

Chapter 2

Robust Constrained Linear Regulator Problem

2.1 Introduction

Generally, for physical, technological, and/or security reasons, any physical system is subject to functional limitations, and moreover, the adopted models are often subject to uncertainties. These uncertainties may find their origin in the modeling and measurement errors or on the computation approximations (see [1–5] and the references therein). As a consequence, the simultaneous presence of uncertainties and constraints in general physical systems has interested many authors to combine the techniques of robust control and constrained control [6, 7]. With the same objective and as an extension to the uncertain case of [8], the first part of this chapter is devoted to design and study robust-constrained regulators. Necessary and sufficient conditions of positive invariance of polyhedral domains with respect to motion of uncertain continuous-time systems are derived.

Further, one has to concede that there are several approaches proposed in the literature to solve this problem [6, 8–15] and others. From these approaches, the positive invariance approach is selected in the second part to calculate robust constant state feedback controllers [6, 10, 16, 17]. But from a different point of view for computation techniques in the design of such controllers, the idea is to translate the results found within the framework of positive invariance to algorithms of linear programming [18–20]. To this end, necessary and sufficient conditions of positive invariance are re-formulated in this part. This enables one to use linear programming algorithms to find robust-constrained regulators for both continuous-time and discrete-time linear systems.

2.2 Robust-Constrained Linear Control

2.2.1 Problem Statement

Consider the linear uncertain continuous-time system represented by the following state space model:

$$\dot{x}(t) = A(q_A(t))x(t) + B(q_B(t))u(t) \quad (2.1)$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $u(t) \in \Omega \subset \mathbb{R}^m$ is the control input. The control input is restricted (by saturation) to evolve in the following polyhedral set:

$$\Omega = \{u(t) \in \mathbb{R}^m / -u_{\min} \leq u \leq u_{\max}; \ u_{\min}, u_{\max} \in \text{int} \mathbb{R}_+^m\} \quad (2.2)$$

$q_A(t) \in \Gamma_A \subset \mathbb{R}^{p_A}$ (resp. $q_B(t) \in \Gamma_B \subset \mathbb{R}^{p_B}$) is the uncertain vector. Γ_A and Γ_B are compact convex sets including the origin in their interiors. These vectors $q_A(t)$ and $q_B(t)$ measure the uncertainty in the model, affecting the parameters of the matrices A and B as follows:

$$A(q(t)) = A_o + \sum_{h=1}^{p_A} A_h q_{Ah}(t) \quad (2.3)$$

$$B(q(t)) = B_o + \sum_{h=1}^{p_B} B_h q_{Bh}(t)$$

where $q_{Ah}(t)$ and $q_{Bh}(t)$ represent the h^{th} component of vectors $q_A(t)$ and $q_B(t)$, respectively:

$$q_A(t) = [q_{A1}(t) \ q_{A2}(t) \ \dots \ q_{Ap_A}(t)]^T \quad (2.4)$$

$$q_B(t) = [q_{B1}(t) \ q_{B2}(t) \ \dots \ q_{Bp_B}(t)]^T$$

Convexity and compactness of the set Γ_A imply that there exist μ_A vertices $v_i, i = 1, \dots, \mu_A$ of Γ_A , such that every $q_A \in \Gamma_A$ can be written as a convex combination of v_i as:

$$q_A = \sum_{i=1}^{\mu_A} \alpha_i v_i \text{ with } \sum_{i=1}^{\mu_A} \alpha_i = 1, \ 0 \leq \alpha_i \leq 1, \text{ for } i = 1, \dots, \mu_A \quad (2.5)$$

The set Γ_B is also convex and compact, so there also exist μ_B vertices $v_j, j = 1, \dots, \mu_B$ of Γ_B such that every $q_B \in \Gamma_B$ can be written as a convex combination of v_j as:

$$q_B = \sum_{j=1}^{\mu_B} \beta_j v_j \text{ with } \sum_{j=1}^{\mu_B} \beta_j = 1, \quad 0 \leq \beta_j \leq 1, \text{ for } j = 1, \dots, \mu_B, \quad (2.6)$$

and the matrix $A(q_A(t))$ (resp. $B(q_B(t))$) as:

$$A(q_A(t)) = \sum_{i=1}^{\mu_A} \alpha_i(t) A_i \quad \left(\text{resp. } B(q_B(t)) = \sum_{j=1}^{\mu_B} \beta_j(t) B_j \right), \quad (2.7)$$

with $A_i = A(v_i)$ (resp. $B_j = B(v_j)$).

Assume that the pair $(A(q_A(t)), B(q_B(t)))$ is controllable for every $q_A \in \Gamma_A$ and $q_B \in \Gamma_B$.

The nominal system is given by:

$$\dot{x}(t) = A_o x(t) + B_o u(t). \quad (2.8)$$

The robust-constrained regulator problem, which will be studied in this chapter, is to find a state feedback control law:

$$u(t) = Fx(t), \quad F \in \mathbb{R}^{m \times n} \text{ with } \text{rank}(F) = m, \quad (2.9)$$

which stabilizes the nominal system

$$\dot{x}(t) = (A_o + B_o F)x(t), \quad (2.10)$$

respecting the control constraints (2.2), such that the feedback system is robustly stable against parametric uncertainty: That is, the feedback stabilizes also the uncertain system (2.1). Application of this control law, while respecting control constraints, leads to the closed-loop system:

$$\dot{x}(t) = (A(q_A(t)) + B(q_B(t))F)x(t) \quad (2.11)$$

$$= \sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \beta_j \alpha_i (A_i + B_j F) \quad (2.12)$$

$$= \sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \beta_j \alpha_i A_{cl}(v_{ij}) \quad (2.13)$$

$$= A_{cl}(q(t)) \quad (2.14)$$

where $q(t) \in \Gamma = (\Gamma_A \times \Gamma_B)$, and v_{ij} denotes the vertices of Γ ($i = 1, \dots, \mu_A$; $j = 1, \dots, \mu_B$).

Also, the proposed control law induces the following set of linear behavior in state space:

$$\mathcal{D} = \{x \in \mathbb{R}^n / -u_{\min} \leq Fx \leq u_{\max}, u_{\min}, u_{\max} \in \text{int}\mathbb{R}_+^m\} \quad (2.15)$$

Note here that as long as the system states remain in the domain $\mathcal{D}$, the linear behavior is guaranteed. Otherwise, the closed-loop system is given by:

$$\dot{x}(t) = A(q_A(t))x(t) + B(q_B(t))\text{sat}(Fx(t)) \quad (2.16)$$

It is worth noticing that positive invariance of domain $\mathcal{D}$ given by (2.15), with respect to (w.r.t.) motion of the closed-loop system (2.11), is the cornerstone to derive robust-constrained regulators.

2.2.2 Design of Robust Controller

This section is devoted to the extension of the preliminary results presented in Chap. 1 to the case of uncertain plants. This extension gives conditions for designing robust controllers in the presence of uncertainty and input limitations. The following lemma is fundamental to extend necessary and sufficient conditions for the positive invariance of the polyhedral domain:

$$\mathcal{D}_p = \{z \in \mathbb{R}^m / -p_2 \leq z \leq p_1, p_1, p_2 \in \text{int}\mathbb{R}_+^m\} \quad (2.17)$$

with respect to motion of the autonomous uncertain system:

$$\dot{z}(t) = H(q(t))z(t) \quad (2.18)$$

Lemma 2.1 *The set $\mathcal{D}_p$ given by (2.17) is positively invariant with respect to motion of system (2.18) if and only if it is positively invariant with respect to motion of system (2.18) at vertices $v_{ij}, i = 1, \dots, \mu_A, j = 1, \dots, \mu_B$ of the set Γ .*

Proof

Necessity: It is obvious that if the domain is positively invariant for every q , it is especially positively invariant at the vertices $v_{ij}, i = 1, \dots, \mu_A, j = 1, \dots, \mu_B$.

Sufficiency: Suppose that $\mathcal{D}_p$ is positively invariant with respect to motion of system (2.18) for $q = v_{ij}, i = 1, \dots, \mu_A, j = 1, \dots, \mu_B$ then:

$$\tilde{H}_c(v_{ij})\varpi \leq 0 \text{ with } \varpi = [p_1^T \ p_2^T]^T,$$

decomposing this inequality, it is possible to write:

$$H_1(v_{ij}) p_1 + H_2(v_{ij}) p_2 \leq 0, \quad (2.19)$$

$$H_2(v_{ij}) p_1 + H_1(v_{ij}) p_2 \leq 0, \quad (2.20)$$

however, according to the definition of matrices H_1 and H_2 given by (1.33), inequality (2.19) can be written as, for $l = 1, \dots, m$,

$$h_{ll}(v_{ij}) p_1^l + \sum_{\substack{k=1 \\ k \neq l}}^m h_{lk}^+(v_{ij}) p_1^k + \sum_{\substack{k=1 \\ k \neq l}}^m h_{lk}^-(v_{ij}) p_2^k \leq 0,$$

multiplying every inequality by α_i, β_j $i = 1, \dots, \mu_A, j = 1, \dots, \mu_B$, and summing on i and j , leads to

$$\sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \beta_j \alpha_i h_{ll}(v_{ij}) p_1^l + \sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \sum_{\substack{k=1 \\ k \neq l}}^m \beta_j \alpha_i h_{lk}^+(v_{ij}) p_1^k + \sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \sum_{\substack{k=1 \\ k \neq l}}^m \beta_j \alpha_i h_{lk}^-(v_{ij}) p_2^k \leq 0,$$

which is equivalent to write

$$\sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \beta_j \alpha_i h_{ll}(v_{ij}) p_1^l + \sum_{\substack{k=1 \\ k \neq l}}^m \sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \beta_j \alpha_i h_{lk}^+(v_{ij}) p_1^k + \sum_{\substack{k=1 \\ k \neq l}}^m \sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \beta_j \alpha_i h_{lk}^-(v_{ij}) p_2^k \leq 0,$$

using the definitions of h_{ij}^+ and h_{ij}^- given in (1.33), it is possible to see that:

$$\left(\sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \beta_j \alpha_i h_{lk}(v_{ij}) \right)^+ \leq \sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \beta_j \alpha_i h_{lk}^+(v_{ij}),$$

and

$$\left(\sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \beta_j \alpha_i h_{lk}(v_{ij}) \right)^- \leq \sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \beta_j \alpha_i h_{lk}^-(v_{ij}),$$

then:

$$\begin{aligned} & \sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \beta_j \alpha_i h_{ll}(v_{ij}) p_1^l + \sum_{\substack{k=1 \\ k \neq l}}^m \left(\sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \beta_j \alpha_i h_{lk}(v_{ij}) \right)^+ \\ & p_1^k + \sum_{\substack{k=1 \\ k \neq l}}^m \left(\sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \beta_j \alpha_i h_{lk}(v_{ij}) \right)^- p_2^k \leq 0. \end{aligned}$$

According to (2.5) and (2.6) the following result is obtained:

$$H(q(t)) = \sum_{j=1}^{\mu_B} \sum_{i=1}^{\mu_A} \beta_j \alpha_i H(v_{ij}),$$

therefore,

$$h_{ll}(q(t)) p_1^l + \sum_{\substack{k=1 \\ k \neq l}}^m h_{lk}^+(q(t)) p_1^k + \sum_{\substack{k=1 \\ k \neq l}}^m h_{lk}^-(q(t)) p_2^k \leq 0,$$

hence,

$$H_1(q(t)) p_1 + H_2(q(t)) p_2 \leq 0.$$

Using a similar reasoning, it is possible to prove that (2.20) leads to:

$$H_2(q(t)) p_1 + H_1(q(t)) p_2 \leq 0,$$

which gives the sufficient condition:

$$\tilde{H}_c(q(t)) \varpi \leq 0.$$

Consequently, the set $\mathcal{D}_P$ is positively invariant with respect to motion of system (2.18) $\forall q(t) \in \Gamma$. $\square$

Now, it is possible to make the desired extension. Hence, let us consider the non-autonomous uncertain system (2.1) with constrained control (2.2). The application of the feedback control law (2.9) leads to the closed-loop system (2.11).

Theorem 2.1 *The subset $\mathcal{D}$ given by (2.15) is positively invariant with respect to motion of the uncertain system (2.11) with constrained control (2.2) if and only if there exist matrices $H(v_{ij})$ for $i = 1, \dots, \mu_A, j = 1, \dots, \mu_B$ such that:*

$$FA_{cl}(v_{ij}) = H(v_{ij}) F \quad (2.21)$$

$$\tilde{H}_c(v_{ij}) U \leq 0 \quad (2.22)$$

Proof

Sufficiency: Assume that conditions (2.21) and (2.22) hold. By virtue of Theorem 1.4, the set $\mathcal{D}$ is positively invariant with respect to motion of system (2.11) at the vertices v_{ij} for $i = 1, \dots, \mu_A, j = 1, \dots, \mu_B$. Further, consider the following change of coordinates

$$z(t) = Fx(t) \quad (2.23)$$

the system (2.11) becomes system (2.18), and $\mathcal{D}$ is transformed as follows:

$$\mathcal{D}_z = \{z \in \mathbb{R}^m / -u_{\min} \leq z \leq u_{\max}, u_{\min}, u_{\max} \in \mathbb{R}_+^m\}.$$

Hence, $\mathcal{D}_z$ is positively invariant with respect to motion of system (2.18) at the vertices v_{ij} , for $i = 1, \dots, \mu_A$, $j = 1, \dots, \mu_B$. Bearing in mind Lemma 2.1, $\mathcal{D}_z$ is positively invariant with respect to system (2.18) $\forall q \in \Gamma$. Consequently, $\mathcal{D}$ is positively invariant with respect to system (2.11) $\forall q \in \Gamma$.

Necessity: Let $\mathcal{D}$ be positively invariant with respect to system (2.11), so it is positively invariant at the vertices by using Lemma 2.1. Therefore, by virtue of Theorem 1.4, there exist matrices $H(v_{ij})$ satisfying conditions (2.21) and (2.22) for every $i = 1, \dots, \mu_A$, $j = 1, \dots, \mu_B$. $\square$

Remark 2.1 For the sequel, and without loss of generality, assume that the system is square, i.e., $n = m$. In fact, if the system has n state and m input $m \leq n$, then the system is augmented with constrained fictitious inputs v as presented above [6]. The resulting system is then given by:

$$\dot{x}(t) = A(q(t))x(t) + B_a(q(t)) \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \quad (2.24)$$

$B_a(q(t))$ is the matrix $B(q(t))$ augmented by $(n - m)$ null columns.

Corollary 2.1 *The positive invariance of domain $\mathcal{D}$ with respect to motions of system (2.24) with state feedback matrix F satisfying condition (2.9) implies the asymptotic stability of system (2.24) for every $x(0) \in \mathcal{D}$.*

Proof

Assume that domain $\mathcal{D}$ is positively invariant with respect to motion of system (2.24), then by virtue of the Theorem 2.1 and using the Lemma 2.1, we have:

$$FA_{cl}(q(t)) = H(q(t)) F \quad (2.25)$$

$$\tilde{H}_c(q(t)) U \leq 0 \quad (2.26)$$

then using Theorem 2.1, the non-symmetric non-quadratic function

$$V(x) = \max_l \max \left\{ \frac{(Fx)_l^+}{u_{\max}^l}, \frac{(Fx)_l^-}{u_{\min}^l} \right\}$$

is a common Lyapunov function of all stationary configuration of the uncertain system (2.11) for every $q \in \Gamma$. Hence, the system (2.11) is asymptotically stable. $\square$

2.3 Robust Control with Constrained State and Input

In this second part, the robustness problem for constrained states and control is addressed. Different writing of the constraints is used. Further, conditions are expressed as linear programming problem.

2.3.1 Problem Statement

Let us consider the linear system represented by:

$$\delta x(t) = Ax(t) + Bu(t) \quad (2.27)$$

where $x(t) \in \mathbb{R}^n$ is the state vector belonging to the polyhedral set of $\mathbb{R}^n$, containing the origin in its interior, defined by :

$$\mathcal{D}(G, \omega) = \{x \in \mathbb{R}^n / Gx \leq \omega, G \in \mathbb{R}^{g \times n}, \omega \in \mathbb{R}_+^{g*}, g \geq n, \text{rank}(G) = n\} \quad (2.28)$$

and $u(t) \in \Omega_1 \subset \mathbb{R}^m$ is the input vector constrained to evolve within the polyhedral set Ω_1 of $\mathbb{R}^m$ where,

$$\Omega_1 = \{u \in \mathbb{R}^m / u \leq \rho, \rho \in \mathbb{R}_+^{m*}\} \quad (2.29)$$

The linear-constrained regulator problem (LCRP) can be stated as: Find a linear state feedback control law given by

$$u(t) = Fx(t), \quad F \in \mathbb{R}^{m \times n}, \quad (2.30)$$

such that all initial states within domain $\mathcal{D}(G, \omega)$ are transferred asymptotically to the origin while the control respects constraints (2.29).

The feedback control law (2.30) induces a domain of admissible states given by

$$\mathcal{D}(F, \rho) = \{x \in \mathbb{R}^n, Fx \leq \rho\} \quad (2.31)$$

The property of control admissibility for all $x \in \mathcal{D}(G, \omega)$ can be expressed as the following polyhedral inclusion

$$\mathcal{D}(G, \omega) \subseteq \mathcal{D}(F, \rho) \quad (2.32)$$

If control constraints are respected, closed-loop behavior is linear and is given by

$$\begin{aligned} \delta x(t) &= (A + BF)x(t) \\ &= A_{cl}x(t) \end{aligned} \quad (2.33)$$

Let us now consider that the system parameters are not perfectly known. The linear uncertain system has the following form:

$$\delta x(t) = A(q_A(t)) x(t) + B(q_B(t)) u(t), \quad (2.34)$$

where $q_A(t) \in \Gamma_A \subset \mathbb{R}^{p_A}$, and $q_B(t) \in \Gamma_B \subset \mathbb{R}^{p_B}$ where Γ_A and Γ_B are compact convex sets including the origin in their interiors. Uncertainties affect the system parameters as detailed in (2.3) and (2.4).

The Robust Linear-Constrained Regulator Problem (RLCRP), as studied hereafter, consists to find a robust regulator that transfers asymptotically all initial states in $\mathcal{D}(G, \omega)$ to the origin while the state and control respect constraints (2.28) and (2.29). The closed-loop system becomes :

$$\begin{aligned} \delta x(t) &= [A(q_A(t)) + B(q_B(t))F] x(t) \\ &= A_{cl}(q(t))x(t) \end{aligned} \quad (2.35)$$

where $q(t) \in \Gamma = \Gamma_A \times \Gamma_B$. The vertices of Γ are noted v_{ij} for $i = 1, \dots, \mu_A$ and $j = 1, \dots, \mu_B$.

Remark 2.2 Without loss generality, it was assumed that $\text{rank}(G) = n$. In fact, if this condition is not satisfied, not penalizing fictitious constraints on the state vector can be imposed. For example, consider that $n = 2$ and that the constraints on the state vector are given by:

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

and we have $\text{rank}(G) = 1$. We impose fictitious suitable constraints, in order to have $\text{rank}(G) = 2$, as follows

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \\ g_{a1} & g_{a2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_a \end{bmatrix}$$

where the row g_a of matrix G and the constraint ω_a are chosen to be not penalizing for the initial problem. It is worth to note here that in [19, 21] for the discrete-time case, the matrix G is required to be square and invertible, which is very restrictive in our sense.

Now, the extension of positive invariance conditions for a polyhedral domain with respect to uncertain systems is studied. Necessary and sufficient conditions are established:

Lemma 2.2 *The set $\mathcal{D}(G, \omega)$, given by (2.28), is positively invariant with respect to motion of system (2.35) if and only if it is positively invariant with respect to motion of system (2.35) at vertices v_{ij} for $i = 1, \dots, \mu_A$, $j = 1, \dots, \mu_B$.*

Proof

Similar to proof of Lemma 2.1. □

Now, with this background, the main result of this section can be worked out.

2.3.2 Robust-Constrained Regulator Problem

As a first step, in this section, conditions to find a solution to the LCRP are formulated as linear inequalities. In a second step, these conditions are extended to the case of uncertain systems. This extension allows to find a solution to the RLCRP. The problem is cast into a linear programming feasibility formulation. Further, the introduction of an objective optimization variable ε enables to solve a linear programming-based algorithm giving stabilizing robust regulators respecting the state and control constraints (2.28) and (2.29).

Proposition 2.1 *A matrix F is solution to the LCRP if there exist matrices $H \in \mathcal{M}_H \subset \mathbb{R}^{g \times g}$, and $T \in \mathbb{R}^{m \times g}$ with nonnegative components ($T \geq 0$) such that:*

$$HG = G(A + BF) \quad (2.36)$$

$$\mathcal{H}\omega < 0 \quad (2.37)$$

$$F = TG \quad (2.38)$$

$$T\omega \leq \rho \quad (2.39)$$

where for a matrix $H \in \mathcal{M}_H$, we use $\mathcal{H}$ to note : $H - \mathbb{I}$ in the DTC, and H in the CTC. Further, the set $\mathcal{M}_H$ denotes the set of nonnegative matrices for the DTC and the set of matrices with nonnegative off-diagonal elements for the CTC (i.e., matrix H is Metzler, such that $h_{ij} \geq 0 \ \forall i \neq j$).

Proof

Assume that conditions (2.36) and (2.37) hold true. By virtue of Proposition 1.1, the domain $\mathcal{D}(G, \omega)$ is positively invariant with respect to motion of system (2.33). The existence of a matrix T , such that conditions (2.38) and (2.39) hold and using the Haar's lemma, leads to the implication $Gx(t) \leq \omega \Rightarrow Fx(t) \leq \rho$ which is equivalent to:

$$\mathcal{D}(G, \omega) \subseteq \mathcal{D}(F, \rho) \quad (2.40)$$

Hence, the closed-loop behavior is linear and from the existence of a matrix H that satisfies (2.36) and (2.37), one can prove that the non-quadratic function:

$$V(x(t)) = \max_i \left\{ \frac{|(Gx(t))_i|}{\omega_i} \right\}, \quad (2.41)$$

is a Lyapunov function for the system. Then, the asymptotic stability is guaranteed (for details about the Lyapunov function, see demonstration of Proposition 2.2). Finally, F is solution to the LCRP. $\square$

Consider now that the system parameters are uncertain. As previously stated, the closed-loop uncertain system is given by (2.35).

Proposition 2.2 *A matrix $F \in \mathbb{R}^{m \times n}$ is solution to the RLCRP if there exist matrices $H_{ij} \in \mathcal{M}_H \subset \mathbb{R}^{g \times g}$, $i = 1, \dots, \mu_A$, $j = 1, \dots, \mu_B$, and $T \in \mathbb{R}^{m \times g}$ with nonnegative components ($T \geq 0$) satisfying conditions*

$$H_{ij}G = G(A_i + B_j F) \quad (2.42)$$

$$\mathcal{H}_{ij}\omega < 0 \quad (2.43)$$

$$F = TG \quad (2.44)$$

$$T\omega \leq \rho \quad (2.45)$$

Proof

First, using the Haar's lemma, conditions (2.44) and (2.45) imply the inclusion (2.40) guaranteeing the admissibility, and hence, the state and control constraints are not violated, and the closed-loop model is always linear.

Further, consider that the conditions (2.42) and (2.43) hold. By virtue of Proposition 1.1, the domain $\mathcal{D}(G, \omega)$ is positively invariant with respect to motion of system (2.35) at the vertices v_{ij} for $i = 1, \dots, \mu_A$, $j = 1, \dots, \mu_B$. Using the Lemma 2.2, the domain is also positively invariant for every $q \in \Gamma$, which enables to write that there exist matrices $H(q(t)) \in \mathcal{M}_H$ such that:

$$H(q(t))G = GA_{cl}(q(t)), \quad (2.46)$$

bearing in mind that:

$$H(q(t)) := \sum_{i=1}^{\mu_A} \sum_{j=1}^{\mu_B} \alpha_i(t) \beta_j(t) H_{ij}, \quad (2.47)$$

one can conclude that:

$$\mathcal{H}(q(t))\omega \leq 0. \quad (2.48)$$

For the discrete-time case, consider the non-quadratic function given by:

$$V(x(t)) = \max_i \left\{ \frac{|(Gx(t))_i|}{\omega_i} \right\}. \quad (2.49)$$

This function is positive definite. Let us compute its rate of increase along the trajectories of our system

$$\begin{aligned}
\Delta V(x(t)) &= V(x(t+1)) - V(x(t)) \\
&= \max_i \left\{ \frac{|(GA_{cl}(q(t))x(t))_i|}{\omega_i} \right\} - \max_i \left\{ \frac{|(Gx(t))_i|}{\omega_i} \right\} \\
&= \max_i \left\{ \frac{|(GA_{cl}(q(t))x(t))_i|}{\omega_i} \right\} - \max_i \left\{ \frac{|(Gx)_i|}{\omega_i} \right\}.
\end{aligned}$$

According to (2.46),

$$\Delta V(x(t)) = \max_i \left\{ \frac{|(H(q(t))Gx(t))_i|}{\omega_i} \right\} - \max_i \left\{ \frac{|(Gx(t))_i|}{\omega_i} \right\},$$

$|(H(q(t))Gx(t))_i|$ can be raised as follows:

$$|(H(q(t))Gx(t))_i| = \left| \sum_{j=1}^g h_{ij}(q(t)) (Gx(t))_j \right|, \quad (2.50)$$

however,

$$|(H(q(t))Gx(t))_i| \leq \sum_{j=1}^g |h_{ij}(q(t))| |(Gx(t))_j|, \quad (2.51)$$

noticing that $H(q(t))$ and ω have non-negative components, then,

$$\begin{aligned}
|(H(q(t))Gx(t))_i| &\leq \sum_{j=1}^g h_{ij}(q(t)) |(Gx(t))_j| \\
&\leq \sum_{j=1}^g \left\{ h_{ij}(q(t)) \omega_j \frac{|(Gx)_j|}{\omega_j} \right\} \\
&\leq \sum_{j=1}^g \{h_{ij}(q(t))\omega_j\} \max_j \left\{ \frac{|(Gx)_j|}{\omega_j} \right\} \\
&\leq (H(q(t))\omega)_i \max_i \left\{ \frac{|(Gx)_i|}{\omega_i} \right\}.
\end{aligned}$$

Substituting into (2.50)

$$\Delta V(x(t)) \leq \max_i \left\{ \left(\frac{(H(q(t))\omega)_i}{\omega_i} - 1 \right) \max_i \left\{ \frac{|(Gx(t))_i|}{\omega_i} \right\} \right\},$$

which is equivalent to

$$\Delta V(x(t)) \leq \left(\frac{(H(q(t))\omega)_i}{\omega_i} - 1 \right) \text{Max}_i \left\{ \frac{|(Gx(t))_i|}{\omega_i} \right\}.$$

Therefore, according to (2.48), the rate of increase $\Delta V(x(t))$ is strictly negative, then the function $V(x(t))$ is a common Lyapunov function for all stationary configurations of the uncertain system (2.35) for every $q \in \Gamma$, and consequently, the system is asymptotically stable.

For the continuous-time case, the non-quadratic function given by (2.49) is positive definite. Let us compute its derivative along the trajectories of the system (2.35)

$$\begin{aligned} \dot{V}(x(t)) &= \lim_{\varepsilon \rightarrow 0^+} \frac{V(x(t) + \varepsilon \dot{x}(t)) - V(x(t))}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \left\{ \text{Max}_i \left\{ \frac{|(Gx(t) + \varepsilon G(A + BF)x(t))_i|}{\omega_i} \right\} - V(x(t)) \right\}. \end{aligned}$$

According to (2.46):

$$\dot{V}(x(t)) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \left\{ \text{Max}_i \left\{ \frac{|(1 + \varepsilon H)Gx(t))_i|}{\omega_i} \right\} - V(x(t)) \right\},$$

this expression is equivalent to (2.50) for the discrete-time case where matrix H is replaced by $1 + \varepsilon H$, and consequently, the same reasoning leads to,

$$\dot{V}(x(t)) \leq \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \left(\text{Max}_i \left\{ \left(\frac{(|1 + \varepsilon H(q(t))\omega)_i}{\omega_i} - 1 \right) V(x(t)) \right\} \right).$$

However, since matrix $H \in \mathcal{M}_H$ and as $\varepsilon \rightarrow 0$:

$$\begin{aligned} |1 + \varepsilon H(q(t))| &= \begin{cases} \varepsilon |h_{ij}| = \varepsilon h_{ij} & \text{for } i \neq j, \\ 1 + \varepsilon h_{ij} & \text{for } i = j, \end{cases} \\ &= 1 + \varepsilon H(q(t)). \end{aligned}$$

Replacing in 2.52, one obtains:

$$\dot{V}(x(t)) \leq \text{Max}_i \left\{ \left(\frac{(H(q(t))\omega)_i}{\omega_i} \right) V(x(t)) \right\}. \quad (2.52)$$

Therefore, according to (2.48), $\dot{V}(x(t))$ is strictly negative definite, and then, the function $V(x(t))$ is a common Lyapunov function for all stationary configuration of the uncertain system (2.35) for every $q \in \Gamma$, and consequently, the system is asymptotically stable. $\square$

The re-formulation of all conditions enabling to solve the RLCRP, in both cases of continuous-time and discrete-time systems, as linear inequalities is now achieved. It is clear now that solving the RLCRP is a feasibility problem. Further, an objective function ε can be introduced as follows:

for the discrete-time case, $H_{ij} \omega < \varepsilon \omega$, $0 < \varepsilon \leq 1$,

for the continuous-time case, $H_{ij} \omega < -\varepsilon \omega$, $\varepsilon \geq 0$,

for $i = 1, \dots, \mu_A$, $j = 1, \dots, \mu_B$. The choice of such objective function plays a double role. In fact, it is directly connected to the assigned spectrum in closed-loop [6, 22], and to the rate of increase in the Lyapunov function (see demonstration of Proposition 2.2 above). Hence, optimizing ε implies that the rate of convergence of the state to the origin is augmented. Solution of the linear programs proposed hereafter allows to find matrices H_{ij} for $i = 1, \dots, \mu_A$, $j = 1, \dots, \mu_B$ together with matrix T . Finally, it is obvious that the gain feedback F solution to the RLCRP is easily deduced from this solution.

Comments 2.1 :

(1) It is worth noting here that solving the RLCRP without linear programming is a complicated problem and needs a number of trial and error with an important computation burden to obtain the adequate regulator [6]. Using linear programming gives nice solution and does not need the trial and error procedure as proposed in previous works. But, it is not, in the other hand, easy to satisfy all required conditions. The fact is looking for a robust static regulator respecting constraints is not a simple problem.

(2) We note also, here, that matrices H_{ij} are used instead of working with the $H_{ij}^+ = \sup(H_{ij}, 0)$ and $H_{ij}^- = \sup(-H_{ij}, 0)$ as several other works [7, 19, 23], which are interested in the same kind of linear programming formulation. The problem in our sense is that there is a fundamental dependence between matrices H_{ij}^+ and H_{ij}^- which is omitted. It comes out that taking into account this fundamental dependence, that is the product $H_{ij}^+ H_{ij}^-$ must be zero since they are representing the $+$ and $-$ part of the same matrix H_{ij} , makes the optimization problem nonlinear. Hence, the linear programming approach can no more be sufficient to solve the problem.

2.4 Examples

Discrete-time case: Consider the linear uncertain discrete-time system given by (2.34) with:

$$A(q_A(t)) = \begin{bmatrix} 0.785 + q_1 & 0.5 \\ -0.4 & 1.2 \end{bmatrix}, \quad B(q_B(t)) = \begin{bmatrix} 0.6 \\ -0.1 + q_2 \end{bmatrix}$$

The vertices of the domain of uncertainties are: $v_{11} = (0.015, 0)$, $v_{12} = (0.015, -0.02)$, $v_{21} = (-0.015, 0)$, $v_{22} = (-0.015, -0.02)$. Matrix G and vector ρ are given by:

$$G = \begin{bmatrix} -2.24 & 1.2 \\ -2.912 & 1.56 \end{bmatrix}, \quad \omega = \begin{bmatrix} 30 \\ 20 \end{bmatrix}$$

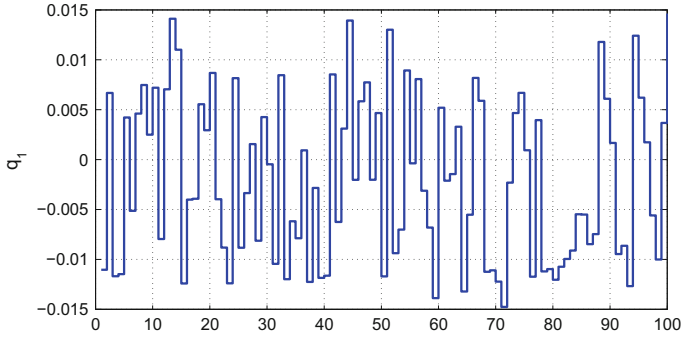


Fig. 2.1 q_1 evolution

and $\rho = 22$, $\text{rank}(G) = 1$.

We impose fictitious suitable constraints, in order to have $\text{rank}(G) = 2$, as follows:

$$G_a = \begin{bmatrix} -2.24 & 1.2 \\ -2.912 & 1.56 \\ 1.14 & -1.96 \end{bmatrix}, \quad \omega_a = \begin{bmatrix} 30 \\ 20 \\ 20 \end{bmatrix}$$

The solution obtained by solving the proposed linear program are $\varepsilon = 0.977$, and

$$H_{11} = \begin{bmatrix} 0.6411 & 0.0000 & 0.2135 \\ 0 & 0.6411 & 0.2775 \\ 0 & 0.0000 & 0.9187 \end{bmatrix}, \quad H_{12} = \begin{bmatrix} 0.6295 & 0 & 0.2061 \\ 0.0653 & 0.5792 & 0.2679 \\ 0 & 0.0147 & 0.9308 \end{bmatrix}$$

$$H_{21} = \begin{bmatrix} 0.5976 & 0.0000 & 0.1868 \\ 0.0832 & 0.5335 & 0.2428 \\ 0 & 0.0171 & 0.9322 \end{bmatrix}, \quad H_{22} = \begin{bmatrix} 0.5859 & 0.0000 & 0.1794 \\ 0.1484 & 0.4717 & 0.2332 \\ 0.0000 & 0.0317 & 0.9443 \end{bmatrix}$$

and

$$T = \begin{bmatrix} 0.3257 & 0.1234 & 0.3083 \end{bmatrix}$$

leading to the following robust controller:

$$F = \begin{bmatrix} -0.7372 & -0.0211 \end{bmatrix}.$$

Evolution of q_1 (resp. q_2) is given in Fig. 2.1 (resp. Fig. 2.2). Figure 2.3 shows the positive invariance and the admissibility of the domain $\mathcal{D}(G, \omega)$ as well as the asymptotic stability of the system motion, from the different initial state $x_o = [80 \ 150]^T$, $x_o = [100 \ 100]^T$, and $x_o = [60 \ 50]^T$. However, Fig. 2.4 shows the control admissibility for $t > 0$.

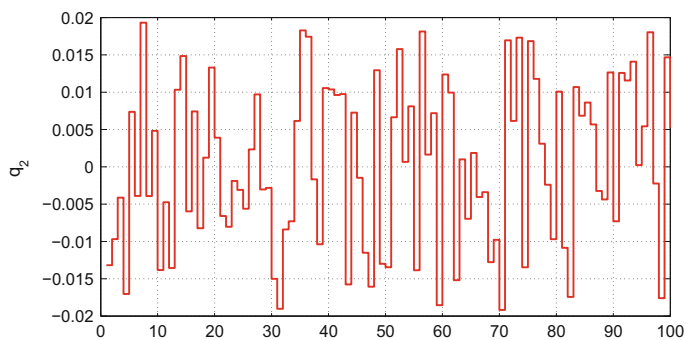


Fig. 2.2 q_2 evolution

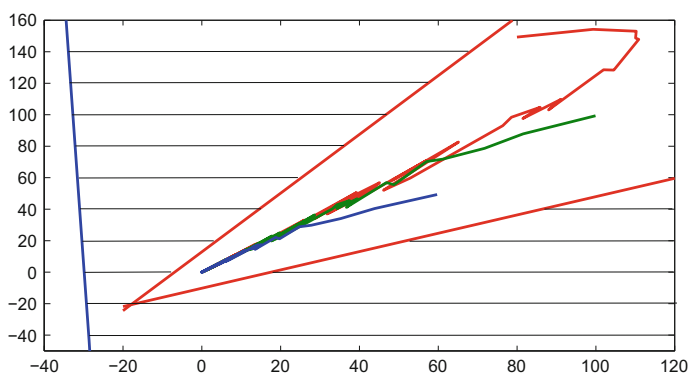


Fig. 2.3 States evolution

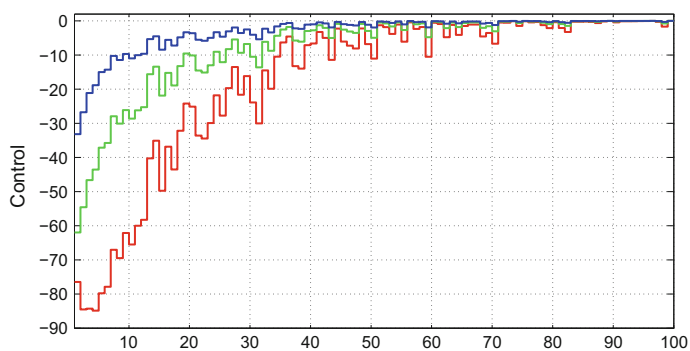


Fig. 2.4 Control evolution

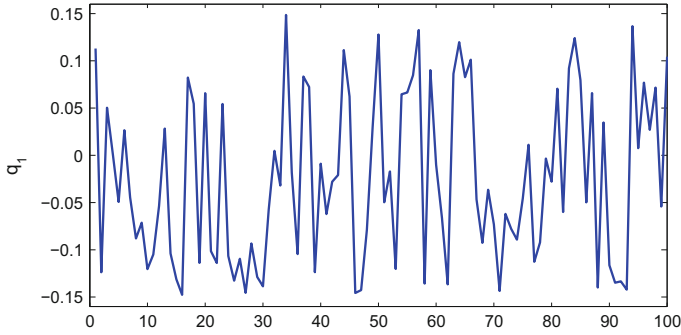


Fig. 2.5 q_1 evolution

Continuous-time case: Consider the linear uncertain continuous-time system given by (2.34) where,

$$A(q_A(t)) = \begin{bmatrix} -1.35 + q_1 & 0.5 \\ -3.5 + q_2 & 2.25 \end{bmatrix}, \quad B = \begin{bmatrix} 0.15 \\ 0.7 \end{bmatrix}$$

$$G = \begin{bmatrix} 2.5 & -1 \\ 1 & -2 \end{bmatrix}, \quad \omega = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

and $\rho = 80$.

The vertices of the domain of uncertainties are:

$$\begin{aligned} v_1 &= (-0.15, -0.1), \quad v_2 = (-0.15, 0.1), \\ v_3 &= (0.15, -0.1) \text{ and } v_4 = (0.15, 0.1). \end{aligned}$$

The solution obtained by the proposed linear program are: $\varepsilon = 0.301$ and

$$H_1 = \begin{bmatrix} -1.7685 & 0.0063 \\ 28.9849 & -0.1115 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -1.9325 & 0.0001 \\ 27.5739 & -0.1475 \end{bmatrix},$$

$$H_3 = \begin{bmatrix} -1.7486 & 0.0035 \\ 28.9837 & -0.1214 \end{bmatrix}, \quad H_4 = \begin{bmatrix} -1.9325 & 0.0001 \\ 27.5739 & -0.1475 \end{bmatrix},$$

and $T = \begin{bmatrix} 1.1000 & 1.4500 \end{bmatrix}$. These results lead to the robust controller

$$F = \begin{bmatrix} 4.2000 & -4.0000 \end{bmatrix}.$$

Evolution of q_1 (resp. q_2) is given by Fig. 2.5 (resp. Fig. 2.6). Fig. 2.7 shows the positive invariance and the admissibility of the domain $\mathcal{D}(G, \omega)$ as well as the asymptotic stability of the system motion, from different initial states $x_o = [10 \ 20]^T$, $x_o = [-20 \ 20]^T$, and $x_o = [-25 \ -20]^T$. However, the Fig. 2.8 shows the control admissibility for all $t > 0$.

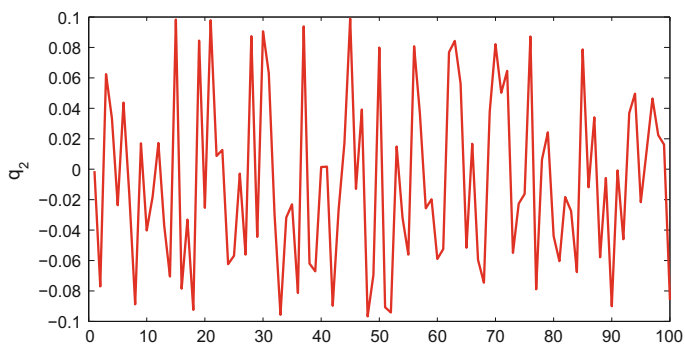


Fig. 2.6 q_2 evolution

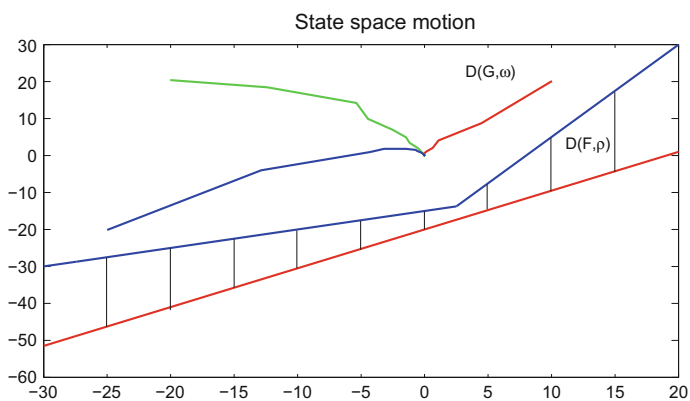


Fig. 2.7 States evolution

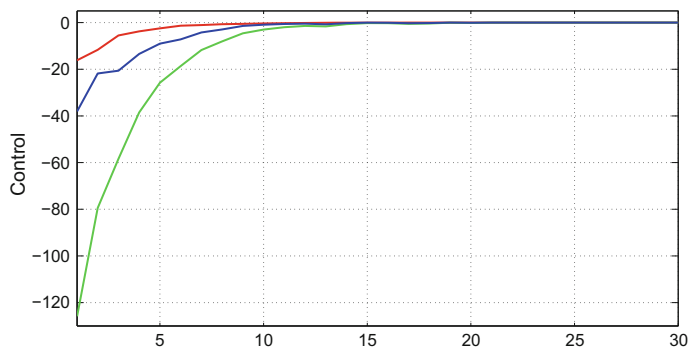


Fig. 2.8 Control evolution

2.5 Conclusion

In this chapter, we have generalized the solution to the LCRP for both continuous-time and discrete-time systems. The extension, when the system parameters are not perfectly known, is obtained in order to study robustness of such regulators. The solution is established using the positive invariance and the admissibility criteria. Hence, the asymptotic stability without constraints state and control violation is guaranteed. The obtained conditions are re-formulated using the Haar's lemma under matrix inequalities. These inequalities enable us to present the solution as linear programming algorithms. Illustrative examples are given in both of continuous-time case and discrete-time cases to show the ease of such approach to hold a complicated problem.

Further, to solve the problem, the weakest assumption for the rank of matrix G is considered, and the problem of dependence between matrices H^+ and H^- is overcome.

It has been shown that the obtained controllers can also be designed by selecting the desired closed-loop performance, followed by computing the solution of an algebraic equation for the nominal system, and finally checking the conditions of robustness at the vertices of the convex set of perturbations.

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