Chapter 2
Dual Numbers

Dual numbers have been introduced in the ninetieth century by *William Clifford* when dealing with the theory of engines which used a nilpotent operator noted ε. Their application to the study of kinematics of rigid articulated bodies has been developed by *Kotelnikov* of Kazan University. More recently several authors (*Yang*, *Ravani*, *Pennock*, *Roth*) have developed computer tools for dual numbers calculus.

Definition of Dual Numbers

The set of dual numbers Δ is the set of R^2 with specific addition and multiplication laws given by:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \tag{2.1}$$

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, x_1 \cdot y_2 + x_2 \cdot y_1) \tag{2.2}$$

The set Δ has a structure of an unity commutative ring with respect to these two laws. Its unitary element is (1, 0). The dual number (0, 1) is nilpotent of order two with respect to multiplication. Since Δ presents divisors of (0, 0), the zero of the multiplication, it is not an integral ring.

The subset of $\Delta\{(x, 0)|x \in R\}$ is a sub-ring of Δ and is isomorph to R.

Adopted notation for a dual number $(x, y)$of Δ:

$$(x, y) = x + \varepsilon\, y \tag{2.3}$$

F. Mora-Camino and C.A. Nunes Cosenza, *Fuzzy Dual Numbers*,
Studies in Fuzziness and Soft Computing 359,
https://doi.org/10.1007/978-3-319-65418-8_2

The inverse of a dual number $x + \varepsilon\, y$ such as $x \neq 0$ is given by:

$$(x + \varepsilon\, y)^{-1} = 1/x_1 - \varepsilon\; y/x^2 \tag{2.4}$$

The ratio between two dual numbers is such as:

$$(x_1, y_1)/(x_2, y_2) = x_1/x_2 + \varepsilon\; (x_2 y_1 - x_1 y_2)/x_2^2 \text{ with } x_2 \neq 0 \tag{2.5}$$

A function f of a dual variable $x + \varepsilon\, y$ is such as:

$$f(x + \varepsilon\, y) = f_1(x, y) + \varepsilon\, f_2(x, y) \tag{2.6}$$

where f_1 and f_2 are two functions of the real variables x and y.

This function has a limit $z_1 + \varepsilon\, z_2$ when $x + \varepsilon\, y$ tends towards $x_1 + \varepsilon\, y_1$ if and only if:

$$\lim_{x \to x_1, y \to y_1} f_1(x, y) = z_1 \quad \text{and} \quad \lim_{x \to y_1, y \to y_1} f_2(x, y) = z_2 \tag{2.7}$$

This function will be continuous at $y_1 + \varepsilon\, y_2$ if:

$$\lim_{x \to x_1, y \to y_1} f(x + \varepsilon\, y) = f(y_1 + \varepsilon\, y_2) \tag{2.8}$$

Such a function of a dual variable will be differentiable at point $X_1 = x_1 + \varepsilon\, y_1$ if there exists a dual number F and a function δ of a dual variable h such as:

$$f(X_1 + h) = f(X_1) + F \cdot h + h \cdot \delta(h) \quad \text{with} \quad \lim_{h \to 0, h \in \Delta} \delta(h) = 0 \tag{2.9}$$

F is the value of the derivative of f at point X_1.

Then the function of the dual variable defined by:

$$f' : X \to F \tag{2.10}$$

is the derivative function of f at point X_1.

It can be easily shown that a necessary and sufficient condition for f to be differentiable at point X_1 of Δ is that f is R^2 differentiable and that at this point:

$$\partial f_1/\partial x = \partial f_2/\partial y \quad \text{and} \quad \partial f_1/\partial y = 0 \tag{2.11}$$

Let O be an open set of R and let g be a function of class C^2 from O to Δ. A dual differential prolongation $\tilde{g}$ of g can be defined as:

$$\tilde{g}(x + \varepsilon\, y) = g(x) + \varepsilon\, y\, g'(x) \quad \text{for} \quad x \in O, y \in R \tag{2.12}$$

Examples of dual differential prolongations:

For $g(x) = \arcsin(x)$, then:

$$\tilde{g}(x + \varepsilon\, y) = \arcsin(x) + \varepsilon\, y/\sqrt{1 - y^2} \tag{2.13}$$

For sin(θ), then

$$\tilde{sin}(\theta + \varepsilon\varphi) = \sin\theta + \varepsilon\,\varphi\, \cos\theta \tag{2.14}$$

For cos(θ), then

$$\tilde{cos}(\theta + \varepsilon\varphi) = \cos\theta - \varepsilon\,\varphi\, \sin\theta \tag{2.15}$$

For tan(θ), then

$$\tilde{tan}(\theta + \varepsilon\varphi) = \tan\theta - \varepsilon\,\varphi/\cos^2\theta \tag{2.16}$$

Dual Vectors and Matrices

Dual Vectors

Let E an Euclidian space over the field of reals of dimension p. A set $\tilde{E}$ composed of de pairs of vectors said dual vectors by considering the Cartesian product $E \times E$ with the following operations:

Addition:

$$(a, b) + (c, d) = (a + c, b + d) \quad \forall\, a, b, c, d \in E \tag{2.17}$$

Multiplication by a scalar $\lambda + \varepsilon\,\mu$:

$$(\lambda + \varepsilon\,\mu)\,(a, b) = (\lambda\, a, \lambda\, b + \mu\, a) \quad \forall\lambda, \mu \in R \quad \forall a, b \in E \tag{2.18}$$

Then:

$$(a, b) = (1 + 0\,\varepsilon)(a, 0) + \varepsilon\,(b, 0) \tag{2.19}$$

or

$$(a, b) = a + \varepsilon\, b \quad \forall\,(a, b) \in \tilde{E} \tag{2.20}$$

The real and dual parts of a dual vector $a+\varepsilon b$ of $\tilde{E}$ are such as:

$$R(a+\varepsilon b) = a \quad \text{and} \quad D(a+\varepsilon b) = b \tag{2.21}$$

A dual scalar product between dual vectors can be defined:

$$u * v = R(u) \cdot R(v) + \varepsilon \left(R(u) \cdot D(v) + D(u) \cdot R(v)\right) \qquad \forall\, u, v \in \tilde{E} \tag{2.22}$$

where (*) represents the dual scalar product of $\tilde{E}$ and $(\cdot)$ represents the scalar product of E.

Two dual vectors u and v are said to be orthogonal for the dual scalar product if:

$$u * v = \tilde{0} \tag{2.23}$$

where $\tilde{0}$ is the zero of the addition of two dual vectors.

For any dual vector u of $\tilde{E}$ such as $R(u) \neq 0$, let the pseudo norm be defined by:

$$\|u\|_D = \sqrt{u * u} \tag{2.24}$$

where $\sqrt{\ }$ is the differential prolongation of the square root function. Here we have:

$$\|u\|_D = \tilde{0} \quad \text{if} \quad u = \tilde{0} \tag{2.25}$$

$$\|u\|_D = \|R(u)\| + \varepsilon\, R(u) \cdot D(u) / \|R(u)\| \quad \text{if} \quad R(u) \neq 0 \tag{2.26}$$

Here $\|\|$ represents the Euclidian norm of E with a space vector structure.

Then it will be possible to define orthogonal bases for $\tilde{E}$.

In the case where p = 3, the dual vector product can be defined as a bilinear antisymmetric application from $\tilde{E} \times \tilde{E}$ to $\tilde{E}$ by:

$$u \otimes v = R(u) \wedge R(v) + \varepsilon \left(R(u) \wedge D(v) + D(u) \wedge R(v)\right) \qquad \forall\, u,\ v \in \tilde{E} \tag{2.27}$$

where $\wedge$ is the vector product over R^3.

If u_1, u_2, u_3 et v_1, v_2, v_3 are the coordinates of dual vectors u and v in an orthonormal basis of $\tilde{E}$ then:

$$u * v = u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3 \tag{2.28}$$

and

$$u \otimes v = \left(u_2 \cdot v_3 - u_3 \cdot v_2, u_3 \cdot v_1 - u_1 \cdot v_3, u_1 \cdot v_2 - u_2 \cdot v_1\right)^t \tag{2.29}$$

In general, vector operations over $\tilde{E}$ inherit their properties from the corresponding vector operations over E. For instance, we have:

$$u * (v \otimes w) = w * (u \otimes v) = v * (w \otimes u) \quad \forall\, u, v, w \in \tilde{E} \tag{2.30}$$

Dual matrices

It can be of interest to introduce the set of square dual matrices $\tilde{M}_3$of order 3 × 3 based on dual numbers as it has been done with dual vectors. Here dual matrix A is such as:

$$A = [a_{ij}] = [R(a_{ij}) + D(a_{ij})] = R(A) + \varepsilon\, D(A) \tag{2.31}$$

and matrix operations over square dual matrices will be such that:

$$A + B = R(A) + R(B) + \varepsilon\ (D(A) + D(B)) \quad \forall A, B \in \tilde{M}_3 \tag{2.32}$$

$$A \cdot B = R(A)\ R(B) + \varepsilon\ (R(A)\, D(B) + D(A)\, R(B)) \quad \forall\, A, B \in \tilde{M}_3 \tag{2.33}$$

$$\lambda A = R(\lambda)\, R(A) + \varepsilon\, (R(\lambda)\, D(A) + D(\lambda)\, R(A)) \quad \forall\, \lambda \in \Delta, \forall A \in \tilde{M}_3 \tag{2.34}$$

The inverse of a square dual matrix A is given by:

$$A^{-1} = R(A)^{-1} - \varepsilon\, R(A)^{-1} D(A)\ R(A)^{-1} \quad A \in \tilde{M}_3, \det(R(A)) \neq 0 \tag{2.35}$$

and its product by a dual vector u is a dual vector v such as:

$$A\ \times u = R(A)R(u) + \varepsilon\ (R(A)\ D(u) + D(A)\ R(u)) \tag{2.36}$$

References

E. Pennestri, R. Stefanelli, "Linear Algebra and Numerical Algorithms using Dual Numbers", *Journal of Multibody Systems Dynamics*, vol. 18, 2007, pp. 323–344.

R. S. Ball, *Theory of screws*, Cambridge University Press, 1900.

J. Duffy, Analysis of mechanisms and robot manipulators, *Halstead Press*, 1980.

Y. L. Gu, Y.L. and J. Y. S. Luh, Dual number transformations and its applications to robotics, *IEEE Journal and Robotics and Automation*, vol. RA-3, December 1987.

K. Sugimoto, K and J. Duffy, Application of linear algebra to screw systems, *Mechanism and Machine Theory*, vol. 17, no. 1, 73–83, 1982.

W. B. Vasantha Kandasamy W.B. and F. Smarandache, Algebraic structures using natural class of intervals, *The Educational Publisher*, Ohio, 2011.

http://www.springer.com/978-3-319-65417-1

Fuzzy Dual Numbers
Theory and Applications
Mora-Camino, F.; Nunes Cosenza, C.A.
2018, XIII, 62 p. 21 illus., Hardcover
ISBN: 978-3-319-65417-1