Chapter 2
Theoretical Competency: For Your Practical Work

After teaching the modeling task, I know why we learned the theoretical background about the perspectives and cycles: you need it to get the whole picture.

Chris, in-service teacher.

This chapter includes a necessary and important theoretical background for your practical work in school. Furthermore several aspects based on theoretical conceptualization and empirical research within the international mathematical modeling discussion in the last decade are discussed in the chapter.

2.1 What Does Mathematical Modeling Mean?: Goals and Aims

Mathematical Modeling became a well-established research field within mathematics education over the last 40–50 years. Particularly over the past 10 years, the research on mathematical modeling has shown a strong increase in the number of empirical studies conducted: both qualitative and quantitative studies. The results of these empirical studies from all over the world have given strong insights, and partially answer how mathematical modeling can be taught and learned effectively. Looking at the international debate, one can find different notions of mathematical modeling beneath others because of various educational approaches in several countries (Borromeo Ferri 2014). However there is a strong consensus that mathematical modeling can be described as an activity involves *transitioning back and forth between reality and mathematics*, because this is an essential feature of mathematical modeling (Fig. 2.1).

Mathematical modeling does not mean having a "pseudo realistic problem", in which all data are given, or you only have to exercise algorithms. Mathematical modeling is a challenge for students on several levels, because the students work on questions out of the reality, to which they have to apply mathematics. As you solve the following modeling task, "Bale of straw" (Blum and Leiß 2007b), you will be

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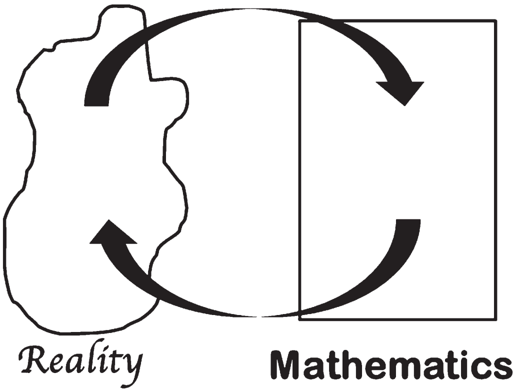

Fig. 2.1 Mathematical modeling: making transitions between reality and mathematics

able to better understand the ways in which a modeling problem can be a challenge, and the process of transitioning between reality and mathematics will become clear.

Please work on this problem first, before you read further. Take notes of possible difficulties and questions your students could raise when they work on the problem.

- Do you have one or more than one result?
- What kind of mathematics did you use to model the real world situation?
- How do you know if your result is correct or not?

For getting a better understanding what mathematical modeling means, a deeper view into the different phases of the modeling cycle is needed. In the modeling problem "Bale of straw" the process is illustrated according to phases of the "Diagnostic modeling cycle" (Borromeo Ferri 2007) used in teacher education (more details in Sect. 2.2) (Fig. 2.2).

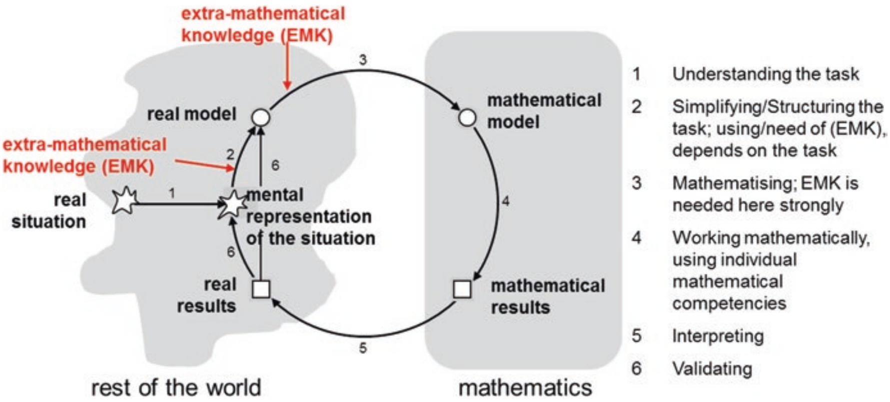

Fig. 2.2 Mathematical modeling cycle from a cognitive perspective (Borromeo Ferri 2007)

2.1.1 *Real Situation (RS)*

The real situation of the "Bale of straw" problem is clear through the picture. Therefore the problem, taken from reality (represented through pictures or text), is called the real situation.

2.1.2 *Mental Representation of the Situation (MRS)*

The mental representation of the situation is very individually and consists of two parts:

1. Associations of the individual, because of the given real problem: If you think back at your solving process perhaps you thought of things associated with summer, referring to your own experiences with straw bales etc.
2. Understanding of the problem: The individuals have to understand the task (that they have to determine the height of the straw bale mountain).

2.1.3 *Real Model (RM)*

To get a real model you have to simplify and structure your mental picture, and thus specify the real situation further. You can think of circles instead of straw bales and perhaps you can draw them as a real model. If you think about the woman sitting on the straw bales and take into account that you need the woman for solving the problem, then she can be simplified as a line. It is important to simplify the real situation and make assumptions; otherwise it is difficult to use or to "find" the mathematics that help you to solve the problem. One assumption is that the height of the woman is approximately 1.7 m. This can be used to estimate the diameter of a straw bale.

2.1.4 Extra-Mathematical Knowledge (EMK)

As you recognized, a lot of relevant data are not given in the task (e.g. the height of the woman or the diameter of a straw bale), therefore extra-mathematical knowledge is needed. The level of extra-mathematical knowledge in this task is not very high compared to other modeling problems, but this always depends on the personal experience one has with the given real context. You can assume that a lot of pupils (living in the city) have seen these straw bales from a distance while driving in the car, but sitting on or touching them is probably not common. Perhaps it is easier to estimate the height of a woman than the height of a straw bale. Both aspects can be used as a step to build a real model and then a mathematical model. You see that modeling problems require everyday knowledge. And if pupils do not know the height of the woman, then they have to learn how to get this information. In this case they can, for example think, about their Mom – how tall is she? Doing this they exercise their competency in estimating and measurement. Using, applying and generating extra-mathematical knowledge makes modeling problems very interesting for pupils. The connection between reality and mathematics and the usefulness of mathematics become apparent.

2.1.5 Mathematical Model (MM)

Because of its complexity, it is necessary to have more than one mathematical model in order to reach a solution of a real problem. For the problem "Bale of straw", two possible mathematical models are:

- Model 1 (Multiple addition of the height of the woman):
  Woman's height of approximately 1.7 m can be piled and then added up to get the height of the straw bale mountain.
- Model 2 (Pythagoras Theorem):
  Using the estimated height of the woman (1.7 m) to get the approximate height of one straw bale (1.5 m), one can use Pythagoras Theorem.

Mathematical competencies such as Pythagoras Theorem, fractional arithmetic, estimating are needed to get mathematical results.

2.1.6 Mathematical Results (MR)

According to the mathematical models, the mathematical results are for both very close with approximately 7 m.

2.1.7 Real Results (RR)

These results must be interpreted concerning the given problem to get real results. In the context of this problem, interpretation means that we are talking not about 7 kg, but about 7 m. Therefore the context of the real problem has to be in the focus, because with the real results, you transition back from mathematics to reality.

Validating means comparing real results with your mental representation and the real model, and thus the assumptions you made at the beginning. The phase of validating is extremely important and has to be guided by the teacher when they first start to use modeling in class. Learners have to think about the question: whether 7 m is the right answer on the basis of their assumptions they made before to formulate a mathematical model. Usually students stop their modeling process with their mathematical results, because this is what they know from solving other mathematical tasks. But mathematical modeling is different. If the reality of the mathematical result is not questioned by the students, then mathematical modeling makes no sense.

Some secondary school students were happy with their result that the height of the straw bale was 123 m and they thought it was a good solution. When I asked them about their assumptions it became clear very quickly, that they made a mistake in their calculation, even though they had made correct assumptions (height of the woman and height of a straw bale). After the students reflected on this aspect they recognized that their mathematical result was complete nonsense and not realistic in the context of the real problem they had to solve. This is why it is necessary to guide your pupils through this whole process very carefully when you first start using modeling activities. Your students have to learn to take the context seriously.

2.1.8 Goals and Justifications for the Inclusion Mathematical Modeling in Everyday Teaching

Goals for implementing mathematical modeling in everyday teaching were formulated early on. Everyone who did research in this field or practiced it at school was aware of the chance to offer students and teachers the possibility to get another, larger perspective on mathematics as a school subject. Because of the fact that it took a long time to eventually include mathematical modeling as part of educational standards in many countries around the world, the necessity to promote it more than other "regular" mathematical topics like algebra or statistics is clear. Kaiser (1995) and Blum (1996) formulated goals for teaching modeling, which included multiple dimensions. These goals do not focus on a specific age or grade, but can be used from primary through high school, and for teacher education, in order to improve individuals' modeling competencies. The following four justifications (see Blum 2011) show the importance of modeling for learning or gaining a deeper understanding of mathematical content, as well as making a contribution to general education:

1. The "pragmatic" justification includes ordering, understanding and mastering real world situations, which means working on applications and modeling examples explicitly and continuously;
2. The "formative" justification means to improve competencies by engaging in modeling activities;
3. The "cultural" justification focuses on the contributions that the real world can make in providing a broader picture of mathematics;
4. The "psychological" justification means that real-world examples should make a contribution towards raising students' interest in mathematics, to motivate them or help them to understand mathematical content better.

The "Straw bale" problem fits nearly all of the above justifications. However, they only can be achieved for students if mathematical modeling is really a frequent part of their education in mathematics. This means in practice that mathematical modeling should not just be an activity once a year, but at least once a week. How this can work will be demonstrated in Chap. 4.

2.1.9 International Perspectives on Mathematical Modeling

Similar to the question "What does mathematical modeling mean?" one can ask "What does the teaching and learning of mathematical modeling mean?" There is a clear consensus that mathematical modeling involves transitions back and forth between reality and mathematics; for more, see also Garfunkel et al. (2016) (GAIMME report) and Hirsch and McDuffie (2016). What is interesting is how mathematical modeling is interpreted in several countries, which have different educational traditions.

The following classification system for different perspectives on modeling was developed during several European Conferences by the group "Mathematical Modeling and Applications", and has gone through several modifications. This is not just the European perspective, because participants came from all over the world. It follows then, that many of the views and the approaches of mathematical modeling are also influenced by researchers from non-European countries. Although these theoretical perspectives are clearly defined as research perspectives, they also help you to understand and to analyze modeling problems much more deeply, and think about the lesson planning and instruction (see Chap. 4).

Theoretical perspectives on mathematical modeling (Kaiser and Sriraman 2006; Borromeo Ferri et al. 2011) include:

- Realistic or applied modeling
- Educational modeling
- Epistemological or theoretical modeling
- Socio-critical modeling

- "Model-Eliciting Activity" approach (MEA)
- Cognitive and affective modeling (as a meta-perspective)

The *realistic or applied modeling perspective* has a strong focus on using realistic and authentic real world examples. Of course there is no modeling without realistic problems and it does not mean that the other perspectives no not use such problems. The "Bale of straw" task is a wonderful problem for a 90 min session, including working in groups, presentations and discussion in the plenum. But the characteristic of this perspective is that the real problems are very complex and generally they can be used for project work, such as modeling days or modeling weeks (see Chap. 6). Examples of complex modeling problems presenting this characteristic are:

- How can the mixture of chemicals in swimming pools be optimized?
- A belt equipped with technical devices to send and detect impulses is fixed to a tree. By means of this equipment we would like to predict the health status of the tree.

The *educational modeling perspective* focuses on two modeling sub-perspectives: didactical and conceptual modeling. The didactical perspective means structuring learning processes for modeling activities. Thus the goal of several research studies was to develop and evaluate lesson units on mathematical modeling. The philosophy for planning and executing lessons with mathematical modeling is in fact different from "normal mathematics lessons", but this depends on the time spent working on modeling problems (see Chap. 4). The conceptual modeling perspective focuses on the introduction or strengthening of a mathematical concept (in the "Bale of straw" task, Pythagoras Theorem) through modeling and meta-level development, such as introducing and clarifying the terms "real model," "mathematical model," and "validation" within the modeling cycle. The educational modeling perspective has strong pedagogical and subject-related goals. Both aspects mentioned are in the focus of many research studies. They represent the goals one has for teaching modeling.

Epistemological modeling (or theoretical modeling) has more theory-oriented goals. The Romance language speaking countries in Europe, primarily, have a unique view on the teaching and learning of mathematical modeling. These countries start from a theory-related background described as epistemological modeling. The focus in lessons based on this perspective does not specifically lie on the transitions from reality to mathematics and back, but on the inner mathematical structures of the problem. Referring to the "Bale of straw" problem, the goal would be to work especially on Pythagoras Theorem, as well as on other concepts like length, angles etc. The difference from the pedagogical modeling perspective is that modeling is used as a tool to work mathematically, rather than to learn and to promote modeling competencies.

The *socio-critical modeling* perspective has pedagogical goals, but is particularly concerned with seeing and understanding the surrounding world critically (see Barbosa 2006; Rosa and Orey 2013). Thoughts underlying this perspective originated primarily from researchers of South-America, in particular from Brazil. Even the "Bale of straw" task, like every kind of modeling problem can be interpreted socio-critically. As well as the question of the height of the straw bales, one could ask how much money a farmer in the USA, in Germany or in Brazil earns with nearly the same size of a field. A lot of questions would be raised by the students comparing different cultures. Additionally they would do a lot of mathematics, such as searching and analyzing data (size of fields, prices of straw, etc.). Based on my experiences teaching modeling at school or in my workshops, both the students and the teachers liked these socio-critical questions. Usually an interesting phenomenon happens, especially with students at school: the involvement in the socio-critical questions is so high that they are doing mathematics without realizing it. This is, of course, a result of the real context and is sometimes to do with their own experiences in life. The meaningfulness of mathematics becomes apparent for them.

Comparing the epistemological modeling and the socio-critical modeling the differences becomes clear, but a very interesting theoretical approach, which seems to combine these two perspectives, is the socioepistemologial approach (Cordero 2008; Buendía and Cordero 2005). On the one hand this approach has the focus on the reconstruction of mathematical knowledge, for example during modeling activities by using tools and on the other hand the role of the people and the social context in which they performed are from great interest.

The *Model-Eliciting Activity (MEA)* approach/perspective can be seen as an important outcome from the debate on modeling in the United States, and from the extensive work of Lesh and his “MEA-group” (see, e.g., Lesh and Doerr 2003). According to Lesh, Model Eliciting Activities should start in the kindergarten, so learners can experience problems typical of various professions, like engineering or economics. This helps them to understand how mathematics is needed in real life. Finally the meta-perspective – *cognitive and affective* modeling – deserves a mention. It is labeled as “meta-perspective” because in most of the research studies this approach is an integral part of the investigation. Hence, the research aim from the cognitive perspective is the analysis of cognitive processes of learners and teachers while undertaking modeling activities (see Borromeo Ferri 2010). Looking at mathematical modeling processes from a cognitive perspective is important for teaching and learning. The cognitive view helps you to analyze cognitive barriers in modeling problems and is thus a basis for diagnosis (see Sects. 2.3 and 5.1).

Although these perspectives are considered as research perspectives, they evolved from the interplay of theory, practical experiences, and empirical research, around the teaching and learning of mathematical modeling. Additionally, it becomes clear how these perspectives are mirrored in the different modeling cycles presented in the next section.

2.2 Modeling Cycle(s): A Multi-faceted Learning Instrument

Looking at the literature on modeling and applications one can find many different modeling cycles. These cycles are different, because they are dependent on various perspectives on modeling and, in some cases, whether complex or non-complex tasks are used (Borromeo Ferri 2006). In the past few years the relevance of the modeling cycle for the teaching and learning of modeling has been investigated theoretically and empirically. On the basis of these results the modeling cycle should be a central part of teacher education and training, and explicitly integrated when starting to use modeling activities in school. A modeling cycle is not only a theoretical model which characterizes the modeling processes, but it is a multi-purpose (meta-) learning instrument for students and a diagnostic instrument for teachers. Similar to the perspectives, a modified classification of modeling cycles is presented based on an earlier classification by Borromeo Ferri (2006). The following classification similarly shows the different aims and purposes of these cycles for research and practice:

- Modeling cycle from applied mathematics
- Didactical or pedagogical modeling cycle
- Psychological modeling cycle
- Diagnostic modeling cycle/modeling cycle from a cognitive perspective

2.2.1 Modeling Cycle from Applied Mathematics

In nearly all books on mathematical modeling you can find modeling cycles which have one thing in common: real situation and mathematical model are the same thing, which directly contradicts what comes after (where there is a clear distinction between the real world and mathematics). This is partly to do with the kind of modeling problems which are used in this context. Mostly these are "realistic and complex" problems (e.g. from industry or economics) in the sense described in the "realistic and applied modeling perspective" mentioned earlier. The complexity of the real problems influences the number of phases within the modeling cycle to some extent, because there is no need to make more distinctions. A prominent researcher, in the field of modeling in general, but especially in the way of considering modeling as a way to understand the real world better, is of course Pollak (1979). Below you can see his modeling cycle, which was used as a prototype for cycles in this group (Fig. 2.3).

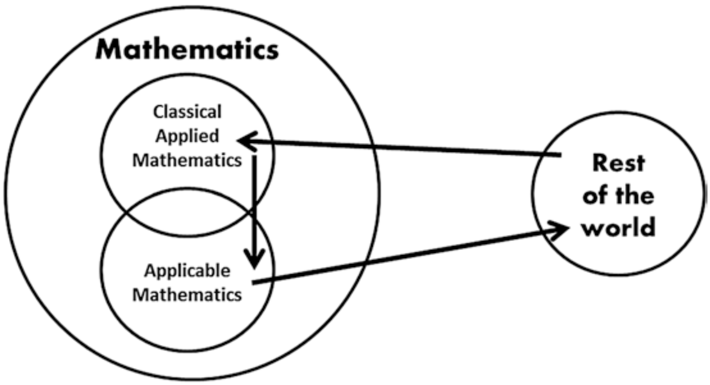

Fig. 2.3 Modeling cycle of Pollak (1979, p. 233)

Pollak's modeling cycle strongly influenced the development of modeling cycles in research on modeling in mathematics education. Thus, most cycles clearly show both worlds: reality and mathematics and the transition processes. In order to understand what mathematical modeling means, it is very helpful, and important, that these two worlds are presented separately. You can find a lot of modeling cycles with four or more steps that are very similar to this three step model of Pollak's ("rest of the world"(step 1) -> "Classical Applied Mathematics" (step 2)-> "Applicable Mathematics" (step 3)-> "rest of the world"), which also have their origin in Applied Mathematics.

When looking at the different cycles, you must always keep in mind the purposes they are used for. An applied mathematician uses and understands Pollak's cycle or the following cycle completely differently to a mathematics teacher or a researcher.

2.2.2 *Didactical or Pedagogical Modeling Cycle*

In this model, reality and mathematics are separated as "two worlds" like in Pollak's model. More transparency is given concerning the steps. Beginning with the real situation, which is the given through the real problem/task, you have to idealize this real situation to build a real model. Referring to the "bale of straw" problem, this means for example, simplifying the straw bales as circles and estimating the height of the woman on the picture. Doing this, you are able to mathematize and building a mathematical model. Investigation of the model simply means inner-mathematical working, such as using the Pythagoras Theorem correctly and so getting mathematical results. The important final step is the interpretation of the mathematical result(s) and, as described in the previous section, this has to be explicitly guided by the teacher (Fig. 2.4).

But why is this cycle named didactical or pedagogical? One focus of research on the teaching and learning of mathematical modeling in the past 10 years was: if, and

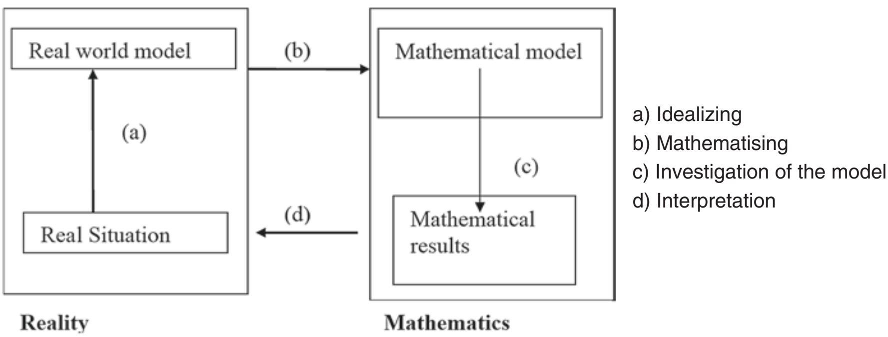

Fig. 2.4 Modeling cycle from Blum (1996) and Kaiser (1995). (**a**) Idealizing. (**b**) Mathematising. (**c**) Investigation of the model. (**d**) Interpretation

how, the modeling cycle can be a tool to promote modeling competencies, and the understanding of modeling in general, of students in middle school, high-school and university (see Blum 2015; Maaß 2007). Firstly, the implementation of the cycle within the modeling lessons offered the students the opportunity to reflect what they had done while solving real problems. Secondly, the students, in this case seven graders within the empirical study of Maaß', learned the notions of "real model" or "mathematical model". Furthermore, this meta-level and the visualization of the modeling process through the cycle is helpful to get an idea of how modeling problems are different from routine problems, because of the transitions between reality and mathematics. Under a didactical and pedagogical viewpoint this cycle is a meaningful tool for modeling lessons: in particular because of the four clearly-arranged steps. The modeling cycle presented in Sect. 2.1 has seven steps and would be too difficult to understand for younger students, as would the "Psychological modeling cycle" shown afterwards.

2.2.3 Psychological Modeling Cycle

These kinds of cycles have their origin research in psychology rather than in applied mathematics or in mathematics education. The following cycle illustrates *the* prototype of a modeling cycle created by the psychologists Verschaffel et al. (2000). When you look at the cycle I would like to draw your attention to the "*situation model*" (Fig. 2.5).

The well-known term "situation model" is mainly used in connection with non-complex modeling problems, specifically with word problems (see Kintsch and Greeno 1985; Nesher et al. 2003 and a lot more), and has its origin in text linguistics. A situation model can be briefly described as a mental representation of the situation that is given in the problem. In Kintsch' and Greeno's work for example (1985), one can find the notion of the *situation model*, sometimes called the "problem model", described as:

> The situation model includes inferences that are made using knowledge about the domain of the text information. It is a representation of the content of a text, independent of how the text was formulated and integrated with other relevant experiences. Its structure is adapted to the demands of whatever tasks the reader expects to perform. (Kintsch and Greeno 1985, 110)

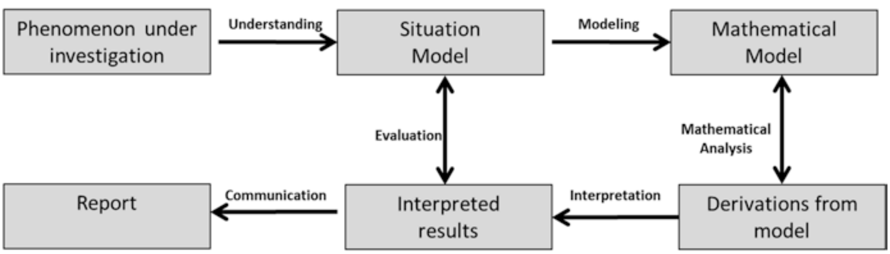

Fig. 2.5 Modeling cycle from Verschaffel et al. (2000)

The phenomenon of the situation model was transferred to the modeling context and brought out modeling cycles (see Blum and Leiß 2007a; Borromeo Ferri 2006) which include the situation model as a further step between the real situation and the real model (see the next cycle). What becomes clear from the cycle presented above is that there is no distinction made between mathematics and reality. This cycle is not used in school and it was not an intention of the developers to do so. But the relevance for including the situation model in the diagnostic modeling cycle offered new ways for research, for practice and particularly for teacher education and training on mathematical modeling.

2.2.4 Diagnostic Modeling Cycle/Modeling Cycle from a Cognitive Perspective

The researchers who "work" with this kind of modeling cycle focus especially on the cognitive processes of individuals during modeling processes. This is one reason why the situation model was included in this cycle, because the researchers suppose that this phase is more or less run through by all individuals during modeling (Fig. 2.6).

Blum and Leiß understand the situation model as an important phase during the modeling process: even as the most important one. That is because they describe the transition between real situation and situation model as a phase of understanding the task. A similar approach (Borromeo Ferri 2007) in the COM^2-project uses the phase of the situation model in an adaptation of the modeling cycle of Blum and Leiß. However I used the name "mental representation of the situation" (MRS) instead of situation model, because this term better describes the kind of internal processes an individual goes through to obtain a corresponding mental picture while/after reading the (complex) modeling task. Besides this aspect, I used this modeling cycle with these different phases to describe and to reconstruct these phases empirically (see next chapter) (Fig. 2.7).

Through the situation model and the mental representation of the situation, a cognitive view of modeling processes is given, which is supported a number of the steps (six and seven). Thus for diagnostic purposes this cycle is a good instrument (see Sect. 5.1).

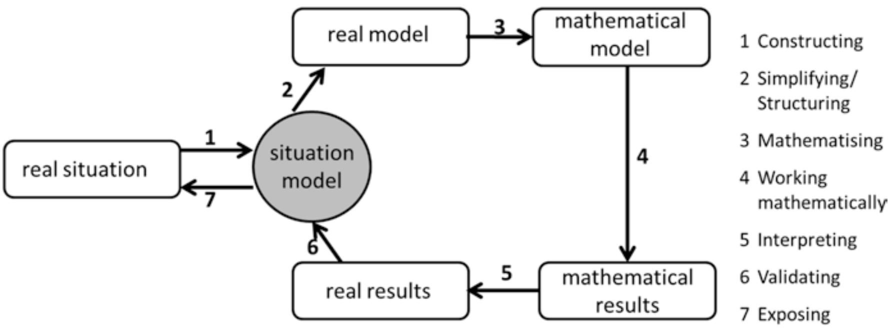

Fig. 2.6 Modeling cycle by Blum and Leiß (2007a)

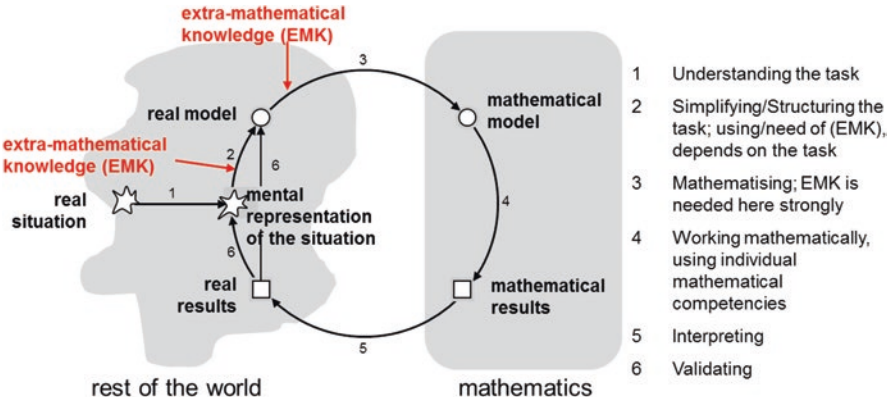

Fig. 2.7 Mathematical modeling cycle from a cognitive perspective (Borromeo Ferri 2007)

If teachers are able to name and to distinguish steps within the modeling cycle then they can diagnose possible cognitive barriers students have while modeling.

In addition to the classification above, you can find further descriptions of modeling cycles, which are used in school or higher education (Cirillo et al. 2016).

2.2.5 Modeling Cycles Presented in Mathematics Standards: Exemplified Along the Common Core State Standards Mathematics of the United States of America

The above classification showed you different kinds of modeling cycles. Perhaps you have not seen these cycles before, because it was not a part of your teacher education or teacher training. Also, in those countries in which mathematical modeling has become a central part of the mathematics standards and the curriculum (e.g. United States, Germany, Chile, etc.) different cycles are used to explain the process and the term of mathematical modeling. When analyzing the mathematics standards or curriculum of countries concerning the aspects on modeling, it becomes clear that the outcomes of research results in the field of modeling are generally not ignored. However, the mathematics standards build a basis for writing school books and are an important guideline for teachers to implement this into practice. Thus a new teaching field such as mathematical modeling should preferably be presented coherently and in connection with the research. Hence it is important to know that there is not only the *one* modeling cycle presented in the present mathematics standard of your country. If you have read the book up to this point you know some modeling cycles and you have an idea what mathematical modeling means, presented on the basis of international research on modeling.

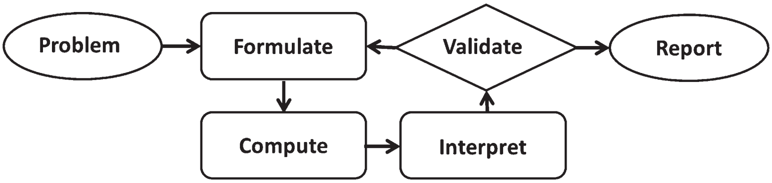

Fig. 2.8 Modeling cycle presented in the CCSSM

In the following I would like to illustrate how mathematical modeling is characterized and which modeling cycle is presented in the Common Core State Standards Mathematics (CCSSM) in the United States. The reason why I use the United States as an example is because the CCSSM are brand new in a lot of States (since 2013), highly discussed and, of course, mathematical modeling is a completely new part. Mathematical Modeling, standards and assessment are not mutually exclusive. This will be explicitly shown in Sect. 5.2 with several examples.

Looking at the NCTM Standards you can see that Mathematical Modeling is not explicitly endorsed as it is now in the Common Core State Standards. Mathematical Modeling, when thinking about the characterization mentioned in Sect. 2.1, was more or less implicitly included in the NCTM-Standards within the expression “connections”. Of course, this description is not satisfying for me as a researcher and teacher educator in mathematical modeling, but the CCSSM shows that mathematical modeling is a central standard now:

> Students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. (NGA Center and CCSSO 2010, pp. 6–8)

This citation makes clear, that the extra-mathematical contexts and situations and real problems have to be solved with the help of mathematics. Transitions take place back and forth between reality and mathematics. Learners should go through the whole modeling cycle as they work on a real life problem. But what does this mean in the context of the modeling cycle described in the CCSSM? Where is the “reality” or the “rest of the world” and where is the “mathematics” you have seen in the other cycles? (Fig. 2.8).

As mentioned earlier, the distinction between reality and mathematics within a cyclic model could be helpful to understand what mathematical modeling means. The cycle presented in the CCSSM does not make clear that problems from real life are the basis for modeling activities. Neither the words “interpret” nor “validate” explain that this cycle deals with modeling. Also, inner-mathematical problems can be interpreted and validated without having any real context. Referring to the above characterization on modeling in the CCSSM one can assume that reality and mathematics are implied in the cycle. It becomes more obvious through the colors green and red:

The problem is always a part of the reality (marked in green) and is called the “real situation” in the other cycles. Learners have to think about the given problem and make assumptions. Then they simplify and structure the real situation (process is shown by the arrow ->) and finally they formulate it. This formulation can be

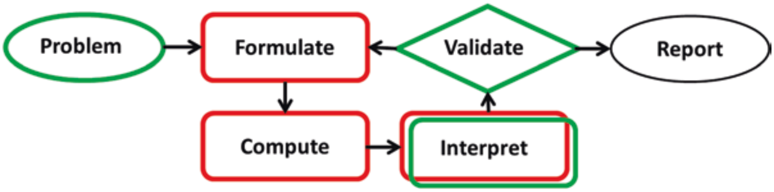

Fig. 2.9 Modeling cycle (CCSM) – *green* ("reality") and *red* ("mathematics")

characterized as a mathematical model and so this is a part of mathematics (marked in red). As you see there is no "real model" shown as a step within this cycle and it goes directly from Problem to Formulate. Within the process from formulation to computation (process is shown by ->) learners use their mathematical competencies and abilities for finally computing the model. To compute is a mathematical step. The mathematical result is included in the step "Compute" in the above model and this result has to be interpreted. For example, if x = 10 is the result and should represent a distance in the context of the problem, then the interpretation is x = 10 miles and not x = 10 kg. According to this we have both mathematics and reality in the Interpretation step. Validating goes one step further than interpretation and is a part of the real world. The term "real result" (as it is called in the other cycles) is included in the step "validate" in the CCSSM cycle. If the result makes sense and if learners also checked their computing process, then they report their results. A report can be verbal expressions or of course written solutions with appropriate reasoning. Another way of looking at the CCSSM modeling cycle is to compare it to the numbered steps in the diagnostic modeling cycle. For example, comparing to Fig. 2.9, Steps 1–3 are included in "Formulate", then Steps 4–6 are Compute, Interpret, Validate respectively.

As long as you know the steps of the modeling process, independent of a specific modeling cycle, then you are well prepared for modeling activities with your students. The importance of the modeling cycle as a learning instrument is described in the following chapter.

2.2.6 Mathematical Modeling Cycle as a Multi-purpose (and Metacognitive) Learning Instrument

On the basis of practical experiences with school students it became clear that cycles with four steps give a better overview in terms of helping them to understand the process of mathematical modeling. Cycles with more steps, which offer a deeper view into cognitive processes, are helpful for teachers. So far, there have been no empirical studies that have investigated which cycle is the best for helping students to learn mathematical modeling. Nevertheless, within the DISUM-project (Blum and Leiß 2007b) one question was, whether or not students' modeling competencies

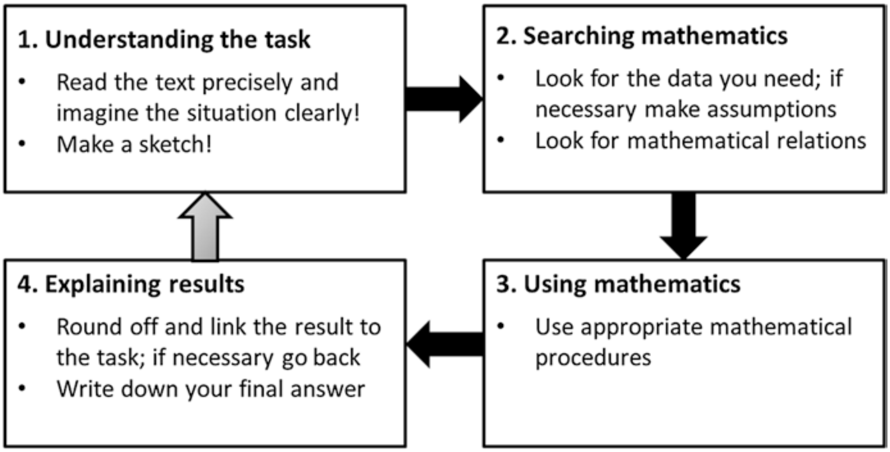

Fig. 2.10 Solution plan (DISUM-project)

increase when they use a modeling cycle during modeling lessons. An experimental and a control group worked on several modeling problems. The experimental group was given one version of a modeling cycle (the solution plan; see Fig. 2.10), and it was integrated in the lessons and students should work with it. The control group did not get the solution plan or any kind of modeling cycle. The results of this intervention study showed that the modeling competencies of the experimental group increased more than the control group.

The solution plan from the DISUM-project tries to make the language used in the other modeling cycles (real model, validation, etc.) more understandable for students in secondary school. At the same time, students get less direction on how they should proceed when working on a modeling problem. This makes sense at the beginning for developing solving strategies. Similar to the modeling cycle of the CCSSM, reality and mathematics are not explicitly shown as two "worlds", and this is a disadvantage of it. Students should become aware of transferring a real world problem into mathematics and backwards. This cycle could be helpful for learners as an introduction when working on modeling problems. Later on cycles that include the concepts "real model", "mathematical model" and so on should be used.

There are three main reasons why the modeling cycle must be a part of the learning and teaching process on mathematical modeling: Modeling cycles

- offer individuals an understanding of what mathematical modeling means (What does it mean?)
- give individuals orientation within their modeling process (Where I am in the process?)
- allow individuals to think about their modeling process retrospectively and on a metacognitive level (Which phases did I go through and which are missing?)

It is not helpful to teach the cycle during the first modeling activity, because for students this kind of model is too complex. They are not familiar with the concepts and thus it will not be motivating for them. Before being introduced to the cycle, they should have worked on at least two modeling problems. Doing this, they get a feeling for this new kind of problem and ultimately you as a teacher will decide when the best point will be for introducing the modeling cycle. There are different teaching approaches for implementing the cycle. I practised them in my workshops with my pre-service and in-service teachers, and finally observed teachers while they were teaching this at school:

- *isolated* or theoretic approach
- *example* bounded approach
- *process* bounded approach

The *isolated teaching approach* describes a procedure that introduces the modeling cycle in isolation rather than in connection to a real problem. The teacher presents a visualization of the cycle on a map, on the board or with a projector and asks the students what they see and understand. Students are able to connect their experiences with the modeling activities they have done before. First give them the model on a sheet of paper and let the students think on their own. While using the teaching method *Think-Pair Share,* students can share their thoughts after a few minutes in pairs to discuss what the different concepts mean and what the cycle represents. After 10 min you can choose a team to present the results. Other teams should add their thoughts as well. Learners often have no problem recognizing that the real problem or question comes out of the "real world", that they need "mathematics" to solve it and then transfer it back to reality. More problems arise with the concepts "real model" or "mathematical model" and with the different arrows explaining the phases like "simplification", "interpretation" or "validation". This phenomenon also comes up with university students or in-service teachers, so is not only a problem for learners at school. One reason is the concept of "model" itself. It helps to clarify first in a practical way that a "model" represents a reduced image of a complex situation. For a better understanding you should directly refer to the modeling activities they have done in the lessons before. This does not means the real problems they solved, but the whole process they went through. In the following modeling lessons, the modeling cycle should always be used in combination with a modeling problem. The learners now have an instrument which gives them orientation during the modeling process and helps them to reflect and to think on a metacognitive level. At the beginning the teacher should explicitly encourage learners to reflect their modeling process and then they will do it more independently.

In the *example bounded teaching approach,* the modeling cycle is introduced by the teacher with a new real problem. The goal is to make the different steps of the cycle transparent while working on the problem. First the students have to think about the problem. Then, on the basis of a plenary discussion, the teacher presents the solving process according to the steps of the cycle. While describing each step of the cycle, the concepts of "real model" or "validation" become more transparent for the learners. After the explanation you should give students time to explain the modeling cycle to each other, and make a second round if things are still unclear for them.

The *process bounded teaching approach* is different from the previous in that learners get a modeling problem first and then while the students start working on a solution. After a while the teacher asks the students to pause and shows the modeling cycle on the board. Then the teacher gives a short explanation only, saying that this cycle represents the modeling process. Then as they continue to work, the students have to brainstorm what the steps, and thus the concepts of the modeling cycle, could mean and which phase they currently are in. During the process the students reflect and understand what the different steps of the cycle are about. At the end, one group can present their results showing the solution process and their ideas, with each step classified according to which part of the modeling cycle it relates to. Thus on a school level the learners do a subject matter analysis of the modeling problem.

It is clear that the modeling cycle is not only used within research but is also important for every day teaching in modeling. There are no specific rules how the modeling cycle should be introduced for learners, but the three teaching approaches described above were successful and can give you ideas on how to do it in your own school.

2.3 Far from Linearity: Individual Modeling Routes

Modeling cycles represent on the one hand a model of a modeling process, and on the other, an idealized and linear description of how the process of modeling should proceed.

In reality the modeling processes of individuals are not linear. This is supported by empirical evidence from several studies (see e.g. Borromeo Ferri 2007, 2010, 2011; Matos and Carreira 1997; Galbraith and Stillman 2006). Investigating these micro processes of learners' modeling activities means taking a cognitive view, to look more deeply into the thinking processes of individuals. Modeling is a complex process – this was mentioned several times, but this aspect is getting more important when analyzing certain steps of the process or especially *individual modeling routes*. On the basis of an empirical study (COM^2-project – **Co**gnitive-psychological analysis of **m**odeling processes in **m**athematics lessons, Borromeo Ferri 2007, 2010) with students of Grade 9 and 10 the reconstruction of the phenomenon individual modeling routes was a central result of the investigation, which I describe as "an individual modeling process on an internal or external level. The individual starts this process during a certain phase, according to his or her preferences, and then goes through different phases several times or only once, focusing on certain phases and/or ignoring others. To be more precise from a cognitive viewpoint, one ought to speak of visible modeling routes, as one can only refer to verbal utterances or external representations for the reconstruction of the starting-point and the course of a modeling route." (Borromeo Ferri 2007, p. 265).

2.3.1 *Visible Modeling Routes and the Influence of Extra-Mathematical Experience*

One main research question of the COM²-project (Cognitive-psychological analysis of Modeling processes in Mathematics lessons) was, if differences between mental representation of the situation, real model, mathematical model and the other phases (as described in the didactic literature on modeling) can be reconstructed from the learners' way of proceeding?

COM²-project was a qualitative study which combined classroom research and analysis of single individuals and corresponding groups of pupils within these classes. For the investigations, three 10th grade classes from different German Grammar Schools were chosen. The sample included 86 pupils (65 pupils in the first phase and additional 26 pupils in the second phase of data collection) and three teachers (two female, one male). The modeling problems used (e.g. "Bales of straw") were of central importance. They were analyzed from three viewpoints: subject matter, cognitive processes, and the ("diagnostic") modeling cycle. All lessons were videotaped and transcribed. Statements of the pupils were analyzed concerning the aspect, in which phase of the modeling cycle they worked on. For every single pupil the statements were coded (Strauss and Corbin 1990), so that an individual modeling route could be reconstructed. For the modeling problem "Bale of straw" (introduced in Sect. 2.1) I will now describe what is meant by an individual modeling route in more detail, and with an example of two pupils from the study (Daniel and Andreas), and also how the influence of extra-mathematical knowledge becomes clear.

Daniel and Andreas worked together in a group of five pupils and were very active within this group. Both made a lot of annotations, although they had less experience with modeling problems. This is why Daniel said at the beginning: "It is not possible to solve it, because we have no numbers to calculate!" Shortly after Daniel made that remark, he had a key idea, and was the first person in the group to formulate it:

> You have to think about the height of the woman.

In this statement one can reconstruct the mental representation he had of the situation. He had not simplified the problem at this point. Then he said to a girl in the group:

> The woman is perhaps as tall as you.

This can be called a real model, because he had a clear idea of how the problem was structured. The interesting thing after this was that Daniel stated: "Yes, but we don't have any numbers!"

As a person with a preference for analytic and formal thinking, he focused on facts and numbers, which were not given in the problem. This was not a problem for Andreas, even in the first few minutes of working on the task. As an integrated thinker, he combined elements of visual and analytic thinking styles. A short time after Daniel repeated the statement, Andreas said to the group members:

> Say, can you imagine that the woman is now standing up? Yes?

This is really a wonderful example for the phase of the mental representation of the situation (MRS). Andreas continued:

> You have to think about the woman, that she stands up now, that means, she must be as big as the straw bale.

It is interesting how Andreas made his mental representation so visual for all group members. Directly after that, Daniel came to an estimated mathematical result:

> I would estimate that the height is 10m altogether, because a straw bale has a diameter of 2m.

So he built a mathematical model on an implicit level and used addition as an inner-mathematical competency. But most of the group members were of the opinion that the height of a bale must be less than 2 m. The group rejected Daniel's result in what was essentially a Validation step and he made again a comparison: "How tall are you Julia?" and started a new modeling process, beginning with a real model. Julia said that she was 1.65 m tall. Daniel concludes that the woman must be 10 cm taller than Julia and tried to convince the others that one straw bale is higher than the woman. Andreas did not share that opinion and answered: "No!" This "No" was the beginning of Andreas' verbalised knowledge about straw bales, which he did not express at first. He then began to measure the straw bale with his ruler. Although Andreas was in the phase of mathematical model, his arguments were on a visual level:

"If you imagine that it is 10 cm then it is 1.50 m." Daniel answered with a good idea and switched back to real situation:

"You have to think that these straw bales sink down! They don't really lie on top of each other, they slide in the gaps." This was again a prompt for Andreas to tell more about his extra-mathematical knowledge. Andreas responded:

> Yes, air must come through the straw bales and they are not stiff! If you cut the straw bale here that will be a quarter.

On the basis of this statement Andreas and Daniel and some of the other group members calculated and discussed about rounding up their results. But Andreas wanted to determine the height more exactly and tried to convince the others that is must be less because of the fact that the straw bales sink down. His knowledge was on an implicit level up to now. The following conversation makes clear that his experiences had an influence on his modeling route and therefore on his transitions between the phases of reality and mathematics:

Julia: "I'm not sure what effect it really has, if these straw bales sink down."
Susi: "I don't think that straw bales sink down so heavily."
Daniel: "Have you ever been on top of one straw bale?"
Julia: "In Grade 5 we made an excursion and I climbed on top of a straw bale like that."

Julia wanted to make clear, that she had "real life experiences". But Andreas had other kinds of experiences and argued: "I grew up on a farm, don't tell me anything!"

After that, Daniel got another result, 6 m, which was not interesting for Andreas. He wanted to talk about what happens if these straw bales become wet. Later on he

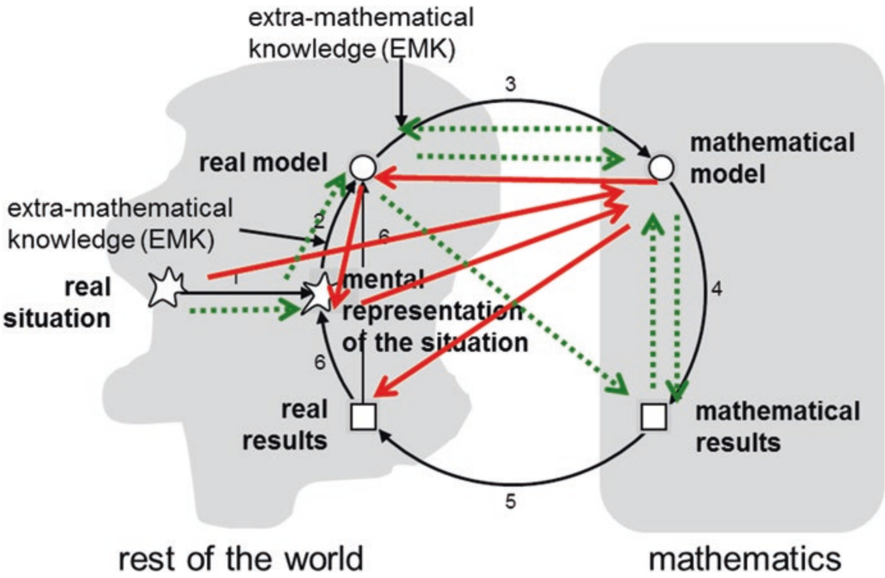

Fig. 2.11 Individual modeling routes of pupils (Borromeo Ferri 2010)

explained to other group members the difference between hay and straw. Through this knowledge and experience it became clear why he wanted to determine the height of the straw bale in such an exact way. For him these were real results, far more than for the others.

These statements and actions of Andreas and Daniel while modeling are illustrated as individual modeling routes within the modeling cycle in Fig. 2.11. The routes demonstrate the changes in the phases, but not the time a student spent within a phase. Remind that only a small part of the whole solving process is shown in the figure.

Due to the time of the solving process, of course further statements were made by Daniel and Andreas while modeling the problem, which could not be integrated into the figure. With the statements of both students in mind, it becomes clear when looking at Andreas' modeling route, that he spent a long time in reality at the beginning by switching from mental representation of the situation (MRS) to the real model (RM) and again to the real situation (RS) before he went into mathematics. However Daniel switched fast from the real model (RM) to mathematical model (MM) and even to mathematical results (MR), as described previously. Then he needed time back in reality to modify his real model for creating a new mathematical model.

Extra-mathematical knowledge/experience can be an influence on the modeling routes of pupils. Andreas often switched back to reality, because he had additional experiences. His clear mental image of the real situation let him determine the result very exactly. Daniel, as an analytic thinker, was more focused on estimating a result. He had fewer experiences with straw bales and didn't have such a clear picture as Andreas. The effect that the context of the task can also have on solving modeling problems is shown more in detail in Sect. 3.3.

2.3.2 Mathematical Thinking Styles and Modeling Routes of Learners and Teachers

Within the COM²-project another focus lay on the analysis of the modeling processes of the learners and teachers, and the research question was:

- What influences do the mathematical thinking styles of the learners' and teachers' have on modeling processes in mathematics lessons?

In the description of the modeling routes of Daniel and Andreas, their preferred mathematical thinking styles were mentioned. Daniel was an analytic thinker and Andreas was an integrated thinker.

But what is a mathematical thinking style and why is this important to know for teaching and learning mathematical modeling?

On the basis of several empirical studies I have done with secondary school students (Borromeo Ferri 2004, 2007, 2010), I characterize a mathematical thinking style as "the way in which an individual prefers to present, to understand and to think through, mathematical facts and connections, by certain internal imaginations and/or externalized representations. Hence, a mathematical style is based on two components: 1) internal imaginations and externalized representations, 2) wholist and dissecting way of proceeding" (Borromeo Ferri 2004, 2010).

A central characteristic of the construct "mathematical thinking style" is the distinction between abilities and preferences. Mathematical thinking styles are about how a person likes to understand and to learn mathematics and not about how well this person understands mathematics. This approach is based on Sternberg's theory of "thinking styles" (1997). Thus according to Sternberg (1997), "A style is a way of thinking. It is not an ability, but rather, a preferred way of using the abilities one has." That means that thinking styles are not viewed as being unchangeable, but they may change depending on time, environment and life demands. If you think about the term *preference* in connection with modeling then an interesting question is: how you and your students *like* to model or proceed along the modeling cycle. Mostly you do this on an unconscious level and normally you cannot easily recognize another person's preferred way of modeling. However, I would like to give you ideas for a deeper view into modeling processes of students and teachers who prefer analytic, visual and integrated thinking styles.

In my empirical studies on mathematical thinking styles (Borromeo Ferri 2004), the goal was to reconstruct and characterize the analytic, visual and integrated thinking styles of students from Grades 9 and 10 during their pair-problem-solving process. The design of the study was very complex, in order to grasp the concept of the construct "style" (preference) itself, as well as the representation (visual, analytic, conceptual) and the way of proceeding (holistic, dissecting). This information was obtained from using stimulated recall and interview. The aim was not only to reconstruct these preferences, but to find explanations of what it means to be a visual or an analytic thinker, inspired by the concepts or classifications of thinking found in the literature (e.g. Hadamard 1945; Skemp 1987; Burton 1995).

Based on this first and follow up studies, an empirically grounded description of the characteristics of the visual, analytic and integrated thinking style could be developed:

- Visual thinking style: Visual thinkers show a preference for distinctive internal pictorial imaginations and externalized pictorial representations, as well as a preference for the understanding of mathematical facts and connections through holistic representations. The internal imaginations are mainly affected by strong associations with situations they have experienced.
- Analytical thinking style: Analytic thinkers show preferences for internal formal imaginations and for externalized formal representations. They are able to comprehend mathematical facts preferably through existing symbolic or verbal representations and prefer to proceed in a sequence of steps.
- Integrated thinking style: These people combine visual and analytic ways of thinking and are able to switch flexibly between different representations or ways of proceeding.

Individuals had preferences for certain mathematical thinking styles, these gave a strong indication that the modeling behavior of individual learners also can be very different because of the influence of the styles. However, the degree to which this is true with more complex modeling tasks and within regular mathematics lessons instead of a laboratory design was still an open question. Answering these questions was one major goal of the COM^2-project.

The individual modeling routes of Daniel and Andreas illustrate parts of the patterns which could be reconstructed for the visual, analytic and integrated thinkers of the study when they worked on the three modeling problems. The results of the micro-analysis showed the following structural characteristics:

- Analytic thinkers usually change to the mathematical model immediately and return to the real model only afterwards when the need arises to understand the task better. They work mainly in a formalistic manner and are better at "perceiving" the mathematical aspects of a given real situation.
- Visual thinkers mostly imagine the situation in pictures and use pictographic drawings. Their reasoning during the modeling process is usually very vivid, even while they are working within the mathematical model. They often follow the given modeling cycle.
- Integrated thinkers showed a balance between reality and mathematics.

The knowledge about these different modeling behaviors, depending on the learners' preferred mathematical thinking styles, helps the teacher to understand why a student, for example, has difficulties building a real model or has problems with validating. Both are necessary sub-competencies for modeling. If an analytic thinker shows very strong inner-mathematical competencies and sees the mathematical model very quickly after reading the task, but cannot imagine the real situation at all, then you, as the teacher, will know how to support this student. Of course you have to know about your students' mathematical thinking styles. In the Appendix you find the scales on mathematical thinking styles which you can use with your class and then match the results with your direct experience with your students (the whole questionnaire including further items see Borromeo Ferri 2015). You will understand the modeling processes much better if you know about the leaner's preferred way of thinking. In Sect. 3.2 we go into more detail concerning modeling competencies and how to foster and elicit them. Then you will also understand that individual modeling routes are needed to diagnose students' missing sub-competencies.

2.3.3 Teachers' Behavior While Modeling Activities in the Classroom

Just as learners prefer a mathematical thinking style which has influence on their modeling process, teacher's behavior of handling modeling problems in the classroom also depends on this effect.

For the COM^2-project three teachers had to be found who preferred different mathematical thinking styles (analytic, visual, integrated). Hence focused interviews were conducted with several teachers to reconstruct the mathematical thinking style for each, and finally three teachers (two females and one male) and their math classes were chosen for the investigation. Additional data were collected, such as biographical questions, mathematical beliefs and questions about their studies of mathematics at university, their current view of mathematics or reasons why they view of mathematics might have changed in the course of their teaching life. During the videotaped lessons the teachers were equipped with a minidisc-recorder strapped to their body in order to document the teacher's help or suggestions during modeling, as this could possibly influence the student's modeling processes. After videotaping the lessons, stimulated recall was done with each of the teachers where they were shown sequences of their behavior in the classroom in order to comment their actions and to ask them why they have acted in certain ways.

Different behaviors of handling with modeling could be reconstructed from this rich source of data, using theoretical coding (Strauss and Corbin 1990) and the reconstruction of ideal types (Weber 1973). Referring to the students' different individual modeling routes dependent on their preferred mathematical thinking style, similar patterns of behavior became evident. The female teacher who was a visual thinker acted and spoke predominantly from a real-world view, thereby helping her students to visualize the real situations given in the problems. The male teacher with a preference for the analytic thinking style communicated on an abstract-mathematical level, emphasizing the mathematics which would be needed for solving the modeling problems. The other female teacher, who was an integrated thinker, acted in a more balanced way. She stressed both – the reality and the mathematics – within her lessons and interventions during students modeling activities. Before giving you a representation of these three types of teachers' modeling behavior so that you can try to classify them for yourself at the end of this chapter, the essential result of the teacher investigation is formulated as follows:

- Teachers have preferences for certain steps while modeling. Different mathematical thinking styles and preferred representations implicitly affect the teacher's focus while teaching mathematical modeling.

Type I: "Retrospective formalizer".

The focus lies on the *formalization* of the solutions as well as enforcement of mathematical aspects with regards to the abstraction level and the formal correctness. Validation and comparison with real facts move into the background.

Type II: "Realistic validator".

Strong *references* to real situations are enriched with verbalized or graphical, vivid and lively pictures and representation. Formalizations only have a low significance.

Type III: "Formalistic-realistic".

This type is *characterized* by a balance between formal-mathematical and reality-based aspects. Real world are phenomena always viewed in connection with the mathematical world.

Before showing the teachers parts of their modeling lessons, they were mostly unaware of their behavior handling modeling problems in the classroom (see also Borromeo Ferri and Blum 2009b), and thus were not conscious of their focus on mathematical or real world aspects. Some illustrative examples of two teachers (with preferences for visual and analytic thinking), to make your teaching of mathematical modeling more consciously balanced between reality and mathematics, are shown below.

2.3.4 Mr. P.: The "Retrospective Formalizer"

Mr. P. was an experienced high school teacher and his second subject after mathematics was physics.[1] He normally used real world problems in physics classes but not in mathematics. His answer to the question "What does mathematics means for you?" was:

> …playing with numbers, playing with variables, logical thinking, building logical connections, yes and there is also a connection to reality. For me, mathematics is the language of physics.

[1] In teacher education in Germany two subjects have to be studied at University as well as pedagogy, psychology and general education and the relevant subject didactics, e.g. mathematics education or biology education etc.

His thoughts on the question “Which kind of view of mathematics do you want to convey to your students?” were very interesting:

> That they mainly learn to recognize structures and, yes, I give them the connection to reality mostly through physics because I am also a physics teacher. But in mathematics I believe that they have to learn to think in structures and be able to ‘move’ within these structures so that they are able to see and to build formulae.

Sometimes the statements of individuals describing what they are doing or thinking do not match with their actual actions in the classroom. The observation of Mr. P. showed that his preferred way of thinking had an impact the way he acted with his students. Normally two groups presented their results for the modeling problems on the board. A typical reaction of Mr. P. was as follows:

> That was really good. […] But what I am missing as a math teacher is that you can use more terms, more abstract terms and that you write down a formula and not only numbers. This way corresponds more to the way of thinking that physicists and mathematicians prefer: when you use and transform terms and get a formula afterwards […]. [Mr. P. developed a formula with the pupils after this statement.]

In particular, when students used a simple model for solving the problem he pushed strongly the mathematical aspects that he thought were missing. Mr. P. as an analytical thinker focused less on interpretation and validation. The formalization of solutions in the form of abstract equations was important for him. Accordingly, the real situation becomes less important.

2.3.5 Mrs. R.: The “Realistic Validator”

Mrs. R. had also been a high school teacher for a long time, and her second subject was English. Mathematics as a scientific discipline and teaching mathematics were for her different areas. For Mrs. R., who enjoys teaching very much, teaching mathematics was emotionally engaged. In the interview she describes what mathematics means for her:

> …an interesting subject, logical thinking, making connections. And…tasks, yes tasks also belong to it.

Concerning the question “Which kind of view of mathematics do you want to convey to your students?”, she goes more in detail, reflecting her own way of thinking:

> That they know that mathematics will be good if they keep the overall view. Often I tell them that I like mathematics. I am not a formalist. When I get a task, the first thing I do is draw a sketch. For me it is not so important that they do everything formally in a correct way, but that they understand that mathematics can help them in their way of thinking.

The aspect that Mrs. R. mentioned last became clear during her modeling lessons. A typical reaction from her after students presented their result was:

> So we have different solutions. But what I recognized and what I missed in our discussion till now is the fact that you are not thinking about what is happening in reality! Start with imagining for yourself the real problem and situation. Two kilometers? Is that much? Is that short for the ray of a light house?

Mrs. R. mainly discussed reality-bounded aspects of the modeling problems. Of course she worked mathematically, but this was not her focus – in contrast to Mr. P. As a visual thinker she interpreted and, above all, validated the modeling processes with the learners.

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