The Classification of All the Subvarieties of $\mathbb{DNMG}$

Stefano Aguzzoli[1], Matteo Bianchi[2(✉)], and Diego Valota[1]

[1] Department of Computer Science, Università degli Studi di Milano, via Comelico 39/41, 20135 Milano, Italy
{aguzzoli,valota}@di.unimi.it

[2] Dipartimento di Scienze Teoriche e Applicate, Università degli Studi dell'Insubria, via Mazzini 5, 21100 Varese, Italy
matteo.bianchi@uninsubria.it

Abstract. MTL is the logic of all left-continuous t-norms and their residua. The equivalent algebraic semantics of MTL is constituted by the variety of MTL-algebras, $\mathbb{MTL}$. The variety $\mathbb{WNM}$ of *weak nilpotent minimum* algebras is a major subvariety of $\mathbb{MTL}$, containing several subvarieties of $\mathbb{MTL}$ which have been subjects of study in the literature, such as Gödel algebras, Nilpotent Minimum algebras, Drastic Product and Revised Drastic Product algebras, NMG-algebras, as well as Boolean algebras. In this paper we introduce and axiomatise $\mathbb{DNMG}$, a proper subvariety of $\mathbb{WNM}$ which contains all the aforementioned varieties. We show that $\mathbb{DNMG}$ is singly generated by a standard algebra. Further, we determine the structure of the lattice of subvarieties of $\mathbb{DNMG}$, and we provide the axiomatisation of every subvariety.

Keywords: WNM-algebras · DNMG-algebras · NM-algebras · Gödel-algebras · DP-algebras · Axiomatisations of subvarieties · Single chain completeness

1 Introduction

Nilpotent minimum t-norm $*_{\mathrm{NM}}$ [14] was one of the first examples of a left-continuous but not continuous t-norm. The logic related to $*_{\mathrm{NM}}$, NM, was introduced by Esteva and Godo in [12]. In the same paper they presented a generalisation of NM, the logic of Weak Nilpotent Minimum, WNM, and the related algebraic semantics, the variety of WNM-algebras $\mathbb{WNM}$. WNM is an extension of MTL, the logic of all left continuous t-norms and their residua [12,20]. Several extensions of WNM have been extensively studied in the literature. In particular, Gödel logic G, Drastic Product DP ([4], firstly introduced as $\mathrm{S_3MTL}$ in [23]), Revised Drastic product RDP ([9,25], based on the t-norm introduced in [19]), NMG [26], NM [12], and classical Boolean logic B. During the years a number of topics concerning WNM and its algebraic semantics has been investigated: the papers [5,13,16,21,24] are only few examples. WNM has been extensively studied in [23], where the problem of axiomatising its extensions has been partially

J. Kacprzyk et al. (eds.), *Advances in Fuzzy Logic and Technology 2017*,
Advances in Intelligent Systems and Computing 641, DOI 10.1007/978-3-319-66830-7_2

solved. The task of characterising and axiomatising the lattice of extensions of a given extension of WNM has been accomplished in some cases. Gispert [17], solved the case for NM. The lattice of extensions of G is well known. The extensions of EMTL, which is the largest common fragment of G and DP, have been axiomatised in [6].

In this article we introduce the variety $\mathbb{DNMG}$, the algebraic semantics of DNMG, which is an extension of WNM which is a particularly tame single-chain complete common fragment of G, NM and DP (and hence, of all the aforementioned extensions of WNM). We prove standard completeness for DNMG. Generalising Gispert's result, we describe the structure of the lattice of subvarieties of $\mathbb{DNMG}$, showing that each one of them is generated by finitely many chains. We further provide a uniform way to axiomatise each one of these subvarieties.

2 Preliminaries

We assume that the reader is acquainted with many-valued logics in Hájek's sense, and with their algebraic semantics. We refer to [11,18] for any unexplained notion. We recall that MTL is the logic, on the language $\{\&, \wedge, \vee, \to, \neg, \bot, \top\}$, of all left-continuous t-norms and their residua, and that its associated algebraic semantics in the sense of Blok and Pigozzi [7] is the variety $\mathbb{MTL}$ of *MTL-algebras*, that is, prelinear, commutative, bounded, integral, residuated lattices [11]. In an MTL-algebra $\mathcal{A} = (A, *, \Rightarrow, \sqcap, \sqcup, \sim, 0, 1)$ the connectives $\&, \to, \wedge, \vee, \neg, \bot, \top$ are interpreted, respectively, by $*, \Rightarrow, \sqcap, \sqcup, \sim, 0, 1$. Totally ordered MTL-algebras are called MTL-chains. In every chain $\sqcap = \min$ and $\sqcup = \max$. An MTL-algebra is called standard whenever its lattice reduct is $([0,1], \leq)$, with the usual order.

Given an MTL-algebra $\mathcal{A}$, with $\mathbf{V}(\mathcal{A})$ we mean the variety generated by $\mathcal{A}$, which is said to be *generic* for $\mathbf{V}(\mathcal{A})$. A logic L is the extension of MTL via a set of axioms $\{\varphi_i\}_{i\in I}$ if and only if $\mathbb{L}$ is the subvariety of MTL-algebras satisfying $\{\bar{\varphi}_i = 1\}_{i\in I}$, where $\bar{\varphi}_i$ is obtained from φ_i by replacing the connectives with the corresponding operations, and every propositional variable in φ with an individual variable. With $\mathcal{A} \models \bar{\varphi} = 1$ we mean that $\mathcal{A}$ satisfies $\bar{\varphi} = 1$.

The logic WNM [12] is axiomatised as MTL plus:

$$\neg(\varphi\&\psi) \vee ((\varphi \wedge \psi) \to (\varphi\&\psi)). \tag{wnm}$$

The logics G, DP, EMTL, RDP, NM, NMG [1,4,6,12,25,26] are axiomatised as WNM plus, respectively:

$$\varphi \to (\varphi\&\varphi). \tag{id}$$
$$\varphi \vee \neg(\varphi\&\varphi). \tag{dp}$$
$$(\varphi \to (\varphi\&\varphi)) \vee (\psi \vee \neg(\psi\&\psi)). \tag{emtl}$$
$$(\varphi \to \neg\varphi) \vee \neg\neg\varphi. \tag{rdp}$$
$$\neg\neg\varphi \to \varphi. \tag{inv}$$
$$\neg\neg\varphi \vee (\neg\neg\varphi \to \varphi). \tag{nmg}$$

NM^- [17] is axiomatised as NM plus $\neg((\neg(\varphi^2))^2) \leftrightarrow (\neg((\neg\varphi)^2))^2$, where φ^2 stands for $\varphi\&\varphi$. B is classical logic, axiomatised as MTL plus $\varphi \vee \neg\varphi$.

As shown in [17], the operations $*, \Rightarrow$ of a WNM-chain $\mathcal{A}$, are:

$$x * y = \begin{cases} 0 & \text{if } x \leq {\sim}y \\ \min(x, y) & \text{Otherwise.} \end{cases} \qquad x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ \max({\sim}x, y) & \text{Otherwise.} \end{cases} \tag{1}$$

Where $\sim$, the negation of $\mathcal{A}$, is a *(generalised) weak negation*, that is a map $\sim\colon A \to A$ such that ${\sim}1 = 0$, ${\sim}{\sim}a \geq a$, and if $a \leq b$, then ${\sim}a \geq {\sim}b$. Each weak negation is the negation of a uniquely determined WNM-chain. $\mathbb{WNM}$ is locally finite, i.e. for every WNM-algebra each one of its finitely generated subalgebras is finite. For each integer $n \geq 2$, with $\mathbb{G}_n$, $\mathbb{DP}_n, \mathbb{NM}^-_{2n}$ we will denote the variety generated, respectively, by the Gödel chain with n elements, the DP-chain with n elements, and the NM^--chain with $2n$ elements.

3 DNMG-algebras

$\mathbb{DNMG}$ is the variety of WNM-algebras satisfying the following identity.

$${\sim}{\sim}x \sqcup ({\sim}{\sim}x \Rightarrow x) \sqcup ({\sim}{\sim}x \Leftrightarrow {\sim}x) = 1. \tag{DNMG}$$

Since each variety of MTL-algebras is generated by its chains, we immediately have that two subvarieties of MTL-algebras coincide iff they have the same class of chains. We are then going to analyse the structure of DNMG-chains.

Definition 1. *Let $\mathcal{A}$ be a WNM-chain. Let us define the following sets.*

- $A^+ = \{a \in A : a > {\sim}a\}$, *and* $A^- = \{a \in A : a < {\sim}a\}$.
- $S(A) = \{a \in A \mid {\sim}{\sim}a = 1, a \neq 1\}$.
- $F(A) = \{a \in A \mid {\sim}a = {\sim}{\sim}a\}$.
- $I^-(A) = \{a \in A \mid {\sim}{\sim}a = a, 0 < a < {\sim}a\}$.
- $I^+(A) = \{a \in A \mid {\sim}{\sim}a = a, 1 > a > {\sim}a\}$.
- $I(A) = I^-(A) \cup I^+(A)$.

Clearly, $I^-(A) \cap I^+(A) = \emptyset$. Further, $\sim$ is a bijection of $I(A)^+$ onto $I(A)^-$. Notice that $S(A)$ is disjunct from $I(A)$ and $F(A)$. Given $B, C \subseteq A$, we write $B \prec C$ whenever $b <_A c$, for every $b \in B$ and $c \in C$. The following is immediate.

Proposition 1. *For every WNM-chain $\mathcal{A}$ it holds that $I^-(A) \prec F(A) \prec I^+(A) \prec S(A)$. In particular, $I(A)^- \cup F(A) \cup \{0\} = A \setminus A^+$, and $I(A)^+ \cup S(A) \cup \{1\} = A^+$.*

By Proposition 1, the sets $S(A), F(A), I^-(A), I^+(A)$ are pairwise disjoint. For any subset $S \subseteq A$, let $\langle S\rangle$ be the subalgebra of $\mathcal{A}$ generated by S.

Proposition 2. *Let $\mathcal{A}$ be a WNM-chain and pick $S \subseteq A$.*

1. *If $S \subseteq I(A)$, then $\langle S\rangle = S \cup \{\sim a \mid a \in S\} \cup \{0,1\}$ is an NM$^-$ algebra.*
2. *If $S \subseteq F(A)$, then $\langle S\rangle = S \cup \{\sim a \mid a \in S\} \cup \{0,1\}$ is a DP-algebra.*
3. *If $S \subseteq S(A)$ then $\langle S\rangle = S \cup \{0,1\}$ is a Gödel algebra.*

Proof. Using (1) an easy check shows that the sets $(S \cap I(A)) \cup \{\sim a \mid a \in S \cap (I(A))\} \cup \{0,1\}$, $(S \cap F(A)) \cup \{\sim a \mid a \in S \cap (F(A))\} \cup \{0,1\}$ and $(S \cap S(A)) \cup \{0,1\}$ are subuniverses of A. The rest follows by Definition 1. □

Theorem 1. *A WNM-chain $\mathcal{A}$ is a DNMG-chain iff $A = S(A) \cup F(A) \cup I(A) \cup \{0,1\}$.*

Proof. Let $\mathcal{A}$ be a WNM-chain such that $A = S(A) \cup F(A) \cup I(A) \cup \{0,1\}$. Then, each element in $a \in S(A)$ satisfies $\sim\sim a = 1$, each element $a \in F(A)$ satisfies $\sim\sim a = \sim a$, and each element $a \in I(A)$ satifies $\sim\sim a = a$. The elements 0 and 1 both satisfy $\sim\sim a = a$. Whence $\mathcal{A}$ satisfies the identity (DNMG). Conversely, let $\mathcal{A}$ be a WNM-chain such that there is $a \in A \setminus (S(A) \cup F(A) \cup I(A) \cup \{0,1\})$. Then $\sim\sim a \neq 1$, $\sim\sim a \neq \sim a$ and $\sim\sim a \neq a$. Whence $\mathcal{A}$ is not a DNMG-chain. □

Lemma 1. *Let $\mathbb{V} \in \{\mathbb{DP}, \mathbb{NM}^-, \mathbb{G}\}$. For each subvariety $\mathbb{W}$ of $\mathbb{V}$, a chain $\mathcal{A}$ is generic for $\mathbb{W}$ iff each finite chain in $\mathbb{W}$ embeds into $\mathcal{A}$. Moreover, $\mathbf{V}(\mathcal{A}) = \mathbb{V}$ iff $\mathcal{A}$ is infinite.*

Proof. Let $\mathbb{V} \in \{\mathbb{DP}, \mathbb{NM}^-, \mathbb{G}\}$, and pick two chains $\mathcal{A}, \mathcal{B} \in \mathbb{V}$. By the results of [6,17] we have two consequences. First, $\mathbf{V}(\mathcal{A}) = \mathbb{V}$ iff $\mathcal{A}$ is infinite. Next, if $|\mathcal{A}| < |\mathcal{B}|$ then $\mathcal{A}$ embeds into $\mathcal{B}$. The claim follows immediately. □

Definition 2. *Let $C \subseteq \{F, I, S\}$, and let $\mathcal{A}$ be a WNM-chain. We say that $\mathcal{A}$ is C-*semigeneric *iff for any $X \in C$ and for any finite chain $\mathcal{B} \in \mathbb{WNM}$, it holds that $\langle X(B)\rangle$ embeds into $\langle X(A)\rangle$.*

In the following, for any $X \in \{F, I, S\}$, by $L(X)$-chain we mean: DP-chain if $X = F$, NM$^-$-chain if $X = I$, and G-chain if $X = S$.

Lemma 2. *For each $X \in \{F, I, S\}$, a WNM-chain $\mathcal{A}$ is X-semigeneric iff $\langle X(A)\rangle$ is generic for the variety generated by $L(X)$-chains.*

Proof. By Lemma 1 and Proposition 2. □

Definition 3. *For each $n > 0$, we let $e_n(F), e_n(I)$ and $e_n(S)$ denote the following terms*

$$e_n(F) = \bigsqcup_{i=1}^{n}((\sim\sim x_i) \sqcup ((\sim\sim x_i \Rightarrow x_i) \sqcap (\sim((\sim(x_i^2))^2) \Leftrightarrow (\sim((\sim x_i)^2)))^2)\,,$$

$$e_n(I) = \bigsqcup_{i=1}^{n}((\sim\sim x_i) \sqcup (\sim x_i) \sqcup (\sim\sim x_i \Leftrightarrow \sim x_i)),$$

$$e_n(S) = \bigsqcup_{i=1}^{n}((\sim\sim x_i \Leftrightarrow \sim x_i) \sqcup (\sim\sim x_i \Rightarrow x_i))\,.$$

Lemma 3. *Let $\mathcal{A}$ be a DNMG-chain, $X \in \{F, I, S\}$ and, for each $n > 0$, let $(a_1, a_2, \ldots, a_n) \in A^n$. Then, $(e_n(X))(a_1, \ldots, a_n) < 1$ iff $(a_1, \ldots, a_n) \in X(A)$.*

Proof. By Definition 1, it is sufficient to note that the ith disjunct of $e_n(X)$ evaluates to 1 iff $a_i \in A \setminus X(A)$. □

Lemma 4. *Let $\mathcal{A}$ be a non-trivial DNMG-chain, and $t(x_1, \ldots, x_n) = 1$ be an equation. Then, for every $X \in \{F, I, S\}$, the equation $t = 1$ holds in $\langle X(A)\rangle$ iff $t \sqcup e_n(X) = 1$ holds in $\mathcal{A}$.*

Proof. If $X(A) = \emptyset$, then $\langle X(A)\rangle$ is isomorphic with the two-element Boolean algebra $\{0, 1\}$, and by Lemma 3 $\mathcal{A} \models e_n(X) = 1$. The claim follows immediately. We then assume $X(A) \neq \emptyset$: by Lemma 3 we have that $\langle X(A)\rangle \not\models e_n(X) = 1$. Assume first that $\mathcal{A} \models t \sqcup e_n(X) = 1$: we must have $\langle X(A)\rangle \models t = 1$. Conversely, suppose $\mathcal{A} \not\models t \sqcup e_n(X) = 1$: by Lemma 3 we have that for some $a_1, \ldots, a_n \in \langle X(A)\rangle$, $e_n(X)(a_1, \ldots, a_n) < 1$ and $t(a_1, \ldots, a_n) < 1$. We conclude that $\mathcal{A} \not\models t = 1$. □

Lemma 5. *A DNMG-chain $\mathcal{A}$ embeds into a DNMG-chain $\mathcal{B}$ iff $\langle X(A)\rangle$ embeds into $\langle X(B)\rangle$ for each $X \in \{F, I, S\}$.*

Proof. One direction is trivial. Assume then that there is $X \in \{F, I, S\}$ such that $\langle X(A)\rangle$ does not embed into $\langle X(B)\rangle$, and assume further, by contradiction, that $f \colon A \to B$ is an embedding of $\mathcal{A}$ into $\mathcal{B}$. Then there is $a \in \langle X(A)\rangle$ such that $f(a) \notin \langle X(B)\rangle$, for otherwise f would embed $\langle X(A)\rangle$ into $\langle X(B)\rangle$. Assume $X = F$ and $f(a) \in B \setminus \langle F(B)\rangle$. Then $\sim\sim a = \sim a$, while $\sim\sim f(a) \neq \sim f(a)$, contradicting the fact that f is an homomorphism. The other cases $X = I$ or $X = S$, are dealt with analogously. □

Lemma 6. *Let $\mathbb{V}$ be a subvariety of $\mathbb{DNMG}$. Then a DNMG-chain $\mathcal{A} \in \mathbb{V}$ is generic for $\mathbb{V}$ iff each finite chain $\mathcal{B}$ in $\mathbb{V}$ embeds into $\mathcal{A}$.*

Proof. Assume that each finite chain $\mathcal{B}$ in $\mathbb{V}$ embeds into $\mathcal{A}$. Since $\mathbb{V}$ has the finite model property, being locally finite, then it is generated by the class of its finite chains. This implies that $\mathbf{V}(\mathcal{A}) = \mathbb{V}$. On the other hand, assume that $\mathcal{B}$ does not embed into $\mathcal{A}$. Then, by Lemma 5, there is $X \in \{F, I, S\}$ such that $\langle X(B)\rangle$ does not embed into $\langle X(A)\rangle$. By Lemma 1, $\langle X(B)\rangle$ is an $L(X)$-chain which does not belong to the variety generated by the $L(X)$-chain $\langle X(A)\rangle$. Whence there is an equation $t(x_1, \ldots, x_n) = 1$ holding in $\langle X(A)\rangle$ and failing in $\langle X(B)\rangle$. Then, by Lemma 4, the equation $t \sqcup e_n(X) = 1$ holds in $\mathcal{A}$ and fails in $\mathcal{B}$, proving that $\mathcal{A}$ is not generic for $\mathbb{V}$. □

Lemma 7. *Let $\mathbb{V} \subseteq \mathbb{DNMG}$ be a single chain generated variety. Then there is a chain $\mathcal{B} \in \mathbb{V}$ such that every countable chain in $\mathbb{V}$ embeds into it, and $\mathbf{V}(\mathcal{B}) = \mathbb{V}$.*

Proof. Immediate by Lemma 6, [10, Theorem 3.8] and [22, Theorems 3 and 5].

Lemma 8. *A DNMG-chain is $\{F, I, S\}$-semigeneric iff it is generic for $\mathbb{DNMG}$.*

Proof. Immediate, from Lemmas 2, 5 and 6. □

Theorem 2. *A chain $\mathcal{A}$ is generic for $\mathbb{DNMG}$ iff the sets $F(A), I(A)$ and $S(A)$ are infinite.*

Proof. By Lemmas 1 and 8. □

We are ready to prove standard completeness for the logic DNMG whose associated algebraic semantics is given by $\mathbb{DNMG}$.

Definition 4. *Let $*\colon [0,1]^2 \to [0,1]$ be defined by:*

$$x * y = \begin{cases} 0 & \textit{if } x + y \leq 3/4 \textit{ or } \max\{x,y\} \leq 1/2 \\ \min\{x,y\} & \textit{otherwise.} \end{cases}$$

It is immediate to check that $*$ is a left-continuous t-norm. Moreover $*$ is the monoidal operation of the standard WNM-chain $[0,1]_*$ determined by the following negation: $\sim 0 = 1$, $\sim x = 3/4 - x$ for all $x \in (0,1/4) \cup (1/2,3/4]$, $\sim x = 1/2$ for all $x \in [1/4,1/2]$, $\sim x = 0$ for all $x \in [3/4,1]$.

Lemma 9. *The WNM-chain $[0,1]_*$ determined by the t-norm in Definition 4 is a DNMG-chain.*

Proof. We just check that any element $a \in [0,1/4) \cup (1/2,3/4)$ is such that $\sim\sim a = a$, while any element $a \in [1/4,1/2]$ is such that $\sim\sim a = \sim a = 1/2$, and finally any element $a \in [3/4,1]$ is such that $\sim\sim a = 1$. □

Theorem 3. *The logic DNMG is standard complete, since the variety $\mathbb{DNMG}$ is generated by $[0,1]_*$.*

Proof. By Lemma 9, $[0,1]_* \in \mathbb{DNMG}$. Now, $F([0,1]_*) = [1/4,1/2]$, $I([0,1]_*) = (0,1/4) \cup (1/2,3/4)$, and $S([0,1]_*) = [3/4,1)$. By Lemma 8 and Theorem 2, the standard chain $[0,1]_*$ is generic for $\mathbb{DNMG}$. □

4 The Lattice of Subvarieties of $\mathbb{DNMG}$

Let ω denote the ordinal $\{0,1,2,\ldots\}$ of the natural numbers, and let $\omega+1$ be the ordinal $\omega \cup \{\omega\}$. For any integer $n > 0$ and any sequence $\kappa_1, \kappa_2, \ldots, \kappa_n$ of ordinals, the *direct product* $\kappa_1 \times \kappa_2 \times \cdots \times \kappa_n$ is the poset obtained equipping the cartesian product with the pointwise order: $(a_1,\ldots,a_n) \leq (b_1,\ldots,b_n)$ iff $a_i \leq b_i$ for all $i \in \{1,2,\ldots,n\}$. We write $\kappa^{(n)}$ to mean the nth *direct power* of the ordinal κ, that is, the direct product of n copies of κ.[1] A subset $S \subseteq \kappa_1 \times \cdots \times \kappa_n$ is an *antichain* if for each $a, b \in S$, neither $a \leq b$ nor $b \leq a$. The set of all antichains of a poset P is denoted $\mathcal{AC}(P)$.

Lemma 10. *For $n > 0$, every antichain of $P = \kappa_1 \times \kappa_2 \times \cdots \times \kappa_n$ is finite.*

[1] We use this notation to distinguish direct powers from ordinal exponentiation.

Proof. By [8, Exercise 2.3.4] a poset is a well-quasi-order (wqo) iff (1) it has no infinite strictly decreasing sequences, i.e. $x_0 > x_1 > \dots$, and (2) it has no infinite antichains. As every ordinal κ_i is well-ordered, then it satisfies (1) and (2): whence it is a wqo. By [8, Lemma 2.3.9] P is also a wqo, whence each one of its antichains is finite. □

Corollary 1. *For each integer $n > 0$, the set $\mathcal{AC}((\omega+1)^{(n)})$ has cardinality $\aleph_0$.*

Proof. By Lemma 10, every antichain of $(\omega+1)^{(n)}$ is a finite set of n-tuples of $\omega \cup \{\omega\}$. Hence it can be coded in the natural numbers, i.e. there is an injective map $\mathcal{AC}((\omega+1)^{(n)}) \to \omega$. Whence, $|\mathcal{AC}((\omega+1)^{(n)})| \leq \aleph_0$. To conclude, note that $\mathcal{AC}((\omega+1)^{(1)}) = \omega+1$, whence, $|\mathcal{AC}((\omega+1)^{(n)})| \geq |\mathcal{AC}((\omega+1)^{(1)})| = \aleph_0$. □

The set $\mathcal{AC}((\omega+1)^{(n)})$ can be equipped with a lattice structure by putting $X \leq Y$ if for each n-tuple $x \in X$ there is $y \in Y$ such that $x \leq y$, for all $X, Y \in \mathcal{AC}((\omega+1)^{(n)})$. In this section we shall prove that the lattice of subvarieties of $\mathbb{DNMG}$ is isomorphic with $\mathcal{AC}((\omega+1)^{(3)})$.

Definition 5. *Let $\mathcal{A}$ be a DNMG-chain. Then the* triplet $T(\mathcal{A})$ *associated with $\mathcal{A}$ is an element $(a, b, c) \in (\omega+1)^{(3)}$ defined as follows.*

1. *If $S(A)$ is infinite then $a = \omega$, otherwise $a = |S(A)|$.*
2. *If $I^-(A)$ is infinite then $b = \omega$, otherwise $b = |I^-(A)|$.*
3. *If $F(A)$ is infinite then $c = \omega$, otherwise $c = |F(A)|$.*

Lemma 11. *Let $\mathcal{A}$ and $\mathcal{B}$ be two DNMG-chains with $\mathcal{A}$ of finite cardinality. Then $\mathcal{A}$ embeds into $\mathcal{B}$ iff $T(\mathcal{A}) \leq T(\mathcal{B})$ in the pointwise order.*

Proof. $\mathcal{A}$ embeds into $\mathcal{B}$ iff, by Lemma 5, $\langle S(A)\rangle$ embeds into $\langle S(B)\rangle$, $\langle I(A)\rangle$ embeds into $\langle I(B)\rangle$ and $\langle F(A)\rangle$ embeds into $\langle F(B)\rangle$, iff $T(\mathcal{A}) \leq T(\mathcal{B})$. □

Lemma 12. *Let $\mathcal{A}$ and $\mathcal{B}$ be two DNMG-chains. Then $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\mathcal{B})$ iff $T(\mathcal{A}) \leq T(\mathcal{B})$. As a consequence, $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\mathcal{B})$ iff $T(\mathcal{A}) = T(\mathcal{B})$.*

Proof. Assume first that $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\mathcal{B})$. Then each chain in $\mathbf{V}(\mathcal{A})$ belongs to $\mathbf{V}(\mathcal{B})$, too. By Lemma 6, each finite chain $\mathcal{C} \in \mathbf{V}(\mathcal{A})$ embeds into both $\mathcal{A}$ and $\mathcal{B}$. By Lemma 11 this occurs only if $T(\mathcal{C}) \leq T(\mathcal{A})$ and $T(\mathcal{C}) \leq T(\mathcal{B})$. But then $\langle X(A)\rangle$ embeds into $\langle X(B)\rangle$ for each $X \in \{F, I, S\}$, or they are both of infinite cardinality. In both cases $T(\mathcal{A}) \leq T(\mathcal{B})$.

For the other way round, assume $T(\mathcal{A}) \leq T(\mathcal{B})$. Take now any finite chain $\mathcal{C} \in \mathbf{V}(\mathcal{A})$. Then by Lemma 6, $\mathcal{C}$ embeds into $\mathcal{A}$, and by Lemma 11, $T(\mathcal{C}) \leq T(\mathcal{A})$. By our standing assumption, $T(\mathcal{C}) \leq T(\mathcal{B})$, too. Whence, again by Lemma 11, $\mathcal{C}$ embeds into $\mathcal{B}$, which implies that each finite chain in $\mathbf{V}(\mathcal{A})$ belongs to $\mathbf{V}(\mathcal{B})$, too. We conclude $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\mathcal{B})$. □

Notice that by Lemmas 2 and 8 and Theorem 2, if C is a set of DNMG-chains containing an infinite chain $\mathcal{A}$ of any cardinality, then $\mathbf{V}(C) = \mathbf{V}(C')$ for $C' = (C \cup \{\mathcal{A}'\}) \setminus \{\mathcal{A}\}$, for a suitable $\mathcal{A}'$. More precisely, $\mathcal{A}'$ is obtained from $\mathcal{A}$ by replacing, for every $X \in \{F, I, S\}$ such that $X(A)$ is infinite, the subalgebra

$\langle X(A)\rangle$ with a fresh copy of a fixed denumerable $L(X)$-chain $\mathcal{B}$, where we safely identify the extremes of $\mathcal{B}$ with those of $\mathcal{A}$. We call $\mathcal{A}'$ a *regular* chain.

Then we may assume that each subvariety of $\mathbb{DNMG}$ is generated by a set of regular chains. Two regular DNMG-chains $\mathcal{A}$ and $\mathcal{B}$ are isomorphic iff $T(\mathcal{A}) = T(\mathcal{B})$. We now fix, once and for all, one *representative* regular chain $\mathcal{A}_T$, for each triple $T \in (\omega+1)^{(3)}$. Notice that given two regular representative DNMG-chains $\mathcal{A}$ and $\mathcal{B}$, we have that $\mathcal{A}$ embeds into $\mathcal{B}$ iff $T(\mathcal{A}) \le T(\mathcal{B})$. A set C of DNMG-chains is *irredundant* if each $\mathcal{A} \in C$ is representative regular and $\mathbf{V}(C \setminus \{\mathcal{A}\}) \subsetneq \mathbf{V}(C)$ for all $\mathcal{A} \in C$. Otherwise C is *redundant*.

Lemma 13. *Every irredundant set C of DNMG-chains is finite. Moreover, the map $\mathcal{A} \mapsto T(\mathcal{A})$ is a bijection between C and $C_T = \{T(\mathcal{A}) : \mathcal{A} \in C\}$, which is the underlying set of a finite antichain of $(\omega+1)^{(3)}$.*

Proof. Let C be an irredundant set of DNMG-chains. Then, for every $\mathcal{A}, \mathcal{B} \in C$, with $\mathcal{A} \neq \mathcal{B}$ we have that $T(\mathcal{A})$ is incomparable with $T(\mathcal{B})$. Indeed, if not, by Lemma 12 either $\mathbf{V}(C \setminus \{\mathcal{A}\}) = \mathbf{V}(C)$ or $\mathbf{V}(C \setminus \{\mathcal{B}\}) = \mathbf{V}(C)$. In both cases we have a contradiction. By the previous observations we have that C_T must be the underlying set of an antichain of $(\omega+1)^{(3)}$, and by Lemma 10 C_T is finite. The proof is settled by noticing that the map $\mathcal{A} \mapsto T(\mathcal{A})$ is a bijection between C and C_T. □

Lemma 14. *The lattice $\Lambda(\mathbb{DNMG})$ of all subvarieties of $\mathbb{DNMG}$, ordered by inclusion, is isomorphic with the lattice $\Gamma(\mathbb{DNMG})$ of all irredundant sets of DNMG-chains, ordered by inclusion.*

Proof. Clearly, each irredundant set of DNMG-chains generates a subvariety of $\mathbb{DNMG}$, and each subvariety is generated by some irredundant set of DNMG-chains. We prove than no subvariety can be generated by two distinct irredundant sets C, D of DNMG-chains, with $C \neq D$: by Lemma 13 we can assume $C = \{\mathcal{A}_1, \dots, \mathcal{A}_h\}$ and $D = \{\mathcal{B}_1, \dots, \mathcal{B}_k\}$. By contradiction, suppose $\mathbf{V}(C) = \mathbf{V}(D)$.

Then, without loss of generality, there is a chain $\mathcal{A}_r \in C \setminus D$. By [6, Theorem 7], we have that the class of chains in $\mathbf{V}(C) = \mathbf{V}(D)$ coincides with the one in $\bigcup_{i=1}^{h} \mathbf{V}(\mathcal{A}_i)$ and with the one in $\bigcup_{i=1}^{k} \mathbf{V}(\mathcal{B}_i)$. This means that there is $\mathcal{B}_s \in D$ such that $\mathbf{V}(\mathcal{A}_r) \subseteq \mathbf{V}(\mathcal{B}_s)$, and clearly $\mathcal{B}_s \notin C$ (otherwise C would be redundant). This implies $T(\mathcal{A}_r) \le T(\mathcal{B}_s)$ and hence $\mathcal{A}_r$ embeds into $\mathcal{B}_s$, being both chains regular representative. On the other hand, for the same reasons, this in turns implies that $\mathcal{B}_s$ embeds into some chain $\mathcal{A}_t \in C$. Note that $t \neq r$, as otherwise $\mathcal{A}_r \simeq \mathcal{B}_s$, and since both chains are regular representative we would conclude $\mathcal{A}_r = \mathcal{B}_s$, in contrast with the fact that $\mathcal{A}_r \notin D$. But then also $\mathcal{A}_r$ embeds into $\mathcal{A}_t$, contradicting the fact that C is irredundant. It is obvious that both the bijective correspondence $C \mapsto \mathbf{V}(C)$ and its inverse are order-preserving. □

Theorem 4. *The lattice $\Lambda(\mathbb{DNMG})$ is isomorphic with $\mathcal{AC}((\omega+1)^{(3)})$.*

Proof. By Lemma 14, we prove that $\Gamma(\mathbb{DNMG})$ is isomorphic with $\mathcal{AC}((\omega+1)^{(3)})$. Consider a set C of pairwise non-isomorphic representative chains in $\mathbb{DNMG}$

such that the set $C_T = \{T(\mathcal{A}) \mid \mathcal{A} \in C\}$ is not an antichain. Then there are chains $\mathcal{A}$ and $\mathcal{B} \in C$ such that $T(\mathcal{A}) \leq T(\mathcal{B})$. By Lemma 12, $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\mathcal{B})$. Whence, $\mathbf{V}(C) = \mathbf{V}(C \setminus \{\mathcal{A}\})$. That is C is redundant. Whence, each element of $\Gamma(\mathbb{DNMG})$ is a set C of chains such that C_T is an antichain, that is T maps $\Gamma(\mathbb{DNMG})$ to $\mathcal{AC}((\omega+1)^{(3)})$. T is injective by construction. For surjectivity, let $A \in \mathcal{AC}((\omega+1)^{(3)})$. With each triple $(a,b,c) \in A$ we associate the representative regular DNMG-chain $\mathcal{A}_{(a,b,c)}$ such that $T(\mathcal{A}_{(a,b,c)}) = (a,b,c)$. Call $C(A)$ the set of chains so obtained. Clearly, $T(C(A)) = A$ and all chains in $C(A)$ are representative and regular. Finally, for any triple $(a,b,c) \in A$ we have that $\mathbf{V}(C(A)\setminus\{\mathcal{A}_{(a,b,c)}\}) \subsetneq \mathbf{V}(C(A))$, for otherwise there is a triple $(d,e,f) \in A$ such that $(a,b,c) \leq (d,e,f)$, contradicting the fact that A is an antichain. Whence $C(A)$ is irredundant, that is, it belongs to $\Gamma(\mathbb{DNMG})$. Surjectivity is proved. □

Corollary 2. *There are countably many subvarieties of* $\mathbb{DNMG}$*. Every subvariety of* $\mathbb{DNMG}$ *is generated by a finite number of DNMG-chains.*

Proof. By Theorem 4, Lemma 10 and Corollary 1. □

5 Uniform Axiomatisations

In this section we axiomatise all the subvarieties of $\mathbb{DNMG}$ in a uniform way.

Definition 6. *For each integer* $n > 0$ *let* Q_n *be the term:*

$$Q_n = \bigsqcup_{1 \leq i \neq j \leq n+1} (x_i \Leftrightarrow x_j)\,.$$

Furthermore let $Q_0 = 0$ *and* $Q_\omega = 1$*, and* $e_{\omega+1}(X) = 1$ *for each* $X \in \{F, I, S\}$*.*

Given $(a,b,c) \in (\omega+1)^{(3)}$, we write $\mathbf{V}(a,b,c)$ to mean the variety generated by a DNMG-chain $\mathcal{A}$ such that $T(\mathcal{A}) = (a,b,c)$.

Theorem 5. *For each* $(a,b,c) \in (\omega+1)^{(3)}$*, the variety* $\mathbf{V}(a,b,c)$ *is the subvariety of* $\mathbb{DNMG}$ *satisfying the following equation.*

$$(Q_a \sqcup e_{a+1}(S)) \sqcap (Q_b \sqcup e_{b+1}(I)) \sqcap (Q_c \sqcup e_{c+1}(F)) = 1\,.$$

Proof. Let $\mathcal{D}$ be a DNMG-chain such that $T(\mathcal{D}) = (a,b,c)$. We show that $\mathcal{D} \models Q_a \sqcup e_{a+1}(S) = 1$. As a matter of fact, if $a = 0$ then by Lemma 3 $1 = e_{a+1}(d_1,\ldots,d_{a+1}) = (Q_a \sqcup e_{a+1}(S))(d_1,\ldots,d_{a+1})$, since $d_i \in D \setminus S(D)$ for all $i \in \{1,2,\ldots,a+1\}$. If $a = \omega$ then $(Q_a \sqcup e_{a+1}(S))(d_1,\ldots,d_{m+1}) = 1$ for $Q_a = 1$. If a is a positive integer, then $(Q_a \sqcup e_{a+1}(S))(d_1,\ldots,d_{a+1}) = 1$. Indeed, if $d_i \in S(D)$ for all $i \in \{1,2,\ldots,a+1\}$ then there are distinct indices i,j such that $d_i \Leftrightarrow d_j = 1$, for $|S(A)| = a$, whence $(Q_a \sqcup e_{a+1}(S))(d_1,\ldots,d_{a+1}) = Q_a(d_1,\ldots,d_{a+1}) = 1$. If on the other hand there is some $d_i \in D \setminus S(D)$, then, by Lemma 3, $e_{a+1}(S)(d_1,\ldots,d_{a+1}) = 1 = (Q_a \sqcup e_{a+1}(S))(d_1,\ldots,d_{a+1})$. We proceed similarly, to show that $\mathcal{D} \models Q_b \sqcup e_{b+1}(I) = 1$ and $\mathcal{D} \models Q_c \sqcup e_{c+1}(F) = 1$, *mutatis mutandis.* Let now $\mathcal{D}$ be a DNMG-chain not in $\mathbf{V}(a,b,c)$, whence $T(\mathcal{D}) = (a',b',c')$

with $(a', b', c') \not\leq (a, b, c)$. If $a' > a$ then picking pairwise distinct elements $a_1, \ldots, a_{a+1} \in S(D)$ we have that $Q_a(d_1, \ldots, d_{a+1}) < 1$. Further, by Lemma 3, $e_{a+1}(S)(d_1, \ldots, d_{a+1}) < 1$, too. We conclude that $\mathcal{D} \not\models Q_a \sqcup e_{a+1}(S) = 1$. The cases $b' > b$ and $c' > c$ are dealt with in the same manner, *mutatis mutandis.* □

Clearly, if some index a, b, c is zero, the associated conjunct in the axiomatising equation can be simplified: $Q_a \sqcup e_{a+1}(S)$ to $e_{a+1}(S)$ and so forth. Analogously, if some index a, b, c is ω, the associated conjunct can be totally disregarded. For instance $\mathbf{V}(\omega, \omega, \omega) = \mathbb{DNMG}$, and as a matter of fact the axiomatising equation in this case is identically $1 = 1$. As $\mathbb{DNMG}$ contains all major subvarieties of $\mathbb{WNM}$, several varieties of the form $\mathbf{V}(a, b, c)$ have already been studied and axiomatised in the literature. The following theorems report on this aspect.

Theorem 6. $\mathbf{V}(0, 0, 0) = \mathbb{B}$. *For each integer* $n > 0$, $\mathbf{V}(n, 0, 0) = \mathbb{G}_{n+2}$, $\mathbf{V}(0, n, 0) = \mathbb{NM}_{2n+2}$, $\mathbf{V}(0, 0, n) = \mathbb{DP}_{n+2}$. *Also,* $\mathbf{V}(\omega, 0, 0) = \mathbb{G}$, $\mathbf{V}(0, \omega, 0) = \mathbb{NM}^-$ *and* $\mathbf{V}(0, 0, \omega) = \mathbb{DP}$.

Proof. Immediate by [6,17], and Definition 5. □

We now show how the axiomatisation provided in Theorem 5 can be simplified, when exactly one element in the triplet (a, b, c) is zero. In this case $\mathbf{V}(a, b, c)$ is either a subvariety of $\mathbb{RDP} = \mathbf{V}(\omega, 0, \omega)$, or of one of the following two subvarieties. The variety $\mathbb{DNM} = \mathbf{V}(0, \omega, \omega)$, axiomatised as $\mathbb{WNM}$ plus:

$$(\sim\sim x \Rightarrow x) \sqcup (\sim\sim x \Leftrightarrow \sim x) = 1. \tag{IF}$$

The variety $\mathbb{NMG}^- = \mathbf{V}(\omega, \omega, 0)$, axiomatised as $\mathbb{NMG}$ plus:

$$\sim((\sim(x^2))^2) \Leftrightarrow (\sim((\sim x)^2))^2 = 1. \tag{NF}$$

Theorem 7. *1. For all* $b, c \in (\omega + 1)^{(2)}$ *with* $b \neq 0 \neq c$, *the variety* $\mathbf{V}(0, b, c)$ *is axiomatised as* $\mathbb{DNM}$ *plus*

$$(Q_b \sqcup e_{b+1}(I)) \sqcap (Q_c \sqcup e_{c+1}(F)) = 1\,.$$

2. For all $a, c \in (\omega + 1)^{(2)}$ *with* $a \neq 0 \neq c$, *the variety* $\mathbf{V}(a, 0, c)$ *is axiomatised as* $\mathbb{RDP}$ *plus*

$$(Q_a \sqcup e_{a+1}(S)) \sqcap (Q_c \sqcup e_{c+1}(F)) = 1\,.$$

3. For all $a, b \in (\omega + 1)^{(2)}$ *with* $b \neq 0 \neq a$, *the variety* $\mathbf{V}(a, b, 0)$ *is axiomatised as* $\mathbb{NMG}^-$ *plus*

$$(Q_a \sqcup e_{a+1}(S)) \sqcap (Q_b \sqcup e_{b+1}(I)) = 1\,.$$

Proof. Immediate by Theorem 5.

We now provide the general criterion to axiomatise the subvarieties of $\mathbb{DNMG}$.

Theorem 8. *Let $C = \{\mathcal{A}_i\}_{i \in I}$ be an irredundant set of DNMG-chains. Let further $t_i(x_1, \ldots, x_{n_i}) = 1$ be the equation axiomatising $\mathbf{V}(\mathcal{A}_i)$ for each $i \in I$, as given by Theorem 5. Then $\mathbf{V}(C)$ contains exactly the DNMG-algebras satisfying the equation*

$$\bigsqcup_{i \in I} t_i(y_{i,1}, \ldots, y_{i,n_i}) = 1\,,$$

where all the variables $y_{i,j}$, for $i \in I$, and $j \in \{1, \ldots, n_i\}$, are pairwise distinct.

Proof. First notice that by Corollary 2, $\bigsqcup_{i \in I} t_i = 1$ is indeed an equation, as I is a finite index set. The proof is settled by noting that $\mathbf{V}(C) = \bigsqcup_{i \in I} \mathbf{V}(\mathcal{A}_i)$, and by using [15, Lemma 5.25]. □

Corollary 3. *Every element of $\Lambda(\mathbb{DNMG})$ is the join of a finite set of join irreducible elements.*

Proof. By [2, Theorem 5.1] a variety of MTL-algebras is join irreducible, in the lattice of the subvarieties of $\mathbb{MTL}$, if and only if it is generated by a single chain. The claim follows by Theorem 8. □

Theorem 9. *$\mathbb{DNMG}$ is the smallest subvariety in $\Lambda(\mathbb{DNMG})$ which contains $\mathbb{DP}$, $\mathbb{NM}^-$, $\mathbb{G}$ and it is generated by a single chain.*

Proof. Immediate by Theorem 6 and Lemma 12, since $\mathbb{DNMG} = \mathbf{V}(\omega, \omega, \omega)$. □

Remark 1. Notice that $\mathbb{NM} = \mathbf{V}(0, \omega, 1)$, and its lattice of subvarieties is given by all antichains $C \in \mathcal{AC}((\omega + 1)^{(3)})$ such that all $T \in C$ have either the form $T = (0, b, 1)$ or $T = (0, b, 0)$ for some integer $b \geq 0$, or $T = (0, \omega, 0)$, whose corresponding variety is $\mathbb{NM}^-$.

The almost minimal subvarieties of $\mathbb{DNMG}$ are exactly $\mathbb{G}_3 = \mathbf{V}(1, 0, 0)$, $\mathbb{NM}_4 = \mathbf{V}(0, 1, 0)$, and $\mathbb{NM}_3 = \mathbb{DP}_3 = \mathbf{V}(0, 0, 1)$. Whence they coincide with the almost minimal subvarieties of $\mathbb{WNM}$ (see [3]).

By Lemma 7 and [10, Theorem 3.5], every variety of DNMG-algebras of the form $\mathbf{V}(a, b, c)$ is such that the corresponding logic has the strong single chain completeness (see [2, 22]).

The subvarieties of $\mathbb{DNMG}$ generated by a standard algebra are exactly $\mathbb{G} = \mathbf{V}(\omega, 0, 0)$, $\mathbb{NM} = \mathbf{V}(0, \omega, 1)$, $\mathbb{NMG} = \mathbf{V}(\omega, \omega, 1)$, $\mathbb{RDP} = \mathbf{V}(\omega, 0, \omega)$, $\mathbb{DNM} = \mathbf{V}(0, \omega, \omega)$, and, clearly, $\mathbb{DNMG} = \mathbf{V}(\omega, \omega, \omega)$.

Finally, $\mathbb{EMTL} = \mathbf{V}(\{(\omega, 0, 0), (0, 0, \omega)\})$ is an example of a subvariety of $\mathbb{DNMG}$ which cannot be generated by a single chain [6].

References

1. Aguzzoli, S., Bianchi, M.: On some questions concerning the axiomatisation of WNM-algebras and their subvarieties. Fuzzy Sets Syst. **292**, 5–31 (2016)
2. Aguzzoli, S., Bianchi, M.: Single chain completeness and some related properties. Fuzzy Sets Syst. **301**, 51–63 (2016)
3. Aguzzoli, S., Bianchi, M.: Minimally many-valued extensions of the monoidal t-norm based logic MTL. In: Lecture Notes in Artificial Intelligence, vol. 10147, pp. 106–115. Springer (2017)
4. Aguzzoli, S., Bianchi, M., Valota, D.: A note on drastic product logic. In: Information Processing and Management of Uncertainty, Communications in Computer and Information Science, vol. 443, pp. 365–374. Springer (2014)
5. Aguzzoli, S., Bova, S., Valota, D.: Free weak nilpotent minimum algebras. Soft. Comput. **21**(1), 79–95 (2017)
6. Bianchi, M.: The logic of the strongest and the weakest t-norms. Fuzzy Sets Syst. **276**, 31–42 (2015)
7. Blok, W., Pigozzi, D.: Algebraizable logics, Memoirs of The American Mathematical Society, vol. 77. American Mathematical Society (1989)
8. Börger, E., Grädel, E., Gurevich, Y.: The Classical Decision Problem. Universitext, Springer, Heidelberg (2001). Reprint of 1997 edn
9. Bova, S., Valota, D.: Finite RDP-algebras: duality. Coproducts Log. J. Log. Comput. **22**(3), 417 (2012)
10. Cintula, P., Esteva, F., Gispert, J., Godo, L., Montagna, F., Noguera, C.: Distinguished algebraic semantics for t-norm based fuzzy logics: methods and algebraic equivalencies. Ann. Pure Appl. Log. **160**(1), 53–81 (2009)
11. Cintula, P., Hájek, P., Noguera, C. (eds.): Handbook of Mathematical Fuzzy Logic, vol. 1, 2, 3. College Publications (2011)
12. Esteva, F., Godo, L.: Monoidal t-norm based logic: towards a logic for left-continuous t-norms. Fuzzy Sets Syst. **124**(3), 271–288 (2001)
13. Esteva, F., Godo, L., Noguera, C.: On expansions of WNM t-norm based logics with truth-constants. Fuzzy Sets Syst. **161**(3), 347–368 (2010)
14. Fodor, J.: Nilpotent minimum and related connectives for fuzzy logic. In: Proceedings of 1995 IEEE International Conference on Fuzzy Systems, pp. 2077–2082. IEEE (1995)
15. Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated lattices: an algebraic glimpse at substructural logics. In: Studies in Logic and The Foundations of Mathematics, vol. 151. Elsevier (2007)
16. García-Cerdaña, À., Noguera, C., Esteva, F.: On the scope of some formulas defining additive connectives in fuzzy logics. Fuzzy Sets Syst. **154**(1), 56–75 (2005)
17. Gispert, J.: Axiomatic extensions of the nilpotent minimum logic. Rep. Math. Log. **37**, 113–123 (2003)
18. Hájek, P.: Metamathematics of Fuzzy Logic, Trends in Logic, vol. 4. Kluwer Academic Publishers, Dordrecht (1998)
19. Jenei, S.: A note on the ordinal sum theorem and its consequence for the construction of triangular norms. Fuzzy Sets Syst. **126**(2), 199–205 (2002)
20. Jenei, S., Montagna, F.: A proof of standard completeness for Esteva and Godo's logic MTL. Stud. Log. **70**(2), 183–192 (2002)
21. Marchioni, E.: On deductive interpolation for the weak nilpotent minimum logic. Fuzzy Sets Syst. **292**, 318–332 (2016)

22. Montagna, F.: Completeness with respect to a chain and universal models in fuzzy logic. Arch. Math. Log. **50**(1–2), 161–183 (2011)
23. Noguera, C.: Algebraic study of axiomatic extensions of triangular norm based fuzzy logics. Ph.D. thesis, IIIA-CSIC (2006)
24. Noguera, C., Esteva, F., Gispert, J.: On triangular norm based axiomatic extensions of the weak nilpotent minimum logic. Math. Log. Q. **54**(4), 387–409 (2008)
25. Wang, S.: A fuzzy logic for the revised drastic product t-norm. Soft. Comput. **11**(6), 585–590 (2007)
26. Wang, S.M., Wang, B.S., Pei, D.W.: A fuzzy logic for an ordinal sum t-norm. Fuzzy Sets Syst. **149**(2), 297–307 (2005)

http://www.springer.com/978-3-319-66829-1

Advances in Fuzzy Logic and Technology 2017
Proceedings of: EUSFLAT-2017 – The 10th Conference of the European Society for Fuzzy Logic and Technology, September 11–15, 2017, Warsaw, Poland IWIFSGN'2017 – The Sixteenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets, September 13–15, 2017, Warsaw, Poland, Volume 1
Kacprzyk, J.; Szmidt, E.; Zadrożny, S.; Atanassov, K.T.; Krawczyk, M. (Eds.)
2018, XII, 712 p. 154 illus., Softcover
ISBN: 978-3-319-66829-1