

Chapter 2

Fault Estimation of Continuous-Time Systems in Finite-Frequency Domain

2.1 Introduction

In this chapter, inspired by the previous work, our objective is to provide a general robust fault estimation observer scheme with finite-frequency specifications for continuous-time systems. Main contributions of this chapter are twofold: (1) Based on the generalized KYP lemma, a multi-constrained FFEO with finite-frequency specifications is proposed to achieve fault estimation, aimed at decreasing the conservatism that results from the entire frequency domain; (2) By using the projection lemma and introducing auxiliary slack variables, we obtain the improved results, which not only design different Lyapunov matrix for each constraint, but also are convenient to calculate FFEO parameters for different frequency domains.

The rest of this chapter is organized as follows. The system description is presented in Sect. 2.2. In Sect. 2.3, based on the generalized KYP lemma, a multi-constrained FFEO design with finite-frequency specifications is proposed to avoid the overdesign problem generated by the entire frequency domain, and improved FFEO results are further obtained by introducing slack variables. Simulation results are presented in Sect. 2.4 to show the effectiveness of the proposed approach, followed by some concluding remarks in Sect. 2.5.

2.2 System Description

Consider the following continuous-time system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ef(t) + D_1d(t) \\ y(t) = Cx(t) + D_2d(t) \end{cases} \quad (2.1)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output, $d(t) \in \mathbb{R}^d$ is the disturbance and noise which belongs to $L_2[0, +\infty)$ and $f(t) \in \mathbb{R}^r$

represents the process and actuator fault. The number of output channels is greater than or equal to the number of fault inputs, i.e. $p \geq r$. A , B , E , C , D_1 and D_2 are constant real matrices of appropriate dimensions. It is supposed that matrices E and C are of full rank. And the pair (A, C) is observable.

Remark 2.1 For the system considered in this chapter, the entire state is not measured, only some states combinations (the p outputs: $y(t) = Cx(t)$) are measured. The r actuator and process faults can be estimated if $p \geq r$ and C , E are of full rank. These conditions mean that the number of independent measured outputs is greater than the number of independent faults to be estimated. If the first condition ($p \geq r$) was not fulfilled, it would not be possible to find independent estimation paths to estimate all faults individually, only combinations of faults would be possible to be estimated. If C is not of full rank, it means that some sensors are redundant. This redundancy is surely useful for sensor fault diagnosis but doesn't bring anything for actuator or process fault estimation. If E is not of full rank, it is not possible to differentiate the corresponding fault effects on the outputs and thus it would not be possible to estimate accurately those faults. The condition that the pair (A, C) is observable is a standard condition for state observer design.

Remark 2.2 In this work, an additive fault is considered in the system description, which can represent a large class of typical faults, such as actuator faults or component faults [15].

Remark 2.3 This work focusses on analysis and design of fault estimation observer, and does not address the controller design. In particular, for unstable systems, the feedback law is firstly needed to be designed to stabilize the system. Therefore, after the occurrence of a bounded fault, the state vector never escapes to infinity in finite time.

For the dynamics (2.1), in order to realize fault estimation, we construct the following FFEO:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + E\hat{f}(t) - L(\hat{y}(t) - y(t)) \\ \hat{y}(t) = C\hat{x}(t) \\ \dot{\hat{f}}(t) = -F(\hat{y}(t) - y(t)) \end{cases} \quad (2.2)$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the observer state, $\hat{y}(t) \in \mathbb{R}^p$ is the observer output, $\hat{f}(t) \in \mathbb{R}^r$ is an estimate of $f(t)$ and $L \in \mathbb{R}^{n \times p}$, $F \in \mathbb{R}^{r \times p}$ are observer gain matrices.

Denote

$$e_x(t) = \hat{x}(t) - x(t), \quad e_y(t) = \hat{y}(t) - y(t), \quad e_f(t) = \hat{f}(t) - f(t),$$

then the error dynamics is written as

$$\begin{cases} \dot{e}_x(t) = (A - LC)e_x(t) + Ee_f(t) + (LD_2 - D_1)d(t) \\ e_y(t) = Ce_x(t) - D_2d(t) \\ \dot{e}_f(t) = -FCe_x(t) + FD_2d(t) - \dot{f}(t) \end{cases} \quad (2.3)$$

Before giving our main results, we first recall the following lemmas which will be used in the sequel.

Lemma 2.1 ([32]). *The eigenvalues of a given matrix $A \in \mathbb{R}^{n \times n}$ belong to the circular region $\mathcal{D}(\alpha, \tau)$ with center $\alpha + j0$ and radius τ if and only if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that the following condition holds:*

$$\begin{bmatrix} -P & P(A - \alpha I_n) \\ * & -\tau^2 P \end{bmatrix} < 0, \quad (2.4)$$

where here and everywhere in the sequel, $*$ denotes the symmetric elements in a symmetric matrix.

Lemma 2.2 ([41, 142]). *Considering the following system*

$$\begin{cases} \dot{X}(t) = AX(t) + BU(t) \\ Y(t) = CX(t) + DU(t) \end{cases} \quad (2.5)$$

with transfer function matrix $G(s) = C(sI - A)^{-1}B + D$. Let a symmetric matrix Π of appropriate dimensions be given, the following statements are equivalent:

(i) *The finite-frequency inequality*

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^T \Pi \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0 \quad (2.6)$$

(ii) *There exists Hermitian matrices P and Q satisfying $Q > 0$, and*

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \Xi \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0, \quad (2.7)$$

where

$$\Xi = \begin{bmatrix} -Q & P \\ P & \varpi_l^2 Q \end{bmatrix}$$

for the low-frequency domain $|\omega| \leq \varpi_l$,

$$\Xi = \begin{bmatrix} -Q & P + j\varpi_c Q \\ P - j\varpi_c Q & -\varpi_1 \varpi_2 Q \end{bmatrix}, \quad \varpi_c = (\varpi_1 + \varpi_2)/2$$

for the middle-frequency domain $\varpi_1 \leq \omega \leq \varpi_2$, and

$$\mathcal{E} = \begin{bmatrix} Q & P \\ P & -\varpi_h^2 Q \end{bmatrix}$$

for the high-frequency domain $|\omega| \geq \varpi_h$.

Lemma 2.3 ([28]) (Projection Lemma). *Let matrices Γ , Λ , Σ be given. There exists a matrix Θ satisfying $\Sigma + \Gamma^T \Theta \Lambda + \Lambda^T \Theta^T \Gamma < 0$ if and only if the following projection inequalities hold:*

$$N_\Gamma^T \Sigma N_\Gamma < 0, \quad N_\Lambda^T \Sigma N_\Lambda < 0 \quad (2.8)$$

where N_Γ and N_Λ respectively are arbitrary matrices whose columns form a basis of the nullspace of Γ and Λ .

2.3 Main Results

Before presenting FFEO design with finite-frequency specifications, an assumption is given here.

Assumption 2.1 The derivative of faults satisfies $\dot{f}(t) \in L_2[0, \infty)$.

Remark 2.4 The fault estimation filter proposed in [29, 73, 77] was developed under the assumption that $f(t) \in L_2[0, \infty)$. This assumption is not satisfied for common step faults, and as a result their asymptotic estimation is impossible. For adaptive and sliding mode observers-based fault estimation methods in [7, 44, 45, 93, 137], the upper bound of $f(t)$ must be known in advance. On another hand, the proportional integral observer-based fault estimation design assumed that $\dot{f}(t) = 0$ after the fault occurrence [50, 86]. Here, we analyze more general cases, i.e. $\dot{f}(t) \in L_2[0, \infty)$, an assumption that is obviously weaker than those used in the above three design methods.

2.3.1 FFEO Design in Finite-Frequency Domain

Now, we are ready to express our main results. After analysing the error dynamics (2.3) in detail, we can obtain the following augmented error system

$$\begin{cases} \begin{bmatrix} \dot{e}_x(t) \\ \dot{e}_f(t) \end{bmatrix} = \begin{bmatrix} A - LC & E \\ -FC & 0_r \end{bmatrix} \begin{bmatrix} e_x(t) \\ e_f(t) \end{bmatrix} + \begin{bmatrix} LD_2 - D_1 \\ FD_2 \end{bmatrix} d(t) - \begin{bmatrix} 0_{n \times r} \\ I_r \end{bmatrix} \dot{f}(t) \\ e_f(t) = \begin{bmatrix} 0_{r \times n} & I_r \end{bmatrix} \begin{bmatrix} e_x(t) \\ e_f(t) \end{bmatrix} \end{cases} \quad (2.9)$$

Denote

$$\begin{aligned} \bar{e}(t) &= \begin{bmatrix} e_x(t) \\ e_f(t) \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & E \\ 0_{r \times n} & 0_r \end{bmatrix}, \quad \bar{L} = \begin{bmatrix} L \\ F \end{bmatrix}, \\ \bar{C} &= \begin{bmatrix} C & 0_{p \times r} \end{bmatrix}, \quad \bar{D}_1 = \begin{bmatrix} D_1 \\ 0_{r \times d} \end{bmatrix}, \quad \bar{I}_r = \begin{bmatrix} 0_{n \times r} \\ I_r \end{bmatrix}, \end{aligned}$$

then it follows that

$$\begin{cases} \dot{\bar{e}}(t) = (\bar{A} - \bar{L}\bar{C})\bar{e}(t) + (\bar{L}D_2 - \bar{D}_1)d(t) - \bar{I}_r\dot{f}(t) \\ e_f(t) = \bar{I}_r^T \bar{e}(t) \end{cases} \quad (2.10)$$

and the subscript of the zero and identity matrices represents the corresponding dimension, for example, $0_{n \times r}$ denotes the zero matrix with dimension $n \times r$, and 0_r denotes the zero matrix with dimension $r \times r$. This dimension definition is also used for the identity matrix.

Remark 2.5 As far as control systems are concerned, the frequency domains of the disturbance and the fault change are usually different. Therefore, they are needed to be separately considered for the design of fault estimation observer with finite-frequency specifications, as shown in (2.10).

Remark 2.6 From the error dynamics (2.10), it can be concluded that the existence condition of FFEO is that the pair (\bar{A}, \bar{C}) is observable, and the FFEO possesses a wider application scope than adaptive and sliding mode observers [7, 44, 45, 93, 137]. Note that the FFEO design can be suitable for open-loop stable and unstable linear systems, as long as this the existence condition can be satisfied. While fault estimation filter is only suitable for open-loop stable systems [29, 73, 77].

Theorem 2.1 gives a multi-constrained FFEO design with finite-frequency specifications to achieve fault estimation.

Theorem 2.1 *Let a circular region $\mathcal{D}(\alpha, \tau)$ and two prescribed H_∞ performance levels γ_1, γ_2 be given. If there exist three symmetric positive definite matrices $\bar{P}_1, \bar{Q}_1, \bar{Q}_2 \in \mathbb{R}^{(n+r) \times (n+r)}$, two symmetric matrices $\bar{P}_2, \bar{P}_3 \in \mathbb{R}^{(n+r) \times (n+r)}$ and a matrix $\bar{L} \in \mathbb{R}^{(n+r) \times p}$ such that the following conditions hold:*

$$\begin{bmatrix} -\bar{P}_1 & \bar{P}_1(\bar{A} - \bar{L}\bar{C}) - \alpha\bar{P}_1 \\ * & -\tau^2\bar{P}_1 \end{bmatrix} < 0, \quad (2.11)$$

$$\begin{bmatrix} \phi_d - (\bar{A} - \bar{L}\bar{C})^T \bar{Q}_1 (\bar{L}D_2 - \bar{D}_1) + \bar{P}_2 (\bar{L}D_2 - \bar{D}_1) & \bar{I}_r \\ * & -(\bar{L}D_2 - \bar{D}_1)^T \bar{Q}_1 (\bar{L}D_2 - \bar{D}_1) - \gamma_1 I_d & 0 \\ * & * & -\gamma_1 I_r \end{bmatrix} < 0$$

for the low-frequency disturbance $|\omega_d| \leq \varpi_{dl}$, (2.12a)

$$\begin{bmatrix} \varphi_{d11} & \varphi_{d12} & \bar{I}_r \\ \varphi_{d21} & -(\bar{L}D_2 - \bar{D}_1)^T \bar{Q}_1 (\bar{L}D_2 - \bar{D}_1) - \gamma_1 I_d & 0 \\ * & * & -\gamma_1 I_r \end{bmatrix} < 0$$

for the middle-frequency disturbance $\varpi_{d1} \leq \omega_d \leq \varpi_{d2}$, (2.12b)

$$\begin{bmatrix} \psi_d & (\bar{A} - \bar{L}\bar{C})^T \bar{Q}_1 (\bar{L}D_2 - \bar{D}_1) + \bar{P}_2 (\bar{L}D_2 - \bar{D}_1) & \bar{I}_r \\ * & (\bar{L}D_2 - \bar{D}_1)^T \bar{Q}_1 (\bar{L}D_2 - \bar{D}_1) - \gamma_1 I_d & 0 \\ * & * & -\gamma_1 I_r \end{bmatrix} < 0$$

for the high-frequency disturbance $|\omega_d| \geq \varpi_{dh}$, (2.12c)

where

$$\begin{aligned} \phi_d &= -(\bar{A} - \bar{L}\bar{C})^T \bar{Q}_1 (\bar{A} - \bar{L}\bar{C}) + \varpi_{dl}^2 \bar{Q}_1 + \bar{P}_2 (\bar{A} - \bar{L}\bar{C}) + (\bar{A} - \bar{L}\bar{C})^T \bar{P}_2, \\ \varphi_{d11} &= -(\bar{A} - \bar{L}\bar{C})^T \bar{Q}_1 (\bar{A} - \bar{L}\bar{C}) - j\varpi_{dc} (\bar{Q}_1 (\bar{A} - \bar{L}\bar{C}) - (\bar{A} - \bar{L}\bar{C})^T \bar{Q}_1) - \\ &\quad \varpi_{d1}\varpi_{d2} \bar{Q}_1 + \bar{P}_2 (\bar{A} - \bar{L}\bar{C}) + (\bar{A} - \bar{L}\bar{C})^T \bar{P}_2, \\ \varphi_{d12} &= -(\bar{A} - \bar{L}\bar{C})^T \bar{Q}_1 (\bar{L}D_2 - \bar{D}_1) - j\varpi_{dc} \bar{Q}_1 (\bar{L}D_2 - \bar{D}_1) + \bar{P}_2 (\bar{L}D_2 - \bar{D}_1), \\ \varphi_{d21} &= -(\bar{L}D_2 - \bar{D}_1)^T \bar{Q}_1 (\bar{A} - \bar{L}\bar{C}) + j\varpi_{dc} (\bar{L}D_2 - \bar{D}_1)^T \bar{Q}_1 + (\bar{L}D_2 - \bar{D}_1)^T \bar{P}_2, \\ \psi_d &= (\bar{A} - \bar{L}\bar{C})^T \bar{Q}_1 (\bar{A} - \bar{L}\bar{C}) - \varpi_{dh}^2 \bar{Q}_1 + \bar{P}_2 (\bar{A} - \bar{L}\bar{C}) + (\bar{A} - \bar{L}\bar{C})^T \bar{P}_2, \text{ and} \end{aligned}$$

$$\begin{bmatrix} \phi_f & (\bar{A} - \bar{L}\bar{C})^T \bar{Q}_2 \bar{I}_r - \bar{P}_3 \bar{I}_r & \bar{I}_r \\ * & -\bar{I}_r^T \bar{Q}_2 \bar{I}_r - \gamma_2 I_r & 0 \\ * & * & -\gamma_2 I_r \end{bmatrix} < 0$$

for the low-frequency fault $|\omega_f| \leq \varpi_{fl}$, (2.13a)

$$\begin{bmatrix} \varphi_{f11} & (\bar{A} - \bar{L}\bar{C})^T \bar{Q}_2 \bar{I}_r + j\varpi_{fc} \bar{Q}_2 \bar{I}_r - \bar{P}_3 \bar{I}_r & \bar{I}_r \\ \varphi_{f21} & -\bar{I}_r^T \bar{Q}_2 \bar{I}_r - \gamma_2 I_r & 0 \\ * & * & -\gamma_2 I_r \end{bmatrix} < 0$$

for the middle-frequency fault $\varpi_{f1} \leq \omega_f \leq \varpi_{f2}$, (2.13b)

$$\begin{bmatrix} \psi_f - (\bar{A} - \bar{L}\bar{C})^T \bar{Q}_2 \bar{I}_r - \bar{P}_3 \bar{I}_r & \bar{I}_r \\ * & \bar{I}_r^T \bar{Q}_2 \bar{I}_r - \gamma_2 I_r & 0 \\ * & * & -\gamma_2 I_r \end{bmatrix} < 0, \quad (2.13c)$$

for the high-frequency fault $|\omega_f| \geq \varpi_{fh}$,

where

$$\begin{aligned} \phi_f &= -(\bar{A} - \bar{L}\bar{C})^T \bar{Q}_2 (\bar{A} - \bar{L}\bar{C}) + \varpi_{fl}^2 \bar{Q}_2 + \bar{P}_3 (\bar{A} - \bar{L}\bar{C}) + (\bar{A} - \bar{L}\bar{C})^T \bar{P}_3, \\ \varphi_{f11} &= -(\bar{A} - \bar{L}\bar{C})^T \bar{Q}_2 (\bar{A} - \bar{L}\bar{C}) - j\varpi_{fc} (\bar{Q}_2 (\bar{A} - \bar{L}\bar{C}) - (\bar{A} - \bar{L}\bar{C})^T \bar{Q}_2) - \\ &\quad \varpi_{f1} \varpi_{f2} \bar{Q}_2 + \bar{P}_3 (\bar{A} - \bar{L}\bar{C}) + (\bar{A} - \bar{L}\bar{C})^T \bar{P}_3, \varphi_{f21} = \bar{I}_r^T \bar{Q}_2 (\bar{A} - \bar{L}\bar{C}) - j\varpi_{fc} \bar{I}_r^T \\ &\quad \bar{Q}_2 - \bar{I}_r^T \bar{P}_2, \psi_f = (\bar{A} - \bar{L}\bar{C})^T \bar{Q}_2 (\bar{A} - \bar{L}\bar{C}) - \varpi_{fh}^2 \bar{Q}_2 + \bar{P}_3 (\bar{A} - \bar{L}\bar{C}) + (\bar{A} - \\ &\quad \bar{L}\bar{C})^T \bar{P}_3, \end{aligned}$$

then the eigenvalues of $(\bar{A} - \bar{L}\bar{C})$ belong to $\mathcal{D}(\alpha, \tau)$, the error dynamics (2.10) satisfies the H_∞ performance indexes $\|e_f(t)\|_2 < \gamma_1 \|d(t)\|_2$ and $\|e_f(t)\|_2 < \gamma_2 \|\dot{f}(t)\|_2$.

Proof Constraint (2.11): Setting $(\bar{A} - \bar{L}\bar{C}) \rightarrow A$ and $\bar{P}_1 \rightarrow P$ in Lemma 2.1, one gets condition (2.11), from which it follows that the eigenvalues of $(\bar{A} - \bar{L}\bar{C})$ belong to $\mathcal{D}(\alpha, \tau)$.

Constraints (2.12a)–(2.12c): Setting $(\bar{A} - \bar{L}\bar{C}) \rightarrow A$, $(\bar{L}D_2 - \bar{D}_1) \rightarrow B$, $\bar{I}_r^T \rightarrow C$, $\bar{P}_2 \rightarrow P$, $\bar{Q}_1 \rightarrow Q$ and $\begin{bmatrix} \frac{1}{\gamma_1} I & 0 \\ 0 & -\gamma_1 I \end{bmatrix} \rightarrow \Pi$ in Lemma 2.2, one gets conditions (2.12a)–(2.12c) respectively according to different matrices \mathcal{E} , from which it follows that the error dynamics (2.10) satisfies the H_∞ performance index $\|e_f(t)\|_2 < \gamma_1 \|d(t)\|_2$.

Constraints (2.13a)–(2.13c): Setting $(\bar{A} - \bar{L}\bar{C}) \rightarrow A$, $-\bar{I}_r \rightarrow B$, $\bar{I}_r^T \rightarrow C$, $\bar{P}_3 \rightarrow P$, $\bar{Q}_2 \rightarrow Q$ and $\begin{bmatrix} \frac{1}{\gamma_2} I & 0 \\ 0 & -\gamma_2 I \end{bmatrix} \rightarrow \Pi$ in Lemma 2.2, one gets conditions (2.13a)–(2.13c) respectively according to different matrices \mathcal{E} , from which it follows that the error dynamics (2.10) satisfies the H_∞ performance index $\|e_f(t)\|_2 < \gamma_2 \|\dot{f}(t)\|_2$. \square

Remark 2.7 In Theorem 2.1, a multi-constrained FFEO design is proposed, where the purpose of introducing the regional pole constraint (2.11) is to control the fault estimation transient performance [32]. While based on robust H_∞ design idea [153] setting the performance indexes $\|e_f(t)\|_2 < \gamma_1 \|d(t)\|_2$ and $\|e_f(t)\|_2 < \gamma_2 \|\dot{f}(t)\|_2$, conditions (2.12a)–(2.12c) and (2.13a)–(2.13c) are used to improve the fault estimation performance by restraining the influence of terms $d(t)$ and $\dot{f}(t)$ with respect to the fault estimation error $e_f(t)$ as much as possible, where the influence of the terms $d(t)$ and $\dot{f}(t)$ on the estimation error $e_f(t)$ are respectively the ratios γ_1 and γ_2 .

Remark 2.8 Multi-constrained design problems may have no solution, due to the increased number of constraints. Furthermore, we consider finite-frequency specifications. Introducing three different Lyapunov matrices \bar{P}_1 , \bar{P}_2 , \bar{P}_3 and finite-

frequency specification matrices \bar{Q}_1, \bar{Q}_2 provide some degrees of design freedom. However, the coupling of the observer gain matrix \bar{L} among (2.11), (2.12a)–(2.12c), (2.13a)–(2.13c) and considering the entire frequency case results in a non-convex problem, which is inconvenient to calculate \bar{L} . A means to recover convexity is to require that all specifications are enforced by a common Lyapunov matrix, i.e. $\bar{P} = \bar{P}_1 = \bar{P}_2 = \bar{P}_3, \bar{P} > 0$ and $\bar{Q}_1 = \bar{Q}_2 = 0$, resulting in the following Corollary 2.1.

Corollary 2.1 *Let a circular region $\mathcal{D}(\alpha, \tau)$ and two prescribed H_∞ performance levels γ_1, γ_2 be given. If there exist a symmetric positive definite matrix $\bar{P} \in \mathbb{R}^{(n+r) \times (n+r)}$ and a matrix $\bar{Y} \in \mathbb{R}^{(n+r) \times p}$ such that the following conditions hold:*

$$\begin{bmatrix} -\bar{P} & \bar{P}\bar{A} - \bar{Y}\bar{C} - \alpha\bar{P} \\ * & -\tau^2\bar{P} \end{bmatrix} < 0, \quad (2.14)$$

$$\begin{bmatrix} \bar{P}\bar{A} + \bar{A}^T\bar{P} - \bar{Y}\bar{C} - \bar{Y}^T\bar{C}^T & \bar{Y}D_2 - \bar{P}\bar{D}_1 & \bar{I}_r \\ * & -\gamma_1 I_d & 0 \\ * & * & -\gamma_1 I_r \end{bmatrix} < 0, \quad (2.15)$$

and

$$\begin{bmatrix} \bar{P}\bar{A} + \bar{A}^T\bar{P} - \bar{Y}\bar{C} - \bar{Y}^T\bar{C}^T & -\bar{P}\bar{I}_r & \bar{I}_r \\ * & -\gamma_2 I_r & 0 \\ * & * & -\gamma_2 I_r \end{bmatrix} < 0, \quad (2.16)$$

then the eigenvalues of $(\bar{A} - \bar{L}\bar{C})$ belong to $\mathcal{D}(\alpha, \tau)$, the error dynamics (2.10) satisfies the H_∞ performance indexes $\|e_f(t)\|_2 < \gamma_1 \|d(t)\|_2$ and $\|e_f(t)\|_2 < \gamma_2 \|\dot{f}(t)\|_2$, and the FFEO gain matrix is given by $\bar{L} = \bar{P}^{-1}\bar{Y}$.

2.3.2 Fault Estimation with Less Conservatism

In order to guarantee both the convergence performance of the FFEO design, the multi-constrained design is involved, leading to non-convex problem and possible conservatism. In this subsection, we are based on Projection lemma, i.e., Lemma 2.3, less restrictive conclusions are obtained in terms of linear matrix inequalities (LMIs).

Theorem 2.2 *Let a circular region $\mathcal{D}(\alpha, \tau)$, two prescribed H_∞ performance levels γ_1, γ_2 and three scalars $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be given. If there exist three symmetric positive definite matrices $\bar{P}_1, \bar{Q}_1, \bar{Q}_2 \in \mathbb{R}^{(n+r) \times (n+r)}$, two symmetric matrices $\bar{P}_2, \bar{P}_3 \in \mathbb{R}^{(n+r) \times (n+r)}$ and two matrices $\bar{S} \in \mathbb{R}^{(n+r) \times (n+r)}, \bar{Y} \in \mathbb{R}^{(n+r) \times p}$ such that the following conditions hold:*

$$\begin{bmatrix} \bar{P}_1 - \bar{S} - \bar{S}^T & \bar{S}\bar{A} - \bar{Y}\bar{C} - \alpha\bar{S} - \varepsilon_1\bar{S}^T \\ * & -\tau^2\bar{P}_1 + \varepsilon_1(\bar{S}\bar{A} + \bar{A}^T\bar{S}^T - \bar{Y}\bar{C} - \bar{Y}^T\bar{C}^T - \alpha\bar{S} - \alpha\bar{S}^T) \end{bmatrix} < 0, \quad (2.17)$$

$$\begin{bmatrix} \phi_d & \bar{P}_2 - \bar{S}^T + \varepsilon_2(\bar{S}\bar{A} - \bar{Y}\bar{C}) & \varepsilon_2(\bar{Y}D_2 - \bar{S}\bar{D}_1) & 0 \\ * & \varpi_{dl}^2\bar{Q}_1 + \bar{S}\bar{A} + \bar{A}^T\bar{S}^T - \bar{Y}\bar{C} - \bar{C}^T\bar{Y}^T & \bar{Y}D_2 - \bar{S}\bar{D}_1 & \bar{I}_r \\ * & * & -\gamma_1 I_d & 0 \\ * & * & * & -\gamma_1 I_r \end{bmatrix} < 0$$

for the low-frequency disturbance $|\omega_d| \leq \varpi_{dl}$,

(2.18a)

$$\begin{bmatrix} -\bar{Q}_1 - \varepsilon_2(\bar{S} + \bar{S}^T) & \varphi_{d12} & \varepsilon_2(\bar{Y}D_2 - \bar{S}\bar{D}_1) & 0 \\ \bar{P}_2 - \bar{S} - j\varpi_{dc}\bar{Q}_1 + \varepsilon_2(\bar{S}\bar{A} - \bar{Y}\bar{C})^T & \varphi_{d22} & \bar{Y}D_2 - \bar{S}\bar{D}_1 & \bar{I}_r \\ * & * & -\gamma_1 I_d & 0 \\ * & * & * & -\gamma_1 I_r \end{bmatrix} < 0$$

for the middle-frequency disturbance $\varpi_{d1} \leq \omega_d \leq \varpi_{d2}$,

(2.18b)

$$\begin{bmatrix} \psi_d & \bar{P}_2 - \bar{S}^T + \varepsilon_2(\bar{S}\bar{A} - \bar{Y}\bar{C}) & \varepsilon_2(\bar{Y}D_2 - \bar{S}\bar{D}_1) & 0 \\ * & -\varpi_{dh}^2\bar{Q}_1 + \bar{S}\bar{A} + \bar{A}^T\bar{S}^T - \bar{Y}\bar{C} - \bar{C}^T\bar{Y}^T & \bar{Y}D_2 - \bar{S}\bar{D}_1 & \bar{I}_r \\ * & * & -\gamma_1 I_d & 0 \\ * & * & * & -\gamma_1 I_r \end{bmatrix} < 0$$

for the high-frequency disturbance $|\omega_d| \geq \varpi_{dh}$,

(2.18c)

where

$$\begin{aligned} \phi_d &= -\bar{Q}_1 - \varepsilon_2(\bar{S} + \bar{S}^T), \\ \varphi_{d12} &= \bar{P}_2 - \bar{S}^T + j\varpi_{dc}\bar{Q}_1 + \varepsilon_2(\bar{S}\bar{A} - \bar{Y}\bar{C}), \\ \varphi_{d22} &= -\varpi_{d1}\varpi_{d2}\bar{Q}_1 + \bar{S}\bar{A} + \bar{A}^T\bar{S}^T - \bar{Y}\bar{C} - \bar{C}^T\bar{Y}^T, \\ \psi_d &= \bar{Q}_1 - \varepsilon_2(\bar{S} + \bar{S}^T), \end{aligned}$$

$$\begin{bmatrix} -\bar{Q}_2 - \varepsilon_3(\bar{S} + \bar{S}^T) & \bar{P}_3 - \bar{S}^T + \varepsilon_3(\bar{S}\bar{A} - \bar{Y}\bar{C}) & -\varepsilon_3\bar{S}\bar{I}_r & 0 \\ * & \varpi_{fl}^2\bar{Q}_2 + \bar{S}\bar{A} + \bar{A}^T\bar{S}^T - \bar{Y}\bar{C} - \bar{C}^T\bar{Y}^T & -\bar{S}\bar{I}_r & \bar{I}_r \\ * & * & -\gamma_2 I_r & 0 \\ * & * & * & -\gamma_2 I_r \end{bmatrix} < 0$$

for the low-frequency fault $|\omega_f| \leq \varpi_{fl}$,

(2.19a)

$$\begin{bmatrix} \bar{P}_3 - \bar{S} - \varepsilon_3(\bar{S} + \bar{S}^T) & \varphi_{f12} - \varepsilon_3\bar{S}\bar{I}_r & 0 \\ \bar{P}_3 - \bar{S} - j\varpi_{fc}\bar{Q}_2 + \varepsilon_3(\bar{S}\bar{A} - \bar{Y}\bar{C})^T & \varphi_{f22} - \bar{S}\bar{I}_r & \bar{I}_r \\ * & * & -\gamma_2 I_r \\ * & * & * & -\gamma_2 I_r \end{bmatrix} < 0$$

for the middle-frequency fault $\varpi_{f1} \leq \omega_f \leq \varpi_{f2}$, (2.19b)

$$\begin{bmatrix} \bar{Q}_2 - \varepsilon_3(\bar{S} + \bar{S}^T) & \bar{P}_3 - \bar{S}^T + \varepsilon_3(\bar{S}\bar{A} - \bar{Y}\bar{C}) & -\varepsilon_3\bar{S}\bar{I}_r & 0 \\ * & -\varpi_{fh}^2\bar{Q}_2 + \bar{S}\bar{A} + \bar{A}^T\bar{S}^T - \bar{Y}\bar{C} - \bar{C}^T\bar{Y}^T & -\bar{S}\bar{I}_r & \bar{I}_r \\ * & * & -\gamma_2 I_r & 0 \\ * & * & * & -\gamma_2 I_r \end{bmatrix} < 0$$

for the high-frequency fault $|\omega_f| \geq \varpi_{fh}$, (2.19c)

where

$\varphi_{f12} = \bar{P}_2 - \bar{S}^T + j\varpi_{fc}\bar{Q} + \varepsilon_2(\bar{S}\bar{A} - \bar{Y}\bar{C})$,
 $\varphi_{f22} = -\varpi_{f1}\varpi_{f2}\bar{Q} + \bar{S}\bar{A} + \bar{A}^T\bar{S}^T - \bar{Y}\bar{C} - \bar{C}^T\bar{Y}^T$,
 then the eigenvalues of $(\bar{A} - \bar{L}\bar{C})$ belong to $\mathcal{D}(\alpha, \tau)$, the error dynamics (2.10) satisfies the H_∞ performance indexes $\|e_f(t)\|_2 < \gamma_1\|d(t)\|_2$ and $\|e_f(t)\|_2 < \gamma_2\|\dot{f}(t)\|_2$, and the FFEO gain matrix is given by $\bar{L} = \bar{S}^{-1}\bar{Y}$.

Proof By introducing the slack variable, different Lyapunov matrices \bar{P}_1 , \bar{P}_2 and \bar{P}_3 can be designed separately. On the basis of the proof of Theorem 2.1, here we need to prove the equivalence of (2.11) and (2.17), the equivalence of (2.12a)–(2.12c) and (2.18a)–(2.18c), and the equivalence of (2.13a)–(2.13c) and (2.19a)–(2.19c).

The equivalence of (2.11) and (2.17): Firstly, (2.17) can be rewritten as

$$\Sigma + \Gamma^T \Theta \Lambda + \Lambda^T \Theta^T \Gamma < 0 \quad (2.20)$$

where

$$\Sigma = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & -\tau^2 \bar{P}_1 \end{bmatrix}, \quad \Gamma = I, \quad \Theta = \begin{bmatrix} \bar{S} \\ \varepsilon_1 \bar{S} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -I_{n+r} & (\bar{A} - \bar{L}\bar{C} - \alpha I_{n+r}) \end{bmatrix}.$$

It is easy to obtain matrix $N_\Lambda = \begin{bmatrix} (\bar{A} - \bar{L}\bar{C} - \alpha I_{n+r}) \\ I_{n+r} \end{bmatrix}$. Based on Lemma 2.3, (2.17) is equivalent to $N_\Lambda^T \Sigma N_\Lambda < 0$, i.e.,

$$\begin{aligned} N_\Lambda^T \Sigma N_\Lambda &= [(\bar{A} - \bar{L}\bar{C} - \alpha I_{n+r})^T \ I_{n+r}] \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & -\tau^2 \bar{P}_1 \end{bmatrix} \begin{bmatrix} (\bar{A} - \bar{L}\bar{C} - \alpha I_{n+r}) \\ I_{n+r} \end{bmatrix} \\ &= [(\bar{A} - \bar{L}\bar{C} - \alpha I)^T \bar{P}_1 - \tau^2 \bar{P}_1] \begin{bmatrix} (\bar{A} - \bar{L}\bar{C} - \alpha I) \\ I_{n+r} \end{bmatrix} \\ &= (\bar{A} - \bar{L}\bar{C} - \alpha I_{n+r})^T \bar{P}_1 (\bar{A} - \bar{L}\bar{C} - \alpha I_{n+r}) - \tau^2 \bar{P}_1 < 0 \end{aligned}$$

which is equivalent to (2.11) by the Schur complement.

The equivalence of (2.12a)–(2.12c) and (2.18a)–(2.18c): Using the Schur complement, (2.18a) can be expressed as

$$\begin{bmatrix} \phi_d & \bar{P}_2 - \bar{S}^T + \varepsilon_2(\bar{S}\bar{A} - \bar{Y}\bar{C}) & \varepsilon_2\bar{S}(\bar{L}D_2 - \bar{D}_1) \\ * & \varpi_{dl}^2\bar{Q}_1 + \bar{S}\bar{A} + \bar{A}^T\bar{S}^T - \bar{Y}\bar{C} - \bar{C}^T\bar{Y}^T + \frac{1}{\gamma_1}\bar{I}_r\bar{I}_r^T & \bar{S}(\bar{L}D_2 - \bar{D}_1) \\ * & * & -\gamma_1 I_d \end{bmatrix} < 0 \quad (2.21)$$

which can be expressed as (2.20), where

$$\Sigma = \begin{bmatrix} -\bar{Q}_1 & \bar{P}_2 & 0 \\ \bar{P}_2 & \varpi_{dl}^2\bar{Q}_1 + \frac{1}{\gamma_1}\bar{I}_r\bar{I}_r^T & 0 \\ 0 & 0 & -\gamma_1 I_d \end{bmatrix},$$

$$\Gamma = I, \quad \Theta = \begin{bmatrix} \varepsilon_2\bar{S} \\ \bar{S} \\ 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -I & (\bar{A} - \bar{L}\bar{C}) & (\bar{L}D_2 - \bar{D}_1) \end{bmatrix}.$$

It is easy to obtain matrix $N_\Lambda = \begin{bmatrix} (\bar{A} - \bar{L}\bar{C}) & (\bar{L}D_2 - \bar{D}_1) \\ I & 0 \\ 0 & I \end{bmatrix}$. Based on Lemma

2.3, (2.18a) is equivalent to $N_\Lambda^T \Sigma N_\Lambda < 0$, i.e.,

$$\begin{aligned} & N_\Lambda^T \Sigma N_\Lambda \\ &= \begin{bmatrix} (\bar{A} - \bar{L}\bar{C})^T & I & 0 \\ (\bar{L}D_2 - \bar{D}_1)^T & 0 & I \end{bmatrix} \begin{bmatrix} -\bar{Q}_1 & \bar{P}_2 & 0 \\ \bar{P}_2 & \varpi_{dl}^2\bar{Q}_1 + \frac{1}{\gamma_1}\bar{I}_r\bar{I}_r^T & 0 \\ 0 & 0 & -\gamma_1 I_d \end{bmatrix} \times \\ & \quad \begin{bmatrix} (\bar{A} - \bar{L}\bar{C}) & (\bar{L}D_2 - \bar{D}_1) \\ I & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} -(\bar{A} - \bar{L}\bar{C})^T \bar{Q}_1 + \bar{P}_2 & (\bar{A} - \bar{L}\bar{C})^T \bar{P}_2 + \varpi_{dl}^2\bar{Q}_1 + \frac{1}{\gamma_1}\bar{I}_r\bar{I}_r^T & 0 \\ -(\bar{L}D_2 - \bar{D}_1)^T \bar{Q}_1 & (\bar{L}D_2 - \bar{D}_1)^T \bar{P}_2 & -\gamma_1 I_d \end{bmatrix} \times \\ & \quad \begin{bmatrix} (\bar{A} - \bar{L}\bar{C}) & (\bar{L}D_2 - \bar{D}_1) \\ I & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} -(\bar{A} - \bar{L}\bar{C})^T \bar{Q}_1 (\bar{A} - \bar{L}\bar{C}) + \varpi_{dl}^2\bar{Q}_1 + \frac{1}{\gamma_1}\bar{I}_r\bar{I}_r^T + \bar{P}_2(\bar{A} - \bar{L}\bar{C}) + (\bar{A} - \bar{L}\bar{C})^T \bar{P}_2 \\ * \\ -(\bar{A} - \bar{L}\bar{C})^T \bar{Q}_1 (\bar{L}D_2 - \bar{D}_1) + \bar{P}_2(\bar{L}D_2 - \bar{D}_1) \\ -(\bar{L}D_2 - \bar{D}_1)^T \bar{Q}_1 (\bar{L}D_2 - \bar{D}_1) - \gamma_1 I_d \end{bmatrix}, \end{aligned}$$

which is equivalent to (2.12a) by the Schur complement.

Similarly, conditions (2.18b) and (2.18c) can respectively obtained by choosing

$$\Sigma = \begin{bmatrix} -\bar{Q}_1 & \bar{P}_2 + j\varpi_{dc}\bar{Q}_1 & 0 \\ \bar{P}_2 - j\varpi_{dc}\bar{Q}_1 - \varpi_{d1}\varpi_{d2}\bar{Q}_1 + \frac{1}{\gamma_1}\bar{I}_r\bar{I}_r^T & 0 \\ 0 & 0 & -\gamma_1 I_d \end{bmatrix}$$

for the equivalence of (2.12b) and (2.18b), and

$$\Sigma = \begin{bmatrix} \bar{Q}_1 & \bar{P}_2 & 0 \\ \bar{P}_2 - \varpi_{dh}^2\bar{Q}_1 + \frac{1}{\gamma_1}\bar{I}_r\bar{I}_r^T & 0 \\ 0 & 0 & -\gamma_1 I_d \end{bmatrix}$$

for the equivalence of (2.12c) and (2.18c).

The equivalence of (2.13a)–(2.13c) and (2.19a)–(2.19c): This proof can refer to that of the equivalence of (2.12a)–(2.12c) and (2.18a)–(2.18c), thus is omitted here. \square

Remark 2.9 From Theorem 2.2, we can see that the conditions are given in terms of LMIs, which is convenient to calculate the FFEO gain matrix. Three scalars ε_1 , ε_2 and ε_3 are given in advance, and an optimal FFEO is obtained such that $(\gamma_1 + \gamma_2)$ are minimal, which boils down to solve the semidefinite program [104]:

minimize $(\gamma_1 + \gamma_2)$ subject to (2.17), (2.18a) – (2.18c) and (2.19a) – (2.19c).

Remark 2.10 In Theorem 2.2, based on the fact that the frequency domain is specified in advance, the proposed method can get a better attenuation performance because information of the disturbance and fault is fully used in the design process, whose advantages will be shown in Sect. 2.4 “Simulation Results”. Note that for conditions (2.18a)–(2.18c) related to the disturbance, since the disturbance is limited into a specific frequency domain (low, middle or high frequency domain) in advance, we only need to choose one of the three conditions, instead of all of them. And the case of conditions (2.19a)–(2.19c) is also similar.

Remark 2.11 On the basis of Theorem 2.1, an improved FFEO design is obtained based on Projection lemma, i.e., Theorem 2.2. Due to the conjunction of the parameters ε_1 , ε_2 , ε_3 and matrix \bar{S} , conditions of Theorem 2.2 are nonconvex. Therefore, the parameters ε_1 , ε_2 and ε_3 in Theorem 2.2 needs to be given in advance such that conditions of Theorem 2.2 becomes a convex optimization problem. It should be noted that the optimal value $(\gamma_1 + \gamma_2)$ in Theorem 2.2 does not monotonously increase or decrease with ε_1 , ε_2 and ε_3 . A global search of the three parameters is needed with the

purpose of looking for the minimum $(\gamma_1 + \gamma_2)$. But note that the computation complexity would not increase much, we generally need to look for the minimum value $(\gamma_1 + \gamma_2)$ within a small region of the three parameters and the computation burden would not increase much. Moreover, it is possible to obtain a smaller $(\gamma_1 + \gamma_2)$ by solving conditions of Theorem 2.2.

Corollary 2.2 *Let a circular region $\mathcal{D}(\alpha, \tau)$, two prescribed H_∞ performance levels γ_1, γ_2 and three scalars $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be given. If there exist three symmetric positive definite matrices $\bar{P}_1, \bar{P}_2, \bar{P}_3 \in \mathbb{R}^{(n+r) \times (n+r)}$, and two matrices $\bar{S} \in \mathbb{R}^{(n+r) \times (n+r)}$, $\bar{Y} \in \mathbb{R}^{(n+r) \times p}$ such that the following conditions hold:*

$$\begin{bmatrix} \bar{P}_1 - \bar{S} - \bar{S}^T & \bar{S}\bar{A} - \bar{Y}\bar{C} - \alpha\bar{S} - \varepsilon_1\bar{S}^T \\ * & -\tau^2\bar{P}_1 + \varepsilon_1(\bar{S}\bar{A} + \bar{A}^T\bar{S}^T - \bar{Y}\bar{C} - \bar{Y}^T\bar{C}^T - \alpha\bar{S} - \alpha\bar{S}^T) \end{bmatrix} < 0, \quad (2.22)$$

$$\begin{bmatrix} -\varepsilon_2(\bar{S} + \bar{S}^T) & \bar{P}_2 - \bar{S}^T + \varepsilon_2(\bar{S}\bar{A} - \bar{Y}\bar{C}) & \varepsilon_2(\bar{Y}D_2 - \bar{S}\bar{D}_1) & 0 \\ * & \bar{S}\bar{A} + \bar{A}^T\bar{S}^T - \bar{Y}\bar{C} - \bar{C}^T\bar{Y}^T & \bar{Y}D_2 - \bar{S}\bar{D}_1 & \bar{I}_r \\ * & * & -\gamma_1 I_d & 0 \\ * & * & * & -\gamma_1 I_r \end{bmatrix} < 0, \quad (2.23)$$

and

$$\begin{bmatrix} -\varepsilon_3(\bar{S} + \bar{S}^T) & \bar{P}_3 - \bar{S}^T + \varepsilon_3(\bar{S}\bar{A} - \bar{Y}\bar{C}) & -\varepsilon_3\bar{S}\bar{I}_r & 0 \\ * & \bar{S}\bar{A} + \bar{A}^T\bar{S}^T - \bar{Y}\bar{C} - \bar{C}^T\bar{Y}^T & -\bar{S}\bar{I}_r & \bar{I}_r \\ * & * & -\gamma_2 I_r & 0 \\ * & * & * & -\gamma_2 I_r \end{bmatrix} < 0, \quad (2.24)$$

then the eigenvalues of $(\bar{A} - \bar{L}\bar{C})$ belong to $\mathcal{D}(\alpha, \tau)$, the error dynamics (2.10) satisfies the H_∞ performance indexes $\|e_f(t)\|_2 < \gamma_1 \|d(t)\|_2$ and $\|e_f(t)\|_2 < \gamma_2 \|\dot{f}(t)\|_2$, and the FFEO gain matrix is given by $\bar{L} = \bar{S}^{-1}\bar{Y}$.

Remark 2.12 When considering the entire frequency case, we have obtained Corollary 2.2, which is similar to results in [145]. We can see that the finite-frequency results can cover the case of the entire frequency domain, and Corollary 2.2 is a special case of Theorem 2.2. Since we introduce slack variables, Corollary 2.2 is less conservative than Corollary 2.1.

2.4 Simulation Results

In this section, the proposed design is illustrated for a linearized dynamic model of a vertical takeoff and landing aircraft in the vertical plane as follows [84]:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

where $x(t) = [V_h, V_v, q, \theta]$ and $u(t) = [\delta_c, \delta_l]$. The states and inputs are horizontal velocity V_h , vertical velocity V_v , pitch rate q , and pitch angle θ ; collective pitch control δ_c and longitudinal cyclic pitch control δ_l . The model parameters are given as follows

$$A = \begin{bmatrix} -0.0336 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.7070 & 1.4200 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ 5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We consider actuator faults. Such faults usually occur in the input channel, so we assume $E = B$. It is assumed that the disturbance distribution matrices are $D_1 = 0.01 [1 \ 1 \ 1 \ 1]^T$ and $D_2 = 0.1 [1 \ 1]^T$. The purpose of presenting matrices D_1 and D_2 that span on the entire state space is utilized to test the robustness of the proposed method. Obviously, because of $\text{rank}(CE) = 1 < r$, the conventional adaptive and sliding mode observers can not be used for this system [7, 44, 45, 93, 137]. Since the system is open-loop unstable, the fault estimation filter design is also not suitable [29, 73, 77]. However, by constructing augmented matrices \bar{A}, \bar{C} , it is easy to verify that the pair (\bar{A}, \bar{C}) is observable, so the proposed FFEO is feasible for such system.

It is assumed that the external disturbance (noises) is high-frequency domain and is greater than $\varpi_{dh} = 200$ rad/s. And the derivative of actuator faults is in low-frequency domain and is less than $\varpi_{fl} = 5$ rad/s. Solving the conditions in Theorem 2.2 with the regional pole constraint $\mathcal{D}(-8, 7.7)$ allows to take into account both the estimation convergence speed and the level of disturbance attenuation. With $\varepsilon_1 = 0$, $\varepsilon_2 = 0.1$ and $\varepsilon_3 = 0.1$, one obtains the optimal value 1.2891 (where $\gamma_1 = 0.8871$, $\gamma_2 = 0.4020$) with

$$\begin{aligned}
\bar{P}_1 &= \begin{bmatrix} 19.4350 & 0.9774 & 1.4434 & 1.6928 & -3.9689 & 0.0009 \\ 0.9774 & 0.1296 & 0.1169 & 0.0137 & -0.1688 & 0.3405 \\ 1.4434 & 0.1169 & 0.4804 & -0.5800 & 0.0553 & -0.1593 \\ 1.6928 & 0.0137 & -0.5800 & 3.5360 & -0.3575 & -0.2104 \\ -3.9689 & -0.1688 & 0.0553 & -0.3575 & 1.7770 & -0.3806 \\ 0.0009 & 0.3405 & -0.1593 & -0.2104 & -0.3806 & 1.963 \end{bmatrix}, \\
\bar{P}_2 &= \begin{bmatrix} 21.7178 & 1.1290 & 1.5841 & 2.5820 & -4.2068 & -0.0488 \\ 1.1290 & 0.1431 & 0.1309 & 0.0671 & -0.1733 & 0.3548 \\ 1.5841 & 0.1309 & 0.5330 & -0.6383 & 0.0789 & -0.1180 \\ 2.5820 & 0.0671 & -0.6383 & 4.2778 & -0.4712 & -0.3706 \\ -4.2068 & -0.1733 & 0.0789 & -0.4712 & 1.9041 & -0.3502 \\ -0.0488 & 0.3548 & -0.1180 & -0.3706 & -0.3502 & 2.1322 \end{bmatrix}, \\
\bar{P}_3 &= \begin{bmatrix} 17.5666 & 1.0353 & 1.5734 & 2.7708 & -2.5146 & -0.4260 \\ 1.0353 & 0.1105 & 0.1301 & 0.1523 & -0.0979 & 0.1568 \\ 1.5734 & 0.1301 & 0.5013 & -0.5945 & 0.0750 & -0.1078 \\ 2.7708 & 0.1523 & -0.5945 & 4.4707 & -0.5502 & -0.1919 \\ -2.5146 & -0.0979 & 0.0750 & -0.5502 & 1.1814 & -0.1117 \\ -0.4260 & 0.1568 & -0.1078 & -0.1919 & -0.1117 & 1.2139 \end{bmatrix}, \\
\bar{Q}_1 &= 10^{-3} \begin{bmatrix} 0.0787 & 0.0046 & -0.0046 & 0.0452 & -0.0181 & 0.0218 \\ 0.0046 & 0.0010 & -0.0009 & 0.0031 & -0.0021 & 0.0060 \\ -0.0046 & -0.0009 & 0.0154 & -0.0425 & 0.0181 & -0.0112 \\ 0.0452 & 0.0031 & -0.0425 & 0.1611 & -0.0488 & 0.0264 \\ -0.0181 & -0.0021 & 0.0181 & -0.0488 & 0.0264 & -0.0200 \\ 0.0218 & 0.0060 & -0.0112 & 0.0264 & -0.0200 & 0.0401 \end{bmatrix}, \\
\bar{Q}_2 &= \begin{bmatrix} 2.1413 & 0.1007 & 0.1772 & -0.0262 & -0.5571 & -0.0385 \\ 0.1007 & 0.0162 & 0.0141 & -0.0074 & -0.0205 & 0.0464 \\ 0.1772 & 0.0141 & 0.0441 & -0.0524 & -0.0137 & -0.0060 \\ -0.0262 & -0.0074 & -0.0524 & 0.3093 & -0.0057 & -0.0622 \\ -0.5571 & -0.0205 & -0.0137 & -0.0057 & 0.2444 & -0.0175 \\ -0.0385 & 0.0464 & -0.0060 & -0.0622 & -0.0175 & 0.2715 \end{bmatrix}, \\
\bar{S} &= \begin{bmatrix} 18.6019 & 0.8900 & 1.4060 & 1.6787 & -4.1003 & -0.5065 \\ 0.9716 & 0.1245 & 0.1295 & -0.0108 & -0.1587 & 0.3230 \\ 1.1097 & 0.0691 & 0.3610 & -0.6485 & 0.0251 & -0.1066 \\ 1.2904 & 0.1014 & -0.2615 & 3.2773 & -0.1135 & -0.3303 \\ -3.8716 & -0.1593 & -0.0582 & -0.4848 & 1.6298 & -0.1150 \\ -0.4341 & 0.2767 & -0.0449 & -0.5826 & -0.2054 & 1.8617 \end{bmatrix}, \\
\bar{Y} &= \begin{bmatrix} 6.4192 & -0.7145 \\ 0.8454 & -0.2260 \\ 1.1297 & -0.1191 \\ 0.8352 & 7.9505 \\ 1.1442 & 0.1446 \\ 4.3353 & -0.4369 \end{bmatrix}.
\end{aligned}$$

Note that, in order to avoid the observer gain is too big, an additional constraint $\|\bar{Y}\| < \delta$ (where $\delta = 8$) has been added when solving the conditions of Theorem 2.2. The constraint can be written as

$$\begin{bmatrix} -\delta I & \bar{Y} \\ * & -\delta I \end{bmatrix} < 0.$$

To solve the above problems we used CVX, a package for specifying and solving convex programs [35, 36].

The FFEO gain matrix is

$$\bar{L} = \bar{S}^{-1} \bar{Y} = \begin{bmatrix} L \\ F \end{bmatrix} = \begin{bmatrix} 6.7855 & 1.7412 \\ -152.5365 & -118.9164 \\ 35.3591 & 43.6843 \\ 8.5188 & 11.1164 \\ 7.8963 & -0.9582 \\ 30.9680 & 22.2690 \end{bmatrix}.$$

where

$$L = \begin{bmatrix} 6.7855 & 1.7412 \\ -152.5365 & -118.9164 \\ 35.3591 & 43.6843 \\ 8.5188 & 11.1164 \end{bmatrix}, \quad F = \begin{bmatrix} 7.8963 & -0.9582 \\ 30.9680 & 22.2690 \end{bmatrix}.$$

The eigenvalues of $(\bar{A} - \bar{L}\bar{C})$ are -10.3869 , -3.5291 , $-1.9824 \pm 2.3251i$, -1.4280 , -0.3436 , and eigenvalue distribution of $(\bar{A} - \bar{L}\bar{C})$ is shown in Fig. 2.1.

Under the same regional pole constraint, it can be checked that Corollaries 2.1 and 2.2 respectively give the optimal values 1.9841 (where $\gamma_1 = 1.0620$, $\gamma_2 = 0.9221$) and 1.6782 (where $\gamma_1 = 0.9497$, $\gamma_2 = 0.7285$) which are larger than 1.2891, thus illustrates the fact that Theorem 2.2 is less conservative.

For simulation, we illustrate the results with two kinds of faults in the low frequency. The first fault $f(t) = [f_1(t) \ f_2(t)]^T$ is assumed to be

$$f_1(t) = 0, \quad f_2(t) = \begin{cases} 0 & 0s \leq t < 30s \\ 7(1 - e^{-0.2(t-30)}) & 30s \leq t \leq 100s \end{cases}.$$

The second fault, i.e. a time-varying case in the low frequency, is simulated as

$$f_1(t) = 0, \quad f_2(t) = \begin{cases} 0 & 0s \leq t < 30s \\ 5 \sin(0.05\pi(t - 30)) & 30s \leq t < 100s \end{cases}.$$

Simulation result of fault estimation for two types of faults are respectively shown in Figs. 2.2 and 2.3.

Fig. 2.1 $(\bar{A} - \bar{L}\bar{C})$ pole distribution

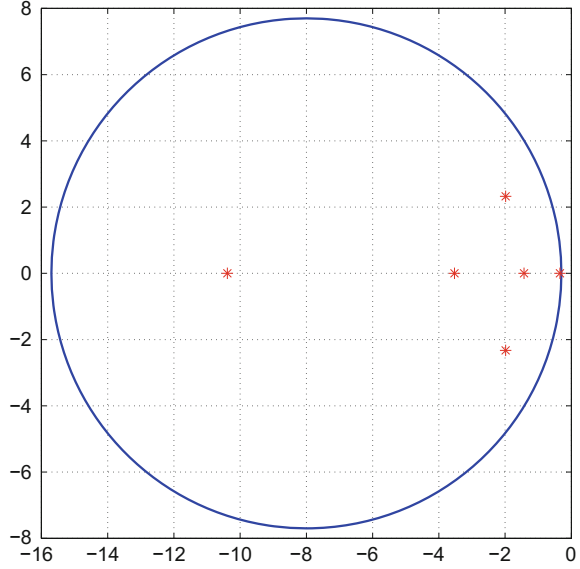
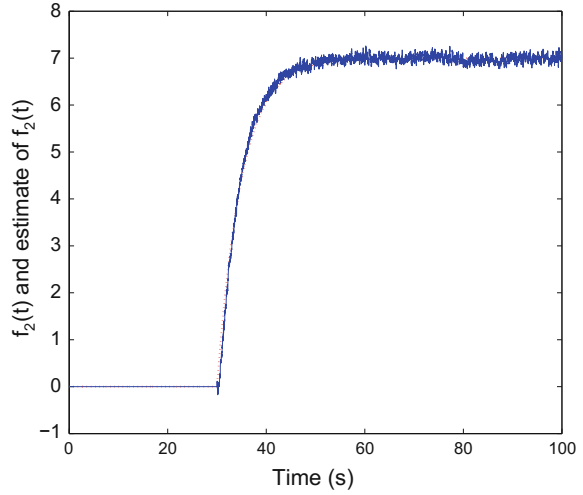
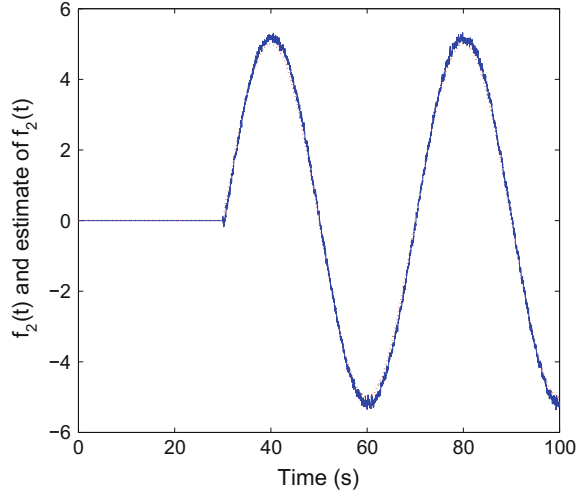


Fig. 2.2 For the first fault, fault $f_2(t)$ (dotted line) and its estimate $\hat{f}_2(t)$ (solid line)



From simulation results, we can see that in spite of the fact that the adaptive and sliding mode observers are not applicable to such system, the proposed FFEO with finite-frequency specifications not only provides a smaller H_∞ performance index, but also achieves quite accurate estimation for constant and time-varying faults.

Fig. 2.3 For the second fault, fault $f_2(t)$ (dotted line) and its estimate $\hat{f}_2(t)$ (solid line)



2.5 Conclusions

In this chapter, a multi-constrained FFEO design in finite-frequency domain has been proposed. For the disturbance and faults occurred in low/middle/high-frequency domains, we present a multi-constrained FFEO in finite-frequency domain to reduce conservatism generated by the entire frequency domain. Furthermore, the conservatism in the design process is further reduced by introducing slack variables. Based on the projection lemma, less conservative FFEO design is obtained in terms of LMIs to calculate the observer parameters conveniently. Finally, simulation results have been given to illustrate the effectiveness of the proposed approach. How to relax the constraint of faults (i.e., Assumption 2.1) and extend the proposed method to nonlinear systems [75, 92, 112, 147] are challenging and interesting issues, which will be investigated in our future work.

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