

Chapter 2

Modeling and Identification

“That’s another thing we’ve learned from your Nation,” said Mein Herr, “map-making. But we’ve carried it much further than you. What do you consider the largest map that would be really useful?”

“About six inches to the mile.”

“Only six inches!” exclaimed Mein Herr. “We very soon got to six yards to the mile. Then we tried a hundred yards to the mile. And then came the grandest idea of all! We actually made a map of the country, on the scale of a mile to the mile!”

“Have you used it much?” I enquired.

“It has never been spread out, yet,” said Mein Herr: “the farmers objected: they said it would cover the whole country, and shut out the sunlight! So we now use the country itself, as its own map, and I assure you it does nearly as well.”

— Lewis Carroll, Sylvie and Bruno Concluded

2.1 Introduction

Most control engineering exercises begin with the phrase “Given is the system with the transfer function ...”. Alas, in engineering practice, a mathematical description of the plant which is to be controlled is seldom available. The plant first needs to be modeled mathematically. In spite of this fact, control engineering courses generally do not spend much time on modeling. Perhaps this is because modeling can be said to be more of an art than a science. This Chapter begins by defining what is meant by the words model and modeling. It then illustrates various types of models, studies the process of modeling and concludes with the problem of parameter identification and related optimization techniques.

2.2 What Is a Model?

Indeed, what is a model? Probably one gets as many different responses as the number of persons one poses this question.¹ Therefore, it is not surprising that Merriam-Webster Dictionary offers 23 different definitions under that one entry. The fourth definition reads “a usually miniature representation of something” as in a “model helicopter”. The ninth definition reads, rather unglamorously, “one who is employed to display clothes or other merchandise” as in Heidi Klum, Naomi Campbell or Adriana Lima depending on which decade you came off age. Well, if your interest in models is limited to these definitions you can close this Book right now.

Definitions 11 and 12 read “a description or analogy used to help visualize something (as an atom) that cannot be directly observed” and “a system of postulates, data and inferences presented as a mathematical description of an entity or state of affairs; also: a computer simulation based on such a system <climate models>”. These definitions are closer to the sense of the word model that is used throughout this Book.

These definitions might be considered too general to be of any practical use. Let us therefore think of a model as an *approximate representation of reality*. One obvious fact, forgotten surprisingly often, is that a model is an abstraction and that any model is, by necessity, an approximation of reality. Consequently, there is no one “true model”, rather there are models which are better than others. But what does “better” mean? This clearly depends on the context and the problem at hand.

Example: An illustrative example which can be found in many high school physics books is dropping an object from the edge of a table and calculating the time it will take for the object to hit the floor. Assuming no air friction, here one can write the well-known Newtonian² equation of motion which states that the object will move with a constant acceleration caused by the weight of the object, which again is caused by the Earth’s gravitational attraction:

$$m\ddot{x}(t) = mg. \quad (2.1)$$

$\ddot{x}(t)$ denotes the second derivative of the distance x (acceleration) of the object from the edge of the table as a function of time after the drop; g is the Earth’s acceleration constant. m is the mass of the object but it is not relevant, because it can be canceled away. Solving the differential Equation (2.1) with the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ results in

$$x(t) = \frac{1}{2}gt^2. \quad (2.2)$$

¹This Chapter is an extended version of Chapter 3 in [51].

²Isaac Newton, English astronomer, physicist, mathematician (1642–1726); widely recognized as one of the most influential scientists of all time and a key figure in the scientific revolution; famous for developing infinitesimal calculus, classical mechanics, a theory of gravitation and a theory of color.

Solving (2.2) for the impact time t_i with the height of the table denoted as h results in the well-known formula

$$t_i = \sqrt{\frac{2h}{g}}. \quad (2.3)$$

Let us now consider the problem of dropping an object attached to a parachute from an aircraft. Here, one can no longer assume there is no air friction. The very reason for deploying a parachute is to make use of the air friction to slow down the impact velocity of the drop. Hence, to calculate the impact time with better accuracy, (2.1) needs to be modified to account for the air friction, a force which increases with the square of velocity:

$$\ddot{x}(t) = g - kS\dot{x}^2(t). \quad (2.4)$$

$\dot{x}(t)$ denotes the first derivative of the distance x (velocity) of the object from the drop height, k is the viscous friction coefficient and S is the effective surface area of the parachute. It is no longer possible to have a simple analytical solution of this non-linear differential equation and one has to revert to numerical means to solve it. When the terminal velocity is reached (*i.e.*, $kS\dot{x}^2 = g \Rightarrow \ddot{x} = 0$) the object stops accelerating and keeps falling with a constant velocity. $\square$

This example shows that the same physical phenomenon can be modeled in different ways with varying degrees of complexity. It is not always possible to know as in this case which model will give better results a priori. Therefore, one often speaks of the “art” of modeling. An engineer’s approach to modeling might be to begin with certain restrictive assumptions leading to a simple model. At further steps, these assumptions can be relaxed to modify the initial simple model and to account for further complications until a model is attained which is appropriate for the intended purpose.

There are many different types of models employed in science and philosophy. The reader might have heard of mental models, conceptual models, epistemological models, statistical models, scientific models, economic models or business models just to name a few. This Book limits itself to scientific and mathematical models. Scientific modeling is the process of generating abstract, conceptual, graphical or mathematical models using a number of methods, techniques and theories. The general purpose of a scientific model is to represent empirical phenomena in a logical and objective way. The process of modeling is presented in the next section.

A mathematical models helps to describe a system using mathematical tools. The use of mathematical models is certainly not limited to engineering or natural sciences applications. Mathematical models are increasingly being used in social sciences like economics, finance, psychology and sociology^{3,4}.

³Isaac Asimov, American writer (1920–1992).

⁴Which science fiction enthusiast is unaffected by the Asimovian character Hari Seldon’s “psychohistory”, which combines history, sociology and statistics to make general predictions about the

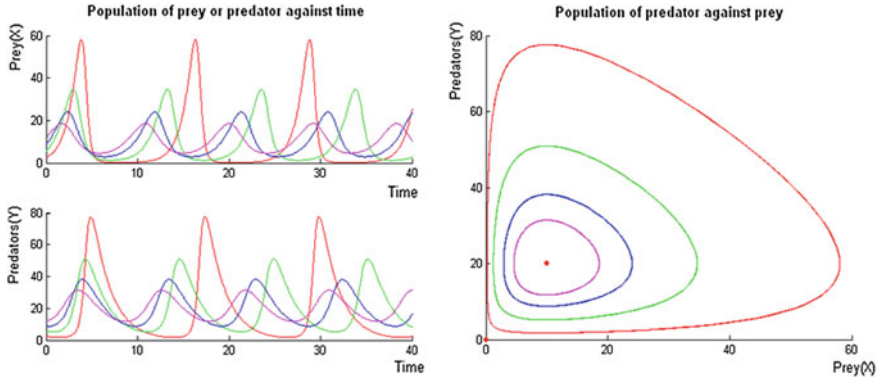


Fig. 2.1 Prey and predator populations as a numerical solution of the set of equations (2.5). Different shades of gray indicate different initial conditions [59]

Mathematical models can be classified under the following dichotomic headings: *Inductive* versus *deductive* models: A deductive model is based on a physical theory. In the example above, Newton's laws of motion were used to model the movement of a falling object. An inductive model, on the other hand, is based on empirical findings and their generalizations without forming a generally applicable law of nature. A well-known example is the set of Lotka-Volterra equations used in modeling the dynamics of biological systems in which two species interact as predator and prey [78]. Here the rate of change of the number of preys, x (say, rabbits) is proportional to the number of preys who can find ample food and who can breed, well, like rabbits. This exponential growth is corrected by prey-predator encounters (say, with foxes). Dually, the number of predators y decreases in proportion to the number of predators either because they starve off or emigrate. This exponential decay is corrected by the encounters of predators with preys. This can be modeled mathematically with the following differential equation system with $\alpha, \beta, \gamma, \delta > 0$ (Figure 2.1).

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta xy, \\ \frac{dy}{dt} &= -\gamma y + \delta xy. \end{aligned} \tag{2.5}$$

Deterministic versus *stochastic* models: In a deterministic model no randomness is involved in the development of future states of the modeled system. In other words, a deterministic model will always produce the same output starting with a given set of initial conditions as in the example above. A stochastic model, on the other hand, includes randomness. The states of the modeled system do not have unique values.

future behavior of large groups of populations—like the Galactic Empire with a quintillion citizens [7]?

They must rather be described by probability distribution functions. A good example in physics is the movement of small particles in a liquid—Brownian motion—resulting from their bombardment by a vast number of fast moving molecules [36]. Finance uses mostly stochastic models for predictions. These are discussed in detail in Chapter 5.

Random or Chaotic?

Webster Dictionary defines “random” as “a haphazard course; without definite aim, direction, rule, or method”. The same source defines “chaos” as “complete confusion and disorder; a state in which behavior and events are not controlled by anything”. Those descriptions sound quite similar. However, in the mathematical or engineering context the word “chaos” has a very specific meaning. Chaos theory is a field in mathematics which studies the behavior of dynamic systems that are extremely sensitive to initial conditions. Tiny differences in initial conditions result in completely different outcomes for such dynamic systems. As the frequently told anecdote goes, Edward Norton Lorenz, a meteorologist with a strong mathematical background in non-linear systems, was making weather predictions using a simplified mathematical model for fluid convection back in 1961. The outcomes of the simulations were wildly different depending on whether he used three digits or six digits to enter the initial conditions of the simulations. Although it was known for a long time that non-linear systems had erratic or unpredictable behavior, this experience of Lorenz became a monumental reminder that even though these systems are deterministic, meaning that their future behavior is fully determined by their initial conditions, with absolutely no random elements involved, they cannot be used to predict the future for any meaningful purpose. Lorenz is reputed to have quipped “Chaos is when the present determines the future, but the approximate present does not approximately determine the future”.

A double pendulum is made of two rods attached to each other with a joint. One of the pendulums is again attached with a joint to a base such that it can revolve freely around that base. Anyone who watches this contraption swing for some time cannot help being mesmerized by its unpredictable slowing downs and speeding ups. The double pendulum can be modeled very accurately by a non-linear ordinary differential equation (see, for instance, [135] for a complete derivation). However, when one tries to simulate (or solve numerically) this equation one will observe the sensitive dependency of its behavior on the initial conditions one chooses.

There is even a measure for chaos. The Lyapunov exponent of a dynamic system is a quantity that characterizes the rate of separation of infinitesimally close trajectories. Two trajectories in state space with initial separation $\delta \mathbf{x}_0$ diverge at a rate given by $|\delta \mathbf{x}(t)| \approx e^{\lambda t} |\delta \mathbf{x}_0|$ where λ denotes the Lyapunov exponent.

Static versus dynamic models: In static models, as the name suggests, the effect of time is not considered. Dynamic models specifically account for time. Such models make use of difference or differential equations with time as a free variable. Looking at the input-output relationship of an amplifier far from saturation, a static model will simply consist of the amplification factor. A dynamic model, on the other hand, will use a time function to describe how the output changes dynamically, including its transient behavior, as a consequence of changes in the input [26].

Discrete versus continuous models: Some recent models of quantum gravity [122] notwithstanding, in the current *Weltbild* time flows smoothly and continuously. Models building on that—often tacit—assumption make use of differential equations, solutions of which are time continuous functions. However, the advent of the digital computer which works with a clock and spews out results at discrete time points made it necessary to work with discrete time models which are best described using difference equations. Whereas the classical speed control by means of a fly ball governor is based upon a continuous time model of a steam engine [130], modern robot movement controllers employing digital processors use discrete time models [106].

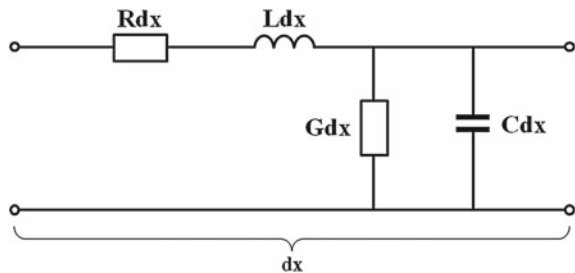
Lumped parameter models versus distributed parameter models: Distributed parameter or distributed element models assume that the attributes of the modeled system are distributed continuously throughout the system. This is in contrast to lumped parameter or lumped element models, which assume that these values are lumped into discrete elements. One example of a distributed parameter model is the transmission line model which begins by looking at the electrical properties of an infinitesimal length of a transmission line and results in the telegrapher's equations. These are partial differential equations involving partial derivatives with respect to both space and time variables [92]. Equations describing the elementary segment of a lossy transmission line developed by Heaviside⁵ in 1880 as shown in Figure 2.2 are

$$\frac{\partial}{\partial x} V(x, t) = -L \frac{\partial}{\partial t} I(x, t) - RI(x, t), \quad (2.6)$$

$$\frac{\partial}{\partial x} I(x, t) = -C \frac{\partial}{\partial t} V(x, t) - GV(x, t). \quad (2.7)$$

$V(x, t)$ and $I(x, t)$ denote the position and time dependent voltage and current respectively. The parameters R, G, L, C are the distributed resistance, conductance, inductance and capacitance per unit length. A lumped parameter model simplifies the description of the behavior of spatially distributed physical systems into a topology consisting of discrete entities that approximate the behavior of the distributed system. The mathematical tools required to analyze such models are ordinary differential equations involving derivatives with respect to the time variable alone. A simple example for a lumped parameter models is an electrical circuit consisting of a resistor and an inductor in series. Such a circuit can adequately be described by the ordinary differential equation

Fig. 2.2 Schematic representation of the elementary components of a transmission line with the infinitesimal length dx



⁵Oliver Heaviside, English mathematician and engineer (1850–1925).

$$V(t) = L \frac{dI(t)}{dt} + RI(t). \quad (2.8)$$

Linear models versus non-linear models: It can be said that a model is linear if the net response caused by two stimuli $u_1(t)$, $u_2(t)$ is the sum of the responses $f(u_1(t))$, $f(u_2(t))$ which would have been caused by each stimulus individually [64]. Formally,

$$f(\alpha_1 u_1(t) + \alpha_2 u_2(t)) = \alpha_1 f(u_1(t)) + \alpha_2 f(u_2(t)). \quad (2.9)$$

Most physical phenomena are inherently non-linear. However, they may legitimately be modeled linearly either in limited operational ranges (e.g., if an amplifier, as in Figure 2.3a, working far from saturation amplifies 1 to 5, it amplifies 2 to 10) or linearized around an operating point (e.g., a diode, as in Figure 2.3b). While writing this Chapter, it was suggested by a student that dividing all models in linear and non-linear models is like dividing the world in bananas and non-bananas. Such a remark is clearly facetious, because there is a closed theory for linear systems and many tools and methods exist for dealing with them. On the other hand, non-linear systems are analyzed mostly with approximative numerical methods specific to the problem at hand [136].

It can be said that nature is inherently dynamic, stochastic, continuous, and non-linear with distributed parameters, hence difficult to model accurately. However, depending on the application area, many simplifying assumptions can be made to make the model more amenable to analytical techniques as indicated above.

To sum it up, a good model should provide some insight which goes beyond what is already known from direct investigation of the phenomenon being studied. The more observations it can explain and the more accurate predictions it can make, the better is the model. Combine that with the quote “Everything should be made as simple as possible, but not simpler” attributed to Einstein,⁶ one has all the elements of a good model. A corollary to this statement is that the level of detail incorporated in the model is determined by the intended purpose of the model itself.

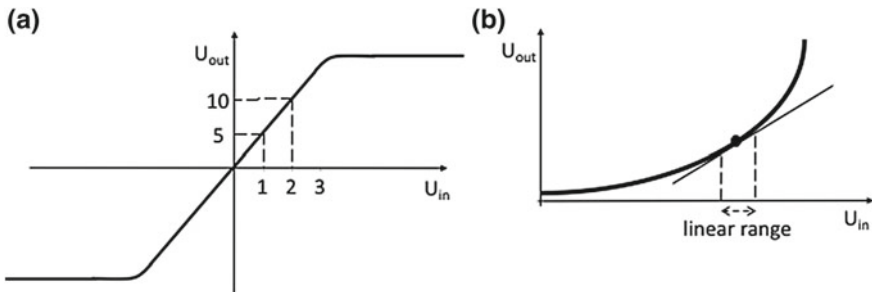


Fig. 2.3 **a** Linear amplifier with saturation. **b** Diode linearized around an operating point

⁶Albert Einstein, German-American physicist (1879–1955); famous for his explanation of the photoelectric effect and the relativity theory which radically changed the understanding of the universe.

The four distinctly different types of models depending on their purposes can be categorized as descriptive, interpretive, predictive or explanatory models. *Descriptive* models represent relationships between various variables using a concise mathematical language. Hooke's law stated as

$$F = -kx, \quad (2.10)$$

is a well-known example here, with F representing the force which lengthens a spring by the distance x . One could describe this law in words or using diagrams, but the expression in (2.10) is a pithy description.

Suppose one wants to model the purchasing power of money in an inflationary economy. One might plot the number of loaves of bread one can buy with a constant salary every month and see that this number drops with time and that the rate of the drop is constant as in Figure 2.4. One can use this exponential decay model to *interpret* the decay rate as a measure of the inflation.

A *predictive* model is required if the model will be used to see how a system will react to a certain stimulus. A typical example is a linear electrical circuit with a resistor, an inductor and a capacitor. The input-output transfer function of such a circuit can be described using the Laplace transform, for example, as

$$G(s) = \frac{2}{s^2 + 3s + 8}. \quad (2.11)$$

The response of the system to a unit step at its input can be simulated using this model as shown in Figure 2.5.

An *explanatory* model is a useful description and explanation why and how something works. It has no claim on accuracy or completeness. An example from finance is how the exchange rate between US Dollars and Turkish Lira affects the inflation in Turkey due to the changing price of imported goods.

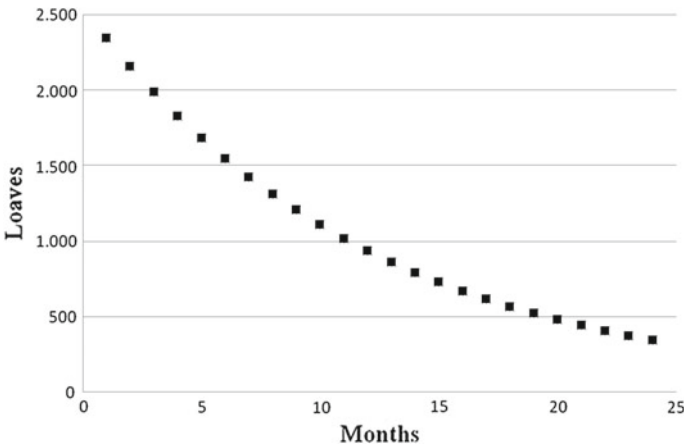


Fig. 2.4 Number of loaves of bread one can buy with a fixed salary in an inflationary economy

A general mathematical model can be represented with a block diagram⁷ as shown in Figure 2.6.

A Philosophical Excursion

Let us consider the Newtonian model of gravity. Newton posits an invisible force able to act over distances which attracts bodies. This force is proportional to the masses of the bodies and inversely proportional to the square of the distance between the bodies. Note that Newton made no attempt at explaining the cause of that attraction. This model is immensely successful. One can even argue that, together with his three laws of motion,

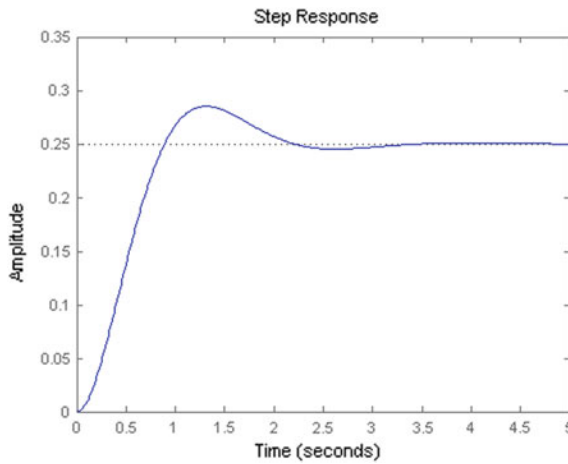


Fig. 2.5 Response of the system in (2.11) to a unit step at its input

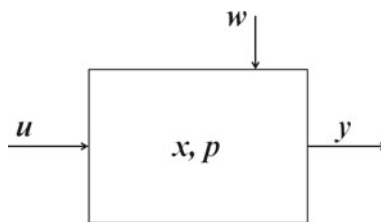


Fig. 2.6 Block diagram representation of a mathematical model has different types of variables: input variables which can be controlled $\mathbf{u}$, input variables which cannot be controlled (often random and called disturbances) $\mathbf{w}$, output variables $\mathbf{y}$, internal state variables $\mathbf{x}$, and model parameters $\mathbf{p}$. Since there can be several of each type of variables, those are aggregated in vectors. The general state space representation of such a system is given by the equations $\dot{\mathbf{R}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}, \mathbf{p}, t)$ and $\mathbf{y}(t) = \mathbf{g}(\mathbf{x}, \mathbf{u}, \mathbf{w}, \mathbf{p}, t)$

⁷Lower case boldface characters are used for vectors and upper case boldface characters are used for matrices throughout the Book unless otherwise specified.

Newtonian mechanics constitutes one of the pillars on which the industrial revolution and the ensuing increase in the welfare of the human population during the last three centuries builds. Construction machines, the houses these machines build, the cars that travel among these houses, the aircraft that move people from one continent to another, the spacecraft that brought the Apollo astronauts to the moon, just to give a couple of examples, are all built using Newtonian mechanics. So, does Newtonian mechanics constitute a correct model of reality?

On the other hand, if one wants to use the satellite based global positioning system (GPS) to determine a position on Earth accurately, Newtonian mechanics will fail. Einstein's general relativity theory does not talk about invisible forces of attraction between bodies but posits the very fabric of space being curved due to the presence of matter. A consequence of this theory is that "time flows faster" in the vicinity of matter. Concretely, this means that the clocks on the surface of the earth and on a satellite in earth orbit do not tick to the same rhythm. Since GPS makes use of accurate time intervals between signals received and sent by satellites, not taking this relativistic effect into account would reduce its accuracy. So, does Einstein's mechanics constitute a correct model of reality?

In case of building an aircraft, one might argue that Newtonian mechanics is demonstrably correct since aircraft fly all the time. In case of building an accurate GPS, one might argue that general relativity is demonstrably correct since, using GPS, hikers or soldiers in unfamiliar terrain know exactly where they are. It follows that the question about which model is better or correct depends on the problem at hand. Clearly, one does not need to deal with the complexity of Einstein's field equations when a Newtonian model suffices to go to the moon.

The pesky question arises inevitably: yes, but how is it in reality? Do objects attract each other or is the space curved due to the presence of matter or is it another yet to be developed theory? The answer is probably none of the above. Depending on the problem at hand it will be more appropriate to use one model of reality or other. The point to remember is that a model of reality is just that: a model. Like all models, that model also has its limitations. If one wants to have a model which incorporates *all* aspects of reality that model will have to be the reality itself as Mein Herr points out in Lewis Carroll's Sylvie and Bruno Concluded: "We actually made a map of the country, on the scale of a mile to the mile!"

Then pops up the next question: which model is closer to reality? Well, even if there is an absolute reality out there, it is highly questionable that science will ever know how it is in reality. As observations and intellectual capability improve, scientists will most likely be able to develop models which will explain more phenomena, more of the observations and make more precise predictions. However, they will probably never know in the epistemological sense whether they are any closer to the reality. Besides, and here comes a rather provocative statement, science is not interested in finding the reality, it is content to find progressively better models of it; search for reality is outside the scope of science!

2.3 Modeling Process

Maybe it will relax the reader to hear that there are no exact rules in mathematical modeling and there are no “correct answers” as should be apparent by now. But how does one get a mathematical model of a process? Let us begin answering this question with a couple of examples.

Example: Consider the inverted pendulum problem: there is a pole mounted with a ball bearing fulcrum on a cart as shown in Figure 2.7. The pole’s vertical equilibrium point is unstable and it will fall down due to the smallest of disturbances. Therefore, the task is to move the cart left and right in such a way as to keep the pole stabilized at its vertical position, much like a jongleur at a country fair who keeps a sword balanced at the tip of his finger by moving his hand back and forth. How could one achieve this target?

The engineers among the readers will immediately recognize this as a typical control problem and will want to apply various techniques they have learned at control engineering classes. They will also remember that a mathematical model of the “plant” they should control is required. Therefore, they will begin to solve the problem by building a mathematical model of the system composed of the cart and the pole. To build the model, it makes sense firstly to analyze the forces involved. A diagram as in Figure 2.8 is very helpful here. Summing the forces in the diagram of the cart in the horizontal direction gives

$$M\ddot{x} + b\dot{x} + N = F, \quad (2.12)$$

with M as the mass of the cart and b as the friction coefficient. Summing the forces in the diagram of the pole in the horizontal direction gives

$$N = m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta, \quad (2.13)$$

with m as the mass of the pole, l as the length of the pole and θ as the angle of the pole from the vertical. Substituting (2.13) in (2.12) yields the first equation of motion for the system:

$$F = (M + m)\ddot{x} + b\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta. \quad (2.14)$$

Fig. 2.7 How to move the cart left and right such that the pole on the cart stays vertical?

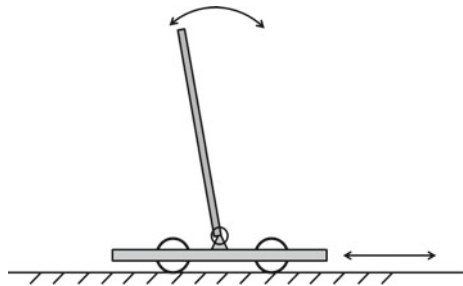
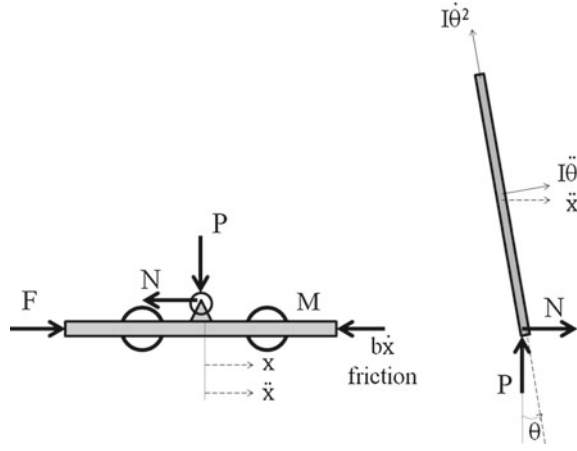


Fig. 2.8 Analyzing the forces in the cart-pole system



Summing up the forces acting on the longitudinal axis of the pole:

$$P \sin \theta + N \cos \theta - mg \sin \theta = m l \ddot{\theta} + m \ddot{x} \cos \theta. \quad (2.15)$$

Then, summing the moments around the center of gravity of the pole to get rid of the terms N and P :

$$- P l \sin \theta - N l \cos \theta = I \ddot{\theta}, \quad (2.16)$$

with I as the moment of inertia of the pole and combining (2.16) with (2.15) yields the second equation of motion as

$$(I + m l^2) \ddot{\theta} + m g l \sin \theta = - m l \ddot{x} \cos \theta. \quad (2.17)$$

At this stage it is assumed that the pole does not deviate too much from the vertical and the equations are linearized about $\theta = \pi$. With $\theta = \pi + \phi$ and ϕ small means $\cos \theta \approx -1$, $\sin \theta \approx -\phi$ and further $(\frac{d\theta}{dt})^2 = (\frac{d\phi}{dt})^2 \approx 0$ results in the linearized equations of motion

$$(I + m l^2) \ddot{\phi} - m g l \phi = m l \ddot{x}, \quad (2.18)$$

$$F = (M + m) \ddot{x} + b \dot{x} - m l \ddot{\phi}. \quad (2.19)$$

Defining the state vector as $\mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4]^T = [x \ \dot{x} \ \phi \ \dot{\phi}]^T$ and assuming that the position of the cart (x) as well as the deviation from the equilibrium point of the pole (ϕ) are measured, the state space representation (see Appendix A) of the model can be written in the form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{d}u\end{aligned}\quad (2.20)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ \frac{(I+ml^2)}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{I(M+m)+Mml^2} \end{bmatrix}, \quad (2.21)$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2.22)$$

Comparing these equations with Figure 2.6 one can see that F , the force moving the cart left and right, is the control vector $\mathbf{u}$ (a scalar in this case), $\mathbf{f}$ and $\mathbf{g}$ are linear functions given by equation (2.20), the parameter vector $\mathbf{p}$ comprises the physical parameters M, m, l, I, g and b , and the disturbance $\mathbf{w}$ is ignored.

These equations represent a natural end of the modeling exercise. The system is modeled with a linear 4th order ordinary differential equation. The rest is a straight forward exercise in control engineering. Possible ways of calculating F are discussed in Chapter 4 and many impressive demonstrations can be found on the YouTube. $\square$

2.3.1 Stock Prices

An interesting problem that arises in financial engineering is how to determine the relative weights of stocks in a diversified portfolio [83] (Chapter 8 addresses this problem in detail). This requires estimates of future stock prices. Most of the time, these estimates are based on past prices of the stocks.⁸

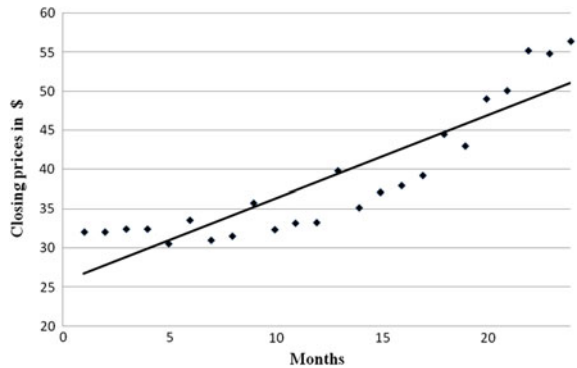
Example: Imagine that the closing prices of a stock during the last 24 months are as shown in Table 2.1.

In order to be able to estimate the closing prices of this stock for the next few months one can naively model the price as a linear function of time and find the equation of the line that best fits the data of the past 24 months. A simple linear regression analysis based on Gaussian⁹ least squares method results in

⁸Louis Jean-Baptiste Alphonse Bachelier, French mathematician (1870–1946) is credited with being the first person who tried modeling stock prices mathematically [134].

⁹Carl Friedrich Gauss, German mathematician (1777–1855); contributed significantly to many fields in mathematics and physics.

Fig. 2.9 Closing prices of a stock during 24 months and their regression line given by equation (2.23)



$$p(t) = 1.0596\, t + 25.635\, , \tag{2.23}$$

with p as the stock price and t as the time in months.

Plotting the equation (2.23) together with the past data is shown in Figure 2.9.

Equation (2.23) is a very rough model and the closeness of its fit to available data measured by Pearson product-moment correlation coefficient is $r^2 = 0.7935$ (the closer r^2 to 1, the better the fit [111]). When (2.23) is used to estimate the closing prices of this stock for months 25–28 (in other words, extrapolation) it results in the values in Table 2.2.

Now, observing the general tendency of the prices in Figure 2.9 to increase faster with time, one can hope to get a better fit of the data by using an exponential rather than a linear model. The regression analysis results in

$$p(t) = 27.593\, e^{0.0258t}\, . \tag{2.24}$$

Table 2.1 Closing prices of a stock during 24 months

Month	Closing price	Month	Closing price	Month	Closing price	Month	Closing price
1	32.12	7	31.03	13	39.85	19	43.01
2	32.12	8	31.56	14	35.21	20	49.05
3	32.45	9	35.75	15	37.11	21	50.07
4	32.48	10	32.39	16	38.00	22	55.22
5	30.37	11	33.19	17	39.25	23	54.82
6	33.61	12	33.29	18	44.52	24	56.46

Table 2.2 Closing prices estimated using (2.23)

Month	Closing price
25	52.13
26	53.18
27	54.24
28	55.30

Fig. 2.10 Closing prices of a stock during 24 months and their regression line given by equation (2.24)

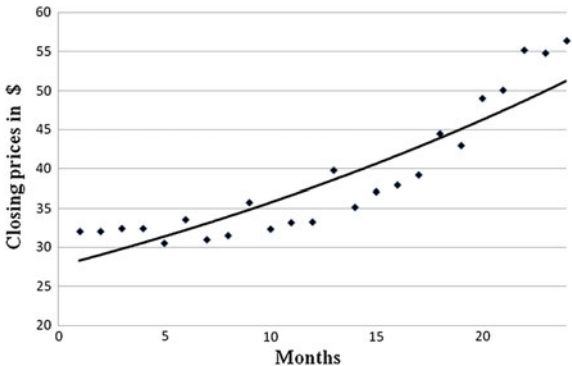


Table 2.3 Closing prices estimated using equation (2.24) are very similar to the result of the previous estimation

Month	Closing price
25	52.59
26	53.97
27	55.38
28	56.82

Table 2.4 Closing prices estimated using equation (2.25) look very different from the previous estimations

Month	Closing price
25	53.20
26	45.36
27	30.33
28	5.21
29	−33.41

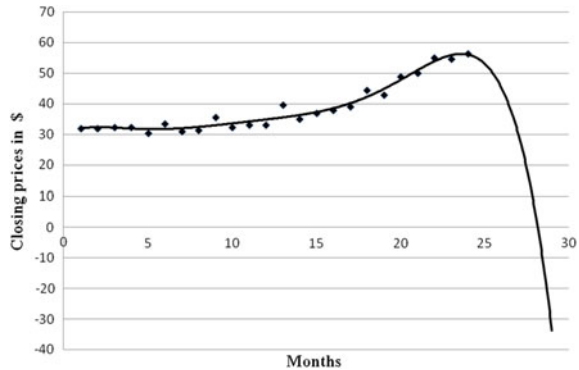
Plotting the equation (2.24) together with the past data is shown in Figure 2.10. Just by looking at their respective graphs, equation (2.24) seems to be a better model than equation (2.23). Also, here $r^2 = 0.8226$. When equation (2.24) is now used to estimate the closing prices of this stock for months 25–28, the values in Table 2.3 are obtained. Can these results be improved? One can try to model the stock prices with a polynomial to get a better fit. Indeed, for a sixth order polynomial the best curve fit results in the equation

$$p(t) = -0.000008 \, t^6 + 0.0006 \, t^5 - 0.0149 \, t^4 + 0.1829 \, t^3 - 1.0506 \, t^2 + 2.489 \, t + 30.408 .$$

(2.25)

The fit is impressive with $r^2 = 0.9666$. Since the fit is better, one might expect to get better estimates for the future values of the stocks. Therefore, (2.25) is now used to estimate the closing prices of this stock for months 25–28 which results in the values in Table 2.4. Plotting the equation (2.25) together with the past data is shown in Figure 2.11.

Fig. 2.11 The model given by equation (2.25) fits available data very well. However, this model results in forecasts with negative values for the stock and is therefore, completely useless for making predictions



Clearly, the model in equation (2.25) is not an appropriate model for forecasting future stock prices. Actually, one can see that there is always at least one $n - 1$ or higher degree polynomial that goes through n points. This means that one can always find a polynomial model that fits the given data points perfectly ($r^2 = 1$ or zero residuals). However, such models are seldom useful for interpolating and hardly ever useful for extrapolating. Actually, the modeling of stock prices has been of great interest to scientists and researchers from fields as diverse as actuary and physics [134]. Stock prices are modeled using stochastic models based on Brownian motion with impressive success as shown in Chapters 5 and 9 [1]. $\square$

2.3.2 Lessons Learned

Two modeling problems were presented and they were attacked with two very different approaches. In the first example, Newton's laws of motion were applied to the problem, deriving the differential equations governing the physical system in a fairly straight forward manner. This approach can be generalized to problems concerning the modeling of physical systems. One can analyze the given system using known physical laws like continuity or conservation of energy and obtain equations pertaining to the behavior of the system. Whenever possible, this is the preferred approach to modeling, because it also gives insight into the workings of the system. This way, what is actually going on can be "understood" and the model can be used to make fairly reliable predictions. The parameters have physical meanings like the mass of a body or the diameter of a pipe which can either be measured or estimated.

In the second example, the relationship between the input of the system (time) and its output (closing price of the stock) was considered as a black-box without any knowledge of its inner workings. This approach does not give any insights and is really an act of desperation. The modeling problem is quickly reduced to a curve fitting problem which can be solved mechanically. For instance, empirical evidence gained over centuries in the stock markets indicates the use of an exponential model [2]. Neither in this case nor in the case of polynomial models, the parameters of the model have any physical meaning. Also, as should be apparent by now, a good curve fit does not necessarily imply a good model which can be used to make predictions.

While modeling, either way, one always has to keep in mind the underlying assumptions. In the first example the equations of motion were linearized close to the vertical position of the pole. That means the model describes the behavior of the system for small deviations of the pole from the vertical in a fairly accurate way. On the other hand, as the pole moves away from its vertical position, the predictive power of the model deteriorates rapidly. Also, a controller based on that linearized model fails miserably. A common error in modeling and while using models is to forget the underlying assumptions and not be aware of the limitations or the applicability range of the model and try to use it to make predictions.

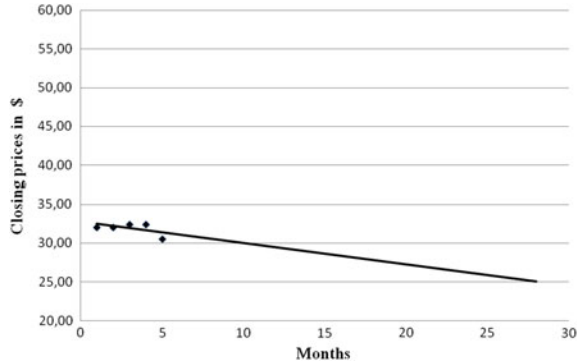
2.4 Parameter Identification

When physical laws are used to model a system, the parameters of the model (*e.g.*, the length of the pole) are often readily measurable. If not directly measurable, they can often be calculated with sufficient precision (*e.g.*, the mass of the pole can be calculated by first measuring its dimensions, using these dimensions to calculate its volume and multiplying the volume with the density of the material used to produce the pole). However, one might still end up with parameters that are neither directly measurable nor calculable (*e.g.*, the friction coefficient). Such parameters must be determined experimentally. The process of determining those model parameters is called *parameter identification*. Actually, while using black-box models, identification is often the only way to determine the model parameters, because the parameters have no physical meaning.

In order to be able to perform a reasonable parameter identification, a sufficiently large number of observations are necessary. Let us consider again the second example above. Imagine that the data available is limited to the first five data points. The result of a linear regression analysis on the available data is shown in Figure 2.12. A comparison with Figure 2.9 shows a radically different course of the regression line. Whereas the regression based on five data points predicts falling stock prices, a similar regression analysis using 24 data points predicts a rising course of the stock prices. This example demonstrates the necessity of using a sufficiently large number of data points for being able to perform parameter identification reliably. In most practical cases the consequence of using a small number of data points is not as dramatic as in this example. Rather, assuming that the model structure is correct, increasing the number of data points will increase the accuracy of the estimated parameters. Most widely used identification methods make use of linear or non-linear least squares methods. For instance [32] gives a detailed analysis of this problem.

How can one identify the parameters of a model when there are no or insufficient data for identification? The solution lies in conducting experiments to obtain sufficient data. Those experiments generally consist of exciting the system with a sufficiently rich set of inputs and observing its output. For that strategy to be successful, the model must be identifiable. A model is said to be identifiable if it is theoretically possible to infer the true value of the model's parameters after obtaining an infinite number of observations from it. In other words there must be a unique solution to the problem of finding the mapping how the input is mapped to the output of the model (see, for instance, [74] for a formal definition of identifiability).

Fig. 2.12 Regression line based on the first five data points predicts falling stock prices



If a system is modeled as a linear system, in an experiment to identify the parameters of the model, in theory, it is sufficient to excite the system with an impulse or a step function and to observe its output at different times.¹⁰ Since measurements cost time and money, one cannot make an infinite number of measurements. Therefore, one has to agree on a number of measurements. The next question to be resolved is when to conduct those measurements. Clearly, it does not help to make many measurements once the transient behavior is over. Last two points make sense intuitively. The general problem of experiment design is more complicated and treated in detail in *e.g.*, [9, 43, 108].

2.5 Mathematics of Parameter Identification

Figure 2.13 illustrates how to identify the parameters of a process. The procedure begins by applying a test input u to the system and measure its output y . The same signal is used to simulate the output of the model, $\hat{y}$. The difference between the measured and simulated outputs is built as $e = y - \hat{y}$ and squared to get $J(\mathbf{p})$ which has to be minimized with respect to the model parameters $\mathbf{p}$. This process can be repeated until the model output approximates the measured output closely enough.

How is $J(\mathbf{p})$ minimized? For that purpose, it is convenient to look at sampled values of the signals such that

$$e_k = y(t_k) - \hat{y}(t_k), \quad (2.26)$$

$$J(\mathbf{p}) = \sum_{k=1}^n e_k^2, \quad (2.27)$$

where t_k are the sampling time points and n is the number of samples. Thus, the problem becomes finding the optimal values of m parameters $p_1, p_2, \dots, p_m$ which are aggregated in the parameter vector $\mathbf{p}$ as

¹⁰A unit impulse function contains all frequencies, because its Laplace transform is simply 1. Therefore, theoretically, it is a sufficiently rich input for identification purposes.

$$\mathbf{p}^* = \arg \min_{\mathbf{p}} J(\mathbf{p}) . \quad (2.28)$$

But how does one numerically find the optimal values of the parameters which minimizes a function? There are scores of different ways to solve that problem. Let us first review some basic concepts before discussing several different algorithms.

2.5.1 Basics of Extremes

Given a continuous function with a single variable, $f(x)$ where x is unbounded and the derivatives $f'(x)$, $f''(x)$, $\dots$ exist and are continuous, the Taylor series expansion can be given as

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}(\Delta x)^2 + \dots$$

$$\Rightarrow \Delta f = f(x + \Delta x) - f(x) = f'(x)\Delta x + \frac{f''(x)}{2!}(\Delta x)^2 + \dots$$

Looking at Figure 2.14 one can see that the points A and B result when $f'(x) = 0$ is solved for x . However, at A $f''(x) < 0$, therefore A is a maximum point and at B $f''(x) > 0$, therefore B is a minimum point.

If, on the other hand, $f(x)$ is monotonous and x is bounded as in Figure 2.15, second derivatives become irrelevant. The extremes are at the boundaries of the definition domain of the function.

When functions with two variables, $f(x, y)$, are now considered again with unbounded x, y , the necessary conditions for extreme points become

$$f_x := \frac{\partial f}{\partial x} = 0, \quad f_y := \frac{\partial f}{\partial y} = 0 .$$

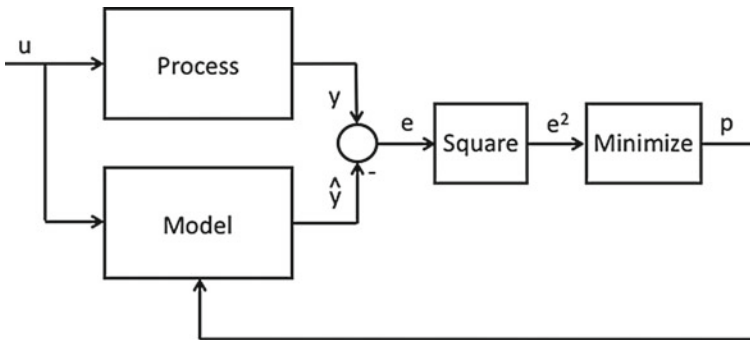


Fig. 2.13 Schematic representation of parameter identification

Furthermore, for a maximum point the conditions

$$f_{xx} < 0, \quad f_{xx}f_{yy} - f_{xy}^2 > 0,$$

and for a minimum point the conditions

$$f_{xx} > 0, \quad f_{xx}f_{yy} - f_{xy}^2 > 0,$$

must be satisfied [19].

Extending these conditions for functions with n variables $x_1, x_2, \dots, x_n$, the necessary conditions for extrema become

$$f_{x_i} = 0, \quad \forall i = 1, 2, \dots, n.$$

Fig. 2.14 Continuous function with an unbounded variable

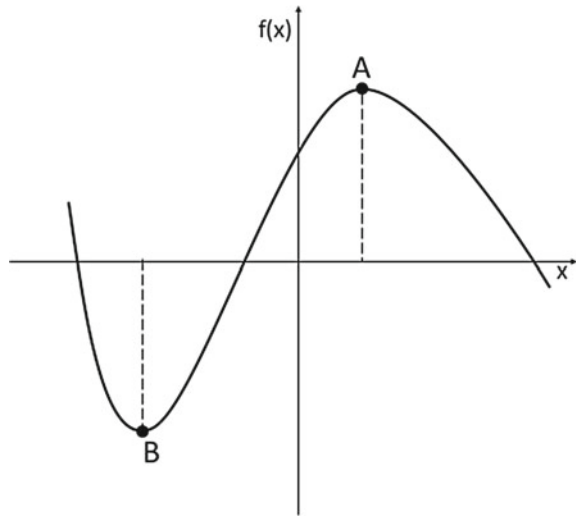


Fig. 2.15 The function $f(x)$ is defined for $a \leq x \leq b$. Absolute minimum is at $x = a$ and absolute maximum is at $x = b$. If the definition domain were open, *i.e.*, $a < x < b$, then the function would not have a minimum or a maximum in the domain. In that case one speaks of an infimum and a supremum

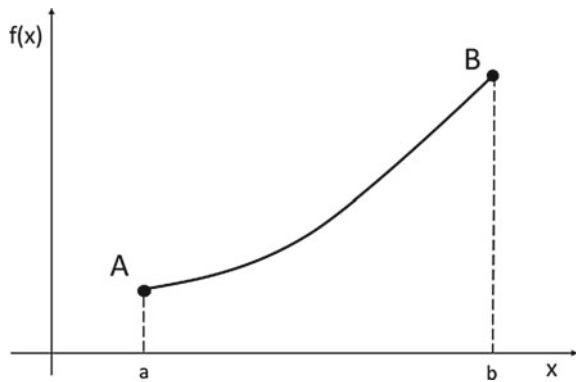
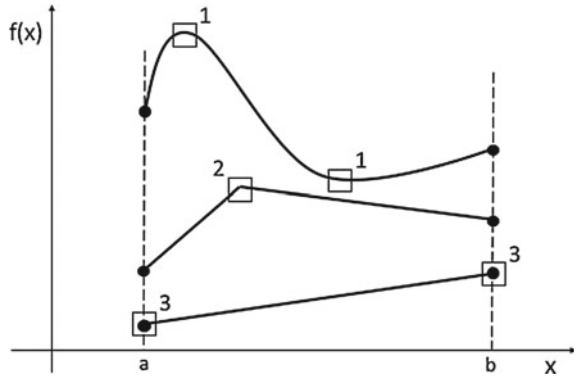


Fig. 2.16 Where to look for extreme values of a function



Furthermore, one has to build the Hessian matrix $\mathbf{H}$ composed of the second-order partial derivatives as

$$\mathbf{H} = \begin{bmatrix} f_{x_1 x_1} & f_{x_1 x_2} & \cdots & f_{x_1 x_n} \\ f_{x_2 x_1} & f_{x_2 x_2} & \cdots & f_{x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_n x_1} & f_{x_n x_2} & \cdots & f_{x_n x_n} \end{bmatrix},$$

calculate the leading first minors¹¹ of $\mathbf{H}$ as D_i and check their signs.¹² For minima

$$D_i > 0 \quad \forall i = 1, 2, \dots, n,$$

and for maxima

$$(-1)^i D_i > 0 \quad \forall i = 1, 2, \dots, n.$$

It is important to keep the Weierstrass extreme value theorem in mind whilst seeking the extreme points of a function. This theorem loosely says that if a function f is continuous in the closed and bounded interval $[a, b]$, then f must have at least a maximum and at least a minimum. Those extreme points can be found either 1) at the stationary points ($f_{x_i} = 0$), or 2) at points where one or more partial derivatives become discontinuous, or 3) at the boundaries of the interval. Figure 2.16 illustrates this theorem for functions with a single variable.

¹¹A minor of a matrix $\mathbf{A}$ is the determinant of some smaller square matrix, cut down from $\mathbf{A}$ by removing one or more of its rows or columns. Minors obtained by removing just one row and one column from square matrices are called first minors. First minors that are obtained by successively removing the last row and the last column of a matrix are called leading first minors.

¹²A similar test checks the positive or negative definiteness of the Hessian matrix via its eigenvalues to determine whether a critical point ($f_{x_i} = 0$) is a minimum or a maximum [90].

2.5.2 Optimization with Constraints

Engineers very often encounter the problem of minimizing a function while at the same time a functional equality constraint must be fulfilled. Formally, for a function with two variables x and y , one is seeking the extreme points of $z = f(x, y)$ with the constraint $g(x, y) = 0$. Using the following total differentials for extrema

$$\begin{aligned} df &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y}, \\ dg &= \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{g_x}{g_y}. \end{aligned}$$

Therefore,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{g_x}{g_y}.$$

Introducing the so-called Lagrange¹³ multiplier

$$\lambda = -\frac{f_x}{g_x} = -\frac{f_y}{g_y},$$

or

$$f_x + \lambda g_x = 0, \quad f_y + \lambda g_y = 0.$$

This means

$$\frac{\partial}{\partial x}(f + \lambda g) = 0, \quad \frac{\partial}{\partial y}(f + \lambda g) = 0.$$

With the new principal function, called the Lagrangian, $F = f + \lambda g$ and F being stationary for optimality, the optimization with constraints problem is now reduced to a simple algebraic problem with three equations

$$\begin{aligned} F_x &= 0, \\ F_y &= 0, \\ g(x, y) &= 0, \end{aligned}$$

with the three unknowns x, y, λ . This result can readily be extended to optimizing a function $z = f(x_1, x_2, \dots, x_n)$ with n variables subject to r constraints

¹³Joseph-Louis Lagrange, French mathematician (1736–1813).

$$\begin{aligned}
g_1(x_1, x_2, \dots, x_n) &= 0, \\
g_2(x_1, x_2, \dots, x_n) &= 0, \\
&\vdots \\
g_r(x_1, x_2, \dots, x_n) &= 0,
\end{aligned}$$

by constructing the Lagrangian $F = f + \lambda_1 g_1 + \lambda_2 g_2 + \dots + \lambda_r g_r$ and the condition for optimality (all derivatives of F must vanish) results in n equations $F_{x_1} = F_{x_2} = \dots = F_{x_n} = 0$. $(n + r)$ algebraic equations with $(n + r)$ unknowns need to be solved now.

Example: Where is the minimum of $f(\mathbf{x}) = 2x_1^2 + x_2^2$ subject to the condition $g(\mathbf{x}) = 1 - x_1 - x_2$?

Firstly, building the Lagrangian as

$$L(x_1, x_2, \lambda) = 2x_1^2 + x_2^2 + \lambda(1 - x_1 - x_2),$$

and taking the derivatives of the Lagrangian and equating them to zero as

$$\begin{aligned}
\frac{\partial L}{\partial x_1}(x_1^*, x_2^*, \lambda^*) &= 4x_1^* - \lambda^* = 0, \\
\frac{\partial L}{\partial x_2}(x_1^*, x_2^*, \lambda^*) &= 2x_2^* - \lambda^* = 0, \\
\frac{\partial L}{\partial \lambda}(x_1^*, x_2^*, \lambda^*) &= 1 - x_1^* - x_2^* = 0,
\end{aligned}$$

and solving these equations results in $x_1^* = \frac{1}{3}$, $x_2^* = \frac{2}{3}$ and $\lambda^* = \frac{4}{3}$. Therefore, the optimal value of the objective function is $\frac{2}{3}$. Since the objective function is convex and the constraint is linear, this optimum solution minimizes the objective function subject to the constraint.

The geometric interpretation of this problem becomes apparent when the objective function and the constraint are plotted as shown in Figure 2.17. The gradients of f and g at the optimum point must point in the same direction but they may have different lengths. In other words, $\nabla f(\mathbf{x}^*) = \lambda \nabla g(\mathbf{x}^*)$. This is exactly the condition for optimality $\nabla L(\mathbf{x}^*, \lambda^*) = 0$.

It is not always easy to find interpretations for Lagrange multipliers in constrained optimization problems. However, in economic problems of the type “Maximize $p(x)$ subject to $g(x) = b$ ” where $p(x)$ is a profit to maximize and b is a limited amount of resource, then the optimal Lagrange multiplier λ^* is the marginal value of the resource. In other words, if b were increased by Δ , the profit would increase by $\lambda^* \Delta$. Similarly, in problems of the type “Minimize $c(x)$ subject to $d(x) = b$ ” where $c(x)$ is the cost to minimize and b is the demand to be met, then the optimal Lagrange multiplier λ^* is the marginal cost of meeting the demand.

Coming back to the example, the minimum value of $f(\mathbf{x}) = 2x_1^2 + x_2^2$ subject to the condition $x_1 + x_2 = 1$ was found to be $\frac{4}{3}$ at $x_1^* = \frac{2}{3}$, $x_2^* = \frac{2}{3}$. If the right hand side is now changed from 1 to 1.05 (i.e., $\Delta = 0.05$), then the optimum objective function value goes from $\frac{2}{3}$ to roughly $\frac{2}{3} + \frac{4}{3} \cdot 0.05 = \frac{2.2}{3}$. $\square$

Example: A common problem in finance is the allocation of funds among different investment instruments. How should an investor divide her savings among three mutual funds with expected returns 10%, 10% and 15% respectively, to minimize her risk while achieving an expected return of 12%? The risk is defined as the variance of the return of an investment (this is discussed in detail in Chapter 6). With the fraction x of the total invested amount being invested in Fund 1, y in Fund 2 and z in Fund 3, where necessarily $x + y + z = 1$, the variance of the return in this example has been calculated to be

$$r(x, y, z) = 400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz.$$

So, the problem is to minimize $r(x, y, z)$ subject to

$$x + y + 1.5z = 1.2,$$

$$x + y + z = 1.$$

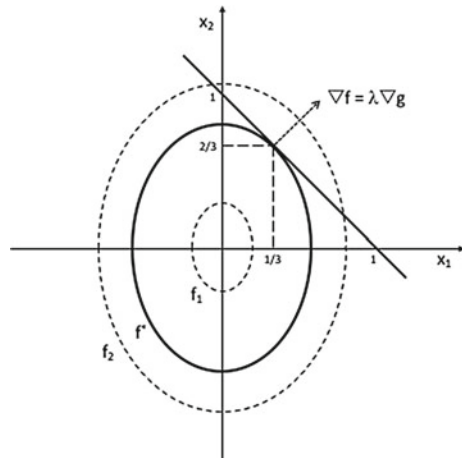
Using the Lagrangian method above, the optimal solution results in

$$x^* = 0.5, \quad y^* = 0.1, \quad z^* = 0.4, \quad \lambda_1^* = 1800, \quad \lambda_2^* = -1380,$$

where λ_1 is the Lagrange multiplier associated with the first constraint and λ_2 with the second constraint. The optimal value of the objective function, in other words the minimum variance of the return, is 390.

It is now possible to answer the following question approximately: If an expected return of 12.5% was desired instead of 12%, what would the corresponding variance of the return be? Δ is 0.05 and the variance would increase by $\Delta\lambda_1 = 0.05 \cdot 1800 = 90$. Therefore, the answer is $390 + 90 = 480$. Intuitively, it is not surprising that a higher risk has to be accepted in order to achieve a higher return. This thinking is the basis of Modern Portfolio Theory which is treated in Chapter 8. $\square$

Fig. 2.17 Geometric interpretation of minimizing $f(x_1, x_2) = 2x_1^2 + x_2^2$ subject to the constraint $g(x_1, x_2) = 1 - x_1 - x_2$



Sometimes the constraints which need to be considered are inequalities. The procedure for finding the constrained optimum in this case becomes a bit more tedious and is conducted by checking so-called Karush-Kuhn-Tucker conditions, often abbreviated as KKT [73].

2.6 Numerical Methods for Parameter Identification

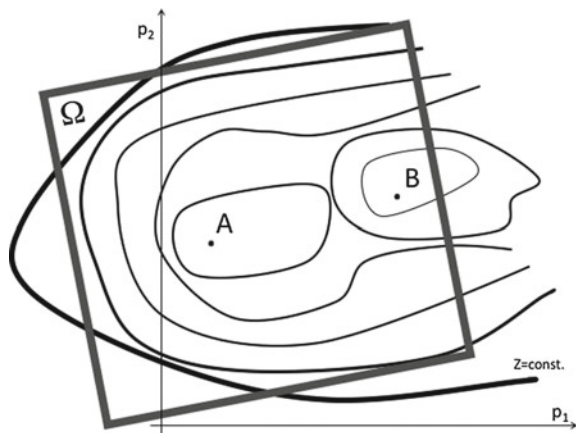
It became clear so far, that parameter identification ultimately boils down to an optimization problem with constraints. The objective function to minimize is $Z = Z(p_1, p_2, \dots, p_m)$ where $p_1, p_2, \dots, p_m$ are the m parameters to be identified and Ω denotes the permissible region in the parameter space. As Figure 2.18 shows for a case with two parameters, the lowest point of Z in Ω is sought. In most cases, an analytical solution of the problem is not possible and solving it numerically is the only option. There are a myriad of different algorithms in the literature. Some of the best books in this area are [14, 16, 21, 100]. Only a couple of representative examples of numerical optimization algorithms are described in this section.

There are two important caveats here: (a) no single numerical method is applicable for all problems, and (b) where the objective function is not convex (multi-modal), meaning that it has several local minima, one can never be sure that the solution the numerical search algorithm delivers is the absolute minimum. Therefore, for most of the numerical optimization algorithms to be successful, several assumptions about the objective function are necessary:

- Z is convex (unimodal) in Ω ,
- Z is “sufficiently smooth” in Ω , meaning that the parameter landscape has no precipitous ravines,
- Z is differentiable with respect to all parameters p_i in Ω ,
- The region Ω is simply connected, meaning that it has no “holes”.

The numerical search algorithms must necessarily have termination criteria. These can be (a) the maximum number of steps exceeded, (b) change in the value of Z from one step to the next is below a threshold, (c) change in parameter values from one step to the next is below a threshold.

Fig. 2.18 The curves denote the counter lines of the objective function Z (points where Z is constant). Ω is the permissible region for the parameters p_1, p_2 . A is a relative minimum in Ω . B is the absolute minimum in Ω



2.6.1 Golden Section

A popular method for numerically minimizing a unimodal function with a single variable derives from the Fibonacci¹⁴ search method. Assuming that the interval (b^1, u^1) entails the minimum, the algorithm can be defined as follows:

1. Find the points $\lambda_1^1 = (1 - \beta)(u^1 - b^1) + b^1$ and $\lambda_2^1 = \beta(u^1 - b^1) + b^1, k = 1$; go to (3)
2. Find the points $\lambda_1^k = (1 - \beta)(u^k - b^k) + b^k$ and $\lambda_2^k = \beta(u^k - b^k) + b^k$; calculate $Z(\lambda_1^k)$ and $Z(\lambda_2^k)$
3. If $Z(\lambda_1^k) > Z(\lambda_2^k)$ $b^{k+1} = \lambda_1^k, \lambda_1^{k+1} = \lambda_2^k, u^{k+1} = u^k, k = k + 1$, go to (2) and calculate λ_2^k .
Else $u^{k+1} = \lambda_2^k, \lambda_2^{k+1} = \lambda_1^k, b^{k+1} = b^k, k = k + 1$, go to (2) and calculate λ_1^k
4. Repeat above steps until $\lambda_2^k - \lambda_1^k < \epsilon$.

This method is called golden section,¹⁵ because

$$\begin{aligned} \frac{b^1 \lambda_2^1}{b^1 u^1} &= \frac{b^1 \lambda_1^1}{b^1 \lambda_2^1} = \beta, & \frac{b^1 \lambda_1^1}{b^1 u^1} &= 1 - \beta \\ \Rightarrow \beta^2 &= 1 - \beta & \Rightarrow \beta &= \frac{-1 + \sqrt{5}}{2} \approx 0.618, \end{aligned}$$

which is known as the golden ratio.

2.6.2 Successive Parameter Optimization

So, what does one do when one has several parameters which must be identified? One possibility is to keep all the parameters constant except for one and optimize it, possibly using the golden section method or something similar. Then go to the next parameter, keep all the rest of the parameters constant and optimize that one. One keeps doing this successively for each parameter.¹⁶ The optimum combination of parameters which minimizes the objective function will result pretty quickly, provided that the objective function is unimodal. However, most of the time, one has no way of knowing whether the objective function is unimodal. Therefore, it is best to assume that it is not and repeat the optimization process, starting with a new set of initial values of parameters until the results are satisfactory. Figure 2.19 illustrates successive parameter optimization.

¹⁴Leonardo Fibonacci, Italian mathematician (c.1175–c.1250).

¹⁵Already the Ancient Greek mathematicians, most notably Euclid, were fascinated by the so-called golden ratio which is recurrent in the nature. This ratio is given as $\frac{a+b}{a} = \frac{a}{b}$.

¹⁶In Germanic literature this method is known as “*achsenparallele Suche*” or “search parallel to the axes”.

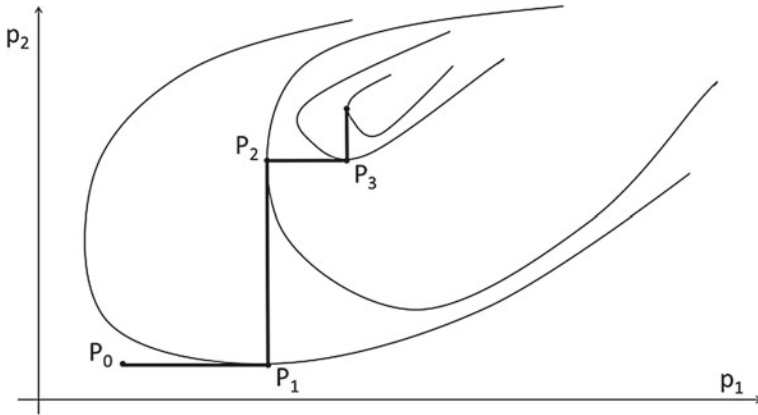


Fig. 2.19 Optimizing one parameter after the other. Note that the contour lines are not known

Let us call the starting set of parameters P_0 and the step size Δ .

$$\begin{aligned}
 P_0 : \quad Z_0 &= Z(p_{10}, p_{20}, \dots, p_{n0}); \\
 P_1 : \quad Z_{01} &= Z(p_{10} + N_1 \Delta, p_{20}, \dots, p_{n0}); \quad p_{11} = p_{10} + N_1 \Delta \\
 P_2 : \quad Z_{02} &= Z(p_{11}, p_{20} + N_2 \Delta, \dots, p_{n0}); \\
 &= Z(p_{11}, p_{21}, p_{30}, \dots, p_{n0}); \\
 &\vdots \\
 P_n : \quad Z_{0n} &= Z(p_{11}, p_{21}, p_{31}, \dots, p_{n0} + N_n \Delta); \\
 &= Z(p_{11}, p_{21}, p_{31}, \dots, p_n); \\
 &= Z_1.
 \end{aligned}$$

P_n is then the starting point of the second search phase. It makes sense to reduce the step size Δ as one comes closer to the minimum.

Another way for optimizing an objective function is to start somewhere, look for the steepest gradient and move a step in that direction. At this new point again look for the steepest gradient and move a step in that direction and so on. This method is known as the *steepest descent*. Several variations of the steepest descent method are explained in detail, for instance, in [123]. It is also possible to proceed without computing any gradients. The so-called *direct search* method looks around the immediate vicinity of the starting point to make a picture of the topography and makes a long step in a descent direction. The procedure is then repeated [71].

The use of so-called *linear programming* is indicated if the objective function and the constraints are all linear [121]. This method of optimization was developed in the middle of the previous century to solve logistics problems with the then nascent digital computers.

A more modern approach is the so-called *particle swarm optimization* which finds an optimum by iteratively improving the result. It begins with a swarm of particles as candidate results and moves these around the parameter space based on their positions and velocities. Each particle moves towards a better solution based not only on their own positions and velocities but also on those of other particles [68].

2.7 Model Validation

Model validation is the process of checking whether a model is appropriate for the intended purpose. Remembering that a model is, by its very nature, but an approximation of reality, it becomes clear that it cannot replicate all the features of the actual system. Therefore, to validate a model means to verify that it can reproduce the features that are essential for the purpose for which the model was developed. One has to keep in mind that model validation is an integral part of the modeling process. In other words, one has to verify the model's appropriateness both during the modeling process and at the end of it. This is done primarily by simulating the system behavior by solving the model equations with the identified parameters. In most cases, the model consists of a set of differential equations which must be integrated numerically. Many techniques and algorithms and even dedicated simulation languages are available [26, 27].

Another important requirement of a model is its parsimony. It is quite possible to develop models that describe the same phenomena appropriately but with varying degrees of complexity. Among such models, the one with the lowest order or smallest number of parameters, the most parsimonious one, is to be preferred. This principle is a direct application of Occam's razor.¹⁷

A common problem in connection with modeling and particularly model validation is model reduction. In many applications, the available model of a system, although appropriate for most purposes of modeling, might be too complex for controller design purposes. It then makes sense to reduce the order of the model. There are several ways of achieving model reduction. Some are obvious, straight forward and simple. Others are more sophisticated and can be rather tedious to perform (e.g., balanced model reduction [50]).

2.8 Summary

This Chapter began by discussing what is meant by a model. A model is always an approximation of reality. Therefore, there is no one "true model". Depending on the context and purpose, there are models which are better than others. Using the example of a falling object it was seen that depending on the context, it might make sense to neglect certain aspects of reality in the modeling process.

¹⁷Occam's razor is the law of parsimony, economy or succinctness. It is a principle urging one to select from among competing hypotheses the one which makes the fewest assumptions. Although the principle was known earlier, Occam's razor is attributed to the 14th-century English Franciscan friar William of Ockham.

2.9 Exercises

A1: Explain in your own words what you understand when you hear the word “model”.

A2: Explain how you would go about modeling an electro-mechanical system.

A3: Why is it difficult to develop a model to predict stock prices?

A4: What are the difficulties in designing experiments for validating a mathematical model of a process?

A5: What are the major difficulties in identifying the parameters of a model numerically and how can those difficulties be dealt with?

B1: A furniture manufacturer produces two sizes of boxes (large, small) that are used to make either a table or a chair. A table makes \$3 profit and a chair makes \$5 profit. If m small blocks and n large blocks are produced, how many tables and chairs should the manufacturer make in order to obtain the greatest profit? What else do you need to know?

B2: Investigate the population dynamics of whales and krill, where krill is assumed to be the main source of food for the whales. We will use the following model:

$$\begin{aligned}\dot{k} &= (a - bw) k, \\ \dot{w} &= (-m + nk) w.\end{aligned}$$

k is the krill population (as a function of the time in years), w is the whale population, and a, b, m and n are positive constants. Firstly, interpret this system of differential equations and the meaning of the constants. Determine if there are any equilibrium points of this system. If so, can you say anything about the stability of these points? Write the equations for simulating this system. Consider the parameter values $a = 0.2$, $b = 0.0001$, $m = 0.5$, $n = 0.000001$. With the starting points $k(0) = 700000$ and $w(0) = 3000$, and time step $\Delta T = 0.3$ years, manually simulate $k(t)$ and $w(t)$ for $t = 0.3$ and $t = 0.6$. Using the parameter values above, use Mathematica (or any other simulation package) to simulate the model from several different starting points. To get interesting results do not just pick arbitrary starting points, but be systematic and think how they relate to the question about the stability of equilibrium points. Discuss your observations. Can you draw general conclusions not dependent on this particular choice of values? Investigate the effect of krill fishing on these populations. To model this, add a term $-rk$ to the equation for $\dot{k}$, where $r < a$. Interpret this term. Try out different values of r , simulate and discuss your observations. Could you suggest how the model could be refined or extended?

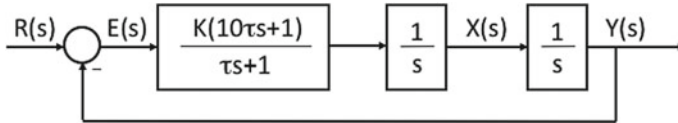
B3: Person A wants to avoid person B , who insists to casually meet A at lunchtime. At the university, there are two restaurants, one with a dining time of 20 min and one with a dining time of 50 min. For simplicity, we can assume that A and B have to go to lunch at 12.00, and that it is not an option not to have lunch. What strategy should A choose to minimize the time spent with B ? Conversely, what is the most successful strategy for B ? How much time will A have to spend with B ? Hint: try first to intuitively understand and solve the problem.

B4: Use the golden section method to find the maximum of the function

$$f(x) = -x^2 + 21.6x + 3$$

in the interval $[0, 20]$ with a tolerance level of 0.001. Solve the problem with different starting points.

B5: The following linear control loop is given in the frequency domain:



Determine the controller parameters K and τ such that $Z = \int_0^\infty e^2 dt$ is minimized with the constraint $J = \int_0^\infty x^2 dt = A$. Assume $r(t)$ to be a unit step. Discuss the case when $J < A$.

B6: Consider the following discrete time system

$$\mathbf{z}[k+1] = \mathbf{A}\mathbf{z}[k] + \mathbf{b}u[k], \quad y[k+1] = \mathbf{c}\mathbf{z}[k+1],$$

where

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

In this problem, we will explore some of the properties of this discrete time system as a function of the parameters, the initial conditions, and the inputs. (a) Assume $a_{12} = 0$ and that there is no input, $u = 0$. Write a closed form expression for the output of the system from a non-zero initial condition $\mathbf{z}[0] = (z_1[0], z_2[0])$ and give conditions on a_{11} and a_{22} under which the output gets smaller as k gets larger. (b) Now assume that $a_{12} \neq 0$ and write a closed form expression for the response of the system from a non-zero initial condition. Give conditions on the elements of $\mathbf{A}$ under which the output gets smaller as k gets larger. (c) Write a MATLAB program to plot the output of the system in response to a unit step input, $u[k] = 1, k \geq 0$. Plot the response of your system with $\mathbf{z}[0] = \mathbf{0}$ and $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.25 \end{bmatrix}.$$

B7: Write an essay arguing for and against the statement in “A Philosophical Excursion” that search for reality is outside the scope of science.

C1: Historically, some concepts like submarines or space travel were first introduced in science fiction literature and became science fact with time. Others, like time travel or Startrek style “beaming” have not (yet?) made that transition. Consider Hari Seldon’s “psychohistory”. Do you think it might one day become a fact? Argue your case using concepts like gas dynamics, stochastic models and chaos.

C2: It is not obvious how to tell just looking at data whether a physical system is random or chaotic. Some have even argued philosophically that these concepts might be observationally equivalent. Develop criteria and methods for distinguishing the two.

C3: A doctor wants to prescribe a safe but effective dosage. How much should she prescribe, how often and when should the patient take the medicine?

C4: Coming back to the example of inverted pendulum in Section 2.3, we modify the problem the following way: let us assume that the cart moves on an elevated track and the pendulum is not a standing pole but a rope hanging down from the cart. The task is to move the cart horizontally in such way that the rope moves from an arbitrary swinging state to a desired position at complete stand still in the least possible time. Note that we are interested in *all* points on the rope being transformed from any position and velocity to a stand still at another position. How would you go about solving this problem? What kind of equations will you need to model the rope and the cart?

C5: In a futuristic bar you are being served your cocktails by robots. The robots are supposed to bring your cocktails as fast as possible without spilling them. How would you model the movement of the drinks in the cocktail glasses? What if the same robot is serving several different glasses at the same time? First, assume the robot moves on a straight line only. Secondly, assume it can move freely in a room. Thirdly, assume that the floor on which the robot moves is not smooth.

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S. Hacisalihzade, S.

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