^{D} as the convex hull representation (2.13) is less conservative than the other treatments.

For the discrete-time counterparts of systems (2.18) and (2.40), similar results on the contractive invariance of ellipsoids and the optimization problems for the largest ellipsoidal estimates of the domains of attraction can be easily obtained. The details are omitted for brevity.

2.5 Numerical Examples

In this section, we present some numerical examples to demonstrate the effectiveness and advantages of the theoretical results obtained by introducing multiple auxiliary feedback matrices to handle saturation/deadzone in the estimation of the domains of attraction of saturated systems.

Example 2.5.1 (Single Layer Saturated Linear Feedback) Consider a second-order continuous-time system (2.1) with

$$A = \begin{bmatrix} 0 & 2 \\ -3 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 1.4 & 3 \\ 0 & -0.7 \end{bmatrix}, \quad F = \begin{bmatrix} -0.4698 & -0.0770 \\ -0.8318 & 0.7640 \end{bmatrix}.$$

Let the shape reference set be $\mathcal{R} = \{x_1\}$, $x_1 = [0 \quad 1]^{\mathrm{T}}$. Then, solving the optimization problems (2.47), (2.49), and (2.50), which are associated with the regional sector condition (2.2), the convex hull representation (2.10) with a single auxiliary matrix and the improved convex hull representation (2.13) with multiple auxiliary matrices, respectively, we can obtain the largest values of α, which measure the sizes of the optimal contractively invariant ellipsoids, and the positive definite matrices that define the shapes of these ellipsoids as follows,

$$\alpha_{\text{opt}}^{\text{regional sector (2.2)}} = 7.7739,$$

$$P_{\text{opt}}^{\text{regional sector (2.2)}} = \begin{bmatrix} 0.0201 & -0.0038 \\ -0.0038 & 0.0165 \end{bmatrix};$$

$$\alpha_{\text{opt}}^{\text{convex hull (2.10)}} = 8.1039,$$

$$P_{\text{opt}}^{\text{convex hull (2.10)}} = \begin{bmatrix} 0.0184 & -0.0034 \\ -0.0034 & 0.0152 \end{bmatrix};$$

$$\alpha_{\text{opt}}^{\text{improved convex hull (2.13)}} = 8.8249,$$

$$P_{\text{opt}}^{\text{improved convex hull (2.13)}} = \begin{bmatrix} 0.0157 & -0.0026 \\ -0.0026 & 0.0128 \end{bmatrix}.$$

We plot these three ellipsoidal estimates in Figure 2.8. The improved convex hull representation (2.13) leads to the largest estimate of the domain of attraction among these three estimates. This implies that the convex hull representation (2.13) with

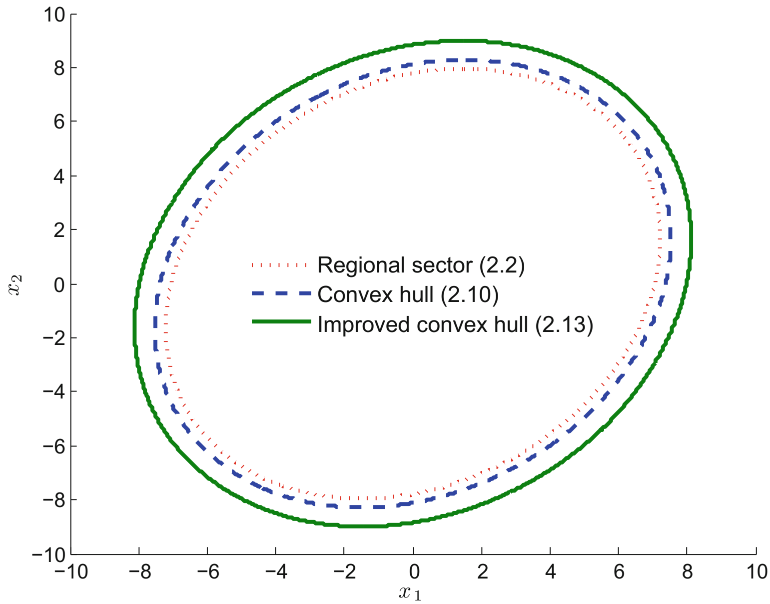

Fig. 2.8 Example 2.5.1: The largest contractively invariant ellipsoids obtained by using the regional sector condition (2.2), the convex hull representation (2.10), and the improved convex hull representation (2.13), respectively.

multiple auxiliary matrices has the least conservativeness among all these treatments of the saturation function introduced in this chapter.

Example 2.5.2 (Nestedly Saturated Linear Feedback) Consider a second-order system (2.18) with $q = 2$,

$$A = \begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix},\ B_1 = \begin{bmatrix} 1.4 & 3 \\ 4.2 & -0.7 \end{bmatrix},\ B_2 = \begin{bmatrix} -2 & 1.9 \\ 4 & -1.3 \end{bmatrix},$$

and

$$F_1 = \begin{bmatrix} -7.7551 & 8.7755 \\ 4.2857 & -3.4286 \end{bmatrix},$$

where F_2 is to be designed later for enlarging the estimate of the domain of attraction of system (2.18). By using the regional sector condition to handle the nested saturation, an LMI-based optimization problem can be formulated as follows,

$$\min_{Q>0,Y,Z_{01},Z_{02},S_1,S_2,W_{21}} \gamma \tag{2.63}$$

$$\text{s.t. a)} \begin{bmatrix} 1 & x_1^{\mathrm{T}} \\ \star & Q \end{bmatrix} \geq 0,$$

$$\text{b)} \begin{bmatrix} 1 & z_{02k} \\ \star & Q \end{bmatrix} \geq 0, \quad k = 1, 2,$$

$$\text{c)} \begin{bmatrix} 1 & z_{01k} & w_{21k} \\ \star & Q & Z_{02}^{\mathrm{T}} - Y^{\mathrm{T}} \\ \star & \star & 2S_2 \end{bmatrix} \geq 0, \quad k = 1, 2,$$

$$\text{d)} \begin{bmatrix} D_1 & -B_1S_1 - Z_{01}^{\mathrm{T}} + QF_1^{\mathrm{T}} + Y^{\mathrm{T}}B_2^{\mathrm{T}} & -B_1B_2S_2 - Z_{02}^{\mathrm{T}} + Y^{\mathrm{T}} \\ \star & -2S_1 & -W_{21} - B_1B_2S_2 \\ \star & \star & -2S_2 \end{bmatrix} \leq 0,$$

where $\gamma = \frac{1}{\alpha^2}$, $Q = P^{-1}$, $Y = F_2Q$, $D_1 = \mathrm{He}((A + B_1F_1)Q + B_1B_2Y)$, and z_{01k}, z_{02k} and w_{21k} are the kth rows of matrices Z_{01}, Z_{02} and W_{21}, respectively. Let $\mathcal{R} = \{x_1\}$, $x_1 = [-0.4274\ -0.9041]^{\mathrm{T}}$. By solving the above optimization problem, we obtain $\alpha_{\text{opt}}^{\text{Regional sector (Tarbouriech et al.)}} = 0.3641$ and

$$P_{\text{opt}}^{\text{Regional sector (Tarbouriech et al.)}} = \begin{bmatrix} 4.4381 & -0.0283 \\ -0.0283 & 8.2079 \end{bmatrix},$$

$$F_{2\text{opt}}^{\text{Regional sector (Tarbouriech et al.)}} = \begin{bmatrix} -1.5892 & 1.8934 \\ 0.5937 & -3.3881 \end{bmatrix}.$$

If we use Lemma 2.2.2, which contains the largest number of auxiliary matrices, to handle the nested saturation function, similarly as the optimization problem (2.56), we can formulate the following LMI-based optimization problem,

$$\min_{Q>0, Z_k(i_1,i_2), Y} \gamma \tag{2.64}$$

$$\text{s.t. a) } \begin{bmatrix} 1 & x_1^{\mathrm{T}} \\ \star & Q \end{bmatrix} \geq 0,$$

$$\text{b) } \begin{cases} \mathrm{He}(AQ+B_1(D_{i_1}(F_1 Q+B_2(D_{i_2}Y + D_{i_2}^{-}Z_2(i_1,i_2))) + D_{i_1}^{-}Z_1(i_1,i_2))) < 0, \\ \qquad i_1, i_2 \in I[1,4], \end{cases}$$

$$\text{c) } \begin{bmatrix} 1 & z_{jk}(i_1,i_2) \\ \star & Q \end{bmatrix} \geq 0, \quad i_1, i_2 \in I[1,4],\ j,k = 1,2,$$

where $z_{jk}(i_1,i_2)$ is the kth row of matrix $Z_j(i_1,i_2)$, $\gamma = \frac{1}{\alpha^2}$, $Q = P^{-1}$ and $Y = F_2 Q$. By solving (2.64), we obtain $\alpha_{\text{opt}}^{\text{Lemma 2.2.7}} = 0.4808$ and

$$P_{\text{opt}}^{\text{Lemma 2.2.7}} = \begin{bmatrix} 3.9029 & -2.1253 \\ -2.1253 & 6.4298 \end{bmatrix},$$

$$F_{2\text{opt}}^{\text{Lemma 2.2.7}} = \begin{bmatrix} -3.0083 & 2.3469 \\ 0.8388 & -3.4434 \end{bmatrix}.$$

On the other hand, if we set $Z_k(i_1,i_2) = Z_k(i_k)$, $k = 1,2$, the optimization problem (2.64) will reduce to that presented in [107]. By solving the optimization problem (2.64) with additional constraints $Z_k(i_1,i_2) = Z_k(i_k)$, $k = 1,2$, we obtain $\alpha_{\text{opt}}^{\text{Lemma 2.2.5 (Zhou et al.)}} = 0.4312$, and

$$P_{\text{opt}}^{\text{Lemma 2.2.5 (Zhou et al.)}} = \begin{bmatrix} 5.4551 & -3.8016 \\ -3.8016 & 8.9551 \end{bmatrix},$$

$$F_{2\text{opt}}^{\text{Lemma 2.2.5 (Zhou et al.)}} = \begin{bmatrix} -2.9265 & 2.1230 \\ 0.4180 & -3.7350 \end{bmatrix}.$$

Moreover, if the treatment of the nestedly saturated linear feedback in [21] is used to establish an optimization problem, that is, (2.64) with additional constraints $Z_1(i_1,i_2) = Z_1(i_1)$, we will obtain $\alpha_{\text{opt}}^{\text{Lemma 2.2.6 (Fiacchini et al.)}} = 0.4700$, and

$$P_{\text{opt}}^{\text{Lemma 2.2.6 (Fiacchini et al.)}} = \begin{bmatrix} 4.3823 & -2.4489 \\ -2.4489 & 6.8751 \end{bmatrix},$$

$$F_{2\text{opt}}^{\text{Lemma 2.2.6 (Fiacchini et al.)}} = \begin{bmatrix} -3.1883 & 2.4894 \\ 0.7261 & -3.5351 \end{bmatrix}.$$

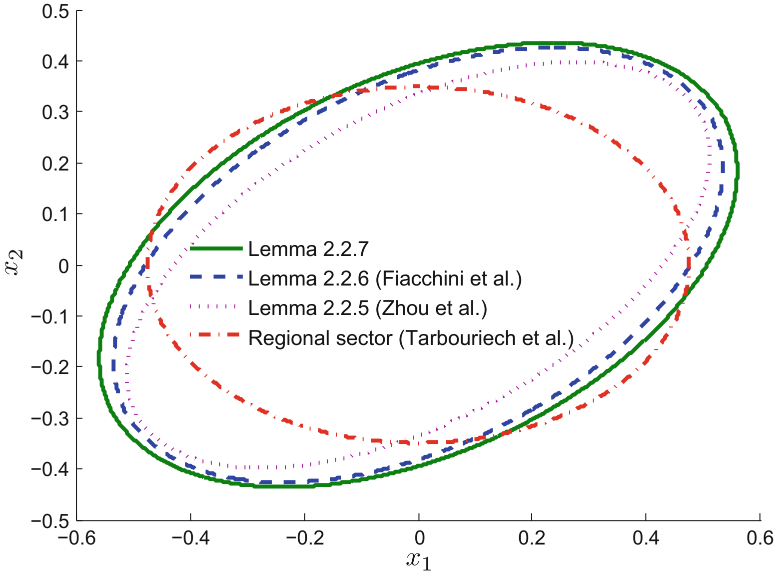

Fig. 2.9 Example 2.5.2: The largest contractively invariant ellipsoids obtained by using the regional sector condition and the convex hull representations of Lemmas 2.2.5–2.2.7, respectively.

The four optimal ellipsoids $\mathcal{E}(P_{\text{opt}}^{\text{Regional sector (Tarbouriech et al.)}})$, $\mathcal{E}(P_{\text{opt}}^{\text{Lemma 2.2.5 (Zhou et al.)}})$, $\mathcal{E}(P_{\text{opt}}^{\text{Lemma 2.2.6 (Fiacchini et al.)}})$ and $\mathcal{E}(P_{\text{opt}}^{\text{Lemma 2.2.7}})$ obtained above give four different estimates of the domain of attraction of system (2.18). These four invariant ellipsoids are plotted in Figure 2.9, which clearly indicates that Lemma 2.2.7 leads to a less conservative result than the methods in [21, 93, 107].

Example 2.5.3 (Piecewise Linear Functions) Consider a second-order continuous-time system (2.40) with

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1.4 & 3 \\ 0 & -0.7 \end{bmatrix}, \quad F = \begin{bmatrix} -0.1566 & -0.0257 \\ -0.4159 & 0.3820 \end{bmatrix},$$

and

$$\psi_1(u_1) = \begin{cases} 3u, & \text{if } u \in [0, 1], \\ 1.5u + 1.5, & \text{if } u \in (1, 2], \\ 0.8u + 3.9, & \text{if } u \in (2, +\infty), \end{cases}$$

$$\psi_2(u_2) = \begin{cases} 2u, & \text{if } u \in [0, 1.5], \\ u + 1.5, & \text{if } u \in (1.5, +\infty), \end{cases}$$

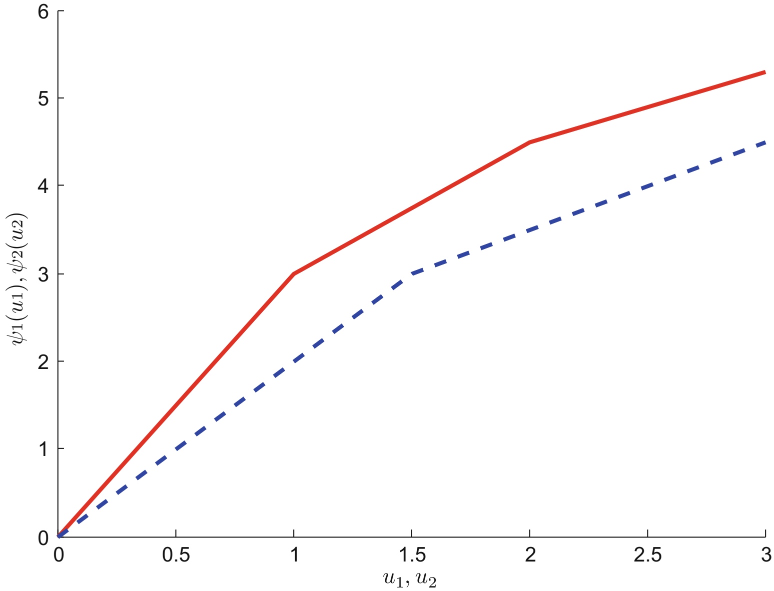

Fig. 2.10 Example 2.5.3: The piecewise linear functions $\psi_1(u_1)$ (solid line) and $\psi_2(u_2)$ (dashed line).

which are depicted in Figure 2.10. Let the shape reference set be $\mathcal{R} = \{x_1\}$, $x_1 = [1 \;\; 1]^{\mathrm{T}}$. Solving the optimization problem (2.58), we can obtain $\alpha_{\mathrm{opt}}^{\mathrm{P}m} = 22.3722$ and

$$P_{\mathrm{opt}}^{\mathrm{P}m} = \begin{bmatrix} 0.0012 & -0.0003 \\ -0.0003 & 0.0014 \end{bmatrix}.$$

Note that the optimization problem (2.58) involves multiple auxiliary matrices. If we use the convex hull representation with a single auxiliary matrix to treat the piecewise linear function and solve the resulting optimization problem, then we can obtain that $\alpha_{\mathrm{opt}}^{\mathrm{P}s} = 19.8105$ and

$$P_{\mathrm{opt}}^{\mathrm{P}s} = \begin{bmatrix} 0.0015 & -0.0004 \\ -0.0004 & 0.0019 \end{bmatrix}.$$

The two optimal ellipsoids $\mathcal{E}(P_{\mathrm{opt}}^{\mathrm{P}m})$ and $\mathcal{E}(P_{\mathrm{opt}}^{\mathrm{P}s})$, each of which can be used as an estimate of the domain of attraction for system (2.40), are plotted in Figure 2.11 for comparison. It is clear that $\mathcal{E}(P_{\mathrm{opt}}^{\mathrm{P}m})$ is significantly larger than $\mathcal{E}(P_{\mathrm{opt}}^{\mathrm{P}s})$. This fact implies that the convex hull representation with multiple auxiliary matrices has the ability to arrive at larger estimates of the domain of attraction.

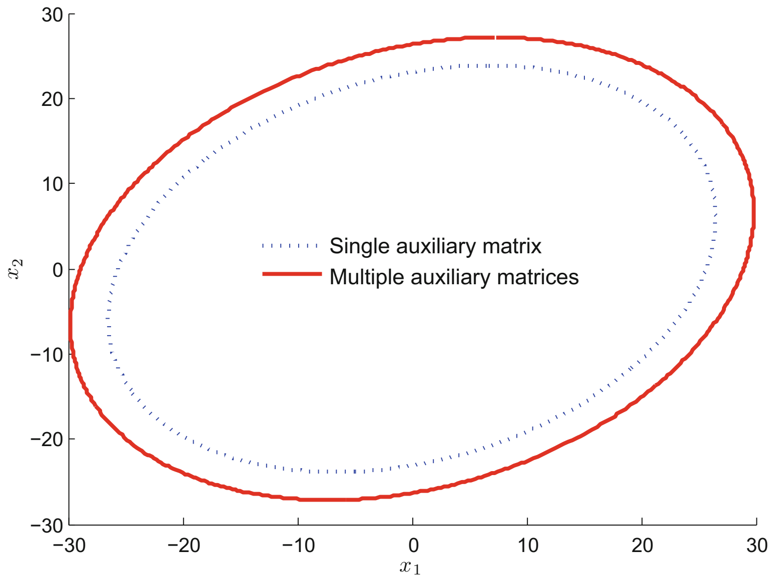

Fig. 2.11 Example 2.5.3: The largest contractively invariant ellipsoids obtained by using the convex hull representations with single auxiliary matrix and multiple auxiliary matrices.

2.6 Conclusions

In this chapter, we have presented treatments of single-layer saturated linear feedbacks, nestedly saturated linear feedbacks, and linear feedbacks subject to a piecewise linear function. These treatments are based on the convex hull representation of saturated linear feedbacks. By introducing more auxiliary matrices, we eliminated the dependence between a group of vertices of the convex hull and a common auxiliary feedback, and establish the improved convex hull representations that are of less conservativeness than the existing convex hull representations and the regional sector conditions. Moreover, based on these treatments of the nonlinearities in the input, we presented conditions that ensure the contractive invariance of an ellipsoid, and, based on these conditions, formulated and solved LMI-based optimization problems to obtain the largest such ellipsoid as the estimate of the domain of attraction. We also discussed the modification of the optimization problems for control designs.

2.7 Notes and References

The introduction of multiple auxiliary matrices to treat saturated linear feedback was earlier given in the literature [2, 4]. The convex hull representation with multiple auxiliary matrices presented in this chapter has the same compact form as the conventional convex hull proposed in [36, 46]. Moreover, for a nestedly saturated linear feedback, its convex hull representation with multiple auxiliary matrices was taken from [58]. The convex hull presentation of piecewise linear functions, which contains as many multiple auxiliary matrices as possible, generalizes the results in [41, 43].

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