

Turbulence and Mixing in Flows Dominated by Buoyancy

Paul F. Linden

Abstract This chapter discusses the physics underlying mixing caused by turbulent flow in a stratified fluid. The ability of the oceans to absorb and redistribute heat from the atmosphere at low latitudes is a crucial aspect of the climate system and accurate quantitative estimates of the mixing rates are critical to the development of reliable climate models. However, mixing occurs at very small scales where molecular diffusion is active and these processes cannot be calculated explicitly in climate models. Thus it is necessary to represent these rates in terms of the larger scale fields and this requires an understanding of the links between these large and small scales. We focus here on laboratory experiments that attempt to make these links and, in particular, represent transport rates in terms of a mixing efficiency. We show that insights can be obtained in this way that point to reasonable representations of mixing in geophysical flows.

1 Introduction

In these lectures we discuss the dynamics of turbulence and mixing in stably stratified fluids. To start we begin by considering what happens when a completely full (spherical) bottle of water is shaken. Shaking a bottle of water dimension L with a speed U inputs energy per unit mass at a rate U^3/L . Shaking at a steady rate implies that the turbulent flow is in a statistically steady state, with eddies ranging in scales from the size of the bottle down to the scale at which the energy input is dissipated by viscosity. This sets up the classical turbulent cascade where energy is input at large scales (the scale of the bottle) and dissipated at some small scale that adjusts to dissipate the energy.

This balance implies that the rate of dissipation per unit mass ϵ is equal to the rate of energy input per unit mass $\epsilon = U^3/L$ and the dimensions of $[\epsilon] = L^2T^{-3}$. This implies that motion within the bottle has large scales comparable with the scale

P.F. Linden (✉)

Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences,
University of Cambridge, Cambridge, UK
e-mail: P.F.Linden@damtp.cam.ac.uk

of the bottle, and with velocity comparable with the boundary speed. This energy is dissipated by viscosity at small scales generated by turbulence that in turn generate large rates of strain.

A main idea put forward by Kolmogorov in 1941 is that the rate of dissipation is independent of viscosity (Kolmogorov 1941a,b,c): that the scales of the flow simply adjust so that the dissipation achieves the required value to balance the rate of energy input into a system. As illustrated by the bottle, a statistical steady state is reached in which the average speeds in the bottle equilibrate and the work applied by shaking the bottle is balanced by the dissipation. The more viscous the fluid in the bottle, the larger the scales become at which dissipation occurs.

This leads to the notion of the Kolmogorov length scale (or dissipation scale) l_η which must depend on ϵ and ν only. Then, on dimensional grounds since $[\epsilon] = \text{L}^2\text{T}^{-3}$ and $[\nu] = \text{L}^2\text{T}^{-1}$, we obtain $l_\eta = (\nu^3/\epsilon)^{1/4}$. By a similar argument we construct an equivalent velocity scale $u_\eta = (\nu\epsilon)^{1/4}$. Note that $Re_\eta = u_\eta l_\eta / \nu = 1$, indicating that viscous stresses provide dissipation at these velocity and length scales.

Suppose, the bottle contains water and is such that $L = 10 \text{ cm}$ and $U = 10 \text{ cm s}^{-1}$. The Reynolds number of the large scale flow $Re \equiv UL/\nu \sim 10^4$, implying that viscous forces are small compared to the fluid inertia at these large scales. Then $\epsilon \sim 100 \text{ cm}^2\text{s}^{-3}$. Water has $\nu \approx 10^{-2} \text{ cm}^2\text{s}^{-1}$ which implies that $u_\eta \sim 1 \text{ cm s}^{-1}$, $l_\eta \sim 10^{-2} \text{ cm}$ (0.1 mm). So dissipation is taking place on very small scales, even in this relatively weakly forced system and $Re_\eta \equiv u_\eta l_\eta / \nu \sim 1$, implying viscous and buoyancy forces balance at these small scales.

Suppose now that the ‘bottle’ is stratified—in this case we imagine heating is applied to the top surface and cooling is applied to the bottom surface to maintain some mean (stable) gradient of density with cool water at the bottom and warm water at the top. Suppose that the rate at which heat is added at the top—the heat flux $\mathcal{H}$ —is the same as that removed at the bottom so that the average temperature of the fluid within the bottle remains constant in time.

If the fluid is at rest the heat transport from the top to the bottom of the bottle is achieved by conduction with $\mathcal{H} = k \frac{\partial T}{\partial z}$, where k is the conductivity of water and $\frac{\partial T}{\partial z}$ is constant.

Now suppose the bottle is shaken as before. In general any motion will result in warm fluid being lowered and cool fluid being raised (since the warmest fluid was at the top and the coolest at the bottom when at rest). These motions require work to be done against the buoyancy forces, so some of the kinetic energy is now used to maintain the imposed heat flux, and presumably the viscous dissipation is reduced. How much of the input energy is used to work against the buoyancy forces is known as the *mixing efficiency*. Further, overall the vertical temperature gradient is reduced—as a result of mixing by the turbulence.

This thought experiment exemplifies the questions addressed in these lectures. What is the rate of mass and momentum transport in a stably stratified fluid subject to a given forcing? How does the change in potential energy that results from mixing actually occur? What parameters govern the dynamics? We begin by defining the governing equations and examine implications for stratified turbulence.

2 Governing Equations

2.1 Mass and Momentum Conservation

The Navier–Stokes equations for mass and momentum conservation are

$$\rho \frac{Du_i}{Dt} = \frac{\partial \tau_{ij}}{\partial x_j} - g\rho\delta_{i3} , \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 , \quad (2)$$

$$\frac{D\rho}{Dt} = \kappa \nabla^2 \rho , \quad (3)$$

where κ is the molecular diffusivity of the scalar responsible for the density variations ρ . The stress tensor τ_{ij} is

$$\tau_{ij} \equiv -p\delta_{ij} + 2\nu S_{ij} , \quad (4)$$

where the rate of strain tensor is

$$S_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) . \quad (5)$$

2.2 Boussinesq Approximation

In most geophysical and environmental flows density variations are small and we invoke the Boussinesq approximation. Write density $\rho(\mathbf{x}, t)$ as the sum of a constant density ρ_0 and a variable field $\rho^*(\mathbf{x}, t)$, such that

$$\rho(\mathbf{x}, t) = \rho_0 + \rho^*(\mathbf{x}, t) , \quad (6)$$

where, under the Boussinesq approximation, $\rho^*(\mathbf{x}, t) \ll \rho_0$. When the fluid is at rest in a gravitational field, lines of constant density are horizontal (normal to the gravity field). Then the density may be written as

$$\rho^*(\mathbf{x}, t) = \bar{\rho}(z) + \rho'(\mathbf{x}, t) , \quad (7)$$

where z is the vertical (antiparallel to gravity) coordinate, $\bar{\rho}$ is the background vertical density variation of the stationary fluid, and $\rho'(\mathbf{x}, t)$ is the density perturbation due to fluid motion.

The corresponding pressure field is

$$p(\mathbf{x}, t) = -g\rho_0 z + \bar{p}(z) + p'(\mathbf{x}, t) , \quad (8)$$

where the hydrostatic pressure field $\bar{p}(z)$ of the stationary fluid is given by

$$\frac{d\bar{p}}{dz} = -g\bar{\rho} . \quad (9)$$

In addition, the pressure consists of a linear variation $-g\rho_0 z$ associated with the hydrostatic component due to the (uniform) reference density ρ_0 and $p'(\mathbf{x}, t)$ is the pressure perturbation corresponding to the fluid motion. Division of Eq.(1) by the density and substitution of Eq.(6) gives

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_0 + \bar{\rho} + \rho'} \nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u} , \quad (10)$$

with gravity defined according to $\mathbf{g} = (0, 0, -g)$. Subtract the hydrostatic pressure field using Eqs.(8) and (9) and obtain

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_0 + \bar{\rho} + \rho'} \nabla p' + \mathbf{g} \frac{\rho'}{\rho_0 + \bar{\rho} + \rho'} + \nu \nabla^2 \mathbf{u} . \quad (11)$$

Define the buoyancy by

$$b \equiv -g \frac{\rho - \rho_0}{\rho_0} , \quad (12)$$

so that

$$b^* = B(z) + b' . \quad (13)$$

The *Boussinesq approximation* assumes that all density variations are small compared to ρ_0 so that $\bar{\rho} + \rho' \ll \rho_0$, but the limit $\mathbf{g} \frac{\rho'}{\rho_0} = b' \delta_{i3}$ is finite. Then the Boussinesq equations become

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_0} \nabla p' + b' \delta_{i3} + \nu \nabla^2 \mathbf{u} . \quad (14)$$

In this limit the buoyancy equation can be written as

$$\frac{Db'}{Dt} - wN^2 = \kappa \nabla^2 b' , \quad (15)$$

where the *buoyancy frequency* is

$$N \equiv \sqrt{-\frac{g}{\rho_0} \frac{\partial \bar{\rho}}{\partial z}} = \sqrt{\frac{\partial B}{\partial z}}, \quad (16)$$

with

$$B \equiv -\frac{g}{\rho_0} \bar{\rho}(z). \quad (17)$$

2.3 Scaling

We now examine the relevant balance of terms in the governing equations by defining a horizontal length scale l_h , a vertical length scale l_v , with $\alpha = l_v/l_h$ and a horizontal velocity scale U . We scale horizontal velocity by U , vertical velocity by $U F_h^2/\alpha = U F_v^2 \alpha$, b' by U^2/l_v , and p' by $\rho_0 U^2$, where the horizontal Froude number $F_h \equiv U/(N l_h)$. The vertical Froude number $F_v \equiv U/(N l_v)$. Then, for $i, j = 1, 2$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{F_h^2}{\alpha^2} u_3 \frac{\partial u_i}{\partial x_3} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re} \left(\frac{1}{\alpha^2} \frac{\partial^2 u_i}{\partial x_3^2} + \frac{\partial^2 u_i}{\partial x_j^2} \right), \quad (18)$$

and

$$F_h^2 \left(\frac{\partial u_3}{\partial t} + u_j \frac{\partial u_3}{\partial x_j} + \frac{F_h^2}{\alpha^2} u_3 \frac{\partial u_3}{\partial x_3} \right) = -\frac{\partial p}{\partial x_3} + b + \frac{F_h^2}{Re} \left(\frac{1}{\alpha^2} \frac{\partial^2 u_3}{\partial x_3^2} + \frac{\partial^2 u_3}{\partial x_j^2} \right), \quad (19)$$

$$\frac{\partial u_i}{\partial x_i} + \frac{F_h^2}{\alpha^2} \frac{\partial u_3}{\partial x_3} = 0, \quad (20)$$

$$\frac{\partial b}{\partial t} + u_j \frac{\partial b}{\partial x_j} + \frac{F_h^2}{\alpha^2} u_3 \frac{\partial b}{\partial x_3} = u_3 + \frac{1}{Re Sc} \left(\frac{1}{\alpha^2} \frac{\partial^2 b}{\partial x_3^2} + \frac{\partial^2 b}{\partial x_j^2} \right). \quad (21)$$

The other dimensionless parameters are the Reynolds number $Re \equiv U l_h/\nu$ —note this is based on the horizontal length scale l_h —and the Schmidt (Prandtl) number $Sc \equiv \nu/\kappa$.

The ratio of magnitude of the buoyancy force in the vertical momentum equation is

$$F_h^2/\alpha^2 = F_v^2 = \frac{U^2}{N^2 l_v^2} = Ri_O^{-1},$$

where the overall Richardson number is defined as

$$Ri_O \equiv \frac{N^2 l_v}{U^2} .$$

In the case of strongly stratified flows $Re \gg 1$, $F_h \ll 1$, the equations reduce to (see Brethouwer et al. 2007)

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{F_h^2}{\alpha^2} u_3 \frac{\partial u_i}{\partial x_3} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re \alpha^2} \frac{\partial^2 u_i}{\partial x_3^2}, \quad i = 1, 2, \quad (22)$$

$$0 = -\frac{\partial p}{\partial x_3} + b, \quad (23)$$

$$\frac{\partial u_i}{\partial x_i} + \frac{F_h^2}{\alpha^2} \frac{\partial u_3}{\partial x_3} = 0, \quad (24)$$

$$\frac{\partial b}{\partial t} + u_j \frac{\partial b}{\partial x_j} + \frac{F_h^2}{\alpha^2} u_3 \frac{\partial b}{\partial x_3} = u_3 + \frac{1}{Re Sc \alpha^2} \frac{\partial^2 b}{\partial x_3^2}. \quad (25)$$

There is a natural length scale for a flow with characteristic speed U and stratification N —the overturning scale

$$l_{ot} = \frac{U}{N}. \quad (26)$$

In the case of large vertical scales $l_v \gg l_{ot}$, then $\alpha \gg F_h$ and, at high Re with $\alpha \gg 1/\sqrt{Re}$, the advective terms can be neglected and the continuity equation becomes $\frac{\partial u_i}{\partial x_i} = 0$, $i = 1, 2$. Thus the governing equation for the horizontal velocity is purely horizontal which implies the motion is similar to 2D turbulence with an inverse cascade of energy (Lilly 1983). This assumes that l_v is not a free parameter and is imposed such that $F_v \ll 1$.

Large buoyancy Reynolds number We define the *buoyancy Reynolds number* as $Re_B \equiv Re F_h^2$. When $Re_B \gg 1 \Rightarrow$ viscous and diffusive terms can be neglected compared to $O(F_h^2/\alpha^2 = F_v^2)$ terms (if $Sc \gtrsim 1$) and the above equations reduce to

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{F_h^2}{\alpha^2} u_3 \frac{\partial u_i}{\partial x_3} = -\frac{\partial p}{\partial x_i}, \quad i = 1, 2, \quad (27)$$

$$0 = -\frac{\partial p}{\partial x_3} + b, \quad (28)$$

$$\frac{\partial u_i}{\partial x_i} + \frac{F_h^2}{\alpha^2} \frac{\partial u_3}{\partial x_3} = 0, \quad (29)$$

$$\frac{\partial b}{\partial t} + u_j \frac{\partial b}{\partial x_j} + \frac{F_h^2}{\alpha^2} u_3 \frac{\partial b}{\partial x_3} = u_3. \quad (30)$$

These equations are invariant under the transformation (constant γ)

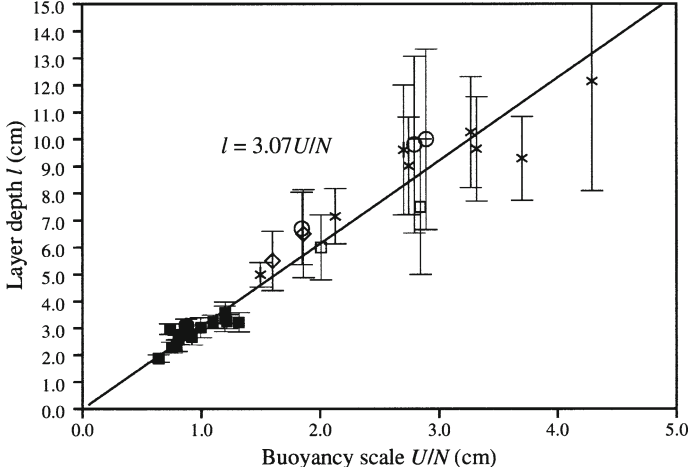


Fig. 1 Layer depths observed in a stably stratified fluid stirred by a rake of vertical bars. Courtesy of Holford and Linden (1999)

$$N = \frac{N^*}{\gamma}; \quad x_3 = x_3^* \gamma; \quad u_3 = u_3^* \gamma; \quad b = \frac{b^*}{\gamma}.$$

So by dimensional analysis these equations are self-similar with respect to $x_3 N/U$, see Billant and Chomaz (2001), suggesting that

$$l_v \sim \frac{U}{N} \Rightarrow \alpha \sim F_h \quad F_v \sim 1, \quad (31)$$

as has been observed in Holford and Linden (1999) (see Fig. 1).

This also suggests a forward cascade of highly anisotropic 3D turbulence (Lindborg 2006) with $\epsilon \sim U^3/l_h$, in which case

$$Re = \frac{U^4}{\nu \epsilon}, \quad F_h = \frac{\epsilon}{N U^2}, \quad Re_B = \frac{\epsilon}{\nu N^2}. \quad (32)$$

Hence for viscous effects to be unimportant $Re^{-1/2} \ll F_h \ll 1$. Similarly, for diffusion to be unimportant $Re^{-1/2} Sc \ll F_h \ll 1$.

The Ozmidov scale, the smallest scale to be affected by buoyancy forces, is defined as $L_O \equiv \sqrt{\frac{\epsilon}{N^3}}$. Thus

$$\frac{l_h}{L_O} = F_h^{-3/2} \quad \text{and} \quad \frac{l_v}{L_O} = F_h^{-1/2}. \quad (33)$$

Since $F_h \ll 1$ (Lindborg 2006 suggests $F_h < 0.02$) this shows that the Ozmidov scale sets lower limits in the horizontal as well as the vertical on scales of stratified turbulence. Note further that

$$Re_B = \left(\frac{L_o}{l_\eta} \right)^{4/3}, \quad (34)$$

so the condition $Re_B \gg 1$ means there is a significant range of scales between the Ozmidov scale and the Kolmogorov scale.

Internal waves Consider the dimensional form of (18) in the limit $F_h \rightarrow 0$ and linearise (with $v = u_2 = 0$):

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}, \quad (35)$$

$$0 = -\frac{\partial p}{\partial z} + b, \quad (36)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (37)$$

$$\frac{\partial b}{\partial t} = wN^2, \quad (38)$$

and write $(u, w, p, b) \propto e^{i(kx+mz-\omega t)}$. Then the dispersion relation is

$$\omega^2 = \frac{k^2}{m^2} N^2, \quad (39)$$

which is the usual dispersion relation of long internal gravity waves when $k \ll m$. Thus these waves are *slow* $c = N/m$, and have large horizontal scales compared to vertical scales. Internal waves have $\omega \in [0, N]$. So even if $N \gg u/l_h$, ($F_h \ll 1$) there are slow internal waves with $\omega \sim u/l_h$ and corresponding vertical wavenumber $m \sim N/U$. This implies that internal waves can interact with horizontal advective motions, and may be responsible for organising the turbulence and promoting the formation of layers (Thorpe 2016). This is likely the reason why the vertical scale of the layers in the grid-stirred experiments of Holford and Linden (1999) (see Fig. 1) have a scale $\sim 3U/N$, which is the vertical half-wavelength of long internal waves travelling at the same speed as the grid.

3 Richardson Number

The stability of stratified shear flow is characterised by a measure of the stabilising effect of stratification and the destabilising effect of shear. There are several measures of this ratio, all called a Richardson number—which can also be considered as a ratio of the potential energy required to mix the fluid to that supplied by the velocity field. In terms of bulk properties, a flow with a velocity difference ΔU , and buoyancy difference ΔB , across a region of height H (Fig. 2), the *overall* (bulk) Richardson number is defined, in terms of the external parameters as

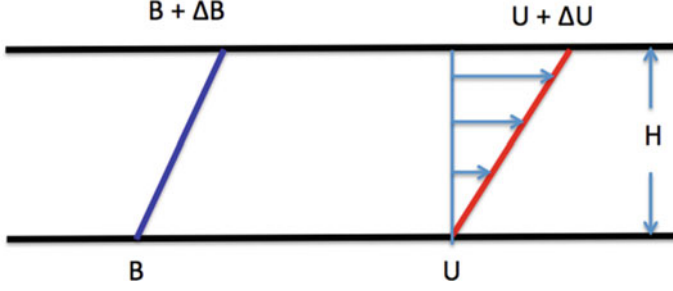


Fig. 2 A schematic of stratified shear flow showing mean constant gradients of buoyancy B and velocity U

$$Ri_O \equiv \frac{\Delta B H}{\Delta U^2} . \quad (40)$$

In many situations $Ri_O \gg 1$, suggesting that there is not sufficient energy to allow for turbulent mixing. For example, considering the ocean as a whole $\Delta U \sim 1 \text{ ms}^{-1}$, $\Delta B \sim 0.25 \text{ ms}^{-2}$, and $H \sim 4000 \text{ m}$, implying $Ri_O \sim 10^3$. The way round this dilemma is to consider the *gradient* Richardson number, defined in terms of the local gradients of buoyancy and velocity

$$Ri_g \equiv \frac{N^2}{\left(\frac{\partial U}{\partial z}\right)^2} . \quad (41)$$

Since the velocity gradient is squared it is possible to choose any value of Ri_g (such as 0.2, say, consistent with the Miles-Howard stability criterion (Miles 1961; Howard 1961 stability criterion) *throughout the fluid* regardless of the value of Ri_O (Turner 1973).

4 Turbulent Kinetic Energy Equation

The turbulent kinetic energy (TKE) equation is formed by multiplying Eq. (1) by u_i to give

$$\frac{1}{2} \frac{\partial u_i u_i}{\partial t} + u_i u_j \frac{\partial u_i}{\partial x_j} = w b + u_i \frac{\partial \tau_{ij}}{\partial x_j} . \quad (42)$$

Define the kinetic energy per unit mass $E \equiv \frac{1}{2} u_i u_i$ and note that with $u_j \frac{\partial u_i^2}{\partial x_j} = 2 u_i u_j \frac{\partial u_i}{\partial x_j}$, then

$$\frac{DE}{Dt} = w b + \frac{1}{\rho_0} u_j \frac{\partial p}{\partial x_j} + \frac{1}{2} \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) . \quad (43)$$

Some elementary algebra gives

$$\begin{aligned} u_i \frac{\partial \tau_{ij}}{\partial x_j} &= \frac{\partial u_i \tau_{ij}}{\partial x_j} - \tau_{ij} \frac{\partial u_i}{\partial x_j} , \\ &= \frac{\partial u_i \tau_{ij}}{\partial x_j} - \frac{1}{2} \tau_{ij} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{\partial u_i \tau_{ij}}{\partial x_j} - \tau_{ij} S_{ij} . \end{aligned}$$

Substitute for τ_{ij} gives

$$\frac{DE}{Dt} + \nabla \cdot \mathbf{T} = wb - 2\nu S_{ij} S_{ij} , \quad (44)$$

where

$$T_i = = \frac{1}{\rho_0} u_i p - 2\nu u_i S_{ij} . \quad (45)$$

Integrate Eq. (44) over a fixed control volume V , with surface ∂V , which gives:

$$\frac{\partial}{\partial t} \int_V E dV + \int_{\partial V} (u_j E + T_j) n_j dA = \int_V w b dV - 2\nu \int_V S_{ij} S_{ij} dV . \quad (46)$$

If V is such that $u_i = 0$ on the boundary, or the domain is periodic (the outward normals on the periodic faces cancel) then

$$\int_{\partial V} (u_j E + T_j) n_j dA = 0 ,$$

and the change in kinetic energy E is due to:

- the buoyancy flux $\int_V w b dV$, which may be positive or negative;
- viscous dissipation $\epsilon \equiv -2\nu \int_V S_{ij} S_{ij} dV$, which is negative definite.

At this point it is convenient to define the average $\langle \cdot \rangle$ as an average over space and/or time, or an ensemble over different realizations. Then any variable A can be written as a sum of the mean $\langle A \rangle$ and the variation a about the mean, such that

$$A = \langle A \rangle + a ,$$

where

$$\langle a \rangle = 0 .$$

Then the mean kinetic energy $\langle E \rangle \equiv \frac{1}{2} \langle \mathbf{U} \rangle \cdot \langle \mathbf{U} \rangle$ and the TKE $k \equiv \frac{1}{2} \langle \mathbf{u} \cdot \mathbf{u} \rangle$ satisfy

$$\frac{\overline{DE}}{\overline{Dt}} + \nabla \cdot \langle T \rangle = -P + WB - \bar{\epsilon} , \quad (47)$$

$$\frac{\overline{Dk}}{\overline{Dt}} + \nabla \cdot \langle t \rangle = P + \langle wb \rangle - \epsilon . \quad (48)$$

The production term P is given by

$$P = -\langle u_i u_j \rangle \frac{\partial \langle U_i \rangle}{\partial x_j} ,$$

which represents the transfer of kinetic energy from the mean flow to the turbulence. The quantity $-\langle u_i u_j \rangle$ is known as the *Reynolds stress*.

In stratified flow there is also production/dissipation of TKE through the *buoyancy flux* $\mathcal{B} = -\langle wb \rangle$, and TKE is generated/lost when $\langle wb \rangle \neq 0$. TKE is generated when buoyant fluid ($b > 0$) is advected upwards ($w > 0$) and heavy fluid ($b < 0$) advected downwards ($w < 0$). In both cases the movement of the fluid lowers the centre of mass of the fluid and so releases potential energy and converts it to kinetic energy. TKE is lost when buoyant fluid ($b > 0$) is advected downwards ($w < 0$) and heavy fluid ($b < 0$) advected upwards ($w > 0$)—as in the shaking bottle. In both cases the movement of the fluid raises the centre of mass of the fluid and so converts kinetic energy to potential energy.

5 Stirring Versus Mixing

Since raising heavy fluid increases the potential energy, then if this fluid is then allowed to drop, the potential energy is lowered and the energy converted back into kinetic energy. Thus the *loss of kinetic energy is only irreversible* if the fluid that is raised is *mixed* with the surrounding fluid so that it cannot fall again. Mixing, like dissipation, is an irreversible molecular process that occurs on small scales and it depends on the diffusivity κ of the scalar field (temperature or concentration of a solute) that is responsible for the density variations in the fluid.

Take b times the buoyancy Eq. (3)

$$b \frac{Db}{Dt} = \kappa b \nabla^2 b , \quad (49)$$

and average to obtain

$$\frac{D\langle b^2 \rangle}{Dt} = 2\kappa \langle b \nabla^2 b \rangle = -2\kappa \langle |\nabla b|^2 \rangle \equiv -\chi , \quad (50)$$

Table 1 Schmidt numbers for heat in air and heat and salt in water

	Sc
Heat in air	0.7
Heat in water	7
Salt in water	700

where we have used $b \frac{\partial^2 b}{\partial x_i^2} = \frac{\partial}{\partial x_i} \left(b \frac{\partial b}{\partial x_i} \right) - \frac{\partial b}{\partial x_i} \frac{\partial b}{\partial x_i}$ and noted that the first term is always zero when averaged over a volume as either $b = 0$ or $\frac{\partial b}{\partial x_i} = 0$ on the boundary of the domain.

5.1 Batchelor Scale

By an argument similar to that for the Kolmogorov length scale irreversible mixing also occurs at small scales. In this case the Batchelor scale l_B (Batchelor 1958) is the size over which a scalar will diffuse during the time $\tau_\eta = l_\eta / u_\eta$ it takes for a turbulent eddy to dissipate. Thus

$$l_B = \sqrt{\kappa \tau_\eta} = \sqrt{\frac{\kappa l_\eta}{u_\eta}} = \left(\frac{\kappa^2 \epsilon}{\nu} \right)^{1/4}.$$

The ratio of the Kolmogorov scale to the Batchelor scale is

$$\frac{l_\eta}{l_B} = \sqrt{\frac{\nu}{\kappa}} \equiv Sc^{1/2},$$

where Sc is the Schmidt/Prandtl number (Table 1 for typical values).

6 Stratified Bottle

We return to the stratified ‘bottle’—with heating applied to the top surface and cooling applied to the bottom surface to maintain some mean gradient of density. We denote the strength of the stratification by the buoyancy frequency $N \equiv \sqrt{\frac{\partial B}{\partial z}}$. (In practice this would be achieved by say imposing a constant heat flux—a so-called Neumann conditions on the boundaries). Suppose in these circumstances it is possible to maintain a constant density (temperature) gradient. Then the buoyancy flux $\mathcal{B}$ would be given by

$$\mathcal{B} = -\langle wb \rangle = K_B N^2, \quad (51)$$

which defines the ‘eddy diffusivity’ K_B of buoyancy. Further the TKE Eq. (48) becomes

$$0 = P - \mathcal{B} - \epsilon. \quad (52)$$

Thus the TKE produced by P either goes into maintaining this buoyancy flux $\mathcal{B} > 0$ or into dissipation.

7 Mixing Efficiency

An important question in stratified turbulence is the amount of energy that goes into *irreversibly* changing the density field. Equation (52) can be interpreted as a division of energy flow into work against buoyancy and dissipation by viscosity and we define the mixing efficiency $\mathcal{E}$

$$\mathcal{M} \equiv \frac{\mathcal{B}}{P} = \frac{\mathcal{B}}{\mathcal{B} + \epsilon}. \quad (53)$$

In Eq. (53) it is assumed that the buoyancy flux represents an irreversible conversion of kinetic energy to potential energy so that it can also be written as

$$\mathcal{M} \equiv \frac{\mathcal{B}}{P} = \frac{\chi}{\chi + \epsilon}. \quad (54)$$

In an unstratified flow $\mathcal{M} = 0$ and for weak stratification $\chi \ll \epsilon$, so that

$$\mathcal{M} \sim \frac{\chi}{\epsilon}.$$

Thus the mixing efficiency increases linearly with the imposed buoyancy flux for small fluxes. This behaviour is confirmed in recent DNS studies in stratified plane Couette flow up to $Ri = 0.2$, see Fig. 3.

A major question is: what happens to $\mathcal{M}$ as the imposed buoyancy flux increases? Does $\mathcal{M}$

- increase monotonically?
- reach an asymptotic value?
- reach a maximum and then decrease for further increases in the buoyancy flux?

Flux Richardson number We now consider a stability parameter in terms of vertical fluxes of buoyancy and momentum. The turbulent kinetic energy for a statistically steady flow can be written as relation (52)

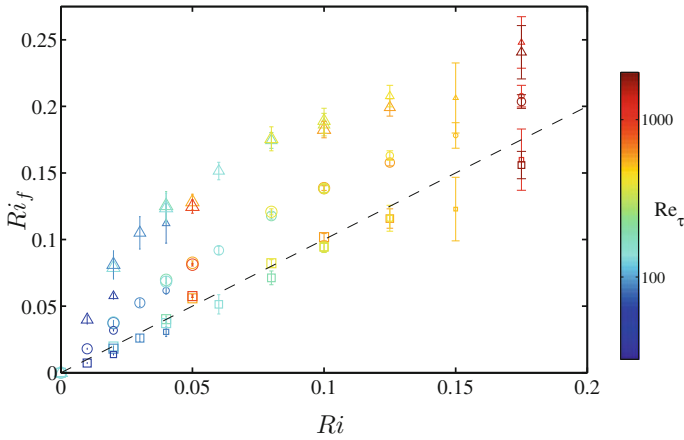


Fig. 3 Flux Richardson number as a function of the gradient Richardson number in a DNS calculation of stratified plane Couette flow. Courtesy of Deusebio et al. (2015)

$$0 = P - \mathcal{B} - \epsilon .$$

The *flux* Richardson number Ri_f is defined by

$$Ri_f \equiv \frac{\mathcal{B}}{P} , \quad (55)$$

and, since $\epsilon > 0$, $Ri_f < 1$. Assuming $\mathcal{B} = \chi$ so that it is associated with irreversible mixing

$$Ri_f = \mathcal{M} , \quad (56)$$

and the flux Richardson number is equivalent to the mixing efficiency.

The form of the ‘flux’ *versus* ‘stability’, defined in a somewhat imprecise sense, has been the subject of significant investigation. As indicated by definition (55) the flux is expected to increase with stability for weakly stratified flows, and it may be reasonable to expect the flux to decrease in very stable flows, because vertical motions are inhibited. This form is sketched in Fig. 4 and this form has been measured in experiments where a horizontal grid falls freely through a sharp density interface, see Linden (1980). However, this form of the curve is not universally agreed upon.

If the form of the curve does look like that shown in Fig. 4, then this implies, as first argued by Phillips (1972), that for weakly stratified flows—on the left of the maximum flux, the ‘left flank’—perturbations in the density gradient will be reduced, while at high stabilities on the ‘right flank’ perturbations will be increased by the flux convergence leading to layered stratification. Consequently, there is continued interest in determining the form of the mixing efficiency curve (Fig. 5).

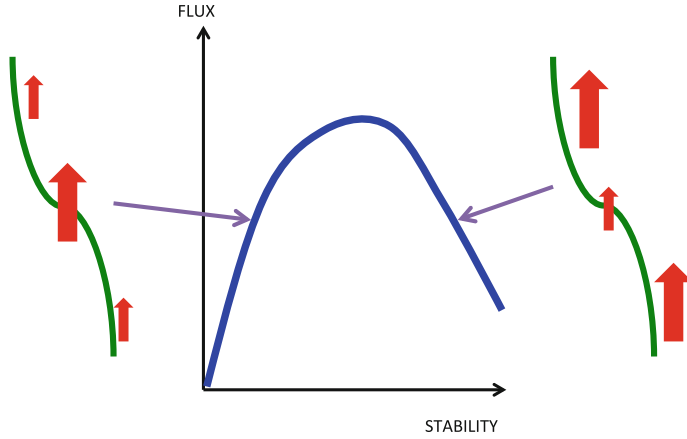
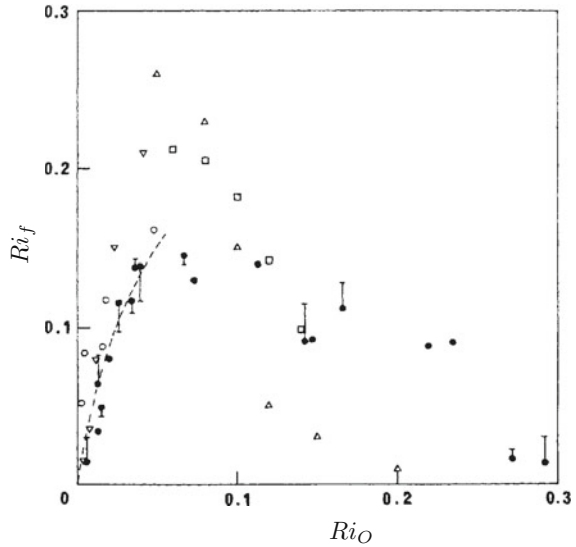


Fig. 4 Schematic of a general flux–stability curve. On the *left* ‘flank’, gradients weaken while on the *right* ‘flank’ gradients increase and layers form (Phillips (1972); Posmentier (1977))

Fig. 5 Mixing efficiency—compilation of different experiments. Courtesy of Linden (1979)



8 Boundary Layer Similarity Theory

8.1 Unstratified Boundary Layer

Consider steady flow $\mathbf{U} = (U(z), 0, 0)$ producing a stress τ on a horizontal surface $z = 0$. The stress gives a velocity scale—the friction velocity

$$u_* \equiv \sqrt{\frac{\tau}{\rho}}.$$

Assuming the stress (and, therefore, u_*) is constant and that mean flow profile depends only on u_* and the distance z from the boundary, dimensional analysis implies

$$\frac{dU}{dz} = \frac{u_*}{kz},$$

where k (dimensionless and ≈ 0.4) is von Kármán's constant. Then

$$U(z) = \frac{u_*}{k} \ln(z/z_0). \quad (57)$$

—the ‘law of the wall’ or logarithmic boundary layer: z_0 is the roughness length.

Eddy viscosity The turbulent transfer of momentum can be characterised by an ‘eddy viscosity’ defined in terms of the stress and the mean velocity gradient

$$\tau \equiv \rho K_M \frac{dU}{dz}. \quad (58)$$

Then in the logarithmic boundary layer

$$K_M = k u_* z, \quad (59)$$

which increases linearly with distance from the boundary.

In the constant-stress layer $-\langle uw \rangle = u_*^2$ and the production term $\mathcal{P} = u_*^2 \frac{dU}{dz}$. Hence

$$\mathcal{P} = K_M \left(\frac{dU}{dz} \right)^2. \quad (60)$$

8.2 Stratified Boundary Layer

Now suppose there is a stabilising buoyancy flux $\mathcal{B}$ imposed on the boundary $z = 0$. This provides another length scale, the *Monin-Obukhov* length L_{MO}

$$L_{MO} \equiv -\frac{u_*^3}{k\mathcal{B}}.$$

If the buoyancy flux is stabilising $\mathcal{B} < 0$ and $L_{MO} > 0$ is the length at which the flow feels the effects of the stratification. Dimensional analysis now gives

$$\frac{dU}{dz} = \frac{u_*}{kz} \phi_M \left(\frac{z}{L_{MO}} \right),$$

where ϕ_M is an unknown dimensionless function and

$$K_M = \frac{ku_*z}{\phi_M}. \quad (61)$$

Stratified boundary layer: weak stratification In the case of weak stratification $z/L_{MO} \ll 1$, then a Taylor series expansion gives $\phi_M \approx 1 + z/L_{MO}$ and

$$U(z) = \frac{u_*}{k} (\ln(z/z_0) + \alpha z/L_{MO}), \quad (62)$$

where $\alpha \approx 5$ is an empirical constant. Then

$$K_M \sim ku_*z(1 - \alpha z/L_{MO}), \quad (63)$$

so momentum diffusivity is reduced. In this case we can define an eddy diffusivity for buoyancy K_B , see Eq. (51), such that

$$\mathcal{B} \equiv -K_B N^2. \quad (64)$$

With Eqs. (60) and (64) the mixing efficiency is

$$\mathcal{M} = Ri_f = \frac{K_B}{K_M} \frac{N^2}{\left(\frac{dU}{dz}\right)^2} = \frac{K_B}{K_M} Ri_g. \quad (65)$$

Stratified boundary layer: moderate stratification We have defined Ri_g , Ri_f , K_M , K_B . In an analogous manner to ϕ_M we can define ϕ_B , and gradient Richardson number can be written as

$$Ri_g \equiv \frac{N^2}{\left(\frac{\partial U}{\partial z}\right)^2} = \frac{z}{L_{MO}} \frac{\phi_B}{\phi_M^2}. \quad (66)$$

Then Eq. (65) implies

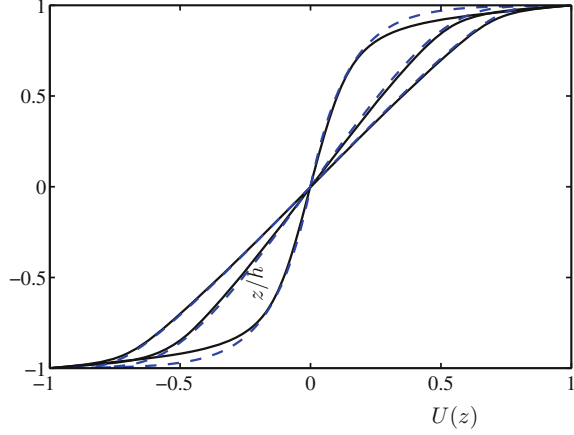
$$Ri_f = \frac{K_M}{ku_*L_{MO}} = \frac{z}{L_{MO}} \frac{1}{\phi_M}. \quad (67)$$

Thus

$$\frac{z}{L_{MO}} = \phi_M Ri_f, \quad \text{and} \quad (68)$$

$$\phi_M = 1 + \alpha \phi_M Ri_f. \quad (69)$$

Fig. 6 Velocity profiles obtained from DNS computations (*solid curves*) and Monin-Obukhov theory (*dashed curves*) of stratified plane Couette flow for $(Re_O, Ri_O) = (2150, 0)$; $(12650, 0.08)$; $(35000, 0.125)$. Note the progression to a linear profile with increasing stability. Courtesy of Deusebio et al. (2015)



In turn, this implies

$$\phi_M = \frac{1}{1 - \alpha Ri_f} . \quad (70)$$

Hence $\phi_M \rightarrow \infty$, $K_M \rightarrow 0$ as $Ri_f \rightarrow 0.2$. This suggests that the limit on mixing efficiency may be significantly smaller than 1.

Recent direct numerical simulations (DNS) show close agreement with the forms predicted by Monin-Obukhov theory, see Fig. 6, in this stability range.

Stratified boundary layer: strong stratification In the case of strong stratification $z/L_{MO} \gg 1$, the distance z from the boundary is no longer relevant. The flow depends only on u_* and L_{MO} , so dimensionally

$$\frac{\partial U}{\partial z} = k_1 \frac{u_*}{L_{MO}} , \quad (71)$$

$$N^2 = k_2^2 \frac{u_*^2}{L_{MO}^2} . \quad (72)$$

Thus the profiles of velocity and density are linear with height and the gradient Richardson number Ri_g

$$Ri_g \equiv \frac{N^2}{\left(\frac{\partial U}{\partial z}\right)^2} = \frac{k_2^2}{k_1^2} = Ri_e , \quad (73)$$

where Ri_e is an ‘equilibrium’ Richardson number (Turner 1973). Again this is consistent with recent DNS calculations (see Fig. 6).

Further

$$K_M = \frac{u_* L_{MO}}{k_1} ; \quad K_B = \frac{u_* L_{MO}}{k_2^2} ; \quad \frac{K_B}{K_M} = \frac{k_1}{k_2^2}, \quad (74)$$

so that the ‘turbulent Prandtl number’ is a constant (not necessarily equal to one).

Large bulk Richardson number Consider, as above, flow in a domain in which $Ri_O \equiv \frac{\Delta B H}{(\Delta U)^2} \gg 1$. Then there are three possibilities:

1. $Ri_O \ll Ri_e$: the shear is too large for very stable scaling to apply so return to z/L_{MO} scaling;
2. $Ri_O = Ri_e$: this case can match with linear gradients of density and velocity throughout the domain;
3. $Ri_O \gg Ri_e$: need non-uniform velocity and density gradients. Can match Ri_e throughout since

$$\left(\frac{\Delta U}{H} \right)^2 \ll \overline{\left(\frac{\partial U}{\partial z} \right)^2},$$

which implies layers with weak shears and small density gradients separated by interfaces with large shears and strong density gradients.

9 Layers and Interfaces

From Eq. (71) the vertical shear can be written as

$$\frac{\partial U}{\partial z} = \frac{u_*}{L_{MO}} = \frac{k_1 \mathcal{B}}{u_*^2} \propto \frac{\text{buoyancy flux}}{\text{momentum flux}}.$$

This implies that:

- interfaces: strong shear $\Rightarrow$ small momentum flux compared to buoyancy flux;
- layers: weak shear $\Rightarrow$ large momentum flux compared to buoyancy flux.

If the buoyancy flux is constant with depth, then there needs to be an alternative way of transferring momentum across the layers. This can be provided by internal waves—which transfer momentum but not buoyancy (unless they break).

9.1 Energetics

Following (Turner 1973) let $r = \frac{\text{interface thickness}}{\text{layer thickness}}$. If the layers have constant density and velocity and $Ri_g = Ri_e$ in the interfaces (see Eq. 73), then

$$r \equiv r_0 = \frac{Ri_e}{Ri_O} . \quad (75)$$

Consider changes in energy from linear gradients $u = \alpha z$, $\rho = \rho_0 - \beta z$ to a well-mixed layer of depth h bounded by sharp steps above and below. Then

$$\begin{aligned} \Delta KE = T &= \frac{1}{24} \alpha^2 h^3 , \\ \Delta PE = V &= -\frac{1}{12} g \beta h^3 , \end{aligned}$$

with KE representing the kinetic energy and PE the potential energy. These balance when $Ri_O = g\beta/\alpha^2 = \frac{1}{2}$. If the interface has a finite thickness (with $r = r_0$ and $Ri_g = Ri_e$) then this requires

$$Ri_O = \frac{1}{2}(Ri_e + 1) . \quad (76)$$

For larger values of Ri_O there is not enough local kinetic energy to mix and some form of external energy supply is needed, which leads to the idea of ‘external mixing processes’ (Turner 1973).

The interfacial Reynolds number Re_i based on the velocity difference and interfacial length scale r is given by

$$Re_i = \frac{Ri_e}{3Ri_e + Ri_O} \frac{Uh}{\nu} . \quad (77)$$

For large $Ri_O \sim N^2 h / U^2 \gg Ri_e$ this gives

$$Re_i \sim \frac{U^3/h}{\nu N^2} \sim \frac{\epsilon}{\nu N^2} = Re_B , \quad (78)$$

which is the buoyancy Reynolds number. Thus the interfacial Reynolds number in this limit is equivalent to the buoyancy Reynolds number.

10 Stratified Mixing Box Experiment

An experiment that exemplifies external mixing is the stratified mixing box first studied by Turner (1968). The experiment consists of a two-layer fluid where one or both of the layers is stirred by an oscillating horizontal grid, see Fig. 7.

In this case fluid is entrained across the interface, reducing the density difference between the two layers with time, and raising the centre of mass of the system. The rate of entrainment is characterised by an entrainment velocity u_e , which is the velocity the interface would move if only one of the layers was stirred. As shown in

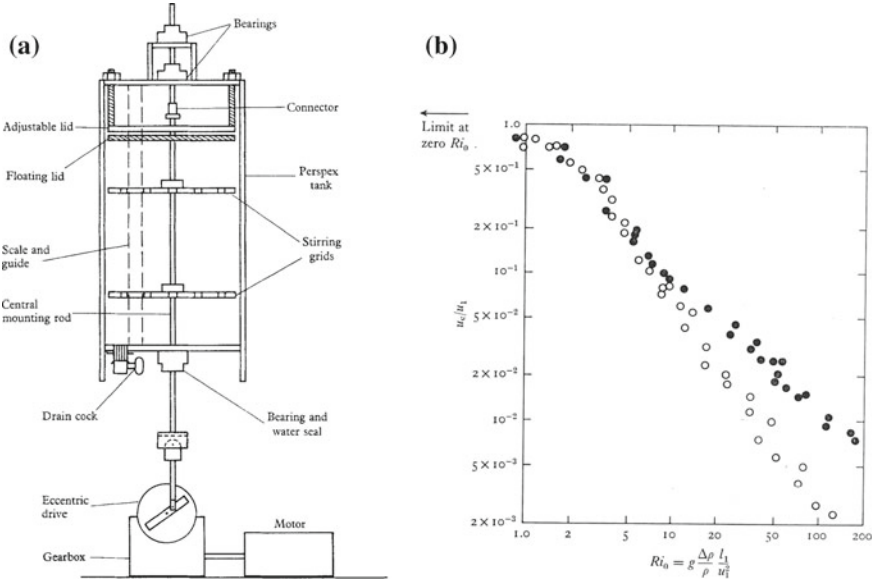


Fig. 7 a A sketch of the stratified mixing box showing the case when both layers are stirred. b The entrainment rate as a function of the interfacial Richardson number. $\circ$ – $Pr = 700$, $\bullet$ – $Pr = 7$. Courtesy of Turner (1968)

Fig. 7 the entrainment rate is a function of the stability of the interface, here described in terms of an interfacial Richardson number

$$Ri_i = \frac{g \Delta \rho l}{\rho_0 u^2},$$

where $\Delta \rho$ is the density jump across the interface and u and l are the turbulent velocity and integral length scale, respectively, *at the interface*.

We can calculate the mixing efficiency by noting that the rate of increase in PE = $\frac{1}{2} b u_e H^2$ and the rate of supply of KE = $\frac{1}{2} u^3 H$. Then the ratio is the flux Richardson number

$$Ri_f = \frac{u_e}{u} \frac{bH}{u^2} \propto \frac{u_e}{u} Ri_i. \quad (79)$$

Given that $u_e/u \propto Ri_i^{-n}$, see Fig. 7,

$$Ri_f \propto Ri_i^{1-n}. \quad (80)$$

Thus for $n < 1$, Ri_f increases with Ri_i and interfaces decay, while for $n > 1$, Ri_f decreases with Ri_i and interfaces sharpen. This behaviour can be seen in Fig. 8 where the interface remains sharp at $Ri_i = 9.4$, but is smeared out at $Ri_i = 3.4$. Note

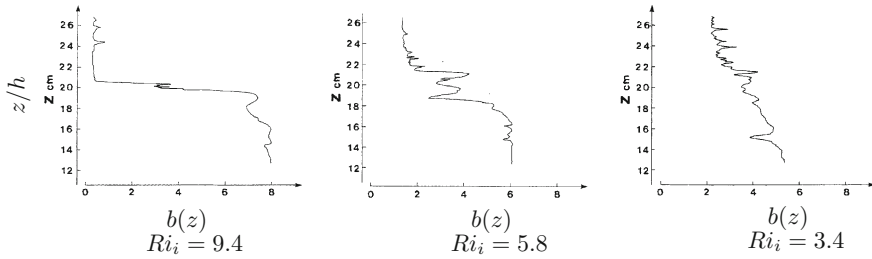


Fig. 8 Density profiles for salt experiments in the stratified mixing box at different interfacial Richardson numbers. Courtesy of Linden (1979)

that $n > 1$ at $Ri_i = 9.4$, but $n < 1$ at $Ri_i = 3.4$ (Fig. 7), consistent with the expected behaviour. It is also worth noting that Turner's experiments show that entrainment is different for fluid stratified with salt ($Pr = 700$) compared to that stratified with heat ($Pr = 7$). Recent work on mixing in a stratified shear flow shows a similar dependence.

11 Mixing Efficiency from Available Potential Energy

In some circumstances it is possible to calculate the mixing efficiency $\mathcal{M}$ from changes in available potential energy (PE_a). In situations where a complete mixing event life-cycle occurs this is probably the most unambiguous way of calculating $\mathcal{M}$. We will discuss two cases: flows driven by Rayleigh-Taylor instability and gravity currents generated by lock exchange.

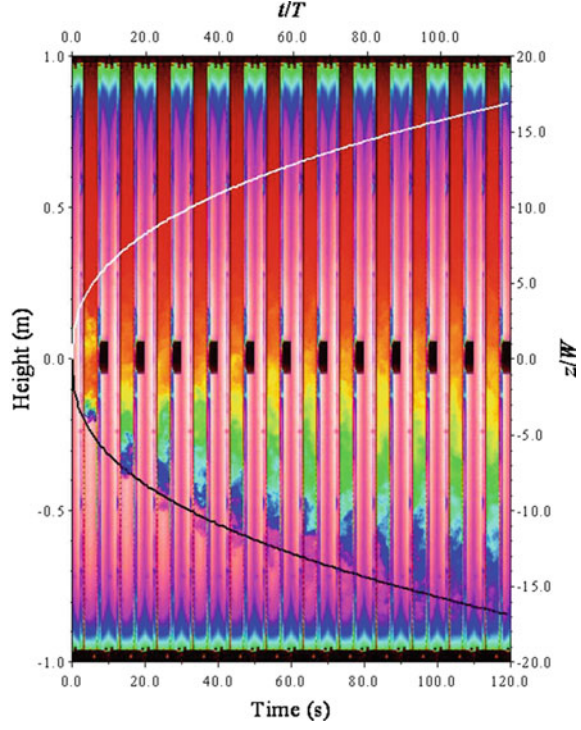
11.1 Tall Rayleigh-Taylor Instability

Rayleigh-Taylor instability (RTI) occurs when dense fluid is placed over less dense fluid under gravity. It drives mixing as the heavy fluid falls and the light fluid rises and it takes a particularly simple form in a tall tube. As can be seen in Fig. 9, the mixing front grows parabolically in time and the final state is one in which the fluid is essentially mixed throughout the tube.

In this case it is possible to calculate the maximum mixing efficiency when all the fluid is mixed, and in the final state all the fluid in the tube is at the mean density of the initial fluid.

The initial potential energy PE_i is

Fig. 9 Mixing observed in a tall tube due to RTI. Courtesy of Lawrie and Dalziel (2011)



$$PE_i = g \int_{-H}^0 \rho_L z dz + g \int_0^H \rho_U z dz = \frac{1}{2} g H^2 (\rho_U - \rho_L), \quad (81)$$

and the background potential energy PE_b , when the fluid is arranged to be adiabatically stable is

$$PE_b = g \int_{-H}^0 \rho_U z dz + g \int_0^H \rho_L z dz = -\frac{1}{2} g H^2 (\rho_U - \rho_L). \quad (82)$$

Consequently, the available potential energy $PE_a = PE_b - PE_i$ is

$$PE_a = g H^2 (\rho_U - \rho_L). \quad (83)$$

If the final state is perfectly mixed with density $\rho = \frac{1}{2}(\rho_L + \rho_U)$ then the final PE_f is

$$PE_f = \frac{1}{2} g \int_{-H}^H (\rho_L + \rho_U) z dz = 0. \quad (84)$$

So the amount of PE_a converted into mixing is $PE_c \equiv PE_i - PE_f = \frac{1}{2}gH^2(\rho_U - \rho_L)$, and the mixing efficiency

$$\mathcal{M} \equiv \frac{PE_c}{PE_a} = \frac{1}{2}. \quad (85)$$

The time-dependent approach to the ultimate state is shown in Fig. 10.

11.2 Mixing Efficiency in Lock Exchange Gravity Currents

This section describes work recently published in Hughes and Linden (2016) and the reader is referred to the original paper for more details about the experiments etc. To calculate the mixing efficiency first calculate the initial potential energy. The initial density distribution is

$$\rho = \begin{cases} \rho_L, & 0 \leq x < L - L_{\text{lock}}, & 0 \leq z \leq H_L, \\ \rho_H, & L - L_{\text{lock}} < x \leq L, & 0 \leq z \leq H_H. \end{cases} \quad (86)$$

Hence the initial potential energy PE_i is

$$PE_i = g \int_0^L \int_0^{H_i} \rho(x, z) z dx dz = \frac{1}{2}gL [\rho_L(1 - \gamma)H_L^2 + \rho_H\gamma H_H^2], \quad (87)$$

where $\gamma \equiv L_{\text{lock}}/L$ (≈ 0.5 in this case) and H_i is the initial depth, H_L or H_H , from Eq. (86). Now consider the final state after the gravity current and all subsequent motion in the channel has ceased. Conservation of volume implies initial and final free surface heights are related by $H = (1 - \gamma)H_L + \gamma H_H$ (Fig. 11).

If there is no mixing and $\rho_L < \rho_H$, the final stratification is

$$\rho = \begin{cases} \rho_H, & 0 \leq x \leq L, & 0 \leq z \leq H_H\gamma, \\ \rho_L, & 0 \leq x \leq L, & H_H\gamma < z \leq H. \end{cases} \quad (88)$$

The final potential energy PE_{nm} in this no-mixing case is

$$PE_{nm} = \frac{1}{2}gH^2L\rho_L + \frac{1}{2}gH_H^2L(\rho_H - \rho_L)\gamma^2. \quad (89)$$

Thus the *maximum* potential energy that can be released in this flow, the available potential energy $PE_a = PE_i - PE_{nm}$, is

$$\begin{aligned} PE_a = & \frac{1}{2}gL(1 - \gamma) [\rho_L(H_L^2 - H^2) + (\rho_L - \rho_H)\gamma^2 H_H^2] \\ & + \frac{1}{2}gL\gamma [\rho_H(H_H^2 - H^2) + (\rho_H - \rho_L)(H^2 - \gamma^2 H_H^2)], \end{aligned} \quad (90)$$

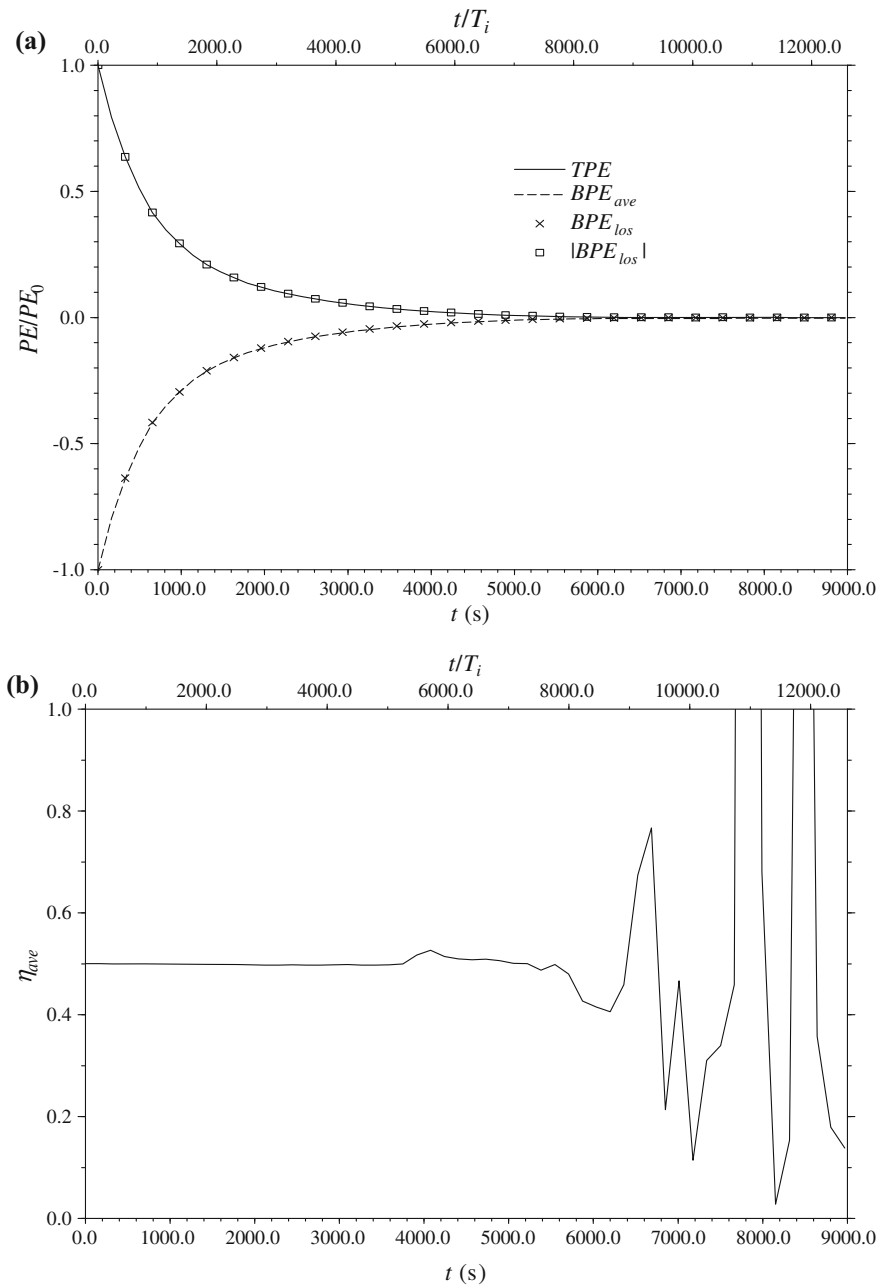


Fig. 10 Measurements of mixing efficiency in time-dependent RTI in a tall tube. Courtesy of Dalziel et al. (2008)

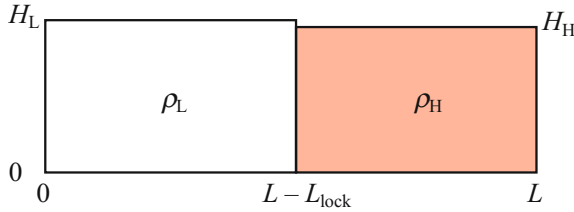


Fig. 11 Schematic of the lock release experiment. Salty water of density ρ_H fills a lock of length L_{lock} , and fresher water of density ρ_L fills the remainder of the channel. The depth $H_L > H_H$ and is set so that the pressures on the two sides of the lock gate are the same at mid-depth of the channel

where the first and third terms on the right are associated with changes in free-surface height, and the second and fourth terms are associated with changes in density between the initial and ‘non-mixed’ states.

For a general final stratification with potential energy PE_f the mixing efficiency $\mathcal{M}$ is defined as

$$\mathcal{M} \equiv \frac{PE_f - PE_{nm}}{PE_a}, \quad (91)$$

and can be calculated from the final density field in the channel after all motion has ceased. Note that if the mixing is complete so that the final density $\rho_f = \rho_L(1 - \gamma)H_L/H + \rho_H\gamma H_H/H$ is uniform throughout the channel, the final potential energy approaches PE_i given by Eq.(87) and the mixing efficiency $\mathcal{M} \rightarrow 1$ in the limit where the initial differential of free surface height across the barrier vanishes, i.e. $H = H_L = H_H$.

An image of a gravity current is shown in Fig. 12. This current has $Re = 72,000$ and $Re_B = 21,000$, and exhibits large scale billow structures on the interface between the current and the counter-flowing current above. These structures are common to all the currents, although the intensity of the turbulence along the interface was noticeably reduced at lower Reynolds numbers.

Figure 13 shows the final density profiles after all motion in the channel has ceased. The profiles are approximately self-similar when normalized by the initial density difference, with a final interfacial region that is symmetrical about mid-depth (defined as $z' = 0$) and significant mixing evident in the region $-0.2 \lesssim z'/H \lesssim 0.2$.



Fig. 12 Visualization of a gravity current. The current is dyed with food colouring and viewed against a translucent lined sheet. Courtesy of Hughes and Linden (2016)

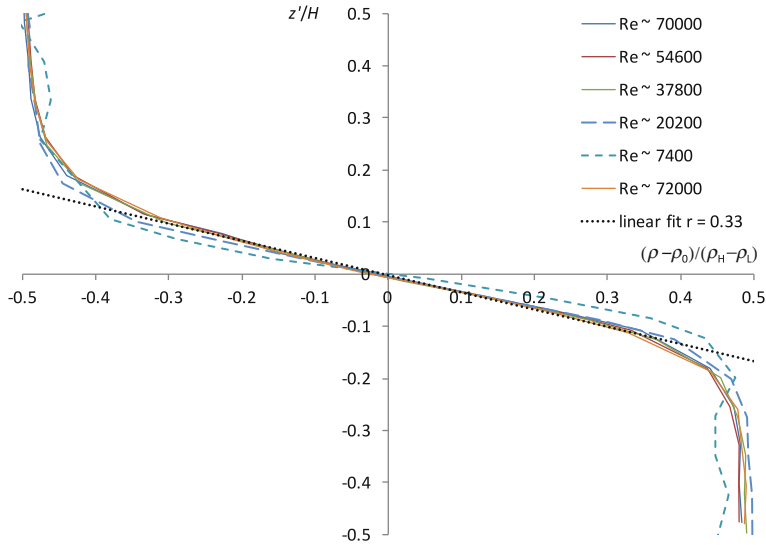


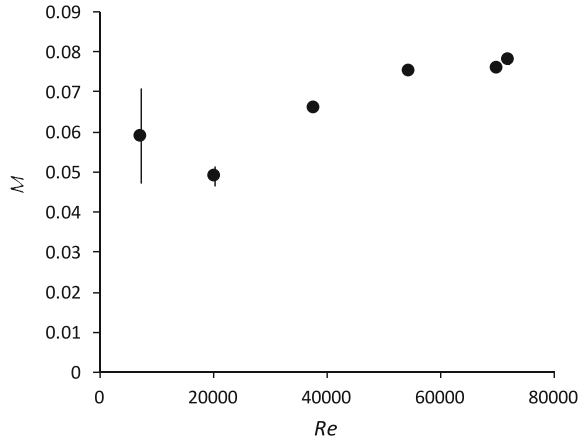
Fig. 13 Final density profiles for the experiments in Table 2 normalized by the initial density difference. The profiles were measured by withdrawing samples at different depths and the profiles are drawn with linear segments between the data points. Also shown is the assumed linear variation of density corresponding to an interfacial region of dimensionless thickness $r = 0.33$ (see Eq. 95). Courtesy of Hughes and Linden (2016)

Table 2 Values of the dimensionless density difference $(\rho_H - \rho_L)/\rho_0$, the overall Richardson number $Ri_O = (g(\rho_H - \rho_L)H/4\rho_0 U_M^2)$, where U_M is the gravity current propagation measured along the bottom, see Hughes and Linden (2016), the dimensionless mass anomaly α transported from one layer into the other by mixing (see Eq. 92), the Reynolds number $Re = (U_M H/2\nu)$ and the buoyancy Reynolds number $Re_B = (C\rho_0 U_M^3 \delta/g(\rho_H - \rho_L)\nu(H/2))$ (where this expression uses an estimate of N^2 based on the interface thickness δ and $C = 0.04$ is the constant in the assumed scaling for dissipation $\epsilon = C U_M^3/(H/2)$). The reference density ρ_0 is taken to be the average of ρ_L and ρ_H , and the uncertainties in Ri_O is based on the finite time taken to withdraw the barrier at the start of the experiment. For further details see Hughes and Linden (2016)

Exp	$(\rho_H - \rho_L)/\rho_0$	Ri_O	α	Re	Re_B
1	0.171	1.28 ± 0.11	0.102 ± 0.005	70, 000	760
2	0.086	1.10 ± 0.07	0.106 ± 0.006	54, 600	680
3	0.043	1.17 ± 0.05	0.109 ± 0.007	37, 800	440
4	0.012	1.20 ± 0.02	0.099 ± 0.007	20, 200	230
5	0.002	1.35 ± 0.02	0.097 ± 0.006	7, 400	70
6	0.172	1.22 ± 0.11	0.106 ± 0.006	72, 000	840

A weak departure from this self-similar form is suggested at the two lowest Reynolds numbers by a larger density gradient at the centre of the interface. The proportions of the less dense and more dense initial fluid volume that are mixed by the current into the other volume,

Fig. 14 Mixing efficiency results plotted as a function of the Reynolds number Re . Error bars are calculated for each experiment and are determined mainly by the relative accuracy with which the changes in free surface height $H_L - H$ and $H_H - H$ in (90) can be measured. The error bars exceed the symbol size only for the lowest two Re experiments. Courtesy of Hughes and Linden (2016)



$$\alpha_L = \frac{\int_0^{\gamma H_H} (\rho_H - \rho) L dz}{(1 - \gamma) H_L L (\rho_H - \rho_L)} \quad \text{and} \quad \alpha_H = \frac{\int_{\gamma H_H}^H (\rho - \rho_L) L dz}{\gamma H_H L (\rho_H - \rho_L)}, \quad (92)$$

respectively, are found to be almost identical, with $\alpha_L \approx \alpha_H \equiv \alpha \approx 0.1$.

The mixing efficiency determined from the initial and final density fields is shown in Fig. 14. The values range from 0.05 to 0.08, and suggest a slight increase with Re to an asymptote at high Re . Unfortunately, we were unable to reach higher Re values with our laboratory facilities and so the asymptotic value cannot be confirmed. However, we have reason to believe this is the high Re limit, as we discuss below.

Lock-exchange gravity current model We now develop a model of a lock-exchange gravity current that accounts for dissipation and stratified mixing. First, we define general conventions applicable to both (idealized) non-dissipative and dissipative currents, and then proceed to use these to characterize the properties and energy budget of the dissipative lock exchange flow.

We assign (see Fig. 11) the reservoir of relatively light (heavy) fluid of density ρ_L (ρ_H) to be initially to the left (right) of the lock. Consider the (coupled) evolution of a column of fluid from each reservoir of height H and width ΔL , such that its volume (per unit spanwise width) is $\Delta Q = H \Delta L$. We denote the volume exchanged (i.e. that carried in each layer) in a time Δt as the exchange volume flux $\dot{Q} = \Delta Q / \Delta t$.

In the idealized limit of inviscid flow (a situation denoted by the subscript i), conservation of energy can be used to predict the flow speed U_i , which is assumed uniform in each layer. Symmetry of the flow about the lock position implies that each layer in the exchange has depth $H/2$, and $\dot{Q}_i = \frac{1}{2} U_i H$.

Following (Yih 1947, 1965) we equate the rate of release of potential energy $\dot{E}$ (which corresponds physically to a raising (lowering) of the height of the centre of mass of the dense (light) fluid by $H/4$) with the rate at which kinetic energy is generated in the flow, i.e.

$$\dot{E} = \frac{1}{4}g\rho_H H \dot{Q}_i - \frac{1}{4}g\rho_L H \dot{Q}_i = 2\dot{Q}_i \frac{1}{2}\rho_0 U_i^2 . \quad (93)$$

This recovers the usual result (e.g. Benjamin 1968 and Simpson 1997) for the speed of an inviscid lock-exchange gravity current,

$$U_i = \frac{1}{2}\sqrt{g'H} , \quad (94)$$

where $g' = g(\rho_H - \rho_L)/\rho_0$ is the reduced gravity.

The idealized two-layer inviscid lock-exchange flow (which requires a step-change in the density and velocity profiles) is unstable to shear at the interface. We assume that instability arises in the vicinity of each gravity current head, and then develops and saturates at some distance behind the head. Thus in a lock-exchange flow with dissipation, we expect instability and turbulent mixing at a given location to be associated with the passage of the gravity current. We also assume that turbulence with sufficient intensity to support mixing is suppressed once the instability has run its course and left behind a stabilized interfacial structure in the wake of the current, consistent with our qualitative observations, previous studies (Thorpe 1973) and the subsequent predictions of this model.

For simplicity we adopt mean velocity and density profiles in the wake of the current that (have evolved via instability from idealized step profiles to) vary linearly with height through the stabilized interfacial region (of thickness δ) in order to match the freestream flow velocities and densities in the layers above and below (Fig. 15). We define the dimensionless thickness r of the stabilized interface in a dissipative current to be

$$r \equiv \frac{\delta}{H} . \quad (95)$$

Volume and mass transport In the dissipative lock-exchange flow we assume that each layer will evolve to consist of a region of depth $\frac{1}{2}H(1-r)$ of unmixed reservoir fluid moving at the freestream speed U , with reduced flow in the interfacial layer sandwiched between, see Fig. 15. Thus the exchange volume transport in the flow with dissipation, $\dot{Q}_d = \Delta Q_d / \Delta t$ ($< \dot{Q}_i$), is

$$\dot{Q}_d = \frac{1}{2}HU(1-r) + \int_0^{\delta/2} \frac{2Uz'}{\delta} dz' = \frac{1}{2}(1-\frac{r}{2})HU . \quad (96)$$

The volume transport in the upper layer can be related to the supply of unmixed fluid originating from the reservoirs by decomposing $\dot{Q}_d$ into the sum of three components (Fig. 15):

(i) an unmodified component from the left reservoir

$$\dot{Q}_L^u = \frac{1}{2}HU(1-r) , \quad (97)$$

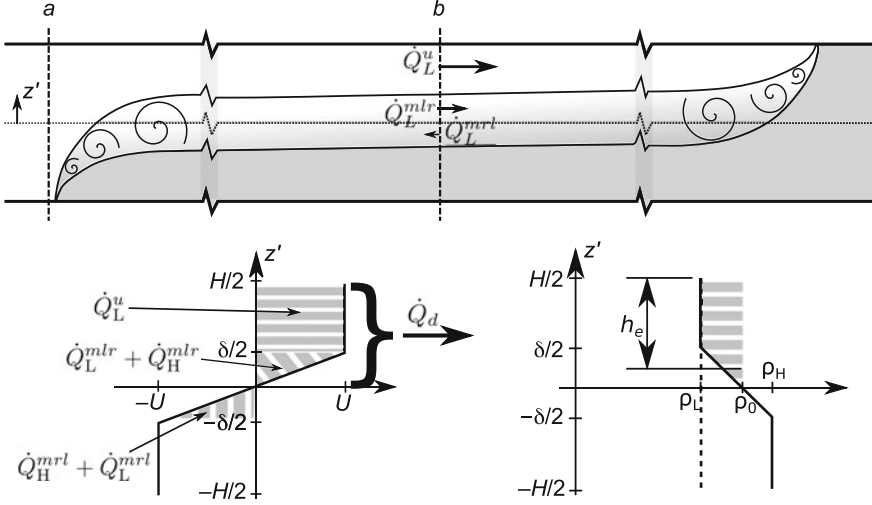


Fig. 15 Schematic diagram of the flow model. The *upper panel* indicates the overall flow structure and the *lower left and right panels* show the assumed (piecewise linear) velocity and density profiles, respectively. The *upper panel* indicates development of shear instability in the vicinity of each gravity current head, with intense turbulent eddies depicted by the swirls. Far enough behind each head, the interface between the counterflowing currents has stabilized. The *darkest grey* shading denotes fluid of density ρ_H , with fluid of intermediate density indicated by lighter shading in the interfacial region between the two currents. Courtesy of Hughes and Linden (2016)

(ii) a component $\dot{Q}_L^{mlr}$ from the left reservoir that is mixed as it flows to the right (in the same direction as $\dot{Q}_L^u$)

$$\dot{Q}_L^{mlr} = \int_0^{\delta/2} \frac{2Uz'}{\delta} c_{\rho_L}(z') dz' = \frac{5}{24} r H U, \quad (98)$$

where $c_{\rho_L}(z') = \frac{1}{2}(1 + 2z'/\delta)$ is the volume fraction of the ρ_L source component in a water parcel at height z' , and

(iii) a component $\dot{Q}_H^{mlr}$ from the right reservoir that is mixed and joins the upper layer flowing from left to right

$$\dot{Q}_H^{mlr} = \int_0^{\delta/2} \frac{2Uz'}{\delta} (1 - c_{\rho_L}(z')) dz = \frac{1}{24} r H U. \quad (99)$$

The volume transport in the lower layer can be decomposed similarly into an unmodified component $\dot{Q}_H^u$ from the right reservoir, a component $\dot{Q}_H^{mrl}$ from the right reservoir that is mixed as it flows to the left and a component $\dot{Q}_L^{mrl}$ from the left reservoir that is mixed and joins the lower layer flowing from right to left. Defining volume transport from left to right as positive and invoking symmetry in the problem, we can write

$$\dot{Q}_d = \dot{Q}_L^u + \dot{Q}_L^{mlr} + \dot{Q}_H^{mlr} = |\dot{Q}_H^u| + |\dot{Q}_H^{mlr}| + |\dot{Q}_L^{mlr}|, \quad (100)$$

and

$$\dot{Q}_L^u = -\dot{Q}_H^u, \quad \dot{Q}_L^{mlr} = -\dot{Q}_H^{mlr} \quad \dot{Q}_H^{mlr} = -\dot{Q}_L^{mlr}. \quad (101)$$

Equations (100) and (101) can be used to account for the volume transport of unmixed fluid that originates from one of the reservoirs, e.g.

$$\dot{Q}_d = \dot{Q}_L^u + \dot{Q}_L^{mlr} + |\dot{Q}_L^{mlr}| \quad (102)$$

for the left reservoir. Furthermore, it follows from Eqs. (97)–(99) and (101) that the *net* left to right transport of fluid that originated from the left reservoir is

$$\dot{Q}_L^u + \dot{Q}_L^{mlr} + \dot{Q}_L^{mlr} = \frac{1}{2} \left(1 - \frac{2r}{3} \right) HU. \quad (103)$$

We define the effective current depth h_e as the depth of unmixed fluid from the appropriate reservoir that would accommodate the buoyancy anomaly present in a layer of the assumed dissipative flow. Taking a layer to be either $-H/2 \leq z' < 0$ or $0 < z' \leq H/2$ and the buoyancy anomaly with respect to the midpoint density ρ_0 , we find $h_e = (1 - r/2)H/2$ (see bottom right panel of Fig. 15). We proceed by assuming that the freestream speed U in each layer will be U_i on the physical basis that dissipation of energy along streamlines outside the interfacial layer will be relatively small.

Mixing and recirculation of fluid in the current head leads to a measured front speed U_M that is somewhat less than U_i , thus we now differentiate between a prediction for the front speed U_e and the freestream speed U . We predict U_e by equating $U_e h_e$ with Eq. (103) and setting $U = U_i$. In physical terms, we expect the net rate of horizontal transport of fluid that has originated from each reservoir to give the volume transport involved in extending each current (of effective depth h_e) in the dissipative exchange flow, i.e. to the right in the upper layer and to left in the lower layer. Hence

$$\frac{U_e}{U_i} = \frac{1 - \frac{2}{3}r}{1 - \frac{1}{2}r}. \quad (104)$$

To enable comparison with the experimental measurements, we can predict the overall Richardson number Ri_O^p for the current by using U_e in place of the measured front speed U_M in the expression for Ri_O , thus

$$Ri_O^p = \frac{g'H}{4U_e^2} = \frac{U_i^2}{U_e^2} = \frac{(1 - \frac{1}{2}r)^2}{(1 - \frac{2}{3}r)^2}, \quad (105)$$

upon substituting Eqs. (94) and (104). (Note that the assumption that the front and freestream speeds are the same and given by Eq. (94), as for an idealized inviscid

gravity current, (i.e. $U_e = U = U_i$) corresponds to $Ri_O^p = 1$). The measurements are consistent with a constant value for $Ri_O = 1.18 (\pm 0.08)$. Thus, equating the measured Ri_O with Eq. (105) is consistent with $r = 0.38 (\pm 0.1)$; however, a more accurate determination (estimated to within ± 0.02) is given below.

In our physical model, mixed fluid is created by the passage of each current at the rate $\dot{Q}_m = \dot{Q}_L^{mlr} + \dot{Q}_H^{mlr} (= \dot{Q}_L^{mlr} + |\dot{Q}_L^{mrl}|)$, which is the sum of the second and third terms on the right hand side of Eqs. (100) and (102). Upon substituting expressions (98) and (99) for $\dot{Q}_L^{mlr}$ and $\dot{Q}_H^{mlr}$, we find that the proportion of the exchange transport involved in mixing is

$$\frac{\dot{Q}_m}{\dot{Q}_d} = \frac{rHU_i/4}{(1-r/2)HU_i/2} = \frac{r}{2-r}. \quad (106)$$

Our model assumes that mixing will occur at a constant rate until each current first reaches the end of the channel (and is zero thereafter). Hence we expect $\dot{Q}_m/\dot{Q}_d$ to be equal to $\alpha_L + \alpha_H \approx 2\alpha$, which is calculated from Eq. (92) and is based on quantities that are measured accurately in experiments. As α is found to take a value close to 0.1, equating the result (106) to 2α yields $r = 0.33 (\pm 0.02)$, a value that is consistent with the final density gradient through the centre of the interfacial region in the self-similar profiles—see Fig. 13.

It is worth remarking that the assumed piecewise linear density profile (lower right panel of Fig. 15) is fully consistent with the value of $r = 0.33$ above. This may be surprising given $r = 0.33$ seems to neglect curvature in the density profile and underestimate the volume of unmixed fluid that is passed to the other layer (as suggested by comparing the areas enclosed between either the measured or piecewise linear profile and the horizontal axis in Fig. 13). Indeed, evaluating expression (92) with the assumed piecewise linear density profile suggests coefficients $\alpha_L^* \approx \alpha_H^* \approx r/4 < \alpha$ for $\gamma = 1/2$ (i.e. for a lock at the channel midpoint)), where the asterisk is used to denote the calculation with the assumed (rather than the measured) profile and Eq. (106) has been equated with 2α . However, we note that determination of r needs to take account of the *rates* of volume transport and creation of mixed fluid. The amount of mixing in the final density profiles is then associated with the exchange volume flux in the currents, which, because of dissipation, is somewhat less than the maximum possible volume flux for an idealized inviscid flow (i.e. $\dot{Q}_d < \dot{Q}_i$). In contrast, the calculation of α_L^* and α_H^* corresponds physically to the proportion of each reservoir volume that has been swapped to obtain the final state and, assuming the exchange flow is steady, would be equal to $\dot{Q}_m/\dot{Q}_i$. Upon comparison with Eq. (106), we reason that α_L^* and α_H^* (and α^*) will be a factor $\dot{Q}_d/\dot{Q}_i = (1-r/2)$ smaller than α_L and α_H (and α), respectively. For $r = 0.33$, we therefore expect $\alpha_L^* \approx \alpha_H^* \approx \alpha^* = 0.83\alpha \approx 0.083$, or approximately $r/4$.

Energy budget for mixing We now consider the energetic consequences of the interfacial mixing, assuming that shear instability and turbulent mixing occur in the vicinity of each gravity current head. The drag associated with the turbulence causes the exchange transport $\dot{Q}_d$ to be less than $\dot{Q}_i$ and, for the same reasons

discussed above, we must analyze the energy budget by comparing the dissipative lock-exchange flow with an idealized non-dissipative counterpart that has the same exchange transport $\dot{Q}_d$. Viewed in this way, dissipation acts to “choke” the rate of release of potential energy $\dot{E}$ driving the flow,

$$\dot{E} = \frac{1}{4} \rho_0 g' H \dot{Q}_d, \quad (107)$$

which is obtained in a similar manner to Eq. (93) ($\dot{Q}_i$ being replaced by $\dot{Q}_d$). We can calculate the rate of mixing that would be associated with the linear variation of density through the interfacial layer, i.e. $\rho(z) = \rho_0 - (\rho_H - \rho_L)z'/\delta$. The density profile if no mixing occurred would be a step from ρ_L to ρ_R at $z' = 0$, thus the rate of change of potential energy owing to mixing at an interface lengthening at a rate $2U_e$ is

$$\begin{aligned} \dot{E}_p &= 2U_e \int_{-\delta/2}^{\delta/2} \rho_0 \frac{g'}{2} \left(\text{sgn}(z') - \frac{2z'}{\delta} \right) z' dz' \\ &= \frac{1}{6} \frac{(1 - \frac{2}{3}r)}{(1 - \frac{1}{2}r)^2} \rho_0 g' H \dot{Q}_d \left(\frac{\delta}{H} \right)^2, \end{aligned} \quad (108)$$

where Eqs. (96) and (104) have been used.

The energy budget can be used to characterize the mixing in terms of a mixing efficiency, and the proportion of total energy released and used for mixing is predicted to be

$$\mathcal{M} = \frac{\dot{E}_p}{\dot{E}} = \frac{2r^2}{3} \frac{(1 - \frac{2}{3}r)}{(1 - \frac{1}{2}r)^2}, \quad (109)$$

which is dependent only upon the parameter r characterizing the self-similar behaviour. The mixing efficiency $\mathcal{M}$ predicted for $r = 0.33$ is 0.081, which corresponds well with the measured asymptotic value, see Fig. 14.

These conceptually simple experiments yield a range of insights into mixing caused by a gravity current. The qualitative observations and measurements are consistent with development of stratified shear instability associated with the passage of the gravity current head. The ensuing turbulence and mixing redistributes momentum and density in the vertical until the interface above or below the current is stabilized. At sufficiently high Reynolds number (of $O(30,000)$ based on the current depth), we find that the resulting density profile becomes self-similar; the thickness of the stabilized interface normalized by the total flow depth r is close to a third. Interestingly, the interfacial signatures resulting from fully-developed Kelvin-Helmholtz instability and mixing are essentially identical (Thorpe 1973).

Simple arguments suggest that the dimensionless interface thickness is a direct indication of the gradient Richardson number that evolves across the interface between the two currents (i.e. $r = Ri_g \equiv g'\delta/4U^2$, from Eq. (95) and with U_e in Eq. (105) replaced by $U = U_i$). Thus $r \approx 0.33$ is consistent with establishment of an interfacial

region that is stable to shear instability ($Ri_g \sim 0.3$). This dimensionless thickness is further consistent with current speed $U_e \approx 0.92U_i$ (see Eq. (104)), and thus a Froude number $Fr = U_e/(g'H)^{1/2} \approx 0.46$ as found in full-depth lock exchange experiments at high Re (Keulegan 1958; Shin et al. 2004).

We find that up to about 0.08 of the energy supplied to the flow is consumed by irreversible mixing. At first glance, this value represents a mixing efficiency that is small compared to values of 0.15–0.2 that are thought to characterize the mixing owing to shear instability. However, it is important to recognise that these efficiencies measure physically different quantities. In this study we include in the energy budget the amount required to sustain the mean flow (i.e. the gravity currents), whereas a variety of measures are instead based on the proportion of energy supplied to turbulence that is consumed by mixing. Furthermore, these measures may rely on some form of averaging (e.g. in a volume, temporal or ensemble sense) or may be applicable at a specific position in the flow. Given that the turbulence in a lock-exchange gravity current is neither homogeneous nor statistically steady, we have chosen to characterize the flow by a bulk mixing efficiency measure that is unambiguous. The results highlight the importance of this consideration in a situation where the mean flow is integral to the location and characteristics of the turbulent mixing.

We have further shown here that the mixing associated with a gravity current only attains a self-similar asymptotic state at Reynolds numbers in excess of about 50,000—well above the range typically considered in previous studies. The results suggest that the stratified turbulence is characterized by buoyancy Reynolds numbers Re_B approaching 700 in this state. If the Ozmidov and Kolmogorov scales,

$$L_o = \left(\frac{\epsilon}{N^3}\right)^{1/2} \quad \text{and} \quad L_k = \left(\frac{v^3}{\epsilon}\right)^{1/4}, \quad (110)$$

respectively, characterize the turbulence spectrum in the lock-exchange gravity current, then the range of scales is given by

$$\frac{L_o}{L_k} = \left(\frac{\epsilon}{vN^2}\right)^{3/4} = Re_B^{3/4}. \quad (111)$$

Hence, these experiments span the range $20 \lesssim L_o/L_k \lesssim 150$, and suggest that asymptotic mixing behaviour owing to shear instability could require a separation of scales $L_o/L_k \gtrsim 130$.

12 Concluding Remarks

These notes are a summary of my lectures on some basic considerations on turbulence and mixing in stratified fluids. This is a subject of strong current interest and the ideas expressed in these notes will undoubtedly be challenged and refined or discarded as our understanding of this fascinating subject continues to develop. What

is clear is that at this time current computational capabilities allow *for the first time* direct numerical simulations at scales relevant to laboratory experiments, along with diagnostics that provide complementary information about the three-dimensional velocity and density fields from these experiments that allow direct comparison with the DNS. Further, and perhaps even more important, is the recognition that both experiments and computations need to be at sufficiently large buoyancy Reynolds numbers for the results to be representative of field scale processes. Both of these considerations make for an exciting new era in the study of stratified turbulence, and I hope these notes provide some inspiration for the reader to explore the field further.

Acknowledgements I would like to thank the organisers of this summer school for inviting me to give the lectures on which these notes are based. I would also like to acknowledge the many discussions I have had on this subject with many colleagues especially Colm Caulfield, Stuart Dalziel, Graham Hughes, Jamie Partridge, and John Taylor. This work is supported by the UK EPSRC, through the Programme Grant EP/K034529/1 and by the Royal Society.

References

- G.K. Batchelor, Small-scale variation of convected quantities like temperature in turbulent fluid. Part 1. general discussion and the case of small conductivity. *J. Fluid Mech.* **5**(1), 113–133 (1958)
- T.B. Benjamin, Gravity currents and related phenomena. *J. Fluid Mech.* **31**(2), 209–248 (1968)
- P. Billant, J.-M. Chomaz, Self-similarity of strongly stratified inviscid flows. *Phys. Fluids* **13**, 1645–1651 (2001)
- G. Brethouwer, P. Billant, E. Lindborg, J.-M. Chomaz, Scaling analysis and simulation of strongly stratified turbulent flow. *J. Fluid Mech.* **585**, 343–368 (2007)
- S.B. Dalziel, M.D. Patterson, C.P. Caulfield, I.A. Coomaraswamy, Mixing efficiency in high-aspect-ratio Rayleigh-Taylor experiments. *Phys. Fluids* **20**, 065106 (2008)
- E. Deusebio, C.P. Caulfield, J.R. Taylor, The intermittency boundary in stratified plane Couette flow. *J. Fluid Mech.* **781**, 298–329 (2015)
- J.M. Holford, P.F. Linden, Turbulent mixing in a stratified fluid. *Dyn. Atmos. Oceans* **30**(2–4), 173–198 (1999)
- L.N. Howard, Note on a paper of John W. Miles. *J. Fluid Mech.* **10**(4), 509–512 (1961)
- G.O. Hughes, P.F. Linden, Mixing in run-down gravity currents. *J. Fluid Mech.* **809**, 691–704 (2016)
- G.H. Keulegan, The motion of saline fronts in still water. National Bureau of Standards Report, 5831, 1958
- A.N. Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. *Comptes Rendus de l'Académie des sciences de l'URSS* **30**, 299–303 (1941a)
- A.N. Kolmogorov, On the degeneration of isotropic turbulence in an incompressible viscous liquid. *Comptes Rendus de l'Académie des sciences de l'URSS* **31**, 319–323 (1941b)
- A.N. Kolmogorov, Dissipation of energy in locally isotropic turbulence. *Comptes Rendus de l'Académie des sciences de l'URSS* **32**, 19–21 (1941c)
- A.G.W. Lawrie, S.B. Dalziel, Turbulent diffusion in tall tubes. I. models for Rayleigh-Taylor instability. *Phys. Fluids* **23**, 085109 (2011)
- D.K. Lilly, Stratified turbulence and the mesoscale variability of the atmosphere. *J. Atmos. Sci.* **40**, 749–761 (1983)
- E. Lindborg, The energy cascade in a strongly stratified fluid. *J. Fluid Mech.* **550**, 207–242 (2006)
- P.F. Linden, Mixing in stratified fluids. *Geophys. Astrophys. Fluid Dyn.* **13**, 3–23 (1979)
- P.F. Linden, Mixing across a density interface produced by grid turbulence. *J. Fluid Mech.* **100**, 691–703 (1980)

- J.W. Miles, On the stability of heterogeneous shear flows. *J. Fluid Mech.* **10**(4), 496–508 (1961)
- O.M. Phillips, Turbulence in a strongly stratified fluid—is it unstable? *Deep Sea Res.* **19**, 79–81 (1972)
- E.S. Posmentier, The generation of salinity finestructure by vertical diffusion. *J. Phys. Oceanogr.* **7**, 298–300 (1977)
- J.O. Shin, S.B. Dalziel, P.F. Linden, Gravity currents produced by lock exchange. *J. Fluid Mech.* **521**, 1–34 (2004)
- J.E. Simpson, *Gravity Currents in the Environment and the Laboratory* (Cambridge University Press, 1997)
- S.A. Thorpe, Experiments on instability and turbulence in a stratified shear flow. *J. Fluid Mech.* **61**(4), 731–751 (1973)
- S.A. Thorpe, Layers and internal waves in uniformly stratified fluids stirred by vertical grids. *J. Fluid Mech.* **793**, 380–413 (2016)
- J.S. Turner, The influence of molecular diffusivity on turbulent entrainment across a density interface. *J. Fluid Mech.* **33**(4), 639–656 (1968)
- J.S. Turner, *Buoyancy Effects in Fluids* (Cambridge University Press, 1973)
- C.S. Yih, A study of the characteristics of gravity waves at a liquid interface. MSc thesis, State University of Iowa, 1947
- C.S. Yih, *Dynamics of Nonhomogeneous Fluids* (Macmillan, 1965)

Mixing and Dispersion in Flows Dominated by Rotation
and Buoyancy

Clercx, H.J.H.; van Heijst, G. (Eds.)

2018, IX, 218 p. 112 illus., 47 illus. in color., Hardcover

ISBN: 978-3-319-66886-4