Sheffer Stroke Fuzzy Implications

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Abstract. A new family of fuzzy implications, motivated by classic Sheffer stroke operator, is introduced. Sheffer stroke, which is a negation of a conjunction and is called NAND as well, is one of the two operators that can be used by itself, without any other logical operators, to constitute a logical formal system. Classical implication can be presented just by Sheffer stroke operator in two ways which leads to two new families of fuzzy implication functions. It turns out that one of them is mainly a subclass of QL-operations, while the other one, called in our paper as SS_{qq}-implications, is independent of other well-known families of fuzzy implications. Basic properties of Sheffer stroke implications are also analysed.

Keywords: Fuzzy connectives · Fuzzy implication · Sheffer stroke · NAND

1 Introduction

The basic fuzzy connectives that perform the role of generalized "And", "Or" and "Not" are t-norms, t-conorms and fuzzy negations, respectively, whereas fuzzy IF-THEN rules are usually managed through multivalued implications called in the literature as fuzzy implications or fuzzy implication functions. Many researchers have devoted their efforts to the study of this wide family of functions (see [1–3,8]). There are three main ways of defining fuzzy implication functions: from basic fuzzy logic operations like t-norms, t-conorms, fuzzy negations, uninorms, copulas, etc.; from unary functions on the unit interval; from other fuzzy implication functions. In this article we focus on yet another two classes of fuzzy implication functions defined from binary operations on the unit interval.

Sheffer stroke [4,9], denoted by "|", is a negation of a conjunction:

$$p|q \equiv \neg(p \wedge q).$$

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Classical implication can be presented just by Sheffer stroke operator in two ways:

$$p \to q \equiv p|(p|q) \equiv \neg(p \wedge \neg(p \wedge q)), \quad \text{(PQ)}$$
$$p \to q \equiv p|(q|q) \equiv \neg(p \wedge \neg(q \wedge q)). \quad \text{(QQ)}$$

The above tautologies may lead to the following two formulas of multivalued implication:

$$I(x,y) = N(C(x,N(C(x,y)))),$$
$$I(x,y) = N(C(x,N(C(y,y)))),$$

for $x, y \in [0,1]$, where C is some generalization of the classical conjunction and N is some generalization of the classical negation.

In this paper we analyse properties of these two new families of functions, based on the above two formulas. Section 2 contains basic notions, definitions and facts used in the sequel. In Sect. 3 we present basic examples of Sheffer stroke implications and we show when these families are fuzzy implications. In Sect. 4 we analyse basic properties of Sheffer stroke implications, in particular we examine the left neutrality property, the exchange principle, the identity principle and the ordering property.

2 Preliminaries

Definition 1 (see [6,7]). *A function $T\colon [0,1]^2 \to [0,1]$ is called a triangular norm (t-norm for short) if it is symmetric, associative and increasing function, and it satisfies $T(x,1) = x$, for all $x \in [0,1]$.*

Definition 2 (see [6,7]). *A function $S\colon [0,1]^2 \to [0,1]$ is called a triangular conorm (t-conorm for short) if it is symmetric, associative and increasing function, and it satisfies $S(x,0) = x$, for all $x \in [0,1]$.*

Definition 3 (see [2,7]). *A non-increasing function $N\colon [0,1] \to [0,1]$ is called a fuzzy negation, if $N(0) = 1$, $N(1) = 0$. Moreover, a fuzzy negation N is called*

(i) strict if it is strictly decreasing and continuous,
(ii) strong if it is an involution, i.e., $N(N(x)) = x$, for all $x \in [0,1]$.

In the sequel we will use basic examples of t-norms and fuzzy negations with the notations from the monograph [2], i.e., $T_{\mathbf{M}}$ is the minimum t-norm, $T_{\mathbf{P}}$ is the algebraic product t-norm, $T_{\mathbf{LK}}$ is the Łukasiewicz t-norm, $T_{\mathbf{D}}$ is the drastic product t-norm, $T_{\mathbf{nM}}$ is the nilpotent minimum t-norm, $N_{\mathbf{C}}$ is the classical fuzzy negation, $N_{\mathbf{D1}}$ is the least fuzzy negation, while $N_{\mathbf{D2}}$ is the greatest fuzzy negation. Their formulas are collected in Table 1.

Definition 4 (see [2,6]). *A function $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication, if it satisfies the following conditions:*

Table 1. Basic t-norms and fuzzy negations.

$T_{\mathbf{M}}(x,y) = \min(x,y)$	$T_{\mathbf{P}}(x,y) = xy$
$T_{\mathbf{D}}(x,y) = \begin{cases} 0, & \text{if } x,y \in [0,1) \\ \min(x,y), & \text{otherwise} \end{cases}$	$T_{\mathbf{nM}}(x,y) = \begin{cases} 0, & \text{if } x+y \leq 1 \\ \min(x,y), & \text{otherwise} \end{cases}$
$T_{\mathbf{LK}}(x,y) = \max(x+y-1,0)$	$N_{\mathbf{C}}(x) = 1-x$
$N_{\mathbf{D1}}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \in (0,1] \end{cases}$	$N_{\mathbf{D2}}(x) = \begin{cases} 1, & \text{if } x \in [0,1) \\ 0, & \text{if } x = 1 \end{cases}$

(I1) *I is non-increasing with respect to the first variable,*
(I2) *I is non-decreasing with respect to the second variable,*
(I3) $I(0,0) = 1$,
(I4) $I(1,1) = 1$,
(I5) $I(1,0) = 0$.

The set of all fuzzy implications will be denoted by $\mathcal{FI}$.

Similarly as for t-norms and fuzzy negations, the notion for basic examples of fuzzy implications is the same as in the book [2]. We recall here the definition of a QL-operation, which is in use in the paper, while the definitions, examples and properties of other well-known families of implications, namely (S,N)-implications, R-implications, f-generated implications and g-generated implications, one may find in [2].

Definition 5. *A function $I\colon [0,1]^2 \to [0,1]$ is called a QL-operation if there exist a t-norm T, a t-conorm S and a fuzzy negation N such that*

$$I(x,y) = S(N(x), T(x,y)), \qquad x,y \in [0,1]. \tag{1}$$

The standard properties of fuzzy implications are listed below.

Definition 6 (see [2]). *We say that a fuzzy implication I satisfies*

(i) the identity principle, if

$$I(x,x) = 1, \qquad x \in [0,1], \tag{IP}$$

(ii) the left neutrality property, if

$$I(1,y) = y, \qquad y \in [0,1], \tag{NP}$$

(iii) the exchange principle, if

$$I(x,I(y,z)) = I(y,I(x,z)), \qquad x,y,z \in [0,1], \tag{EP}$$

(iv) the ordering property, if

$$x \leq y \iff I(x,y) = 1, \qquad x,y \in [0,1]. \tag{OP}$$

Lemma 1 (see [2, Sects. 2 and 3]). *If I is an (S,N)-implication, then it satisfies* (NP) *and* (EP). *If I is an R-implication, then it satisfies* (NP) *and* (IP). *If I is a QL-operation, then it satisfies* (NP). *If I is an f-generated implication or a g-generated implication, then it satisfies* (NP) *and* (EP).

3 Sheffer Stroke Implications

As we have noted in Introduction, there are two representations of implications that use only Sheffer stroke operator (see Eqs. (PQ) and (QQ)), which leads us to two families of implications. We introduce them in the next two subsections and call them SS_{pq}-implications and SS_{qq}-implications, respectively.

3.1 SS_{pq}-Implications

Now, we define new family of implications based on the equivalence (PQ).

Definition 7. *A function $I\colon [0,1]^2 \to [0,1]$ is called an SS_{pq} - implication if there exist a fuzzy negation N and a t-norm T such that*

$$I(x,y) = N(T(x, N(T(x,y)))), \qquad x,y \in [0,1]. \tag{2}$$

If I is an SS_{pq} - implication generated from the couple (N,T), then we will often denote it by $I^{pq}_{N,T}$.

Example 1. Probably the best known SS_{pq}-implication would be the one generated from $N_{\mathbf{C}}$ and $T_{\mathbf{LK}}$, which is a fuzzy implication $I_{\mathbf{KD}}$, i.e.,

$$\begin{aligned} I^{pq}_{N_{\mathbf{C}},T_{\mathbf{LK}}}(x,y) &= N_{\mathbf{C}}(T_{\mathbf{LK}}(x, N_{\mathbf{C}}(T_{\mathbf{LK}}(x,y)))) \\ &= 1 - \max(x + (1 - \max(x+y-1,0)) - 1, 0) \\ &= 1 - \max(\min(1-y,x),0) = 1 - \min(1-y,x) = \max(y, 1-x) \\ &= I_{\mathbf{KD}}(x,y), \end{aligned}$$

for all $x,y \in [0,1]$.

Let us now examine some basic properties of the function $I^{pq}_{N,T}$ to check whether it actually could serve as a fuzzy implication operation.

Lemma 2. *If $I^{pq}_{N,T}$ is an SS_{pq}-implication, then it satisfies (I2)–(I5).*

Proof. Property (I3) holds since

$$I^{pq}_{N,T}(0,0) = N(T(0, N(T(0,0)))) = N(T(0, N(0))) = N(T(0,1)) = N(0) = 1.$$

The next two properties are direct conclusions from Definition 7 as well. To prove (I2) it is enough to observe that N is decreasing and T is increasing with respect to the second variable. □

The immediate question arises if the requirement (I1), i.e., $I^{pq}_{N,T}$ is decreasing with respect to the first variable, is fulfilled as well. Unfortunately, this is not always true. Consider, e.g. the classical negation $N_{\mathbf{C}}$ and a product t-norm $T_{\mathbf{P}}$. Then we obtain

$$I^{pq}_{N_{\mathbf{C}},T_{\mathbf{P}}}(x,y) = 1 - x + x^2 y, \qquad x,y \in [0,1],$$

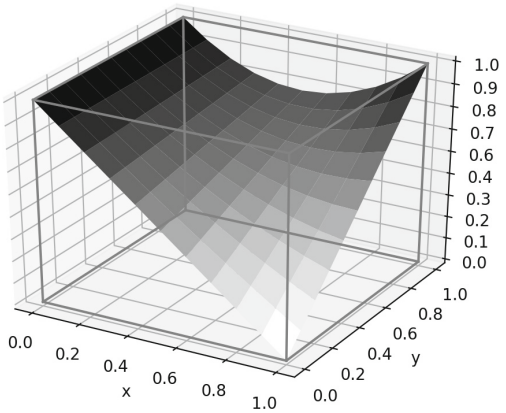


Fig. 1. The plot of $I^{pq}_{N_{\mathbf{C}},T_{\mathbf{P}}}(x,y) = 1 - x + x^2y$, which is a QL-operation, but not a fuzzy implication.

and this function is decreasing for $x \in [0, \frac{1}{2y}]$ and increasing for $x \in [\frac{1}{2y}, 1]$, for any specific $y \in (0.5, 1]$ (see Fig. 1). Therefore, however there are fuzzy implications among this new family of operations (e.g. $I^{pq}_{N_{\mathbf{C}},T_{\mathbf{LK}}} = I_{\mathbf{KD}} \in \mathcal{FI}$), an SS_{pq}-implication is not – in general – a fuzzy implication.

Theorem 1. *An SS_{pq}-implication $I^{pq}_{N,T}$ is a fuzzy implication if and only if it satisfies (I1).*

It is an open problem to characterize property (I1) in the terms of N and T. It turns out that the family of SS_{pq}-implications is basically a subclass of QL-operations, i.e., all SS_{pq}-implications generated from strong negations are QL-operations.

Theorem 2. *Let N be a strong fuzzy negation and T be a t-norm. The SS_{pq}-implication $I^{pq}_{N,T}$ is a QL-operation.*

Proof. Consider the t-conorm S which is N-dual to the t-norm T, i.e., let

$$S(x,y) := N(T(N(x), N(y))), \qquad x, y \in [0,1].$$

Then for all $x, y \in [0,1]$ we have

$$\begin{aligned} I^{pq}_{N,T}(x,y) &= N(T(x, N(T(x,y)))) = N(T(N(N(x)), N(T(x,y)))) \\ &= S(N(x), T(x,y)), \end{aligned}$$

which is the representation (1) of the QL-operation $I_{T,S,N}$. □

The following Table 2 introduces basic examples of SS_{pq}-implications and indicates which of them are fuzzy implications or QL-operations. The last four implications are not QL-operations, because they do not satisfy (NP), what will be proven in the next section, while due to Lemma 1 all QL-operations do have this property. Please note that the first implication is indicated as I_{-1} in [2], while the last implication is indicated as I_4 in [5].

Table 2. Basic SS_{pq}-implications.

Negation	T-norm	Sheffer stroke SS_{pq}-implication	$I \in \mathcal{FI}$	$I \in$QL-oper.
$N_{\mathbf{C}}$	$T_{\mathbf{M}}$	$I(x,y) = \max(1-x, \min(x,y))$	✗	✓
$N_{\mathbf{C}}$	$T_{\mathbf{LK}}$	$I_{\mathbf{KD}}$	✓	✓
$N_{\mathbf{C}}$	$T_{\mathbf{P}}$	$I(x,y) = 1 - x + x^2 y$	✗	✓
$N_{\mathbf{C}}$	$T_{\mathbf{D}}$	$I(x,y) = \begin{cases} 1, & \text{if } y = 1 \\ y, & \text{if } x = 1 \\ 1-x, & \text{otherwise} \end{cases}$	✗	✓
$N_{\mathbf{C}}$	$T_{\mathbf{nM}}$	$I(x,y) = \begin{cases} 1, & \text{if } x \le y \text{ and } y > 1-x \\ y, & \text{if } x > y \text{ and } y > 1-x \\ 1-x, & \text{otherwise} \end{cases}$	✗	✓
$N_{\mathbf{D1}}$	$T_{\mathbf{M}}, T_{\mathbf{P}}$	$I(x,y) = \begin{cases} 0, & \text{if } y = 0 \text{ and } x > 0 \\ 1, & \text{otherwise} \end{cases}$	✓	✗
$N_{\mathbf{D1}}$	$T_{\mathbf{LK}}, T_{\mathbf{nM}}$	$I(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } x + y > 1 \\ 0, & \text{otherwise} \end{cases}$	✗	✗
$N_{\mathbf{D1}}$	$T_{\mathbf{D}}$	$I(x,y) = \begin{cases} 0, & \text{if } x > 0 \text{ and} \\ & (y = 0 \text{ or } \max(x,y) < 1) \\ 1, & \text{otherwise} \end{cases}$	✗	✗
$N_{\mathbf{D2}}$	any T	$I(x,y) = \begin{cases} 0, & \text{if } x = 1 \text{ and } y < 1 \\ 1, & \text{otherwise} \end{cases}$	✓	✗

3.2 SS_{qq}-Implications

Now, we define the second family of Sheffer stroke implications, based on the equivalence (QQ).

Definition 8. *A function $I\colon [0,1]^2 \to [0,1]$ is called an SS_{qq} - implication if there exist a fuzzy negation N and a t-norm T such that*

$$I(x,y) = N(T(x, N(T(y,y)))), \qquad x, y \in [0,1]. \tag{3}$$

If I is an SS_{qq} - implication generated from the couple (N,T), then we will often denote it by $I^{qq}_{N,T}$.

Let us check if $I^{qq}_{N,T}$ is a fuzzy implication or if some additional requirements are necessary, as in the case of SS_{pq}-implications.

Theorem 3. *If $I^{qq}_{N,T}$ is an SS_{qq} - implication, then it is a fuzzy implication, thus $I^{qq}_{N,T} \in \mathcal{FI}$.*

Proof. The conditions $I^{qq}_{N,T}(0,0) = 1$, $I^{qq}_{N,T}(1,1) = 1$, $I^{qq}_{N,T}(1,0) = 0$ are direct conclusions from Definition 8, the fact that $T(0,x) = 0, T(1,x) = x$ for all $x \in [0,1]$, and the property (N1) satisfied by N. The function $I^{qq}_{N,T}(x,\cdot)$, for any fixed $x \in [0,1]$, is increasing since T is increasing and N is decreasing. For the same reason, the function $I^{qq}_{N,T}(\cdot,y)$, for any fixed $y \in [0,1]$, is decreasing. □

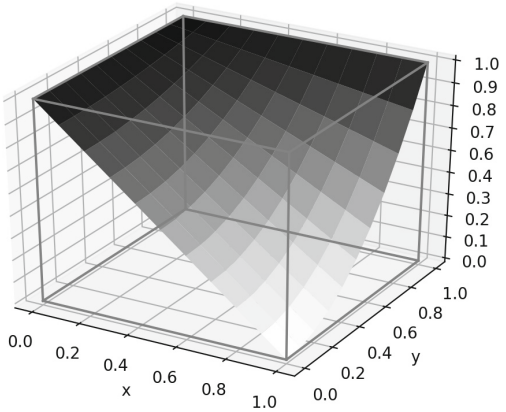


Fig. 2. The plot of $I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}}}(x,y) = 1 - x + xy^2$ fuzzy implication.

Example 2. Let us consider the SS_{qq}-implication $I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}}}$ generated from the classical negation $N_{\mathbf{C}}$ and the product t-norm $T_{\mathbf{P}}$:

$$I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}}}(x,y) = 1 - x + xy^2, \qquad x,y \in [0,1]. \tag{4}$$

The function $I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}}}$ is a continuous fuzzy implication (see Fig. 2), which does not belong to any well-known family of fuzzy implications, i.e., it is none of an R-implication, an (S,N)-implication, a QL-operation, an f- or g-generated implication. This is because $I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}}}$ does not satisfy (NP), i.e., $I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}}}(1,y) = y^2 \neq y$ for $y \in (0,1)$, but due to Lemma 1, implications from all the above mentioned families of fuzzy implications do satisfy that property.

Remark 1. Actually we may consider more general family of functions

$$I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},p}(x,y) = 1 - x + xy^p, \qquad x,y \in [0,1],$$

for any $p > 0$. When $p = 1$ we have $I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},1} = I_{\mathbf{RC}}$ the Reichenbach implication (see [2]). It turns out that all functions $I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},p}$ are continuous fuzzy implications, but none of them satisfies (IP) or (OP), and $I_{\mathbf{RC}}$ is the only one that satisfies (NP) or (EP).

Indeed, firstly we have $I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},p}(1,y) = y^p$, which is equal to y for all $y \in [0,1]$ only if $p = 1$, thus (NP) is satisfied only for $I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},1} = I_{\mathbf{RC}}$. Let us consider $x = 1, y = 0.5, z = 0$ to examine (EP),

$$I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},p}(1, I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},p}(0.5,0)) = I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},p}(1,0.5) = 0.5^p,$$
$$I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},p}(0.5, I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},p}(1,0)) = I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},p}(0.5,0) = 0.5,$$

and $0.5^p = 0.5$ if and only if $p = 1$. Furthermore

$$\begin{aligned} I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},1}(x, I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},1}(y,z)) &= I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},1}(x, 1-y+yz) \\ &= 1 - xy(1-z) \\ &= I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},1}(y, I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},1}(x,z)), \end{aligned}$$

thus $I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},1} = I_{\mathbf{RC}}$ is the only one satisfying (EP) (cf. [2]).

Finally, assume that (IP) is satisfied, so $I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},p}(x,x) = 1 - x + x^{p+1} = 1$, for all $x \in [0,1]$ and some $p > 0$. Then $x^{p+1} - x = 0$, thus $x(x^p - 1) = 0$, so either $x = 0$ or $x^p = 1$, for all $x \in (0,1]$. Thus $p = 0$, but we have an assumption that $p > 0$, so $I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}},p}$ does not satisfy (IP) for any $p > 0$ and consequently also (OP) is not satisfied.

Example 2 with the function $I^{qq}_{N_{\mathbf{C}},T_{\mathbf{P}}}$ indicates that this new family of SS$_{qq}$-implications is independent from the other well-known families of fuzzy implications.

Table 3 presents basic examples of SS_{qq}-implications which in most cases are not QL-operations.

Table 3. Basic SS$_{qq}$-implications.

Negation	T-norm	Sheffer stroke SS$_{qq}$-implication	$I \in \mathcal{FI}$	$I \in$QL-oper
$N_{\mathbf{C}}$	$T_{\mathbf{M}}$	$I_{\mathbf{KD}}$	✓	✓
$N_{\mathbf{C}}$	$T_{\mathbf{LK}}$	$I(x,y) = \begin{cases} 1-x, & \text{if } y \le 0.5 \\ 1, & \text{if } x \le 2y-1 \\ 2y-x, & \text{otherwise} \end{cases}$	✓	✗
$N_{\mathbf{C}}$	$T_{\mathbf{P}}$	$I(x,y) = 1 - x + xy^2$	✓	✗
$N_{\mathbf{C}}$	$T_{\mathbf{D}}$	$I(x,y) = \begin{cases} 1, & \text{if } y = 1 \\ 1-x, & \text{otherwise} \end{cases}$	✓	✗
$N_{\mathbf{C}}$	$T_{\mathbf{nM}}$	$I(x,y) = \begin{cases} 1-x, & \text{if } y \le 0.5 \\ y, & \text{if } x > y > 0.5 \\ 1, & \text{otherwise} \end{cases}$	✓	✗
$N_{\mathbf{D1}}$	$T_{\mathbf{M}}$, $T_{\mathbf{P}}$	$I(x,y) = \begin{cases} 0, & \text{if } y = 0 \text{ and } x > 0 \\ 1, & \text{otherwise} \end{cases}$	✓	✗
$N_{\mathbf{D1}}$	$T_{\mathbf{LK}}$, $T_{\mathbf{nM}}$	$I(x,y) = \begin{cases} 0, & \text{if } x > 0 \text{ and } y \le 0.5 \\ 1, & \text{otherwise} \end{cases}$	✓	✗
$N_{\mathbf{D1}}$	$T_{\mathbf{D}}$	$I_0(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1 \\ 0, & \text{if } x > 0 \text{ and } y < 1 \end{cases}$	✓	✗
$N_{\mathbf{D2}}$	any T	$I(x,y) = \begin{cases} 0, & \text{if } x = 1 \text{ and } y < 1 \\ 1, & \text{otherwise} \end{cases}$	✓	✗

Remark 2. Please note that $I_{\mathbf{KD}}$ is the only fuzzy implication which is both SS_{pq}-implication and SS_{qq}-implication, with the assumption that a negation used in both definitions is classical. Indeed, assume that N is a strong negation and put $x = 1$ in (2). Then we obtain that $I^{pq}_{N,T_1}(1, y) = N(N(y)) = y$, for all $y \in [0, 1]$. Now, putting $x = 1$ in (3) we have $I^{qq}_{N,T_2}(1, y) = N(N(T_2(y, y))) = T_2(y, y)$, for all $y \in [0, 1]$. Therefore t-norm T_2 has to be idempotent, which implies $T_2 = T_{\mathbf{M}}$, and in the case when $N = N_{\mathbf{C}}$ we obtain Kleene-Dienes implication $I_{\mathbf{KD}}$.

4 Basic Properties of Sheffer Stroke Implications

This section is devoted to study basic properties of both types of Sheffer stroke implications, namely (NP), (EP), (IP) and (OP). We introduce just partial results which we plan to complete in our future works.

Proposition 1. *An SS_{pq}-implication $I^{pq}_{N,T}$ satisfies* (NP) *if and only if N is strong.*

Proof. For any $y \in [0, 1]$ we have

$$I^{pq}_{N,T}(1, y) = N(T(1, N(T(1, y)))) = N(N(y)),$$

which is equal to y for all $y \in [0, 1]$ if and only if N is a strong negation. □

Proposition 2. *Let N be a strong negation. An SS_{qq}-implication $I^{qq}_{N,T}$ satisfies* (NP) *if and only if $T = T_M$.*

Proof. For any $y \in [0, 1]$ we have

$$I^{qq}_{N,T}(1, y) = N(T(1, N(T(y, y)))) = N(N(T(y, y))) = T(y, y),$$

which is equal to y for all $y \in [0, 1]$ if and only if T is an idempotent t-norm, thus $T = T_{\mathbf{M}}$. □

Proposition 3. *Let $T = T_{\mathbf{M}}$. An SS_{qq}-implication $I^{qq}_{N,T}$ satisfies* (NP) *if and only if N is strong.*

Proof. For any $y \in [0, 1]$ we have

$$I^{qq}_{N,T}(1, y) = N(T(1, N(T(y, y)))) = N(N(T(y, y))) = N(N(y)),$$

which is equal to y for all $y \in [0, 1]$ if and only if N is a strong negation. □

We suppose that implication $I^{qq}_{N,T}$ satisfies (NP) only if N is strong and T is the minimum t-norm. However, the following result states what we can prove for now.

Proposition 4. *If an SS_{qq}-implication $I^{qq}_{N,T}$ satisfies* (NP)*, then N is continuous, while T is not continuous and Archimedean simultaneously, but $T(x, x)$ is an injective function.*

Proof. Let us assume that $I^{qq}_{N,T}$ satisfies (NP), i.e.,

$$I^{qq}_{N,T}(1,y) = N(T(1,N(T(y,y)))) = N(N(T(y,y))) = y, \qquad y \in [0,1].$$

This directly implies that N is surjective, which, with its monotonicity, gives continuity of N. Continuous fuzzy negation N has a unique fixed point $e \in (0,1)$, i.e., $N(e) = e$. Let $f(x) := T(x,x)$, $x \in [0,1]$. If T is continuous, then f is continuous and from Darboux property there exists $x_e \in (0,1)$ such that $f(x_e) = e$. If T is continuous and Archimedean, then $f(x_e) = T(x_e,x_e) < x_e$ (see [2, Remark 2.1.4]), thus $e < x_e$. Then we obtain a contradiction by

$$x_e = I^{qq}_{N,T}(1,x_e) = N(N(T(x_e,x_e))) = N(N(e)) = N(e) = e \neq x_e.$$

Thus T cannot be continuous and Archimedean simultaneously.

Finally, let us assume that $T(x,x)$ is not an injective function, i.e., there exist $x_1, x_2 \in [0,1], x_1 \neq x_2$, such that $T(x_1,x_1) = T(x_2,x_2)$. Then

$$x_1 = I^{qq}_{N,T}(1,x_1) = N(N(T(x_1,x_1))) = N(N(T(x_2,x_2))) = I^{qq}_{N,T}(1,x_2) = x_2,$$

which is a contradiction. □

Finally, we present some partial results connected to (IP) and (OP).

Proposition 5. *Let $I_{N,T}$ be an SS_{pq}-implication or an SS_{qq}-implication, generated from a strong fuzzy negation N and a strict t-norm T. Then $I_{N,T}$ does not satisfy* (OP) *nor* (IP).

Proof. Firstly, notice that if T is a strict t-norm, then $T(x,y) = 0$ implies $x = 0$ or $y = 0$. Indeed, because of the representation of a strict t-norm by an additive generator (see [7]) we have

$$\begin{aligned} T(x,y) = & f^{-1}(f(x) + f(y)) = 0 \;\Leftrightarrow\; f(x) + f(y) = \infty \\ & \Leftrightarrow\; f(x) = \infty \text{ or } f(y) = \infty \Leftrightarrow x = 0 \text{ or } y = 0. \end{aligned}$$

If $I_{N,T}$ satisfies (IP), then $I_{N,T}(x,x) = N(T(x,N(T(x,x)))) = 1$, for all $x \in [0,1]$. Applying the negation N to both sides of the above equation we obtain $T(x,N(T(x,x))) = 0$, for all $x \in [0,1]$, which due to the fact just showed above implies that $x = 0$ or $N(T(x,x)) = 0$ for all $x \in (0,1]$. Since T is continuous, the set of values of $T(x,x)$ for $x \in (0,1]$ consists the set $(0,1]$. Thus $N = N_{\mathbf{D1}}$, which is not a strong negation, a contradiction. Of course, if $I_{N,T}$ does not satisfy (IP), then it does not satisfy (OP), too. □

Example 3. An SS_{pq}-implication $I^{pq}_{N_{\mathbf{C}},T_{\mathbf{LK}}} = I_{\mathbf{KD}}$ satisfies (EP), while other SS_{pq}-implications generated from the classical negation $N_{\mathbf{C}}$ and the following t-norms $T_{\mathbf{M}}$, $T_{\mathbf{P}}$, $T_{\mathbf{D}}$, $T_{\mathbf{nM}}$ do not satisfy (EP). In the case of SS_{qq}-implications it seems that all ot them generated from the least or the greatest fuzzy negations $N_{\mathbf{D1}}, N_{\mathbf{D2}}$ do satisfy (EP), while most of implications generated from the classical fuzzy negation $N_{\mathbf{C}}$ do not (see Table 4). However, the full characterization of solutions to (EP) among Sheffer stroke implications remains an open problem.

Table 4. Sheffer stroke implications and their properties.

Negation	T-norm	(NP)	(EP)	(IP)	(OP)
SS_{pq}-implications					
$N_{\mathbf{C}}$	$T_{\mathbf{M}}, T_{\mathbf{P}}, T_{\mathbf{D}}, T_{\mathbf{nM}}$	✓	✗	✗	✗
$N_{\mathbf{C}}$	$T_{\mathbf{LK}}$	✓	✓	✗	✗
$N_{\mathbf{D1}}$	$T_{\mathbf{M}}, T_{\mathbf{P}}$	✗	✓	✓	✗
$N_{\mathbf{D1}}$	$T_{\mathbf{LK}}, T_{\mathbf{nM}}, T_{\mathbf{D}}$	✗	✗	✗	✗
$N_{\mathbf{D2}}$	any T	✗	✓	✓	✗
SS_{qq}-implications					
$N_{\mathbf{C}}$	$T_{\mathbf{M}}$	✓	✓	✗	✗
$N_{\mathbf{C}}$	$T_{\mathbf{LK}}, T_{\mathbf{P}}, T_{\mathbf{D}}, T_{\mathbf{nM}}$	✗	✗	✗	✗
$N_{\mathbf{D1}}$	$T_{\mathbf{M}}, T_{\mathbf{P}}$	✗	✓	✓	✗
$N_{\mathbf{D1}}$	$T_{\mathbf{LK}}, T_{\mathbf{nM}}, T_{\mathbf{D}}$	✗	✓	✗	✗
$N_{\mathbf{D2}}$	any T	✗	✓	✓	✗

5 Conclusions

According to their name Sheffer stroke implications have a basic genesis related to Sheffer stroke which is one of the two operators that can be used by itself, without any other logical operators, to constitute a logical formal system. Such a genesis is in and of itself a good enough reason to study this new family of implications. In the paper we introduced two subfamilies of Sheffer stroke implications, namely SS_{pq}-implications and SS_{qq}-implications, and we showed that while the first one is mainly a subfamily of QL-operations, the second one is in fact a new family of fuzzy implications. We introduced some basic properties of both families. However it seems that Sheffer stroke implications usually do not satisfy (NP), (EP), (IP) or (OP), we have got so far only partial results. The questions about other properties, among them t-conditionality, distributivity laws and laws of contraposition, are in our future plans. We believe that answering them in the future would show the potential in applications of Sheffer stroke implications.

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