

## Chapter 2

# Optimal Power Allocation for Kalman Filtering over Fading Channels

Kalman filtering with random packet drops has been studied extensively since the work of [1], which showed that for i.i.d. Bernoulli packet drops, there exists a critical threshold such that if the packet arrival rate exceeds this threshold, then the expected error covariance remains bounded, but diverges otherwise. This work has been extended in various directions such as: multiple sensors [2, 3], further characterizations of the critical threshold [4, 5], probabilistic notions of performance [6, 7], performing local processing before transmission [8], consideration of delays [9] and Markovian packet drops [10, 11].

As mentioned in Chap. 1, in wireless communications, power control is regularly used to improve system performance and reliability [12, 13]. The primary focus of the previously mentioned works is on deriving conditions for stability of the estimator, and power control is not explicitly considered. However, power control can also be used in Kalman filtering to improve the estimator stability and estimation performance. For Kalman filtering over continuous fading channels, the use of power control for outage minimization and expected error covariance minimization has been studied in [14]. The works of [15, 16] consider the use of power control at the sensor over a continuous fading channel, with the data being sent over this channel after digital modulation, which would then give a corresponding packet loss probability dependent on the transmit power at the sensor. Power allocation using model predictive control techniques is considered in [15], while optimal power allocation schemes to guarantee stability are investigated in [16].

In conventional wireless communication systems, the sensors have access either to a fixed energy supply or have batteries that may be easily recharged/replaced. In contrast, when energy harvesting capabilities are available, then the sensors can recharge their batteries by collecting energy from the environment, e.g. solar, thermal, mechanical vibrations, or electromagnetic radiation [17, 18]. In the context of wireless sensor networks, the use of energy harvesting may be especially useful, e.g. in remote locations with restricted access to an energy supply, and even mandatory

where it is dangerous or impossible to change the batteries. The amount of energy harvested is random as most renewable energy sources are unreliable. Clearly, the energy expenditure at every time slot is constrained by the amount of stored energy currently available. This, however, complicates the design of suitable transmission power allocation policies. Communication schemes for optimizing throughput or minimizing transmission delay for transmitters with energy harvesting capability have been studied in [19–23], while a remote estimation problem with an energy harvesting sensor was considered in [24], which minimized a cost consisting of both the distortion and the number of sensor transmissions.

In this chapter, we adopt the channel model of [15, 16], but instead of using power allocation to achieve filter stability, we are interested in the use of power allocation to improve the estimation performance of the Kalman filter. Section 2.1 first studies optimal power allocation for sensors without energy harvesting capabilities. Here, we focus on minimizing the trace of the expected error covariance subject to an average transmit power constraint. The problem is formulated as a Markov decision process (MDP) problem that can be solved numerically with dynamic programming techniques. Two simpler suboptimal schemes are also investigated, namely a constant power allocation scheme and a truncated channel inversion policy. Section 2.2 then investigates the situation with an energy harvesting sensor. An important issue is to address the trade-off between the use of available stored energy to improve the current transmission reliability (and thus state estimation accuracy), or the storing of energy for future transmissions which may be affected by higher packet loss probabilities due to severe fading. The optimal transmission energy allocation policies are obtained by the use of dynamic programming techniques. Using the concept of submodularity [25], the structure of the optimal transmission energy policies is also studied.

## 2.1 Optimal Power Allocation for Remote State Estimation

### 2.1.1 System Model

A diagram of the system model for this section is given in Fig. 2.1. Consider a linear system

$$x_{k+1} = Ax_k + w_k \quad (2.1)$$

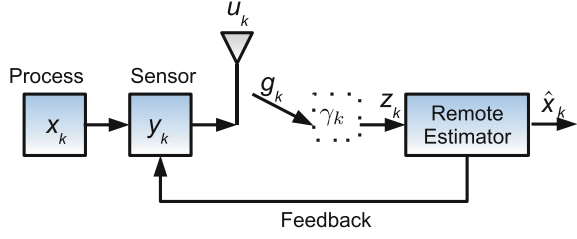
where  $x_k \in \mathbb{R}^n$ , and  $w_k$  is i.i.d. Gaussian with zero mean and covariance matrix  $Q > 0$ .<sup>1</sup> The sensor makes a measurement

$$y_k = Cx_k + v_k \quad (2.2)$$

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<sup>1</sup>We say that a matrix  $X > 0$  if  $X$  is positive definite, and  $X \geq 0$  if  $X$  is positive semi-definite.

**Fig. 2.1** Transmission power control for remote state estimation



where  $y_k \in \mathbb{R}^m$ , and  $v_k$  is i.i.d. Gaussian with zero mean and covariance matrix  $R > 0$ . We assume that the pair  $(A, C)$  is detectable and the pair  $(A, Q^{1/2})$  is stabilizable.

The measurement is then sent to a remote estimator over a packet dropping link, which can be modelled as

$$z_k = \gamma_k y_k,$$

where  $z_k$  is the quantity received at the remote estimator. Here, the measurement  $y_k$  is assumed to be encoded to form a single packet, and  $\gamma_k = 1$  denotes that the measurement packet is received (i.e. correctly decoded), while  $\gamma_k = 0$  denotes that the packet is lost (i.e. corrupted).<sup>2</sup>

### Kalman Filter with Random Packet Drops

In order to estimate the state  $x_k$ , the remote estimator runs a Kalman filter, which also takes into account the random packet drops [1]. The Kalman filter state estimates and error covariances are defined as:

$$\begin{aligned}\hat{x}_{k|k} &= \mathbb{E}[x_k | z_0, \dots, z_k, \gamma_0, \dots, \gamma_k] \\ \hat{x}_{k+1|k} &= \mathbb{E}[x_{k+1} | z_0, \dots, z_k, \gamma_0, \dots, \gamma_k] \\ P_{k|k} &= \mathbb{E}[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T | z_0, \dots, z_k, \gamma_0, \dots, \gamma_k] \\ P_{k+1|k} &= \mathbb{E}[(x_{k+1} - \hat{x}_{k+1|k})(x_{k+1} - \hat{x}_{k+1|k})^T | z_0, \dots, z_k, \gamma_0, \dots, \gamma_k].\end{aligned}$$

The Kalman filtering equations with packet drops are given by:

$$\begin{aligned}\hat{x}_{k|k} &= \hat{x}_{k|k-1} + \gamma_k K_k (y_k - C \hat{x}_{k|k-1}) \\ \hat{x}_{k+1|k} &= A \hat{x}_{k|k} \\ P_{k|k} &= P_{k|k-1} - \gamma_k P_{k|k-1} C^T (C P_{k|k-1} C^T + R)^{-1} C P_{k|k-1} \\ P_{k+1|k} &= A P_{k|k} A^T + Q,\end{aligned}\tag{2.3}$$

where  $K_k = P_{k|k-1} C^T (C P_{k|k-1} C^T + R)^{-1}$ . In this chapter, we will also use the shorthand  $P_k \triangleq P_{k|k-1}$ . Then  $\{P_k\}$  satisfies

$$P_{k+1} = A P_k A^T + Q - \gamma_k A P_k C^T (C P_k C^T + R)^{-1} C P_k A^T.$$

<sup>2</sup>In practice this can be determined using simple error detecting codes.

### Packet Drop Model

In this chapter, we will adopt a model from [15, 16] for the packet loss process  $\{\gamma_k\}$  that is governed by a time-varying wireless fading channel  $\{g_k\}$  and the sensor transmit power control  $\{u_k\}$  over this channel. In this model, the conditional packet reception probabilities are given by

$$\mathbb{P}(\gamma_k = 1 | g_k, u_k) \triangleq f(g_k u_k) \quad (2.4)$$

where  $f(\cdot) : [0, \infty) \rightarrow [0, 1]$  is a monotonically increasing continuous function. The form of  $f(\cdot)$  will depend on the particular digital modulation scheme being used [26], see e.g. (2.12) for the case of binary phase shift keying (BPSK) transmission.

We will consider the case where  $\{g_k\}$  is an i.i.d. block fading process [27], where the channel remains constant over a fading block (representing the coherence time of the channel [28]) but can vary from block to block in an i.i.d. manner.

### Kalman Filter Stability

We assume that channel state information (CSI) is available at the remote estimator, such that the remote estimator knows the values of the channel gains  $g_k$  at time  $k$ .<sup>3</sup> Since CSI is assumed to be available, we will allow the sensor transmit power  $u_k$  to depend on both  $g_k$  and  $P_k$ . Note that if the energy allocation  $u_k$  is computed based on the estimation error covariance (and not the state  $x_k$ ), then the optimal estimator is still given by the Kalman filter (2.3). In the next section, we consider optimal power allocation to minimize the trace of the expected error covariance. Due to limited computational resources at the sensor, the optimal sensor transmit powers are computed at the remote estimator and fed back to the sensor.<sup>4</sup>

Using techniques from [29], we can obtain the following sufficient condition for stability of the Kalman filter, for power control schemes  $\{u_k\}$  which are allowed to depend on the channel gains  $g_k$  and error covariances  $P_k$ .

**Theorem 2.1** *Let  $\|A\|$  denote the spectral norm of  $A$ . If there exists an  $r \in [0, 1)$  such that:*

$$\mathbb{P}(\gamma_k = 1) \geq 1 - \frac{r}{\|A\|^2}, \quad \forall k \in \mathbb{N},$$

*then  $\{P_k\}$  satisfies*

$$\mathbb{E}[\text{tr}(P_k)] \leq \alpha r^k + \beta, \quad \forall k \in \mathbb{N} \quad (2.5)$$

*for some  $\alpha, \beta \in \mathbb{R}$ .*

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<sup>3</sup>In practice, this can be achieved by periodically sending pilot signals either from the sensor to the remote estimator to allow the remote estimator to estimate the channel, or from the remote estimator to the sensor under channel reciprocity.

<sup>4</sup>In wireless communications, online computation of powers at the base station, which is then fed back to the mobile transmitters, is commonly done in practice [12], at time scales on the order of milliseconds.

### 2.1.2 Optimal Power Allocation

The problem we consider in this subsection is to determine the optimal sensor transmit power allocation, in order to minimize the trace of the expected error covariance subject to an average transmit power constraint  $\mathcal{P}$ , i.e. we are interested in solving

$$\begin{aligned} \min_{\{u_k\}} \limsup_{K \rightarrow \infty} \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\text{tr}(P_{k+1})] \\ \text{s.t. } \limsup_{K \rightarrow \infty} \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[u_k] \leq \mathcal{P}. \end{aligned} \quad (2.6)$$

*Remark 2.1* When the system matrix  $A$  is unstable (i.e. has eigenvalues outside the unit circle), Kalman filtering with packet losses can have unbounded expected error covariances in certain situations [1]. This then raises the question as to whether problem (2.6) is well posed. In [16], we studied the problem of determining the minimum average power required for guaranteeing that Theorem 2.1 is satisfied. Choosing  $\mathcal{P}$  in the average power constraint of problem (2.6) to be greater than this minimum average power (see [16] for details on how to compute this minimum average power) will be sufficient to make (2.6) well posed.

The optimization problem (2.6) can be regarded as a constrained average cost Markov decision process (MDP) problem [30] with  $(P_k, g_k)$  as the ‘state’ and  $u_k$  as the ‘action’ of the MDP. To solve this problem, we will use a Lagrangian technique similar to [14, 30, 31] that considers instead the following unconstrained MDP problem:

$$\begin{aligned} \min_{\{u_k\}} \limsup_{K \rightarrow \infty} \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\text{tr}(P_{k+1}) + \beta u_k] \\ = \min_{\{u_k\}} \limsup_{K \rightarrow \infty} \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[\mathbb{E}[\text{tr}(P_{k+1}) | P_k, g_k, u_k] + \beta u_k], \end{aligned} \quad (2.7)$$

where  $\beta \geq 0$  specifies the trade-off between the average transmit power and expected error covariance. Solving (2.7) for different values of  $\beta$  will then correspond to minimizing the trace of the expected error covariance for different average transmit power constraints in (2.6).

The average cost optimality equation or Bellman equation [32] associated with problem (2.7) can be written as

$$\begin{aligned} \rho + h(P_k, g_k) = \min_{u_k} \left[ \mathbb{E}[\text{tr}(P_{k+1}) | P_k, g_k, u_k] + \beta u_k \right. \\ \left. + \int h(P_{k+1}, g_{k+1}) F(d(P_{k+1}, g_{k+1}) | P_k, g_k, u_k) \right], \end{aligned} \quad (2.8)$$

$g_{k+1}, P_{k+1}$

where  $\rho$  is the optimal average cost per stage,  $h$  the differential cost and  $F$  the probability transition law of  $(P_k, g_k)$ .

We first show that there exist stationary solutions to the MDP (2.7). We will make the following additional assumption:

**Assumption 2.1.1** The range of  $u_k$  is bounded, i.e.  $u_k \in [0, u_{\max}]$ ,  $\forall k$ .

Such an assumption is obviously justified from a practical point of view.

**Lemma 2.1** *Under Assumption 2.1.1, there exists a stationary solution to the Bellman equation (2.8) which solves the MDP (2.7).*

*Proof* The proof involves verifying the conditions from [33] that guarantee the existence of stationary solutions for MDPs with Borel state and action spaces. The verification of these conditions is very similar to the proof of Lemma 3 in [14], see also the proof of Theorem 2.3 in the appendix to this chapter. The details are omitted for brevity.  $\square$

For computational purposes, the Bellman equation can be further simplified as follows:

$$\begin{aligned}
& \rho + h(P_k, g_k) \\
&= \min_{u_k} \left[ \mathbb{E}[\text{tr}(P_{k+1}) | P_k, g_k, u_k] + \beta u_k \int_{g_{k+1}, P_{k+1}} h(P_{k+1}, g_{k+1}) F(d(P_{k+1}, g_{k+1}) | P_k, g_k, u_k) \right] \\
&= \min_{u_k} \left\{ \text{tr}(A P_k A^T + Q) + \beta u_k - f(g_k u_k) \text{tr} \left( A P_k C^T (C P_k C^T + R)^{-1} C P_k A^T \right) \right. \\
&\quad \left. + \int_{P_{k+1}, g_{k+1}} h(P_{k+1}, g_{k+1}) F(d(P_{k+1}, g_{k+1}) | P_k, g_k, u_k) \right\} \\
&\stackrel{(a)}{=} \min_{u_k} \left\{ \text{tr}(A P_k A^T + Q) + \beta u_k - f(g_k u_k) \text{tr} \left( A P_k C^T (C P_k C^T + R)^{-1} C P_k A^T \right) \right. \\
&\quad \left. + \int_{P_{k+1}, g_{k+1}} h(P_{k+1}, g_{k+1}) F(d P_{k+1} | P_k, g_k, u_k) F(d g_{k+1}) \right\} \\
&\stackrel{(b)}{=} \min_{u_k} \left\{ \text{tr}(A P_k A^T + Q) + \beta u_k - f(g_k u_k) \text{tr} \left( A P_k C^T (C P_k C^T + R)^{-1} C P_k A^T \right) \right. \\
&\quad \left. + \int_{g_{k+1}} \left[ h(A P_k A^T + Q, g_{k+1}) (1 - f(g_k u_k)) \right. \right. \\
&\quad \left. \left. + h \left( A P_k A^T + Q - A P_k C^T (C P_k C^T + R)^{-1} C P_k A^T, g_{k+1} \right) f(g_k u_k) \right] F(d g_{k+1}) \right\} \tag{2.9}
\end{aligned}$$

where (a) follows from the fact that  $g_{k+1}$  is independent of  $P_{k+1}$ , and (b) follows from writing out the conditional expectation  $\mathbb{E}[h(P_{k+1}, g_{k+1}) | P_k, g_k, u_k]$ . For numerical implementation, a discretized version of the Bellman equation (2.9) can then be solved using, e.g. the relative value iteration algorithm [32] to find solutions to the MDP (2.7).

*Remark 2.2* The discretized solution is, strictly speaking, a suboptimal approximation to the true optimal solution, however, the use of discretization is generally unavoidable for MDPs with continuous state and action spaces. As the number of discretization levels increases, the discretized solution usually converges to the optimal solution [34].

Now let  $p^*(u)$  be the minimum trace of the expected error covariance such that the average transmit power is less than  $u$ . By solving the MDP (2.7) for different values of  $\beta$ , one can obtain points of the function  $p^*(u)$ , corresponding to different trade-offs between the average transmit power and trace of the expected error covariance, see Fig. 2.2. We have the following characterization of the function  $p^*(u)$ :

**Lemma 2.2** *Suppose  $f(\cdot)$  in (2.4) is a strictly concave function. Then  $p^*(u)$  is a decreasing strictly convex function of  $u$ .*

*Proof* See Appendix.

An example of a strictly concave  $f(\cdot)$  is given by (2.12) in Sect. 2.1.4. Using Lemma 2.2, one can conclude from the theory of Pareto optimality that all points on the curve  $p^*(u)$  can be obtained by solving the MDP (2.7) for an appropriate choice of  $\beta$ , see [35, 36] for further details.

### 2.1.3 Suboptimal Power Allocation Policies

The optimal solution considered in the previous section requires the solution of an MDP, which is computationally demanding, particularly for vector systems. In this section, we consider two suboptimal policies which are simpler to compute and implement than the optimal solution of Sect. 2.1.2.

#### Constant Power Allocation

One very simple scheme is to use constant power allocation, where  $u_k = u_{\text{const}}, \forall k$ . With this policy, the conditional packet reception probabilities  $f(g_k u_{\text{const}})$  will depend only on the channel gain  $g_k$ .

#### Truncated Channel Inversion

Another suboptimal scheme is based on the concept of channel inversion, which is a simple but quite commonly used technique in wireless communications, that attempts to invert the channel at every time instance to maintain a constant quality of service. However, it is known that for certain fading distributions such as Rayleigh fading, channel inversion actually requires infinite average power, so some modifications to the scheme such as truncation (where channel inversion is only carried out if the channel gain is sufficiently large) are necessary [37]. The power allocation policy we consider here is of the following form:

$$u_k = \begin{cases} \alpha/g_k, & \text{if } g_k > g^* \\ \alpha/g^*, & \text{otherwise} \end{cases} \quad (2.10)$$

where  $\alpha$  and  $g^*$  are values which can be designed. This scheme inverts the channel  $g_k$  and multiplies it by a gain  $\alpha$  if  $g_k$  is greater than some threshold  $g^*$ , otherwise it transmits with the constant power  $\frac{\alpha}{g^*}$ . The average transmit power using this scheme is

$$\begin{aligned} \mathbb{E}[u_k] &= \int_{g^*}^{\infty} \frac{\alpha}{g_k} F(dg_k) + \int_0^{g^*} \frac{\alpha}{g^*} F(dg_k) \\ &= \alpha E(g^*) + \frac{\alpha}{g^*} F_G(g^*), \forall k \end{aligned}$$

where

$$E(g^*) \triangleq \int_{g^*}^{\infty} \frac{1}{g_k} F(dg_k),$$

and  $F_G(\cdot)$  is the cumulative distribution function of  $g_k$ . For instance, if  $g_k \sim \text{Exp}(1)$ , which is an example of Rayleigh fading [28], we have  $E(g^*) = \int_{g^*}^{\infty} \exp(-g_k)/g_k dg_k = E_1(g^*)$  (i.e. the exponential integral), and  $F_G(g^*) = 1 - \exp(-g^*)$ .

In terms of the packet loss process  $\{\gamma_k\}$ , under this power allocation scheme,  $\gamma_k = 1$  with conditional probability  $f(\alpha)$  when  $g_k > g^*$ , and  $\gamma_k = 1$  with conditional probability  $f(\frac{\alpha g_k}{g^*})$  when  $g_k \leq g^*$ . That is, we have

$$\gamma_k = \begin{cases} 1, & \text{w.p. } f(\alpha)(1 - F_G(g^*)) + \int_0^{g^*} f\left(\frac{\alpha g_k}{g^*}\right) F(dg_k) \\ 0, & \text{w.p. } (1 - f(\alpha))(1 - F_G(g^*)) + \int_0^{g^*} \left(1 - f\left(\frac{\alpha g_k}{g^*}\right)\right) F(dg_k). \end{cases}$$

Therefore, using this scheme,  $\gamma_k$  becomes an i.i.d. Bernoulli process with probability of successful packet reception  $f(\alpha)(1 - F_G(g^*)) + \int_0^{g^*} f(\frac{\alpha g_k}{g^*}) F(dg_k)$ .

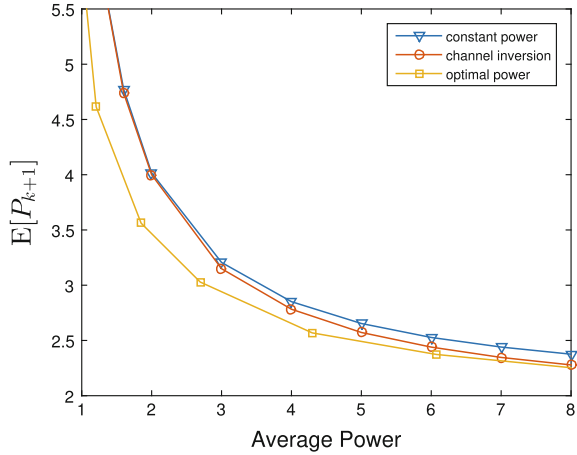
As the values  $\alpha$  and  $g^*$  can be chosen by us, we can optimize  $\alpha$  and  $g^*$  to minimize the trace of the expected error covariance subject to an average power constraint, i.e. solving problem (2.6) but with  $u_k$  restricted to be of the form (2.10). For i.i.d. packet losses, it is known that the expected error covariance is a decreasing function of the packet reception probability [1]. Hence, the problem is equivalent to minimizing the probability of packet loss subject to an average power constraint  $\mathcal{P}$ , i.e.

$$\begin{aligned} \min_{\alpha, g^*} & (1 - f(\alpha))(1 - F_G(g^*)) + \int_0^{g^*} \left(1 - f\left(\frac{\alpha g_k}{g^*}\right)\right) F(dg_k) \\ \text{s.t. } & \alpha E(g^*) + \frac{\alpha}{g^*} F_G(g^*) = \mathcal{P}. \end{aligned} \quad (2.11)$$

We can further simplify problem (2.11) by rearranging the constraint to express  $\alpha$  in terms of  $g^*$ , i.e.



**Fig. 2.2** Average transmit power versus expected error covariance



$$\alpha = \frac{\mathcal{P}}{E(g^*) + \frac{1}{g^*} F_G(g^*)}.$$

The optimization problem (2.11) then becomes a one-dimensional line search over  $g^*$ , which can be easily solved numerically.

### 2.1.4 Numerical Studies

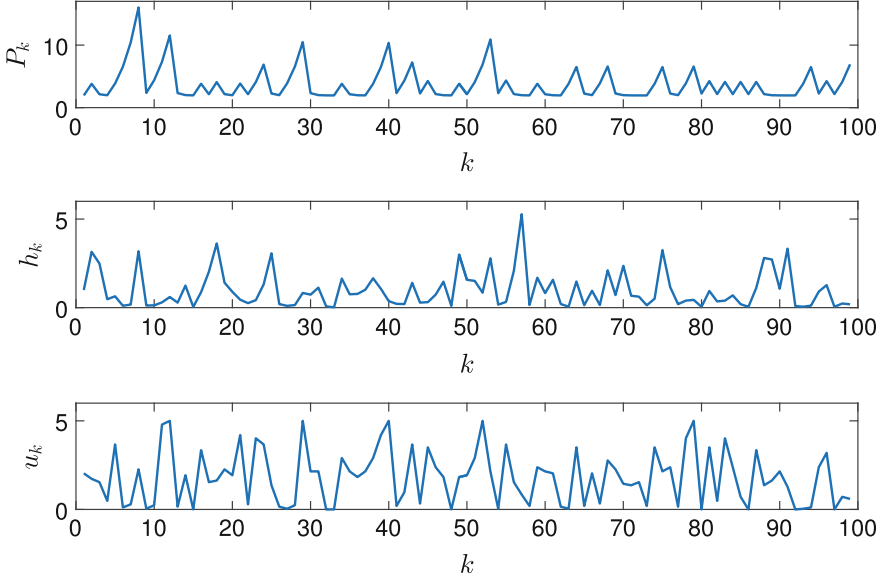
We present here numerical results for a scalar system with parameters  $A = 1.2$ ,  $C = 1$ ,  $Q = 1$ ,  $R = 1$ . We consider the case where the digital communication uses binary phase shift keying (BPSK) transmission [26] with  $b$  bits per packet, so that we have

$$\mathbb{P}(\gamma_k = 1 | g_k, u_k) = f(g_k u_k) = \left( \int_{-\infty}^{\sqrt{g_k} u_k} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \right)^b \quad (2.12)$$

One can verify that  $f(\cdot)$  is a strictly concave function for  $b \in \{1, 2, 3, 4, 5\}$ . In the simulations below we use  $b = 4$ . The fading channel is taken to be Rayleigh [28], so that  $g_k$  is exponentially distributed with p.d.f.

$$p(g_k) = \frac{1}{\bar{g}} \exp(-g_k/\bar{g}), \quad g_k \geq 0$$

with  $\bar{g}$  being its mean. Here, we will use  $\bar{g} = 1$ . In solving the Bellman equation (2.9), we use 50 discretization points for each of the quantities  $P_k$ ,  $g_k$ ,  $u_k$ , see Remark 2.2.



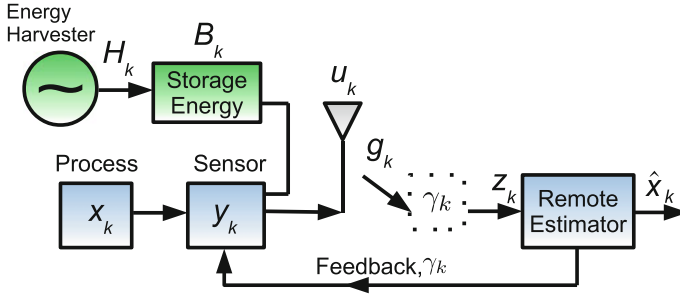
**Fig. 2.3** Optimal power allocations

In Fig. 2.2, we plot the average transmit power versus expected error covariance trade-off, for the cases of optimal power allocation of Sect. 2.1.2, and the constant power allocation and truncated channel inversion policies of Sect. 2.1.3. We see that optimal power allocation has significant performance gains over the simpler suboptimal policies of Sect. 2.1.3 for low average transmit powers, with the performance of the constant power allocation and channel inversion policies being almost identical. While for higher average transmit powers, the truncated channel inversion policy has performance approaching that of the optimal power allocation policy.

In Fig. 2.3, we plot a single simulation run of  $P_k$  and  $g_k$ , together with the corresponding optimal power allocations  $u_k$ . We can see that in the optimal power allocation scheme, the allocated powers will depend on both the current channel gain  $g_k$  and error covariance  $P_k$ . The allocated power  $u_k$  tends to be higher when the error covariance  $P_k$  is larger, provided the corresponding channel gain  $g_k$  is not too small.

## 2.2 Optimal Power Allocation with Energy Harvesting

A diagram of the system architecture for this section is shown in Fig. 2.4. The model for the process (2.1) and (2.2) and packet drops (2.4) is the same as that of Sect. 2.1. We assume that the packet loss process  $\{\gamma_k\}$  is fed back to the sensor, which allows the sensor to reconstruct the error covariances  $\{P_k\}$  at the remote estimator.



**Fig. 2.4** Transmission power control with energy harvesting

In contrast to Sect. 2.1, here, the sensor is equipped with energy harvesting capabilities. Let the energy harvesting process be denoted by  $\{H_k\}$ , where  $H_k$  is the energy harvested between the discrete time instants  $k-1$  and  $k$ . The process  $\{H_k\}$  is modelled as a stationary, first-order, homogeneous Markov process, which is independent of the fading process  $\{g_k\}$ . This modelling for the harvested energy process is justified by empirical measurements in, e.g. the case of solar energy [38].

We assume that the dynamics of the stored battery energy  $B_{(\cdot)}$  is given by the following first-order Markov model

$$B_{k+1} = \min\{B_k - u_k + H_{k+1}, B_{\max}\}, \quad k \geq 0, \quad (2.13)$$

where  $u_k$  is the transmission energy at time  $k$ , and  $B_{\max}$  is the maximum stored energy in the battery.

### 2.2.1 Optimal Energy Allocation Problems

In this subsection, we formulate optimal transmission energy<sup>5</sup> allocation problems in order to minimize the trace of the receiver's expected estimation error covariance. Unlike the problem formulation in Sect. 2.1, here, the optimal energy policies are computed at the sensor, since the sensor has information about the energy harvesting and instantaneous battery levels, as well as knowledge of  $\{P_k\}$  from the feedback of  $\{\gamma_k\}$ .

We consider the scenario of causal information, where the realizations of future wireless fading channel gains and harvested energies are not a priori known to the transmitter, see also Remark 2.4. More precisely, the information available at the sensor at any time  $k \geq 1$  is given by

<sup>5</sup>We measure energy on a per channel use basis and we will refer to energy and power interchangeably in this chapter.

$$\mathbb{I}_k = \{s_t := (\gamma_{t-1}, g_t, H_t, B_t) : 1 \leq t \leq k\} \cup \mathbb{I}_0 \quad (2.14)$$

where  $\mathbb{I}_0 := \{g_0, H_0, B_0, P_0\}$  is the initial condition.

The information  $\mathbb{I}_k$  is used at the sensor to decide  $u_k$ , the amount of transmission energy to use at time  $k$ . This quantity affects both the packet loss process and the amount of energy in the battery. A policy  $\{u_k\}$  is feasible if the energy harvesting constraint  $0 \leq u_k \leq B_k$  is satisfied for all  $k \geq 1$ . The admissible control set is then given by  $\mathcal{U} := \{u_{(\cdot)} : u_k \text{ is adapted to sigma-field } \sigma(\mathbb{I}_k) \text{ and } 0 \leq u_k \leq B_k\}$ .

The optimization problems are now formulated as Markov decision processes for the following two cases:

(i) *Finite-time horizon*:

$$\begin{aligned} \min_{\{u_k : 0 \leq k \leq T-1\}} & \sum_{k=0}^{T-1} \mathbb{E}[\text{tr}(P_{k+1})] \\ \text{s.t. } & 0 \leq u_k \leq B_k \quad 0 \leq k \leq T-1 \end{aligned} \quad (2.15)$$

(ii) *Infinite-time horizon*:

$$\begin{aligned} \min_{\{u_k : k \geq 0\}} & \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}[\text{tr}(P_{k+1})] \\ \text{s.t. } & 0 \leq u_k \leq B_k \quad k \geq 0 \end{aligned} \quad (2.16)$$

where  $B_k$  is the stored battery energy available at time  $k$ , which satisfies the battery dynamics (2.13). It is evident that the transmission energy  $u_k$  at time  $k$  not only affects the amount of stored energy  $B_{k+1}$  available at time  $k+1$ , but thereby also the transmission energy  $u_{k+1}$ , since  $0 \leq u_{k+1} \leq B_{k+1} = \min\{B_k - u_k + H_{k+1}, B_{\max}\}$  by (2.13). One of the key issues in solving problems (2.15) and (2.16) is to determine if one should use a lot of energy at time  $k$ , or save up some of the energy for use at future times.

We now give sufficient conditions under which the infinite horizon stochastic control problem (2.16) is well posed when the system matrix  $A$  is unstable. With well posed we, here, mean that an exponential boundedness condition for the expected estimation error covariance is satisfied. Let  $\mathbb{G}$  and  $\mathbb{H}$  be the time-invariant probability transition laws of the Markovian channel fading process  $\{g_k\}$  and the Markovian harvested energy process  $\{H_k\}$ , respectively.

We introduce the following assumption:

**Assumption 2.2.1** The channel fading process  $\{g_k\}$ , harvested energy process  $\{H_k\}$  and the maximum battery storage  $B_{\max}$  satisfy the following:

$$\begin{aligned}
& \sup_{(g,H)} \int_{g_k} \int_{H_k} (1 - h(g_k \min\{H_k, B_{\max}\})) \mathbb{P}(g_k | g_{k-1} = g) \mathbb{P}(H_k | H_{k-1} = H) dg_k dH_k \\
& \leq \frac{r}{\|A\|^2}, \quad k \geq 0
\end{aligned} \tag{2.17}$$

for some  $r \in [0, 1)$ , where  $\|A\|$  denotes the spectral norm of  $A$ .

**Theorem 2.2** Assume that Assumption 2.2.1 holds. Then there exist energy allocations  $\{u_k\}$  such that  $\mathbb{E}[P_k]$  satisfies;

$$\mathbb{E}[\text{tr}(P_k)] \leq \alpha r^k + \beta, \quad k \geq 0 \tag{2.18}$$

for some nonnegative scalars  $\alpha$  and  $\beta$ , and  $r \in [0, 1)$ . As a result, the stochastic optimal control problem (2.16) is well posed.

*Proof* Based on Theorem 1 of [39], a sufficient condition for exponential stability in the sense of (2.18) is that

$$\begin{aligned}
& \sup_{(g,H)} \mathbb{P}(\gamma_k = 0 | g_{k-1} = g, H_{k-1} = H) \\
& = \sup_{(g,H)} \int_{g_k} \int_{H_k} \mathbb{P}(\gamma_k = 0 | g_k = g', H_k = H', g_{k-1} = g, H_{k-1} = H) \\
& \quad \times \mathbb{P}(g_k, H_k | g_{k-1} = g, H_{k-1} = H) dg_k dH_k \\
& = \sup_{(g,H)} \int_{g_k} \int_{H_k} \mathbb{P}(\gamma_k = 0 | g_k = g', H_k = H', g_{k-1} = g, H_{k-1} = H) \\
& \quad \times \mathbb{P}(g_k | g_{k-1} = g) \mathbb{P}(H_k | H_{k-1} = H) dg_k dH_k \\
& = \sup_{(g,H)} \int_{g_k} \int_{H_k} (1 - h(g_k u_k)) \mathbb{P}(g_k | g_{k-1} = g) \mathbb{P}(H_k | H_{k-1} = H) dg_k dH_k \leq \frac{r}{\|A\|^2}
\end{aligned}$$

for some  $r \in [0, 1)$ . We now consider a suboptimal solution to the stochastic optimal control problem (2.16), where the full amount of energy harvested at each time step is used, i.e.  $u_0 = B_0$  and  $u_k = \min\{H_k, B_{\max}\}$  for  $k \geq 1$ . Then with this policy (2.17) will be a sufficient condition for (2.18) in terms of the channel fading process, harvested energy process and the maximum battery storage. Therefore, Assumption 2.2.1 provides a sufficient condition for the exponential boundedness (2.18) of the expected estimation error covariance.  $\square$

*Remark 2.3* In general, condition (2.17) given by Assumption 2.2.1 may be difficult to verify for all values of  $g, H$  and  $k$ . However, if we assume that the channel fading and harvested energy processes are stationary, then it would not be necessary to verify the condition for all  $k$ . Furthermore, in the two most commonly used models, namely i.i.d. processes and finite state Markov chains, the condition (2.17) can be simplified as follows:

(i) If  $\{g_k\}$  and  $\{H_k\}$  are i.i.d., then (2.17) amounts to

$$\int_{g_k} \int_{H_k} (1 - h(g_k \min\{H_k, B_{\max}\})) \mathbb{P}(g_k) \mathbb{P}(H_k) dg_k dH_k \leq \frac{r}{\|A\|^2}.$$

(ii) If  $\{g_k\}$  and  $\{H_k\}$  are stationary finite state Markov chains with state spaces  $\{1, \dots, M\}$  and  $\{1, \dots, N\}$  respectively, then (2.17) becomes

$$\max_{(i,j)} \sum_{i'=1}^M \sum_{j'=1}^N (1 - h(i \min\{j, B_{\max}\})) \mathbb{P}(g_k = i' | g_{k-1} = i) \mathbb{P}(H_k = j' | H_{k-1} = j) \leq \frac{r}{\|A\|^2}.$$

### 2.2.2 Solutions to the Optimal Energy Allocation Problems

The stochastic control problems (2.15) and (2.16) can be regarded as constrained Markov decision process (MDP) [30] problems with  $s_k := (P_k, g_k, H_k, B_k)$  as the state and  $u_k$  as the control action. We will approach the constrained MDPs (2.15) and (2.16) by the use of dynamic programming techniques.

Note that due to the existence of a perfect feedback link the sensor has knowledge about whether its transmissions have been received at the receiver or not. Hence, at time  $k$  the sensor knows  $\{P_t : 0 \leq t \leq k\}$ . The information available at the sensor at time instant  $k \geq 0$  is given by (2.14), which can be easily shown to be equivalent to

$$\mathbb{I}_k := \{s_t = (P_t, g_t, H_t, B_t) : 0 \leq t \leq k\}.$$

The causal information  $\mathbb{I}_k$  is used to decide the amount of transmit energy  $u_k$  to be used at time  $k$ . The transmit energy policy is computed offline using dynamic programming. We recall that a policy  $u_k$  is feasible if the energy harvesting constraints  $0 \leq u_k \leq B_k = \min\{B_{k-1} - u_{k-1} + H_{k-1}, B_{\max}\}$  are satisfied for all  $k \geq 1$ .

For the finite-time horizon problem (2.15), we may define the value function at time  $k$  as

$$V_k(s) := \min_{\{u_t\}_{t=k}^{T-1}} \sum_{t=k}^{T-1} \mathbb{E}[\text{tr}(P_{t+1}) | s_t, u_t], \text{ s.t. } s_k = s.$$

The optimality equation or Bellman dynamic programming equation associated with the constrained stochastic control problem (2.15) is then given by

$$V_k(s_k) = \min_{0 \leq u_k \leq B_k} \left\{ \mathbb{E}[\text{tr}(P_{k+1}) | s_k, u_k] + \mathbb{E}[V_{k+1}(s_{k+1}) | s_k, u_k] \right\} \quad (2.19)$$

with the terminal condition

$$V_T(s_T) := \min_{0 \leq u_T \leq B_T} \mathbb{E}[\text{tr}(P_{T+1}) | s_T, u_T] = \mathbb{E}[\text{tr}(P_{T+1}) | s_T, B_T],$$

where we use all available energy for transmission at the final time  $T$ .

The optimal transmission energy at time instant  $k \geq 0$  is

$$u_k^*(s_k) = \arg \min_{0 \leq u_k \leq B_k} \left\{ \mathbb{E}[\text{tr}(P_{k+1})|s_k, u_k] + \mathbb{E}[V_{k+1}(s_{k+1})|s_k, u_k] \right\} \quad (2.20)$$

where  $V_{k+1}(\cdot)$  is the solution to the Bellman equation (2.19).

We now simplify the terms in (2.19). First, we have

$$\mathbb{E}[\text{tr}(P_{k+1})|s_k, u_k] = \text{tr}(AP_k A^T + Q) - f(g_k u_k) \text{tr}(AP_k C^T [C P_k C^T + R]^{-1} C P_k A^T)$$

with the constraint that  $0 \leq u_k \leq B_k$ . On the other hand,

$$\begin{aligned} \mathbb{E}[V_{k+1}(s_{k+1})|s_k, u_k] &= \int_{s_{k+1}} V_{k+1}(s_{k+1}) F(ds_{k+1}|s_k, u_k) \\ &= \int_{P_{k+1}, g_{k+1}, H_{k+1}, B_{k+1}} V_{k+1}(P_{k+1}, g_{k+1}, H_{k+1}, B_{k+1}) F(d(P_{k+1}, g_{k+1}, H_{k+1}, B_{k+1})|P_k, g_k, H_k, B_k, u_k) \end{aligned}$$

where  $F$  is the probability transition law. But this together with (2.13) implies that

$$\begin{aligned} \mathbb{E}[V_{k+1}(s_{k+1})|s_k, u_k] &= \int_{P_{k+1}, g_{k+1}, H_{k+1}} V_{k+1}(P_{k+1}, g_{k+1}, H_{k+1}, \min\{B_k - u_k + H_k, B_{\max}\}) \\ &\quad \times F(dP_{k+1}|P_k, g_k, u_k) \mathbb{G}(g_{k+1}|g_k) \mathbb{H}(H_{k+1}|H_k) \end{aligned}$$

which follows from the fact that the mutually independent Markovian processes  $g_{k+1}$  and  $H_{k+1}$  are independent of  $P_{k+1}$ . This gives

$$\begin{aligned} &\mathbb{E}[V_{k+1}(s_{k+1})|s_k, u_k] \quad (2.21) \\ &= \int_{g_{k+1}, H_{k+1}} \left\{ V_{k+1}(AP_k A^T + Q, g_{k+1}, H_{k+1}, \min\{B_k - u_k + H_k, B_{\max}\}) \right. \\ &\quad \times (1 - f(g_k u_k)) \end{aligned}$$

$$\begin{aligned} &+ V_{k+1}(AP_k A^T + Q - AP_k C^T [C P_k C^T + R]^{-1} C P_k A^T, g_{k+1}, H_{k+1}, \\ &\quad \min\{B_k - u_k + H_k, B_{\max}\}) \\ &\quad \times f(g_k u_k) \Big\} \mathbb{G}(g_{k+1}|g_k) \mathbb{H}(H_{k+1}|H_k). \end{aligned} \quad (2.22)$$

Define

$$\mathcal{L}(P, \gamma) \triangleq APA^T + Q - \gamma APC^T (CPC^T + R)^{-1} CPA^T \quad (2.23)$$

For the infinite-time horizon problem (2.16), we have the following:

**Theorem 2.3** *Independent of the initial condition  $\mathbb{I}_0 = \{g_0, H_0, B_0, P_0\}$ , the value of the infinite-time horizon minimization problem (2.16) is given by  $\rho$ , which is the solution of the average cost optimality (Bellman) equation*

$$\begin{aligned} \rho + V(P, g, H, B) = \min_{0 \leq u \leq B} & \left\{ \mathbb{E}[\text{tr}(\mathcal{L}(P, \gamma)) | P, g, u] \right. \\ & \left. + \mathbb{E}[V(\mathcal{L}(P, \gamma), \tilde{g}, \tilde{H}, \min\{B - u + \tilde{H}, B_{\max}\}) | P, g, H, u] \right\}, \end{aligned} \quad (2.24)$$

where  $V$  is the relative value function.

*Proof* See Appendix.  $\square$

We note that a discretized version of the Bellman equations (2.19) or (2.24) can be used for numerical computation to find solutions to the MDP problems (2.15) and (2.16).

**Remark 2.4** The causal information pattern is clearly relevant to most practical scenarios. However, it is also instructive to consider the non-causal information scenario where the sensor has a priori information about the energy harvesting  $\{H_k\}$  process and the fading channel gains  $\{g_k\}$  for all time periods, including the future ones. This may be feasible in the situation of known environment where the wireless channel fading gains and the harvested energies are predictable with high accuracy [22]. Furthermore, the performance of the non-causal information case can serve as a benchmark (a lower bound) for the causal case. Indeed, we will present some performance comparisons between the causal and non-causal cases in Sect. 2.2.4. Note that the energy allocation problems for the non-causal case can be solved using similar techniques as in the current subsection, thus the details are omitted for brevity.  $\square$

### 2.2.3 Structural Results on the Optimal Energy Allocation Policies

In this section, the structure of the optimal transmission energy policy (2.20) is studied for the case of the finite-time horizon stochastic control problem (2.15) with causal information. Following similar arguments, one can show similar structural results for the infinite-time horizon problem (2.16). We begin with a preliminary result, which will be needed for the proof of Theorem 2.4.

**Lemma 2.3** *Suppose  $f(\cdot)$  in (2.4) is a concave function in  $u_k$  given  $g_k$ . Then, for given  $P_k, g_k$  and  $H_k$ , the value function  $V_k(P_k, g_k, H_k, B_k)$  in (2.19) is convex in  $B_k$  for  $0 \leq k \leq T$ . As a result,*



$$V_0(P_0, g_0, H_0, B_0) = \min_{\{0 \leq u_k \leq B_k\}_{k=0}^{T-1}} \sum_{k=0}^{T-1} \mathbb{E}[\text{tr}(P_{k+1})]$$

is convex in  $B_0$ .

*Proof* Recall that  $s_k = (P_k, g_k, H_k, B_k)$ . First, note that, for given  $P_T, g_T$  and  $H_T$ , the final time value function

$$V_T(s_T) = \min_{0 \leq u_T \leq B_T} \mathbb{E}[\text{tr}(P_{T+1})|s_T, u_T] = \mathbb{E}[\text{tr}(P_{T+1})|s_T, B_T]$$

is a convex function in  $B_T$ , due to the fact that  $f(\cdot)$  is a concave function in  $u_k$  given  $g_k$  (see Lemma 2.2). Assume that  $V_{k+1}(s_{k+1})$  is convex in  $B_{k+1}$  for given  $P_{k+1}, g_{k+1}$  and  $H_{k+1}$ . Then, for given  $H_k$  and  $u_k$ , the function

$$V_{k+1}(P_{k+1}, g_{k+1}, H_{k+1}, \min\{B_k - u_k + H_k, B_{\max}\})$$

is convex in  $B_k$ , since it is the minimum of the constant  $V_{k+1}(P_{k+1}, g_{k+1}, H_{k+1}, B_{\max})$  and (by the induction hypothesis) the convex function  $V_{k+1}(P_{k+1}, g_{k+1}, H_{k+1}, B_k - u_k + H_k)$ . Since the expectation operator preserves convexity,  $\mathbb{E}[V_{k+1}(s_{k+1})|s_k, u_k]$  given in (2.22) is a convex function in  $B_k$ . As  $V_k(s_k)$  in (2.19) is the infimal convolution of two convex functions in  $B_k$  for given  $P_k, g_k$  and  $H_k$ , it is also convex in  $B_k$  (see the proof of Theorem 1 in [22]).  $\square$

The following result shows that for fixed  $P_k, g_k$  and  $H_k$ , the optimal energy allocated is increasing with the battery level.

**Theorem 2.4** Suppose  $f(\cdot)$  in (2.4) is a concave function in  $u_k$  given  $g_k$ . Then, given  $P_k, g_k$  and  $H_k$ , the optimal energy policy  $u_k^o(P_k, g_k, H_k, B_k)$  in (2.20) is non-decreasing in  $B_k$  for  $0 \leq k \leq T$ .

*Proof* Assume  $P_k, g_k$  and  $H_k$  are fixed. Define

$$\begin{aligned} L(B, u) &= \mathbb{E}[\text{tr}(P_{k+1})|P_k, g_k, u] \\ &\quad + \mathbb{E}[V_{k+1}(P_{k+1}, g_{k+1}, H_{k+1}, \min\{B - u + H_k, B_{\max}\})|P_k, g_k, H_k, u]. \end{aligned}$$

We wish to show that  $L(B, u)$  is submodular in  $(B, u)$ , i.e. for every  $u' \geq u$  and  $B' \geq B$ , we have [25]:

$$L(B', u') - L(B, u') \leq L(B', u) - L(B, u). \quad (2.25)$$

It is evident that  $\mathbb{E}[\text{tr}(P_{k+1})|P_k, g_k, u]$  is submodular in  $(B, u)$  since it is independent of  $B$ . Let

$$Z(x) := \mathbb{E}[V_{k+1}(P_{k+1}, g_{k+1}, H_{k+1}, \min\{x + H_k, B_{\max}\})|P_k, g_k, H_k, u].$$

Since  $Z(x)$  is convex in  $x$  by Lemma 2.3, we have

$$Z(x + \varepsilon) - Z(x) \leq Z(y + \varepsilon) - Z(y), \quad x \leq y, \quad \varepsilon \geq 0$$

(see Proposition 2.2.6 in [40]). Letting  $x = B - u'$ ,  $y = B - u$  and  $\varepsilon = B' - B$ , we then have the submodularity condition (2.25) for  $\tilde{Z}(B, u) \triangleq Z(B - u)$  [22]. Therefore,  $L(B, u)$  is submodular in  $(B, u)$ . We then note that submodularity is a sufficient condition for optimality of monotone increasing policies [25], i.e. since  $L(B, u)$  is submodular in  $(B, u)$ , then  $u^*(B) = \arg \min_u L(B, u)$  is non-decreasing in  $B$ .  $\square$

As discussed in [22], the structural result of Theorem 2.4 implies that if  $u_k^{\text{uc}}$  is the unique solution to the convex unconstrained minimization problem

$$\begin{aligned} & u_k^{\text{uc}}(P_k, g_k, H_k) \\ &= \arg \min_{u_k} \left\{ \mathbb{E}[\text{tr}(P_{k+1}) | P_k, g_k, u_k] + \mathbb{E}[V_{k+1}(P_{k+1}, g_{k+1}, H_{k+1}) | P_k, g_k, H_k, u_k] \right\}, \end{aligned}$$

then the solution to the constrained problem (2.20), where  $0 \leq u_k \leq B_k$ , will be of the form

$$u_k^*(P_k, g_k, H_k, B_k) = \begin{cases} 0, & \text{if } u_k^{\text{uc}} \leq 0 \\ u_k^{\text{uc}}, & \text{if } 0 < u_k^{\text{uc}} < B_k \\ B_k, & \text{if } u_k^{\text{uc}} \geq B_k. \end{cases}$$

In the case that the transmission energy allocation  $u_k$  belongs to a two element set  $\{E_0, E_1\}$ , the monotonicity of Theorem 2.4 yields a threshold structure. This threshold structure implies that, for fixed  $P_k, g_k$  and  $H_k$ , the optimal transmission energy allocation is of the form

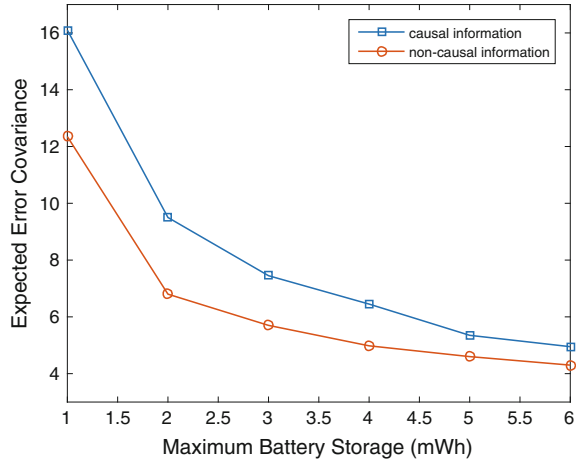
$$u_k^*(P_k, g_k, H_k, B_k) = \begin{cases} E_0, & \text{if } B_k \leq B^* \\ E_1, & \text{otherwise,} \end{cases}$$

where  $B^*$  is the corresponding battery storage threshold. The threshold structure of the optimal energy allocation policy in the case of a binary energy allocation set simplifies the implementation of the optimal energy allocation significantly. A stochastic gradient algorithm for computing  $B^*$  is presented in [41].

### 2.2.4 Numerical Studies

We present here numerical results for a scalar process with the following parameters:  $A = 1.2$ ,  $C = 1$ ,  $Q = 1$ ,  $R = 1$ . We assume that the sensor uses a binary phase shift keying (BPSK) transmission scheme with  $b$  bits per packet. Therefore, (2.4) is of the form [26]:

**Fig. 2.5** Infinite-time horizon average error covariance versus maximum battery storage



$$\mathbb{P}(\gamma_k = 1 | g_k, u_k) = f(g_k u_k) = \left( \int_{-\infty}^{\sqrt{g_k u_k}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \right)^b$$

where we use  $b = 4$  in the simulations.

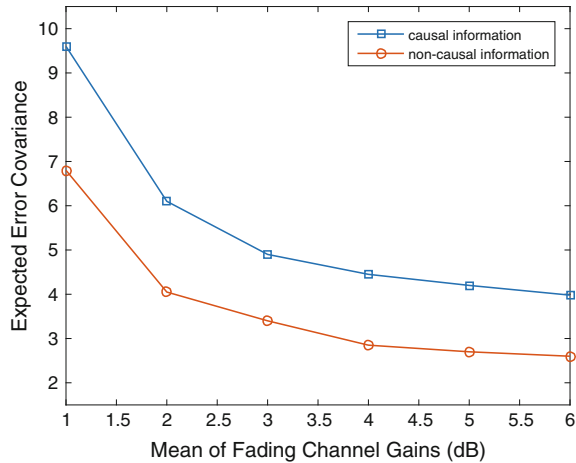
The fading channel is taken to be Rayleigh [28], so that  $\{g_k\}$  is i.i.d. exponentially distributed with probability density function (p.d.f) of the form  $\mathbb{P}(g_k) = \frac{1}{\bar{g}} \exp(-g_k/\bar{g})$ , with  $\bar{g}$  being its mean. We also assume that the harvested energy process  $\{H_k\}$  is i.i.d. and exponentially distributed, with p.d.f.  $\mathbb{P}(H_k) = \frac{1}{\bar{H}} \exp(-H_k/\bar{H})$ , with  $\bar{H}$  being its mean.

For the following simulation results, we use 50 discretization points for each of the quantities  $P_k$ ,  $g_k$ ,  $B_k$ ,  $u_k$  in the Bellman equations.

We first fix the mean of the fading channel gains to  $\bar{g} = 1$  decibel (dB) and the mean of the harvested energy to  $\bar{H} = 1$  milliwatt hour (mWh). Then, we plot in Fig. 2.5 the expected error covariance versus the maximum battery storage energy for the infinite-time horizon problem (2.16), where both cases of causal and non-causal fading channel gains and energy harvesting information are shown, see Remark 2.4. We see that the performance gets better as the maximum battery storage energy increases in both cases. Figure 2.5 also shows that, as expected, the performance for the non-causal information case is generally better than the performance of the system with only causal information.

Finally, we fix the mean of the harvested energy to  $\bar{H} = 1$  (mWh), and the maximum battery storage energy to 2 (mWh). For the infinite-time horizon formulation (2.16), the expected error covariance versus the mean of the fading channel gains is plotted in Fig. 2.6, for both cases of causal and non-causal information. As shown in Fig. 2.6, in both cases, the performance improves as the mean of the fading channel gain increases.

**Fig. 2.6** Infinite-time horizon average error covariance versus mean of fading channel gains



## 2.3 Conclusion

In this chapter, we have investigated transmission power control for Kalman filtering with random packet drops over a fading channel, where the packet reception probability depends on both the time-varying fading channel gain and the sensor transmit power. We first studied the problem of minimizing the trace of the expected error covariance subject to an average power constraint. The resulting Markov decision process problems are solved by the use of dynamic programming techniques. Simpler suboptimal power allocation policies such as a constant power allocation policy and a truncated channel inversion policy have also been considered. Numerical studies suggest that, for low average transmit powers, optimal power allocation significantly outperforms the suboptimal policies, while for higher average transmit powers, the performance of the truncated channel inversion policy approaches the performance of the optimal policy.

We then studied the problem of optimal transmission energy allocation for estimation error covariance minimization, when the sensor is equipped with energy harvesting capabilities. In this problem formulation, the trace of the expected estimation error covariance of the Kalman filter is minimized, subject to energy harvesting constraints. Using the concept of submodularity, some structural results on the optimal transmission energy allocation policy have also been obtained.

*Notes:* Section 2.1 is based on [42], while Sect. 2.2 is based on [41]. The case of imperfect feedback acknowledgements, and a stochastic gradient algorithm for computing the threshold in the case of binary energy levels, is also considered in [41]. The work of [41] has since been extended to control with an energy harvesting in [43]. Energy harvesting in the context of estimation and control has also been subsequently studied in [44, 45], see also Sect. 3.2.

In this book, power allocation decisions are often made at the remote estimator (this is analogous to the situation in wireless communications, where power allocation is often done at the base station and fed back to the mobiles), which motivates us to consider decisions based on the estimation error covariance. When power allocation decisions are made at the sensor, researchers have tried to make use of additional state (or measurement) information [46, 47].

## Appendix

### *Proof of Lemma 2.2*

*Proof* The proof uses similar ideas to the proof of Proposition 3.1 in [36]. The decreasing property follows from the relation

$$\begin{aligned}\mathbb{E}[P_{k+1}] &= \mathbb{E}[P_{k+1}|P_k, g_k, u_k] \\ &= \mathbb{E}[AP_k A^T + Q - f(g_k u_k)AP_k C^T (C P_k C^T + R)^{-1} C P_k A^T]\end{aligned}$$

and the assumption that  $f(\cdot)$  is an increasing function.

For the proof of convexity, let  $u^1$  and  $u^2$  be two average transmit powers, where  $u^1 \neq u^2$ , with  $\mathbf{p}^*(u^1)$  and  $\mathbf{p}^*(u^2)$  the corresponding traces of the expected error covariances. We want to show that

$$\mathbf{p}^*(\lambda u^1 + (1 - \lambda)u^2) < \lambda \mathbf{p}^*(u^1) + (1 - \lambda)\mathbf{p}^*(u^2), \forall \lambda \in (0, 1).$$

Let  $\{u_k^1(P_k, g_k)\}$  be the optimal power allocation policy that achieves  $\mathbf{p}^*(u^1)$ , and  $\{u_k^2(P_k, g_k)\}$  be the optimal power allocation policy that achieves  $\mathbf{p}^*(u^2)$ . Define a new policy  $\{u_k^\lambda(P_k, g_k)\}$  such that

$$u_k^\lambda(P_k, g_k) = \lambda u_k^1(P_k, g_k) + (1 - \lambda)u_k^2(P_k, g_k), \forall P_k, g_k.$$

We will first show that for a given  $P_k$ , we have:

- (1)  $\mathbb{E}[u_k^\lambda|P_k] \leq \lambda \mathbb{E}[u_k^1|P_k] + (1 - \lambda)\mathbb{E}[u_k^2|P_k]$ , and
- (2)  $\mathbb{E}[\text{tr}(P_{k+1}^\lambda)|P_k] < \lambda \mathbb{E}[\text{tr}(P_{k+1}^1)|P_k] + (1 - \lambda)\mathbb{E}[\text{tr}(P_{k+1}^2)|P_k]$ ,

where  $P_{k+1}^j$  is the value of  $P_{k+1}$  that follows from using policy  $\{u_k^j(\cdot)\}$ , for  $j = 1, 2, \lambda$ , respectively. For (1), this clearly follows from the definition of  $u_k^\lambda$ . For (2), we have

$$\begin{aligned}
& \mathbb{E}[\text{tr}(P_{k+1}^\lambda) | P_k] \\
&= \int \left( \text{tr}(A P_k A^T + Q) - f(g_k u_k^\lambda) \text{tr}(A P_k C^T (C P_k C^T + R)^{-1} C P_k A^T) \right) F(dg_k) \\
&< \int \left( \text{tr}(A P_k A^T + Q) - (\lambda f(g_k u_k^1) + (1 - \lambda) f(g_k u_k^2)) \right. \\
&\quad \left. \times \text{tr}(A P_k C^T (C P_k C^T + R)^{-1} C P_k A^T) \right) F(dg_k) \\
&= \lambda \mathbb{E}[\text{tr}(P_{k+1}^1) | P_k] + (1 - \lambda) \mathbb{E}[\text{tr}(P_{k+1}^2) | P_k]
\end{aligned}$$

where the inequality comes from the strict concavity of  $f(\cdot)$ .

From (1) and (2), we have

$$\begin{aligned}
\lim_{K \rightarrow \infty} \frac{1}{K} \sum_{k=1}^K \mathbb{E}[u_k^\lambda] &= \lim_{K \rightarrow \infty} \frac{1}{K} \sum_{k=1}^K \mathbb{E}[\mathbb{E}[u_k^\lambda | P_k]] \\
&\leq \lim_{K \rightarrow \infty} \frac{1}{K} \sum_{k=1}^K \mathbb{E}[\lambda \mathbb{E}[u_k^1 | P_k] + (1 - \lambda) \mathbb{E}[u_k^2 | P_k]] \\
&= \lambda u^1 + (1 - \lambda) u^2
\end{aligned}$$

and

$$\begin{aligned}
\lim_{K \rightarrow \infty} \frac{1}{K} \sum_{k=1}^K \mathbb{E}[\text{tr}(P_{k+1}^\lambda)] &= \lim_{K \rightarrow \infty} \frac{1}{K} \sum_{k=1}^K \mathbb{E}[\mathbb{E}[\text{tr}(P_{k+1}^\lambda) | P_k]] \\
&< \lim_{K \rightarrow \infty} \frac{1}{K} \sum_{k=1}^K \mathbb{E}[\lambda \mathbb{E}[\text{tr}(P_{k+1}^1) | P_k] + (1 - \lambda) \mathbb{E}[\text{tr}(P_{k+1}^2) | P_k]] \\
&= \lambda \mathbf{p}^*(u^1) + (1 - \lambda) \mathbf{p}^*(u^2).
\end{aligned}$$

By the definition of  $\mathbf{p}^*(u)$  being the minimum expected error covariance such that the average transmit power is less than or equal to  $u$ , we then have  $\mathbf{p}^*(\lambda u^1 + (1 - \lambda) u^2) \leq \frac{1}{K} \sum_{k=1}^K \mathbb{E}[\text{tr}(P_{k+1}^\lambda)] < \lambda \mathbf{p}^*(u^1) + (1 - \lambda) \mathbf{p}^*(u^2)$ .

### ***Proof of Theorem 2.3***

We first establish the inequality

$$\begin{aligned}
\rho + V(P, g, H, B) &\geq \min_{0 \leq u \leq B} \left\{ \mathbb{E}[\text{tr}(\mathcal{L}(P, \gamma)) | P, g, u] \right. \\
&\quad \left. + \mathbb{E}[V(\mathcal{L}(P, \gamma), \tilde{g}, \tilde{H}, \min\{B - u + \tilde{H}, B_{\max}\}) | P, g, H, u] \right\} \quad (2.26)
\end{aligned}$$

by verifying conditions (W) and (B) of [48], that guarantee the existence of solutions to (2.26) for MDPs with general state space. Denote the state space by  $\mathcal{S}$  and action space by  $\mathcal{A}$ , i.e.  $(P_k, g_k, H_k, B_k) \in \mathcal{S}$  and  $u_k \in \mathcal{A}$ . Condition (W) of [48] in our notation says that:

- (1) The state space  $\mathcal{S}$  is locally compact.
- (2) Let  $U(\cdot)$  be the mapping that assigns to each  $(P_k, g_k, H_k, B_k)$  the nonempty set of available actions. Then  $U(P_k, g_k, H_k, B_k)$  lies in a compact subset of  $\mathcal{A}$  and  $U(\cdot)$  is upper semicontinuous.
- (3) The transition probabilities are weakly continuous.
- (4)  $\mathbb{E}[\text{tr}(\mathcal{L}(P, \gamma)) | P, g, u]$  is lower semicontinuous.

By our assumption that  $u_k \leq B_k \leq B_{\max}$ , (0) and (1) of (W) can be easily verified. The conditions (2) and (3) follow from the definition (2.23).

Define  $w_\delta(P_0, g_0, H_0, B_0) = v_\delta(P_0, g_0, H_0, B_0) - m_\delta$ , where

$$v_\delta(P_0, g_0, H_0, B_0) = \inf_{\{u_k: k \geq 0\}} \mathbb{E} \left[ \sum_{k=0}^{\infty} \delta^k \mathbb{E}[\text{tr}(\mathcal{L}(P_k, \gamma_k)) | P_k, g_k, u_k] | P_0, g_0, H_0, B_0 \right]$$

and  $m_\delta = \inf_{(P_0, g_0, H_0, B_0)} v_\delta(P_0, g_0, H_0, B_0)$ . Condition (B) of [48] in our notation says that

$$\sup_{\delta < 1} w_\delta(P_0, g_0, H_0, B_0) < \infty, \quad \forall (P_0, g_0, H_0, B_0).$$

Following Sect. 4 of [48], define the stopping time

$$\tau = \inf\{k \geq 0 : v_\delta(P_k, g_k, H_k, B_k) \leq m_\delta + \varsigma\}$$

for some  $\varsigma \geq 0$ . Given  $\varsigma > 0$  and an arbitrary  $(P_0, g_0, H_0, B_0)$ , consider a suboptimal power allocation policy where the sensor transmits based on the same policy as the one that achieves  $m_\delta$  (with a different initial condition) until  $v_\delta(P_N, g_N, H_N, B_N) \leq m_\delta + \varsigma$  is satisfied at some time  $N$ . By the exponential forgetting property of initial conditions for Kalman filtering, we have  $N < \infty$  with probability 1 and  $\mathbb{E}[N] < \infty$ . Since  $\tau \leq N$ , we have  $\mathbb{E}[\tau] < \infty$ . Then by Lemma 4.1 of [48],

$$\begin{aligned} w_\delta(P_0, g_0, H_0, B_0) &\leq \varsigma + \inf_{\{\gamma_k\}} \mathbb{E} \left[ \sum_{k=0}^{\tau-1} \mathbb{E}[\text{tr}(\mathcal{L}(P_k, \gamma_k)) | P_k, g_k, u_k] | P_0, g_0, H_0, B_0 \right] \\ &\leq \varsigma + \mathbb{E}[\tau] \times Z < \infty \end{aligned} \tag{2.27}$$

where the second inequality uses Wald's equation, with  $Z$  being an upper bound to the expected error covariance, which exists by Theorem 2.2. Hence, condition (B) of [48] is satisfied and a solution to the inequality (2.26) exists.

To show equality in (2.26), we will require a further equicontinuity property of the optimal cost for the related discounted cost MDP to be satisfied. This can be shown

by a similar argument as in the proof of Proposition 3.2 of [49]. The assumptions in Sects. 5.4 and 5.5 of [33] may then be verified to conclude the existence of a solution to the average cost optimality equation (2.24).

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