Preface

This book has been written to be an agile but accurate introduction to the dynamics over network usually known as *averaging* or *consensus dynamics*. Mathematically, the simplest averaging dynamics is a linear dynamical system driven by a fixed stochastic matrix, whose zero pattern is determined by the network topology. The main goal of this book is proposing a unified and self-contained theoretical framework that is suitable to analyze not only this simple instance, but also several related dynamical models that feature time-variance, randomness, and heterogeneities. Even if most emphasis will be put on the methodological and general aspects of the subject, we will also treat applications to inferential problems in sensor networks, rendezvous of mobile robots, and opinion dynamics in social networks.

This book originated from our lectures for the graduate courses *Control of/over networks* and *Dynamics over networks*, taught at the Politecnico di Torino since 2011. The treatment is completely self-contained, with only standard linear algebra, calculus, and probability as prerequisites. For this reason and for the abundance of exercises, we hope that this text can be effectively used as a resource for teaching and self-study at graduate or advanced undergraduate level.

This preface has for us a threefold goal: concisely introducing the topic of this book, explaining our perspective in writing it, and outlining its contents.

Averaging Dynamics and Multi-agent Systems

Multi-agent systems constitute one of the fundamental paradigms of science and technology in the present century. Their key feature is that complex dynamical evolutions originate from the interactions of a large number of simple units. Not only such collective behaviors are evident in biological and social systems, but the digital revolution and the miniaturization in electronics have also made possible the creation of man-made complex architectures of interconnected devices, including computers, sensors, and cameras. Moreover, the creation of the Internet has enabled totally new forms of social and economic aggregation. These technological and social evolutions have strongly pushed researchers toward a deeper and more systematic study of multi-agent dynamical systems.

The mathematical structure of multi-agent systems is that of a graph (typically of large scale), where the nodes are agents or units endowed with some (typically simple) dynamical system. These dynamical systems are coupled through the edges of the graph. Complexity is thus the outcome of the topology and the nature of the interconnections, which may often be of stochastic nature. The typical mathematical issue is understanding how the topology of the graph affects the transient and asymptotic behavior of the intercoupled dynamics, relating graph-theoretical concepts (such as diameter, degrees, connectivity, presence of bottlenecks) to the dynamic behavior in a quantitative way.

In the applications, these dynamical systems can represent a multitude of different situations. For instance, the graph can be an infrastructure network (e.g., sensor or computer network) and the dynamics be an algorithm designed to fuse information and eventually reach a preassigned goal, such as estimation or synchronization, through cooperation. In other situations, the network may represent relationships between socioeconomic or financial units (people, companies) and the state variables represent opinions or other economic indicators. Finally, the units may be physically positioned in the space (such as animals, pedestrians, or vehicles) and have as state variables their positions and velocities: The dynamics then represent some collective motion, such as platooning for automated vehicles, formation flight for drones, or flocking for animals.

Averaging dynamics is one of the most popular and maybe the simplest multi-agent dynamics. It may be convenient to introduce it by the language of social sciences. Suppose that a number of individuals possess some information represented by a real number: For instance, such numbers can represent their opinions on a certain matter. The individuals interact and change their opinions by averaging them with the opinions of other individuals to which they are connected. Under certain assumptions, these updates will lead all the community to converge to a common opinion that depends on the initial opinions of all individuals. Because of its intrinsic push toward consensus, the averaging dynamics is also known as consensus dynamics.

Aims and Scope

While teaching this topic and preparing this manuscript, we have tried to follow four guidelines.

(i) We concentrate on *linear discrete-time averaging dynamics*, which we identify as the core theoretical issue. We present the fundamental results on averaging dynamics and a unified viewpoint of various models and results scattered in the literature. Starting from the classical evolution of the powers of a fixed stochastic matrix, we then consider more general products of a sequence of stochastic matrices.

(ii) We keep the discourse *simple and self-contained*. The necessary theory is constructed in this book without, in particular, assuming any knowledge of Markov chains or of Perron–Frobenius theory. All convergence results presented are proposed as derivations of two different principles. The first one is a "contraction" principle, prescribing that the convex hull of the states shrinks as time elapses. This principle applies to time-invariant dynamics and to time-varying dynamics where information is able to flow between any two units within a bounded time. The second principle, instead, postulates a "reciprocity" in the network dynamics, namely that if information can flow from a set of units to another, also the converse must be possible. This second principle also permits to prove convergence in settings where consensus is not necessarily reached.

(iii) We constantly aim to relate the properties of the information flow (essentially determined by a suitable graph) with the properties of dynamics. Dynamical properties of interest include not only mere convergence but also "performance," broadly intended. We indeed consider different notions of performance, including the rate of convergence, the accuracy in approximating the average of the initial states, and the robustness against noise and communication errors. In all these cases, the *relation between graph and performance* is made explicit by the spectral analysis of the update matrix.

(iv) We develop our approach in the perspective of *large-scale networks*: Even though our theory is valid for networks of any size, we pay special attention to how dynamical properties depend on the size of the network. Concretely, this leads us to specialize our results to specific families of graphs (for instance, grids) and to take limits where the number of nodes grows to infinity.

Contents

We now outline the contents of this book, even though the reader will be provided with more detailed summaries at the beginning of each of its five chapters.

The initial Chap. 1 presents the graph-theoretical background that is essential to our work, with particular emphasis on connectivity properties and on algebraic graph theory. We also define several graphs that are used as leading examples in the rest of this book.

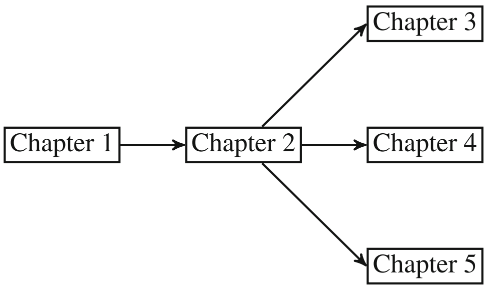
Chapter 2 is the core of this book and is entirely devoted to time-invariant averaging dynamics. Actually, we jointly study the convergence of averaging dynamics and the properties of stochastic matrices, by means of a basic and fundamental contraction principle. We also pay attention to reversible matrices and present the classical Cheeger bound to estimate the second eigenvalue.

In Chap. 3, we study time-varying dynamics; that is, we allow the graph to change with time. We present an extension to the contraction principle already used in Chap. 2 and then propose a different convergence analysis based on a principle of reciprocity in the flow of information. A number of applications of these general results are presented including models where time-variance is due to a random mechanism, e.g., gossiping, and models exhibiting nonconsensus behaviors, e.g., bounded-confidence opinion dynamics in social networks.

Chapter 4 is devoted to a finer analysis of the time-invariant averaging dynamics. We define various performance metrics that quantify, for instance, convergence speed and robustness with respect to noise. Performance metrics are then evaluated as functions of the eigenvalues of the graph. We also present the application to distributed inferential estimation.

Finally, in Chap. 5, we develop the theory of electrical networks of resistors, which has important connections with reversible stochastic matrices. Using just linear algebra techniques and no probability, we present some basic concepts like that of voltage, Green matrix, harmonic extension, and effective resistance. This machinery is then used to address two different problems: (i) estimation from relative measurements and (ii) averaging dynamics in the presence of stubborn agents.

The main dependences between these chapters are straightforward to describe by a graph.



All chapters include extended examples and are concluded by a selection of Exercises (about 100 in total) and by some Bibliographical Notes, which have no ambition to be exhaustive but simply to provide some context and propose some additional readings.

Last but not least, we would like to acknowledge that our perspective on these topics has been shaped by fruitful collaborations with students and colleagues: A nonexhaustive list includes Ruggero Carli, Giacomo Como, Jean-Charles Delvenne, Federica Garin, Julien Hendrickx, Chiara Ravazzi, Wilbert Samuel Rossi, and Sandro Zampieri.

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