Chapter 2
Power System Fundamentals: Balanced Three-Phase Circuits

This chapter reviews the fundamentals of balanced three-phase alternating current (ac) circuits. First, we define positive and negative balanced three-phase sequences. Second, we analyze balanced three-phase voltages and currents. Third, the different types of power are defined and measurements techniques for power are briefly reviewed. Fourth, we provide an overview of the analysis of balanced three-phase circuits using the per-unit system. This chapter provides an appropriate background of three-phase power for the unfamiliar reader, establishing the link between the physical reality and analytical techniques. It can be skipped by readers with knowledge of three-phase circuit analysis.

2.1 Introduction

Power systems are generally based on three-phase alternating current (ac) circuits. This chapter describes the fundamentals of this type of circuits and is organized as follows. Section 2.2 defines balanced three-phase sequences. Section 2.3 describes balanced three-phase voltage and currents, as well as the two different symmetrical connections of system components and the equivalence among them. Section 2.4 defines instantaneous, active, reactive, and apparent powers and explains how to measure them. Section 2.5 clarifies why three-phase power is generally preferred over single phase-phase power. Section 2.6 defines the per-unit system, which is used in the remaining chapters of this book. Section 2.7 summarizes the chapter and suggests some references for further study. Finally, Sect. 2.8 proposes some exercises for further comprehending the concepts addressed in this chapter.

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2.2 Balanced Three-Phase Sequences

There are two ways of representing an ac source:

1. Using a sinusoidal representation:

$$a(t) = \sqrt{2}A\sin(\omega t + \psi), \tag{2.1}$$

where:

- A is the root mean square (RMS) value of the source,
- ω is its angular frequency (also known as angular speed) measured in radians per second, and
- ψ is its initial phase angle.

The RMS value of the source is computed as:

$$A = \sqrt{\frac{1}{T}\int_0^T a^2(t)\mathrm{d}t}, \tag{2.2}$$

where T is the period (measured in seconds).

The angular frequency ω is defined as the rate of change of the phase of the sinusoidal source and is computed as:

$$\omega = \frac{2\pi}{T} = 2\pi f, \tag{2.3}$$

where f is the ordinary frequency (measured in Hertz).

2. Using a phasorial representation:

$$\bar{A} = A\angle\psi. \tag{2.4}$$

Figure 2.1 illustrates the relationship between a sinusoidal ac source (left plot) and a rotating vector or phasor (right plot). Observe that the projection of the rotating vector on the imaginary axis (right-hand-side of the figure) renders the sinusoidal form of the source: $a(t) = \sqrt{2}A\sin(\omega t + \psi)$, shown on the left-hand side of the figure.

If three ac sinusoidal sources (or phasors) have equal magnitude and equal angle separation ($\frac{2\pi}{3}$ rad or120°), then they constitute a balanced three-phase sequence. For example, the following three ac sources constitute a balanced three-phase sequence:

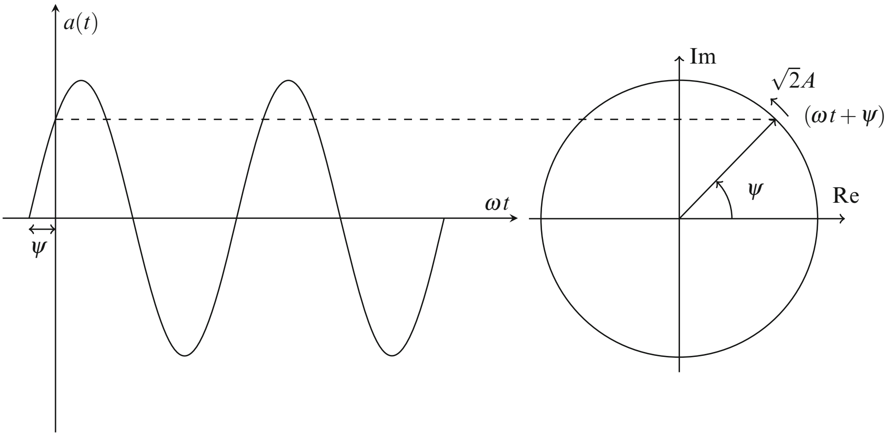


Fig. 2.1 Relationship between a sinusoidal AC source (left) and a rotating vector (right)

$$\begin{cases} a_A(t) = \sqrt{2}A\sin\left(\omega t + \psi\right), \\ a_B(t) = \sqrt{2}A\sin\left(\omega t + \psi - \frac{2\pi}{3}\right), \\ a_C(t) = \sqrt{2}A\sin\left(\omega t + \psi + \frac{2\pi}{3}\right). \end{cases} \tag{2.5}$$

Since $a_A(t)$, $a_B(t)$, and $a_C(t)$ constitute a balanced three-phase sequence, then we have:

$$a_A(t) + a_B(t) + a_C(t) = 0. \tag{2.6}$$

The reader is encouraged to verify that (2.6) is correct.

We may also represent the balanced sinusoidal sources in (2.5) using phasors, i.e.:

$$\begin{cases} \bar{A}_A = A\angle\psi, \\ \bar{A}_B = A\angle\psi - \frac{2\pi}{3}, \\ \bar{A}_C = A\angle\psi + \frac{2\pi}{3}. \end{cases} \tag{2.7}$$

Figure 2.2 shows a balanced three-phase sequence using phasors, where the initial phase ψ is 0. Note that the phasor denoted by $\bar{A}_A$ is leading $2\pi/3$ rad the phasor denoted by $\bar{A}_B$ and lagging $2\pi/3$ rad the phasor denoted by $\bar{A}_C$. In this case, the balanced three-phase sequence is denominated *positive* sequence.

If phases B and C are swapped, we obtain the so-called *negative* sequence that is shown in Fig. 2.3 and represented by the following phasors:

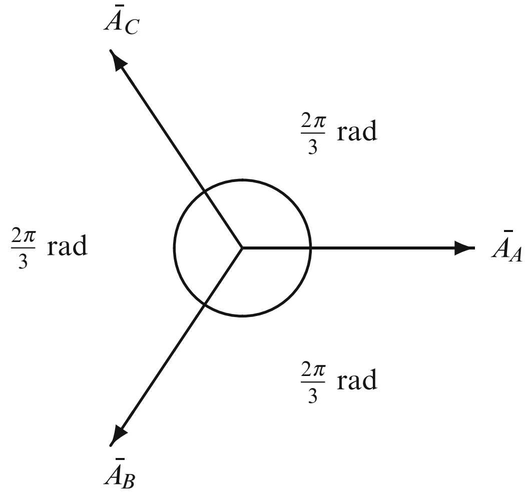


Fig. 2.2 Balanced three-phase positive sequence

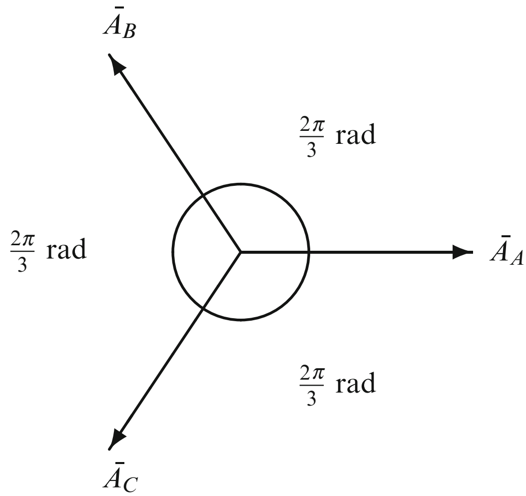


Fig. 2.3 Balanced three-phase negative sequence

$$\begin{cases} \bar{A}_A = A\angle 0, \\ \bar{A}_B = A\angle + \frac{2\pi}{3}, \\ \bar{A}_C = A\angle - \frac{2\pi}{3}. \end{cases} \tag{2.8}$$

In power systems, the reference phasor is generally indicated using the letter R, the phasor lagging 120° (or $\frac{2\pi}{3}$ rad) using the letter S, and the phasor leading 120° (or $\frac{2\pi}{3}$ rad) using the letter T. That is:

$$\begin{cases} \bar{A}_R = A\angle 0°, \\ \bar{A}_S = A\angle -120°, \\ \bar{A}_T = A\angle +120°. \end{cases}$$

Therefore, in power systems, a balanced three-phase positive sequence is generally represented as RST, while a negative one as RTS.

2.3 Balanced Three-Phase Voltages and Currents

In this section, we analyze voltages and currents in balanced three-phase circuits with different connections.

2.3.1 Balanced Three-Phase Voltages

In a balanced three-phase circuit, we identify two balanced voltage sequences, namely *phase voltages* and *line voltages*, as described below and illustrated in Fig. 2.4.

Phase voltages are defined as the voltages between each phase and a reference point known as "common star point," usually denoted as N (upper plot of Fig. 2.4). That is:

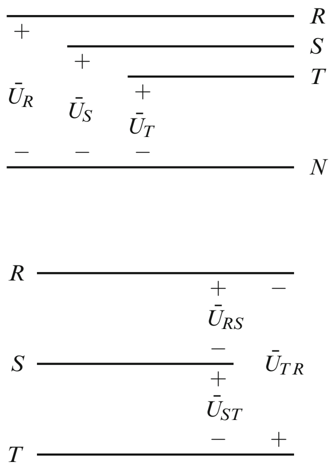


Fig. 2.4 Phase (upper plot) and line (lower plot) voltages in a balanced three-phase network

$$\begin{cases} \bar{U}_R = \bar{U}_{RN} = U_\mathrm{F}\angle 0°, \\ \bar{U}_S = \bar{U}_{SN} = U_\mathrm{F}\angle -120°, \\ \bar{U}_T = \bar{U}_{TN} = U_\mathrm{F}\angle +120°, \end{cases} \tag{2.9}$$

where U_F is the magnitude of the phase voltage.

Since phase voltages constitute a balanced sequence, we have:

$$\bar{U}_R + \bar{U}_S + \bar{U}_T = 0. \tag{2.10}$$

On the other hand, each line voltage is defined as the difference of two phase voltages (lower plot of Fig. 2.4). That is:

$$\begin{cases} \bar{U}_{RS} = \bar{U}_R - \bar{U}_S = \sqrt{3}U_\mathrm{F}\angle 30°, \\ \bar{U}_{ST} = \bar{U}_S - \bar{U}_T = \sqrt{3}U_\mathrm{F}\angle -90°, \\ \bar{U}_{TR} = \bar{U}_T - \bar{U}_R = \sqrt{3}U_\mathrm{F}\angle 150°. \end{cases} \tag{2.11}$$

Note that:

$$\bar{U}_{RS} + \bar{U}_{ST} + \bar{U}_{TR} = 0, \tag{2.12}$$

i.e., the line voltages also constitute a balanced sequence.

Phasor diagrams for both phase voltages and line voltages are shown in Fig. 2.5.

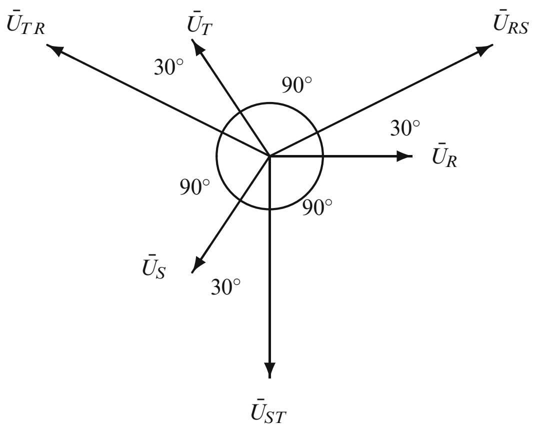


Fig. 2.5 Balanced phase and line voltages

2.3.2 Balanced Three-Phase Currents

In a balanced three-phase network, the line currents constitute a balanced sequence, i.e.:

$$\begin{cases} \bar{I}_R = I_\mathrm{L}\angle(-\varphi - 0)^\circ, \\ \bar{I}_S = I_\mathrm{L}\angle(-\varphi - 120)^\circ, \\ \bar{I}_T = I_\mathrm{L}\angle(-\varphi + 120)^\circ, \end{cases} \tag{2.13}$$

where:

- I_L is the magnitude of the line current and
- φ is the angle of a phase voltage with respect to the corresponding line current.

Line currents are shown in Fig. 2.6.

Note that:

$$\bar{I}_R + \bar{I}_S + \bar{I}_T = 0. \tag{2.14}$$

Illustrative Example 2.1 *Currents in a balanced three-phase delta-connected load*

We consider the balanced three-phase delta-connected load (impedance $\bar{Z}$ per phase) depicted in Fig. 2.7. We show below that if this load is supplied by a balanced three-phase line-current sequence ($\bar{I}_R, \bar{I}_S, \bar{I}_T$), then the delta currents, i.e., the currents "inside" the delta ($\bar{I}_{RS}, \bar{I}_{ST}, \bar{I}_{TR}$), constitute a balanced sequence as well.

From Fig. 2.7, we obtain:

Fig. 2.6 Balanced line currents

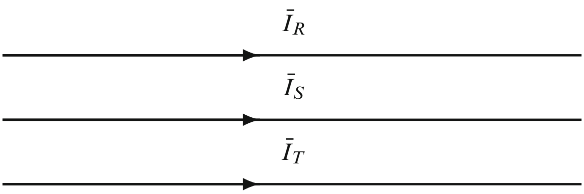
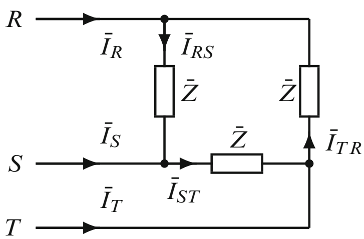


Fig. 2.7 Balanced delta currents



$$\begin{cases} \bar{I}_R + \bar{I}_{TR} = \bar{I}_{RS}, \\ \bar{I}_S + \bar{I}_{RS} = \bar{I}_{ST}, \\ \bar{I}_T + \bar{I}_{ST} = \bar{I}_{TR}. \end{cases} \tag{2.15}$$

Since:

$$\bar{I}_R + \bar{I}_S + \bar{I}_T = 0$$

and:

$$\bar{I}_S + \bar{I}_T + \bar{I}_R = 0,$$

subtracting the above two expressions renders:

$$\bar{I}_{RS} + \bar{I}_{ST} + \bar{I}_{TR} = 0. \tag{2.16}$$

Considering (2.15) and (2.16), we obtain:

$$\begin{cases} -\bar{I}_{RS} + \bar{I}_{ST} \phantom{+ \bar{I}_{TR}} = \bar{I}_S, \\ \phantom{-\bar{I}_{RS}} - \bar{I}_{ST} + \bar{I}_{TR} = \bar{I}_T, \\ \bar{I}_{RS} + \bar{I}_{ST} + \bar{I}_{TR} = 0, \end{cases}$$

which in matrix form is:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \bar{I}_{RS} \\ \bar{I}_{ST} \\ \bar{I}_{TR} \end{bmatrix} = \begin{bmatrix} \bar{I}_S \\ \bar{I}_T \\ 0 \end{bmatrix}.$$

Solving for the delta currents yields:

$$\begin{bmatrix} \bar{I}_{RS} \\ \bar{I}_{ST} \\ \bar{I}_{TR} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \bar{I}_S \\ \bar{I}_T \\ 0 \end{bmatrix}$$

or:

$$\begin{bmatrix} \bar{I}_{RS} \\ \bar{I}_{ST} \\ \bar{I}_{TR} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \bar{I}_S \\ \bar{I}_T \\ 0 \end{bmatrix}.$$

Thus, we can express delta currents as a function of line currents as:

$$\begin{cases} \bar{I}_{RS} = \dfrac{1}{3}\left(-2\bar{I}_S - \bar{I}_T\right), \\ I_{ST} = \dfrac{1}{3}\left(\bar{I}_S - \bar{I}_T\right), \\ I_{TR} = \dfrac{1}{3}\left(\bar{I}_S + 2\bar{I}_T\right). \end{cases}$$

Since $\bar{I}_R + \bar{I}_S + \bar{I}_T = 0$, we get:

$$\begin{cases} \bar{I}_{RS} = \dfrac{1}{3}\left(\bar{I}_R - \bar{I}_S\right), \\ \bar{I}_{ST} = \dfrac{1}{3}\left(\bar{I}_S - \bar{I}_T\right), \\ \bar{I}_{TR} = \dfrac{1}{3}\left(\bar{I}_T - \bar{I}_R\right), \end{cases}$$

or:

$$\begin{cases} \bar{I}_{RS} = \dfrac{1}{3}\sqrt{3}\bar{I}_R\angle + 30°, \\ \bar{I}_{ST} = \dfrac{1}{3}\sqrt{3}\bar{I}_S\angle + 30°, \\ \bar{I}_{TR} = \dfrac{1}{3}\sqrt{3}\bar{I}_T\angle + 30°, \end{cases}$$

or finally:

$$\begin{cases} \bar{I}_{RS} = \dfrac{1}{\sqrt{3}}I_{\mathrm{L}}\angle\left(-\varphi + 30\right)°, \\ \bar{I}_{ST} = \dfrac{1}{\sqrt{3}}I_{\mathrm{L}}\angle\left(-\varphi - 90\right)°, \\ \bar{I}_{TR} = \dfrac{1}{\sqrt{3}}I_{\mathrm{L}}\angle\left(-\varphi + 150\right)°. \end{cases}$$

We conclude that if the line currents used to supplied a balanced delta-connected load constitute a balanced three-phase sequence, then the delta currents $\bar{I}_{RS}$, $\bar{I}_{ST}$, and $\bar{I}_{TR}$ have equal magnitude and equal angle separation, i.e., the delta currents constitute a balanced three-phase sequence as well.

Figure 2.8 visualizes the relationship between line currents and delta currents in a balanced three-phase delta-connected load.

□

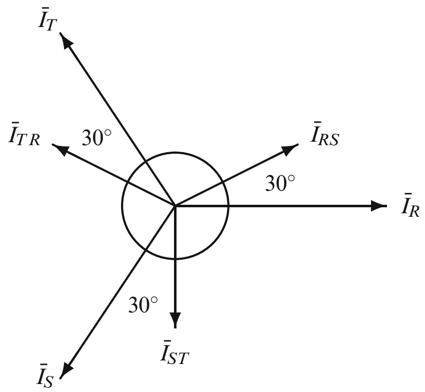


Fig. 2.8 Balanced line and delta currents in a balanced delta-connected load

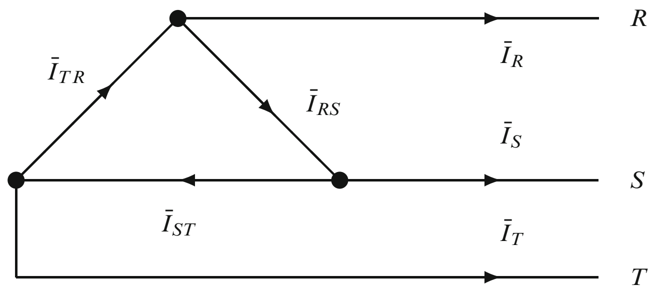


Fig. 2.9 Balanced three-phase delta-connected generator

Illustrative Example 2.2 *Currents in a balanced three-phase delta-connected generator*

We obtain below the relationships between the line and delta currents in a balanced delta-connected generator as the one in Fig. 2.9. Note that a simplified circuit diagram is used to show exclusively the reference directions for the currents.

If instead of a delta-connected load we consider a delta-connected generator, the current balance equations become (Fig. 2.9):

$$\begin{cases} -\bar{I}_R + \bar{I}_{TR} = \bar{I}_{RS}, \\ -\bar{I}_S + \bar{I}_{RS} = \bar{I}_{ST}, \\ -\bar{I}_T + \bar{I}_{ST} = \bar{I}_{TR}. \end{cases}$$

Since $\bar{I}_{RS} + \bar{I}_{ST} + \bar{I}_{TR} = 0$, we get (see (2.16)):

$$\begin{cases} \bar{I}_{RS} = \frac{1}{3}\left(\bar{I}_S - \bar{I}_R\right), \\ \bar{I}_{ST} = \frac{1}{3}\left(\bar{I}_T - \bar{I}_S\right), \\ \bar{I}_{TR} = \frac{1}{3}\left(\bar{I}_R - \bar{I}_T\right), \end{cases}$$

or:

$$\begin{cases} \bar{I}_{RS} = \frac{1}{3}\sqrt{3}\left(-\bar{I}_R\right)\angle + 30^\circ, \\ \bar{I}_{ST} = \frac{1}{3}\sqrt{3}\left(-\bar{I}_S\right)\angle + 30^\circ, \\ \bar{I}_{TR} = \frac{1}{3}\sqrt{3}\left(-\bar{I}_T\right)\angle + 30^\circ, \end{cases}$$

or similarly:

$$\begin{cases} \bar{I}_{RS} = \frac{1}{3}\sqrt{3}\bar{I}_R\angle - 150^\circ, \\ \bar{I}_{ST} = \frac{1}{3}\sqrt{3}\bar{I}_S\angle - 150^\circ, \\ \bar{I}_{TR} = \frac{1}{3}\sqrt{3}\bar{I}_T\angle - 150^\circ, \end{cases} \tag{2.17}$$

and finally:

$$\begin{cases} \bar{I}_{RS} = \frac{1}{3}\sqrt{3}\bar{I}_S\angle - 30^\circ, \\ \bar{I}_{ST} = \frac{1}{3}\sqrt{3}\bar{I}_T\angle - 30^\circ, \\ \bar{I}_{TR} = \frac{1}{3}\sqrt{3}\bar{I}_R\angle - 30^\circ. \end{cases}$$

Considering:

$$\begin{cases} \bar{I}_R = I_{\mathrm{G}}\angle 0^\circ, \\ \bar{I}_S = I_{\mathrm{G}}\angle - 120^\circ, \\ \bar{I}_T = I_{\mathrm{G}}\angle + 120^\circ, \end{cases} \tag{2.18}$$

where I_{G} is the magnitude of the line current, we have:

$$\begin{cases} \bar{I}_{RS} = \frac{1}{\sqrt{3}}I_{\mathrm{G}}\angle - 150^\circ, \\ \bar{I}_{ST} = \frac{1}{\sqrt{3}}I_{\mathrm{G}}\angle + 90^\circ, \\ \bar{I}_{TR} = \frac{1}{\sqrt{3}}I_{\mathrm{G}}\angle - 30^\circ. \end{cases} \tag{2.19}$$

From Eqs. (2.18) and (2.19) above, we conclude that the line and delta currents in a balanced three-phase delta-connected generator are balanced sequences.

□

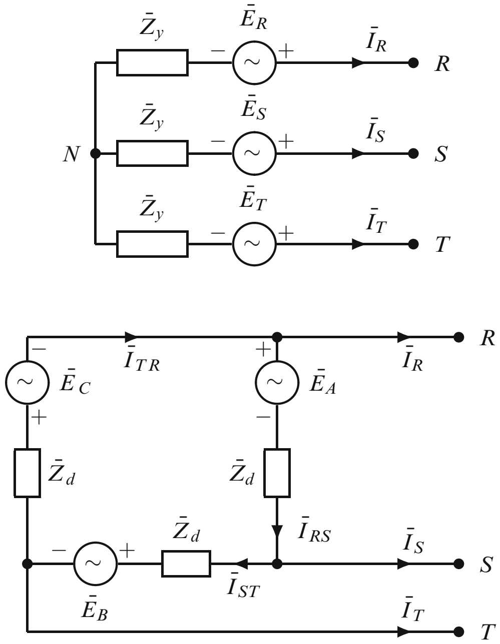


Fig. 2.10 Voltage sources: wye (upper plot) and delta (lower plot) connected

2.3.3 Equivalence Wye-Delta

To preserve balance, three balanced voltage sources can be either wye (y) or delta (d) connected, as shown in the upper and lower plots of Fig. 2.10, respectively. In the wye connection, point N, known as the common start point, is considered as the reference for phase voltages.

We consider the balanced voltage source sequence:

$$\begin{cases} \bar{E}_R = E_\mathrm{F}\angle 0^\circ, \\ \bar{E}_S = E_\mathrm{F}\angle -120^\circ, \\ \bar{E}_T = E_\mathrm{F}\angle +120^\circ, \end{cases} \tag{2.20}$$

where E_F is the magnitude of each voltage source, while the balanced voltage-source current sequence is:

$$\begin{cases} \bar{I}_R = I_\mathrm{L}\angle\left(-\varphi - 0\right)^\circ, \\ \bar{I}_S = I_\mathrm{L}\angle\left(-\varphi - 120\right)^\circ, \\ \bar{I}_T = I_\mathrm{L}\angle\left(-\varphi + 120\right)^\circ, \end{cases} \tag{2.21}$$

where I_{L} is the magnitude of the line current. We derive below equivalence conditions for the wye and delta connections considering the circuits in Fig. 2.10.

On the one hand, these circuits should be equivalent under no load conditions, i.e., if currents are null:

$$\bar{I}_R = \bar{I}_S = \bar{I}_T = 0. \tag{2.22}$$

Then:

$$\begin{cases} \bar{E}_R - \bar{E}_S = \bar{E}_A = \sqrt{3}\bar{E}_R\angle 30^\circ = \sqrt{3}E_{\mathrm{F}}\angle 30^\circ, \\ \bar{E}_S - \bar{E}_T = \bar{E}_B = \sqrt{3}\bar{E}_S\angle 30^\circ = \sqrt{3}E_{\mathrm{F}}\angle -90^\circ, \\ \bar{E}_T - \bar{E}_R = \bar{E}_C = \sqrt{3}\bar{E}_T\angle 30^\circ = \sqrt{3}E_{\mathrm{F}}\angle 150^\circ. \end{cases} \tag{2.23}$$

On the other hand, these circuits should be equivalent under load conditions as well. Thus:

$$\begin{cases} \bar{I}_{TR} - \bar{I}_{RS} = \bar{I}_R, \\ \bar{I}_{RS} - \bar{I}_{ST} = \bar{I}_S, \\ \bar{I}_{ST} - \bar{I}_{TR} = \bar{I}_T. \end{cases} \tag{2.24}$$

Considering (2.17) and (2.18) renders:

$$\begin{cases} \bar{I}_{RS} = \dfrac{1}{\sqrt{3}}\bar{I}_S\angle -30^\circ, \\ \bar{I}_{ST} = \dfrac{1}{\sqrt{3}}\bar{I}_T\angle -30^\circ, \\ \bar{I}_{TR} = \dfrac{1}{\sqrt{3}}\bar{I}_R\angle -30^\circ, \end{cases} \tag{2.25}$$

or:

$$\begin{cases} \bar{I}_{RS} = \dfrac{1}{\sqrt{3}}I_{\mathrm{G}}\angle -150^\circ, \\ \bar{I}_{ST} = \dfrac{1}{\sqrt{3}}I_{\mathrm{G}}\angle +90^\circ, \\ \bar{I}_{TR} = \dfrac{1}{\sqrt{3}}I_{\mathrm{G}}\angle -30^\circ. \end{cases} \tag{2.26}$$

Under load conditions, line voltages in both connections should be equal. Thus, from Fig. 2.10, we have:

$$\bar{E}_A + \bar{I}_{RS}\bar{Z}_d = -\bar{E}_S + \bar{Z}_y\bar{I}_S - \bar{Z}_y\bar{I}_R + \bar{E}_R, \tag{2.27}$$

or:

$$\bar{E}_A + \bar{I}_{RS}\bar{Z}_d = \bar{E}_R - \bar{E}_S + \left(\bar{I}_S - \bar{I}_R\right)\bar{Z}_y. \tag{2.28}$$

Considering (2.23), we obtain:

$$\bar{E}_A + \bar{I}_{RS}\bar{Z}_d = \bar{E}_A + \sqrt{3}\bar{I}_R\angle-150^\circ\bar{Z}_y, \tag{2.29}$$

or:

$$\frac{1}{\sqrt{3}}\bar{I}_R\angle{-150^\circ}\bar{Z}_d = \sqrt{3}\bar{I}_R\angle-150^\circ\bar{Z}_y, \tag{2.30}$$

as well as:

$$\frac{1}{\sqrt{3}}\bar{Z}_d = \sqrt{3}\bar{Z}_y, \tag{2.31}$$

and finally:

$$\bar{Z}_d = 3\bar{Z}_y. \tag{2.32}$$

We conclude that the equivalence conditions for the wye and delta connections in the circuits in Fig. 2.10 are as follows:

$$\begin{cases} \bar{E}_A = \sqrt{3}\bar{E}_R\angle30^\circ, \\ \bar{E}_B = \sqrt{3}\bar{E}_S\angle30^\circ, \\ \bar{E}_C = \sqrt{3}\bar{E}_T\angle30^\circ, \end{cases} \tag{2.33}$$

and:

$$\bar{Z}_d = 3\bar{Z}_y, \tag{2.34}$$

or alternatively:

$$\begin{cases} \bar{E}_R = \dfrac{1}{\sqrt{3}}\bar{E}_A\angle-30^\circ, \\ \bar{E}_S = \dfrac{1}{\sqrt{3}}\bar{E}_B\angle-30^\circ, \\ \bar{E}_T = \dfrac{1}{\sqrt{3}}\bar{E}_C\angle-30^\circ, \end{cases} \tag{2.35}$$

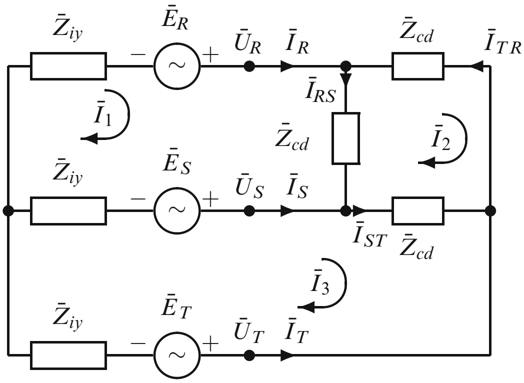


Fig. 2.11 Illustrative Example 2.3: balanced three-phase circuit

and:

$$\bar{Z}_y = \frac{1}{3}\bar{Z}_d. \tag{2.36}$$

Illustrative Example 2.3 *Balanced three-phase circuit*

We consider the circuit depicted in Fig. 2.11 in which voltage sources constitute a known balanced three-phase positive sequence and impedances $\bar{Z}_{iy}$ and $\bar{Z}_{cd}$ are also known. We compute below:

1. Currents $\bar{I}_R$, $\bar{I}_S$, and $\bar{I}_T$.
2. Currents $\bar{I}_{RS}$, $\bar{I}_{ST}$, and $\bar{I}_{TR}$.
3. Voltages $\bar{U}_R$, $\bar{U}_S$, and $\bar{U}_T$.

Since voltage sources constitute a balanced three-phase positive sequence, we have:

$$\begin{cases} \bar{E}_R, \\ \bar{E}_S = \bar{\alpha}^2 \bar{E}_R, \\ \bar{E}_T = \bar{\alpha}\bar{E}_R, \end{cases}$$

where $\bar{\alpha} = 1\angle 120°$. Additionally note that $\bar{\alpha}^2 = -\bar{\alpha} = -1\angle 120°$

The circuit in Fig. 2.11 is solved below using the mesh-current method [5]:

$$\begin{bmatrix} 2\bar{Z}_{iy} + \bar{Z}_{cd} & -\bar{Z}_{cd} & -\bar{Z}_{iy} \\ -\bar{Z}_{cd} & 3\bar{Z}_{cd} & -\bar{Z}_{cd} \\ -\bar{Z}_{iy} & -\bar{Z}_{cd} & 2\bar{Z}_{iy} + \bar{Z}_{cd} \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} \bar{E}_R - \bar{E}_S \\ 0 \\ \bar{E}_S - \bar{E}_T \end{bmatrix}$$

or:

$$\begin{bmatrix} 2\bar{Z}_{iy}+\bar{Z}_{cd} & -\bar{Z}_{cd} & -\bar{Z}_{iy} \\ -\bar{Z}_{cd} & 3\bar{Z}_{cd} & -\bar{Z}_{cd} \\ -\bar{Z}_{iy} & -\bar{Z}_{cd} & 2\bar{Z}_{iy}+\bar{Z}_{cd} \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} \bar{E}_R\left(1-\bar{\alpha}^2\right) \\ 0 \\ \bar{E}_R\left(\bar{\alpha}^2-\bar{\alpha}\right) \end{bmatrix}.$$

Solving for the currents:

$$\begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} 2\bar{Z}_{iy}+\bar{Z}_{cd} & -\bar{Z}_{cd} & -\bar{Z}_{iy} \\ -\bar{Z}_{cd} & 3\bar{Z}_{cd} & -\bar{Z}_{cd} \\ -\bar{Z}_{iy} & -\bar{Z}_{cd} & 2\bar{Z}_{iy}+\bar{Z}_{cd} \end{bmatrix}^{-1} \begin{bmatrix} \bar{E}_R\left(1-\bar{\alpha}^2\right) \\ 0 \\ \bar{E}_R\left(\alpha^2-\alpha\right) \end{bmatrix},$$

and:

$$\begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \frac{1}{3\bar{Z}_{iy}+\bar{Z}_{cd}} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1+\dfrac{\bar{Z}_{iy}}{\bar{Z}_{cd}} & 1 \\ 1 & 1 & 2 \end{bmatrix} \bar{E}_R \begin{bmatrix} 1-\bar{\alpha}^2 \\ 0 \\ \bar{\alpha}^2-\bar{\alpha} \end{bmatrix},$$

or:

$$\begin{cases} \bar{I}_1 = \dfrac{3\bar{E}_R}{3\bar{Z}_{iy}+\bar{Z}_{cd}}, \\ \bar{I}_2 = \dfrac{(1-\bar{\alpha})\,\bar{E}_R}{3\bar{Z}_{iy}+\bar{Z}_{cd}}, \\ \bar{I}_3 = \dfrac{-3\bar{\alpha}\bar{E}_R}{3\bar{Z}_{iy}+\bar{Z}_{cd}}. \end{cases}$$

Thus, currents $\bar{I}_R$, $\bar{I}_S$, and $\bar{I}_T$ are:

$$\begin{cases} \bar{I}_R = \bar{I}_1 = \dfrac{\bar{E}_R}{\bar{Z}_{iy}+\frac{1}{3}\bar{Z}_{cd}}, \\ \bar{I}_S = \bar{I}_3 - \bar{I}_1 = \dfrac{(-\bar{\alpha}-1)\,\bar{E}_R}{\bar{Z}_{iy}+\frac{1}{3}\bar{Z}_{cd}} = \dfrac{\bar{\alpha}^2\bar{E}_R}{\bar{Z}_{iy}+\frac{1}{3}\bar{Z}_{cd}}, \\ \bar{I}_T = -\bar{I}_3 = \dfrac{\bar{\alpha}\bar{E}_R}{\bar{Z}_{iy}+\frac{1}{3}\bar{Z}_{cd}}, \end{cases} \tag{2.37}$$

or:

$$\begin{cases} \bar{I}_R = \dfrac{\bar{E}_R}{\bar{Z}_{iy} + \frac{1}{3}\bar{Z}_{cd}}, \\ \bar{I}_S = \dfrac{\bar{E}_S}{\bar{Z}_{iy} + \frac{1}{3}\bar{Z}_{cd}}, \\ \bar{I}_T = \dfrac{\bar{E}_T}{\bar{Z}_{iy} + \frac{1}{3}\bar{Z}_{cd}}. \end{cases}$$

On the other hand, currents $\bar{I}_{RS}$, $\bar{I}_{ST}$, and $\bar{I}_{TR}$ are computed as:

$$\begin{cases} \bar{I}_{RS} = \bar{I}_1 - \bar{I}_2 = \dfrac{E_R\,(3 - 1 + \bar{\alpha})}{3\bar{Z}_{iy} + \bar{Z}_{cd}}, \\ \bar{I}_{ST} = \bar{I}_3 - \bar{I}_2 = \dfrac{\bar{E}_R\,(-3\bar{\alpha} - 1 + \bar{\alpha})}{3\bar{Z}_{iy} + \bar{Z}_{cd}}, \\ \bar{I}_{TR} = -\bar{I}_2 = \dfrac{\bar{E}_R\,(\bar{\alpha} - 1)}{3\bar{Z}_{iy} + \bar{Z}_{cd}}, \end{cases}$$

or:

$$\begin{cases} \bar{I}_{RS} = \dfrac{\bar{E}_R\left(1 - \bar{\alpha}^2\right)}{3\bar{Z}_{iy} + \bar{Z}_{cd}}, \\ \bar{I}_{ST} = \dfrac{\bar{E}_R\left(\bar{\alpha}^2 - \bar{\alpha}\right)}{3\bar{Z}_{iy} + \bar{Z}_{cd}} = \dfrac{\bar{\alpha}^2\bar{E}_R\left(1 - \frac{1}{\bar{\alpha}}\right)}{3\bar{Z}_{iy} + \bar{Z}_{cd}} = \dfrac{\bar{E}_S\left(1 - \bar{\alpha}^2\right)}{3\bar{Z}_{iy} + \bar{Z}_{cd}}, \\ \bar{I}_{TR} = \dfrac{\bar{E}_R\,(\bar{\alpha} - 1)}{3\bar{Z}_{iy} + \bar{Z}_{cd}} = \dfrac{\bar{\alpha}\bar{E}_R\left(1 - \frac{1}{\bar{\alpha}}\right)}{3\bar{Z}_{iy} + \bar{Z}_{cd}} = \dfrac{\bar{E}_T\left(1 - \bar{\alpha}^2\right)}{3\bar{Z}_{iy} + \bar{Z}_{cd}}. \end{cases}$$

Note that the relationship between currents $\bar{I}_{RS}, \bar{I}_{ST}, \bar{I}_{TR}$ and $\bar{I}_R, \bar{I}_S, \bar{I}_T$ is as follows:

$$\frac{\bar{I}_{RS}}{\bar{I}_R} = \frac{\bar{I}_{ST}}{\bar{I}_S} = \frac{\bar{I}_{TR}}{\bar{I}_T} = \frac{1 - \bar{\alpha}^2}{3} = \frac{\sqrt{3}\angle 30^\circ}{3} = \frac{1}{\sqrt{3}}\angle 30^\circ.$$

Finally, voltages $\bar{U}_R$, $\bar{U}_S$, and $\bar{U}_T$ are computed as:

$$\begin{cases} \bar{U}_R = \bar{E}_R - \bar{I}_R\bar{Z}_{iy}, \\ \bar{U}_S = \bar{E}_S - \bar{I}_S\bar{Z}_{iy}, \\ \bar{U}_T = \bar{E}_T - \bar{I}_T\bar{Z}_{iy}, \end{cases} \tag{2.38}$$

or:

$$\begin{cases} \bar{U}_R = \bar{E}_R - \dfrac{\bar{E}_R \bar{Z}_{iy}}{\bar{Z}_{iy} + \frac{1}{3}\bar{Z}_{cd}}, \\ \bar{U}_S = \bar{E}_S - \dfrac{\bar{E}_S \bar{Z}_{iy}}{\bar{Z}_{iy} + \frac{1}{3}\bar{Z}_{cd}}, \\ \bar{U}_T = \bar{E}_T - \dfrac{\bar{E}_T \bar{Z}_{iy}}{\bar{Z}_{iy} + \frac{1}{3}\bar{Z}_{cd}}, \end{cases}$$

and finally:

$$\begin{cases} \bar{U}_R = \bar{E}_R \left(1 - \dfrac{\bar{Z}_{iy}}{\bar{Z}_{iy} + \frac{1}{3}\bar{Z}_{cd}} \right), \\ \bar{U}_S = \bar{E}_S \left(1 - \dfrac{\bar{Z}_{iy}}{\bar{Z}_{iy} + \frac{1}{3}\bar{Z}_{cd}} \right), \\ \bar{U}_T = \bar{E}_T \left(1 - \dfrac{\bar{Z}_{iy}}{\bar{Z}_{iy} + \frac{1}{3}\bar{Z}_{cd}} \right). \end{cases}$$

Note that $(\bar{I}_R, \bar{I}_S, \bar{I}_T)$, $(\bar{I}_{RS}, \bar{I}_{ST}, \bar{I}_{TR})$, and $(\bar{U}_R, \bar{U}_S, \bar{U}_T)$ constitute balanced three-phase positive sequences.

Currents and voltages have been obtained by analyzing the three-phase circuit depicted in Fig. 2.11. However, note that the resulting equations for line currents (2.37) and phase voltages (2.38) are decoupled per phase. Thus, instead of using the three-phase circuit in Fig. 2.11, it is possible to use the three equivalent single-phase circuits, one per phase, depicted in Fig. 2.12.

□

Illustrative Example 2.4 *Equivalent single-phase circuits*

We consider the balanced three-phase circuit depicted in Fig. 2.13. Impedances $\bar{Z}_{id}$ and $\bar{Z}_{cy}$, as well as voltage sources are known. Moreover, voltage sources constitute a balanced three-phase positive sequence and, thus:

$$\begin{cases} \bar{E}_A, \\ \bar{E}_B = \bar{\alpha}^2 \bar{E}_A, \\ \bar{E}_C = \bar{\alpha} \bar{E}_A. \end{cases} \tag{2.39}$$

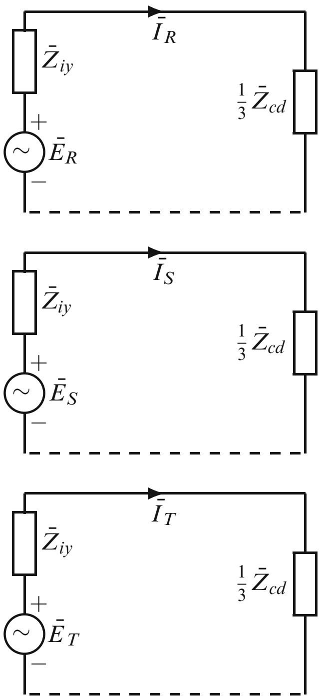


Fig. 2.12 Illustrative Example 2.3: equivalent single-phase circuits

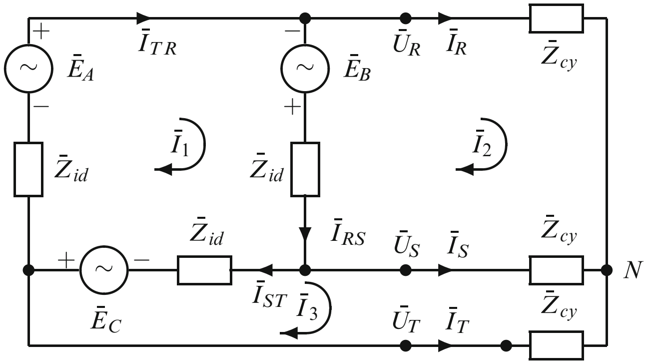


Fig. 2.13 Illustrative Example 2.4: balanced three-phase circuit

We compute below:

1. Currents $\bar{I}_R$, $\bar{I}_S$, and $\bar{I}_T$.
2. Currents $\bar{I}_{RS}$, $\bar{I}_{ST}$, and $\bar{I}_{TR}$.
3. Phase voltages $\bar{U}_R$, $\bar{U}_S$, and $\bar{U}_T$.

We solve the circuit in Fig. 2.13 using the mesh-current method [5]:

$$\begin{bmatrix} 3\bar{Z}_{id} & -\bar{Z}_{id} & -\bar{Z}_{id} \\ -\bar{Z}_{id} & 2\bar{Z}_{cy}+\bar{Z}_{id} & -\bar{Z}_{cy} \\ -\bar{Z}_{id} & -\bar{Z}_{cy} & 2\bar{Z}_{cy}+\bar{Z}_{id} \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\bar{\alpha}^2\bar{E}_A \\ -\bar{\alpha}\bar{E}_A \end{bmatrix}.$$

Solving for the currents:

$$\begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} 3\bar{Z}_{id} & -\bar{Z}_{id} & -\bar{Z}_{id} \\ -\bar{Z}_{id} & 2\bar{Z}_{cy}+\bar{Z}_{id} & -\bar{Z}_{cy} \\ -\bar{Z}_{id} & -\bar{Z}_{cy} & 2\bar{Z}_{cy}+\bar{Z}_{id} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\bar{\alpha}^2\bar{E}_A \\ -\bar{\alpha}\bar{E}_A \end{bmatrix},$$

and:

$$\begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \frac{1}{\bar{Z}_{id}+3\bar{Z}_{cy}} \begin{bmatrix} 1+\dfrac{\bar{Z}_{cy}}{\bar{Z}_{id}} & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -\bar{\alpha}^2\bar{E}_A \\ -\bar{\alpha}\bar{E}_A \end{bmatrix},$$

and finally:

$$\begin{cases} \bar{I}_1 = \dfrac{1}{\bar{Z}_{id}+3\bar{Z}_{cy}}\bar{E}_A, \\ \bar{I}_2 = \dfrac{1}{\bar{Z}_{id}+3\bar{Z}_{cy}}\bar{E}_A\left(1-\bar{\alpha}^2\right), \\ \bar{I}_3 = \dfrac{1}{\bar{Z}_{id}+3\bar{Z}_{cy}}\bar{E}_A\left(1-\bar{\alpha}\right). \end{cases}$$

Then, we can compute currents $\bar{I}_R$, $\bar{I}_S$, and $\bar{I}_T$ as follows:

$$\begin{cases} \bar{I}_R = \bar{I}_2 = \dfrac{\bar{E}_A\left(1-\bar{\alpha}^2\right)}{\bar{Z}_{id}+3\bar{Z}_{cy}}, \\ \bar{I}_S = \bar{I}_3 - \bar{I}_2 = \dfrac{\bar{E}_A\left(\bar{\alpha}^2-\bar{\alpha}\right)}{\bar{Z}_{id}+3\bar{Z}_{cy}}, \\ \bar{I}_T = -\bar{I}_3 = \dfrac{\bar{E}_A\left(\bar{\alpha}-1\right)}{\bar{Z}_{id}+3\bar{Z}_{cy}}, \end{cases}$$

or:

$$\begin{cases} \bar{I}_R = \dfrac{\bar{E}_A\left(1-\bar{\alpha}^2\right)}{\bar{Z}_{id}+3\bar{Z}_{cy}}, \\ \bar{I}_S = \dfrac{\bar{E}_B\left(1-\bar{\alpha}^2\right)}{\bar{Z}_{id}+3\bar{Z}_{cy}}, \\ \bar{I}_T = \dfrac{\bar{E}_C\left(1-\bar{\alpha}^2\right)}{\bar{Z}_{id}+3\bar{Z}_{cy}}. \end{cases}$$

Next, mesh currents $\bar{I}_{RS}$, $\bar{I}_{ST}$, and $\bar{I}_{TR}$ are computed as:

$$\begin{cases} \bar{I}_{RS} = \bar{I}_1 - \bar{I}_2 = \dfrac{\bar{\alpha}^2\bar{E}_A}{\bar{Z}_{id}+3\bar{Z}_{cy}}, \\ \bar{I}_{ST} = \bar{I}_1 - \bar{I}_3 = \dfrac{\bar{\alpha}\bar{E}_A}{\bar{Z}_{id}+3\bar{Z}_{cy}}, \\ \bar{I}_{TR} = \bar{I}_1 = \dfrac{\bar{E}_A}{\bar{Z}_{id}+3\bar{Z}_{cy}}, \end{cases}$$

or:

$$\begin{cases} \bar{I}_{RS} = \dfrac{\bar{E}_B}{\bar{Z}_{id}+3\bar{Z}_{cy}}, \\ \bar{I}_{ST} = \dfrac{\bar{E}_C}{\bar{Z}_{id}+3\bar{Z}_{cy}}, \\ \bar{I}_{TR} = \dfrac{\bar{E}_A}{\bar{Z}_{id}+3\bar{Z}_{cy}}. \end{cases}$$

Note that line currents ($\bar{I}_R$, $\bar{I}_S$, $\bar{I}_T$) and mesh currents ($\bar{I}_{RS}$, $\bar{I}_{ST}$, $\bar{I}_{TR}$) constitute balanced three-phase positive sequences. The relationship between line and mesh currents is as follows:

$$\frac{\bar{I}_{RS}}{\bar{I}_R} = \frac{\bar{I}_{ST}}{\bar{I}_S} = \frac{\bar{I}_{TR}}{\bar{I}_T} = \frac{\bar{\alpha}^2}{1-\bar{\alpha}^2} = \frac{1}{\sqrt{3}}\angle -150^\circ.$$

Note that phases R, S, and T are decoupled. Thus, instead of analyzing the three-phase circuit in Fig. 2.13, it is possible to consider the three equivalent single-phase circuits depicted in Fig. 2.14. Using these equivalent single-phase circuits, we obtain that phase voltages are equal to:

$$\begin{cases} \bar{U}_R = \bar{I}_R\bar{Z}_{cy}, \\ \bar{U}_S = \bar{I}_S\bar{Z}_{cy}, \\ \bar{U}_T = \bar{I}_T\bar{Z}_{cy}, \end{cases}$$

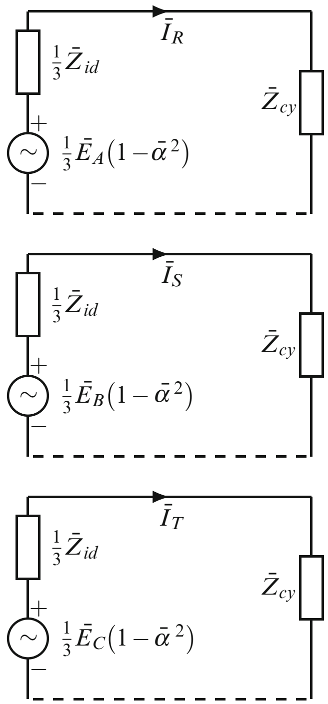


Fig. 2.14 Illustrative Example 2.4: equivalent single-phase circuits

or:

$$\begin{cases} \bar{U}_R = \quad \bar{E}_A\left(1-\bar{\alpha}^2\right)\dfrac{\bar{Z}_{cy}}{\bar{Z}_{id}+3\bar{Z}_{cy}}, \\ \bar{U}_S = \bar{\alpha}^2\bar{E}_A\left(1-\bar{\alpha}^2\right)\dfrac{\bar{Z}_{cy}}{\bar{Z}_{id}+3\bar{Z}_{cy}}, \\ \bar{U}_T = \bar{\alpha}\bar{E}_A\left(1-\bar{\alpha}^2\right)\dfrac{\bar{Z}_{cy}}{\bar{Z}_{id}+3\bar{Z}_{cy}}. \end{cases}$$

As in the case of line and mesh currents, the phase voltages $\bar{U}_R$, $\bar{U}_S$, $\bar{U}_T$ constitute also a balanced three-phase positive sequence.

□

2.3.4 Common Star Connection

We consider the balanced three-phase network depicted in Fig. 2.15. In this circuit, we have:

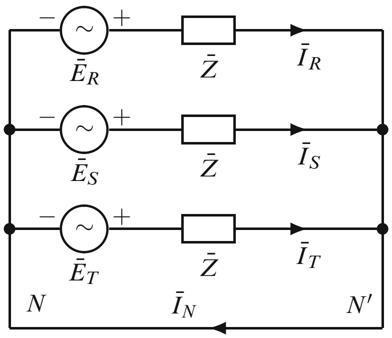


Fig. 2.15 Common star connection: analysis of circuit with connection NN'

$$\begin{cases} \bar{E}_R - \bar{Z}\bar{I}_R = 0, \\ \bar{E}_S - \bar{Z}\bar{I}_S = 0, \\ \bar{E}_T - \bar{Z}\bar{I}_T = 0, \end{cases} \tag{2.40}$$

and:

$$\begin{cases} \bar{I}_R = \dfrac{\bar{E}_R}{\bar{Z}}, \\ \bar{I}_S = \dfrac{\bar{E}_S}{\bar{Z}}, \\ \bar{I}_T = \dfrac{\bar{E}_T}{\bar{Z}}. \end{cases} \tag{2.41}$$

Adding these currents, we obtain:

$$\bar{I}_N = \bar{I}_R + \bar{I}_S + \bar{I}_T = \frac{1}{\bar{Z}}(\bar{E}_R + \bar{E}_S + \bar{E}_T). \tag{2.42}$$

Since voltage sources constitute a balanced sequence, we obtain:

$$\bar{I}_N = 0 \tag{2.43}$$

and, thus:

$$\bar{U}_{NN'} = 0. \tag{2.44}$$

That is, connection NN' is immaterial.

Next, we analyze the same network, but in this case without connection NN', as depicted in Fig. 2.16. Solving this circuit by the mesh-current method [5], we obtain:

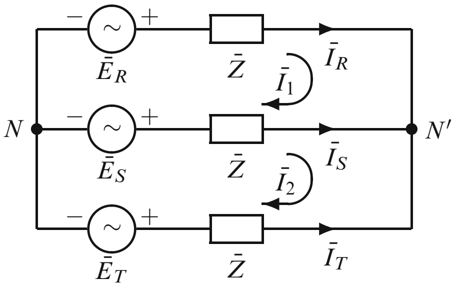


Fig. 2.16 Common star connection: analysis of circuit without connection NN'

$$\bar{Z}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \bar{E}_R\begin{bmatrix} 1-\bar{\alpha}^2 \\ \bar{\alpha}^2-\bar{\alpha} \end{bmatrix}. \tag{2.45}$$

Thus:

$$\begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \frac{\bar{E}_R}{\bar{Z}}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1-\bar{\alpha}^2 \\ \bar{\alpha}^2-\bar{\alpha} \end{bmatrix}, \tag{2.46}$$

and:

$$\begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \frac{\bar{E}_R}{3\bar{Z}}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1-\bar{\alpha}^2 \\ \bar{\alpha}^2-\bar{\alpha} \end{bmatrix}, \tag{2.47}$$

and finally:

$$\begin{cases} \bar{I}_1 = \dfrac{\bar{E}_R}{\bar{Z}}, \\ \bar{I}_2 = -\bar{\alpha}\dfrac{\bar{E}_R}{Z} = -\dfrac{\bar{E}_T}{\bar{Z}}. \end{cases} \tag{2.48}$$

Then, line currents can be computed as:

$$\begin{cases} \bar{I}_R = \bar{I}_1 = \dfrac{1}{\bar{Z}}\bar{E}_R, \\ \bar{I}_S = \bar{I}_2 - \bar{I}_1 = \dfrac{1}{\bar{Z}}\bar{E}_R\left(-\bar{\alpha}-1\right), \\ \bar{I}_T = -\bar{I}_2 = \dfrac{1}{\bar{Z}}\bar{E}_T, \end{cases} \tag{2.49}$$

or:

$$\begin{cases} \bar{I}_R = \dfrac{1}{\bar{\bar{Z}}}\bar{E}_R, \\ \bar{I}_S = \dfrac{1}{\bar{\bar{Z}}}\bar{E}_S, \\ \bar{I}_T = \dfrac{1}{\bar{\bar{Z}}}\bar{E}_T, \end{cases} \tag{2.50}$$

which is the same result previously obtained in Eq. (2.41) for the circuit with NN' connection.

Note also that:

$$\bar{U}_{N'N} = \bar{E}_R - \bar{I}_R\bar{Z} = \bar{E}_R - \frac{1}{\bar{\bar{Z}}}\bar{E}_R\bar{Z} = 0, \tag{2.51}$$

as expected.

We note once more that phase equations are decoupled: the equation of a given phase depends only on variables and constants of that phase. That is:

$$\begin{cases} \bar{E}_R = \bar{Z}\bar{I}_R, \\ \bar{E}_S = \bar{Z}\bar{I}_S, \\ \bar{E}_T = \bar{Z}\bar{I}_T, \end{cases} \tag{2.52}$$

or, in matrix form:

$$\begin{bmatrix} \bar{E}_R \\ \bar{E}_S \\ \bar{E}_T \end{bmatrix} = \begin{bmatrix} \bar{Z} & 0 & 0 \\ 0 & \bar{Z} & 0 \\ 0 & 0 & \bar{Z} \end{bmatrix} \begin{bmatrix} \bar{I}_R \\ \bar{I}_S \\ \bar{I}_T \end{bmatrix}. \tag{2.53}$$

In other words, the impedance matrix is diagonal, which verifies phase decoupling. Thus, we can consider three independent single-phase networks as depicted in Fig. 2.17.

Typically, the R equivalent single-phase circuit is used to represent the balanced three-phase circuit. Note that such single-phase circuit includes all required information to characterize the balanced three-phase circuit. The other two equivalent single-phase circuits replicate the R one; circuit S lagging 120°, and circuit T leading 120°.

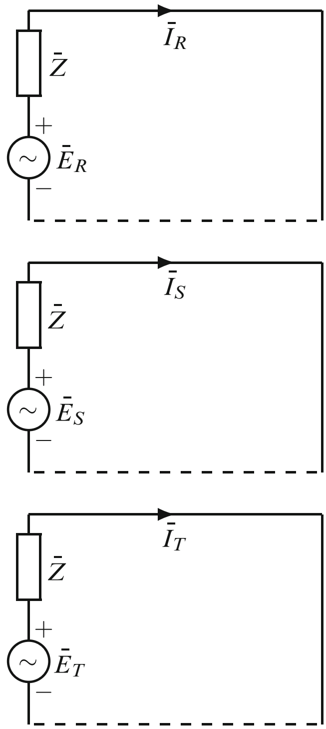


Fig. 2.17 Equivalent single-phase circuits

2.4 Instantaneous, Active, Reactive, and Apparent Power

One of the most important magnitudes in three-phase circuits is the power, which is analyzed in this section.

2.4.1 *Definitions*

The instantaneous power at any point of a three-phase circuit is defined as:

$$\begin{aligned} p(t) &\triangleq p_R(t) + p_S(t) + p_T(t) \\ &= u_R(t)i_R(t) + u_S(t)i_S(t) + u_T(t)i_T(t). \end{aligned} \tag{2.54}$$

Thus, considering a balanced three-phase circuit, $p(t)$ can be computed as:

$$\begin{aligned} p(t) = &\sqrt{2}U_{\mathrm{F}}\sin(\omega t)\sqrt{2}I_{\mathrm{L}}\sin(\omega t - \varphi) \\ &+\sqrt{2}U_{\mathrm{F}}\sin\left(\omega t - \frac{2\pi}{3}\right)\sqrt{2}I_{\mathrm{L}}\sin\left(\omega t - \frac{2\pi}{3} - \varphi\right) \end{aligned}$$

$$+\sqrt{2}U_{\mathrm{F}}\sin\left(\omega t+\frac{2\pi}{3}\right)\sqrt{2}I_{\mathrm{L}}\sin\left(\omega t+\frac{2\pi}{3}-\varphi\right), \tag{2.55}$$

where:

- U_{F} is the RMS value of the phase voltage and
- I_{L} is the RMS value of the line current.

Rearranging terms:

$$\begin{aligned} p(t) = {} & U_{\mathrm{F}}I_{\mathrm{L}}\left[\cos\varphi-\cos\left(2[\omega t-0]-\varphi\right)\quad\right] \\ & + U_{\mathrm{F}}I_{\mathrm{L}}\left[\cos\varphi-\cos\left(2\left[\omega t-\frac{2\pi}{3}\right]-\varphi\right)\right] \\ & + U_{\mathrm{F}}I_{\mathrm{L}}\left[\cos\varphi-\cos\left(2\left[\omega t+\frac{2\pi}{3}\right]-\varphi\right)\right] \end{aligned} \tag{2.56}$$

and:

$$p(t)=3U_{\mathrm{F}}I_{\mathrm{L}}\cos\varphi=3\frac{U_{\mathrm{L}}}{\sqrt{3}}I_{\mathrm{L}}\cos\varphi. \tag{2.57}$$

That is:

$$p(t)=\sqrt{3}U_{\mathrm{L}}I_{\mathrm{L}}\cos\varphi, \tag{2.58}$$

where U_{L} is the RMS value of the line voltage. Thus, the instantaneous power at any point of a balanced three-phase circuit is time invariant.

The three-phase active power, denoted by P, is equal to the instantaneous power and, thus:

$$P\triangleq p(t)=\sqrt{3}U_{\mathrm{L}}I_{\mathrm{L}}\cos\varphi. \tag{2.59}$$

The fact that in three-phase power systems the three-phase active power is time-invariant makes these systems preferable over single-phase systems, in which the active power has a nonzero average value, but is alternating. Alternating active power results in vibrations and long-term mechanical issues, while time-invariant active power does not. This is indeed a key reason for using three-phase systems instead of single-phase ones.

The per-phase complex power is computed as:

$$\begin{cases} \bar{S}_R=\bar{U}_R\bar{I}_R^*=\dfrac{U_{\mathrm{L}}}{\sqrt{3}}I_{\mathrm{L}}\angle\varphi, \\ \bar{S}_S=\bar{U}_S\bar{I}_S^*=\dfrac{U_{\mathrm{L}}}{\sqrt{3}}I_{\mathrm{L}}\angle\left(-\dfrac{2\pi}{3}+\dfrac{2\pi}{3}+\varphi\right)=\dfrac{U_{\mathrm{L}}}{\sqrt{3}}I_{\mathrm{L}}\angle\varphi, \\ \bar{S}_T=\bar{U}_T\bar{I}_T^*=\dfrac{U_{\mathrm{L}}}{\sqrt{3}}I_{\mathrm{L}}\angle\left(\dfrac{2\pi}{3}-\dfrac{2\pi}{3}+\varphi\right)=\dfrac{U_{\mathrm{L}}}{\sqrt{3}}I_{\mathrm{L}}\angle\varphi. \end{cases} \tag{2.60}$$

Then, the three-phase complex power, $\bar{S}$, is computed as:

$$\begin{aligned}\bar{S} = \bar{S}_R + \bar{S}_S + \bar{S}_T &= \sqrt{3}U_{\mathrm{L}}I_{\mathrm{L}}\angle\varphi \\ &= \sqrt{3}U_{\mathrm{L}}I_{\mathrm{L}}\cos\varphi + j\sqrt{3}U_{\mathrm{L}}I_{\mathrm{L}}\sin\varphi \\ &= P + jQ,\end{aligned} \tag{2.61}$$

where:

$$Q \triangleq \sqrt{3}U_{\mathrm{L}}I_{\mathrm{L}}\sin\varphi \tag{2.62}$$

is the three-phase reactive power.

The magnitude of the three-phase complex power (S) is the so-called three-phase apparent power.

2.4.2 *How to Measure Power?*

Note that the active power can be computed using either of the two expressions below:

$$P = U_R I_R \cos\varphi_R + U_S I_S \cos\varphi_S + U_T I_T \cos\varphi_T, \tag{2.63}$$

or:

$$P = \sqrt{3}U_{\mathrm{L}}I_{\mathrm{L}}\cos\varphi. \tag{2.64}$$

Equation (2.64) requires the system to be balanced, while Eq. (2.63) does not.

Active power is generally measured using a watt-meter that multiplies three terms: the RMS value of current, the RMS value of the voltage, and the cosine of the angle between these two signals (see (2.64)).

On the other hand, reactive power is measured as active power, but using var-meters that, instead of multiplying by the cosine, multiply by the sine.

Finally, apparent power is measured using a volt-meter and an amp-meter.

If we need to measure energy, then we should use an energy meter, which is a watt-meter that integrates over time.

2.5 Why Three-Phase Power?

We illustrate below the economic advantage of three-phase power versus single-phase power through an example.

We consider the transfer of apparent power S [MVA] over a distance d [km] with a phase-to-neutral voltage U [kV]. Additionally, we consider that the available conductor admits a maximum current density δ [A/cm^2].

Using single-phase ac power, we have:

$$\begin{aligned} S_1 &= S, \\ U_1 &= U, \\ I_1 &\approx \frac{S}{U}. \end{aligned} \tag{2.65}$$

Then, the conductor section should be:

$$A_1 = \frac{I_1}{\delta} = \frac{S}{\delta U} \tag{2.66}$$

and the required material is:

$$M_1 = 2A_1 d = 2\frac{I_1}{\delta}d = 2\frac{Sd}{\delta U}, \tag{2.67}$$

while losses are:

$$P_1^{\mathrm{L}} \approx 2I_1^2\rho\frac{d}{A_1} = 2\frac{S^2}{U^2}\rho\frac{\delta U d}{S} = 2\frac{S}{U}\rho\delta d, \tag{2.68}$$

where ρ is the resistivity of the material used.

Using three-phase ac power, we have:

$$\begin{aligned} S_3 &= S, \\ U_3 &= \sqrt{3}U, \\ I_3 &\approx \frac{S_3}{\sqrt{3}U_3} = \frac{S}{3U}. \end{aligned} \tag{2.69}$$

In this case, the conductor section should be:

$$A_3 = \frac{I_3}{\delta} = \frac{S}{3\delta U} \tag{2.70}$$

and the required material is:

$$M_3 = 3A_3 d = 3\frac{S}{3\delta U}d = \frac{Sd}{\delta U}, \tag{2.71}$$

while losses in this case are:

$$P_3^{\mathrm{L}} \approx 3I_3^2\rho\frac{d}{A_3} = 3\frac{S^2}{9U^2}\rho\frac{3\delta U d}{S} = \frac{S}{U}\rho\delta d. \tag{2.72}$$

Note that, on the one hand:

$$\frac{M_1}{M_3} = 2, \tag{2.73}$$

i.e., the material required to transmit apparent power S [MVA] over a distance d [km] with a phase-to-neutral voltage U [kV] considering a single-phase ac line is about twice the material needed if a three-phase ac line is used.

On the other hand, we have:

$$\frac{P_1^{\mathrm{L}}}{P_3^{\mathrm{L}}} = 2, \tag{2.74}$$

i.e., the losses of transmitting apparent power S [MVA] over a distance d [km] with a phase-to-neutral voltage U [kV] considering a single-phase ac line are about twice the losses if a three-phase ac line is used.

This simple back-of-the-envelope analysis illustrates the economic advantage of building/using a three-phase transmission line over a single-phase one.

2.6 Per-Unit System

This section defines and describes the per-unit system, which is important in power systems spanning different voltage levels.

2.6.1 Motivation

Power transformers interconnect power system areas with different voltage levels. This is a problem at the time of analyzing these systems since all magnitudes need to be transformed to a single voltage level. However, if a per-unit analysis is performed, this problem disappears and a unique voltage level is obtained. This greatly simplifies the subsequent analysis.

2.6.2 Per-Unit System Definition

Any electrical variable or parameter (voltage, current, power, impedance) can be expressed as a function of its own units or with respect to a reference value known as base value, i.e.:

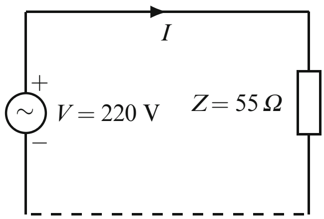


Fig. 2.18 Illustrative Example 2.5: single-phase circuit using real magnitudes

$$m = \frac{M}{M^{\mathrm{B}}}, \tag{2.75}$$

where:

- m is the per-unit value,
- M is the value of the variable/parameter in its own units, and
- M^{B} is the base value.

Then, instead of analyzing a circuit using actual values, it is possible to analyze it using per-unit values. This generally simplifies the subsequent analysis.

Illustrative Example 2.5 *Illustration of per-unit analysis*

We consider the single-phase circuit depicted in Fig. 2.18. Taking into account that $V = 220\,\mathrm{V}$ and $Z = 55\,\Omega$, we obtain the current I. To do so, we analyze the circuit using the per-unit system considering a base-voltage value of 220 V and a base-current value of 2 A.

First, we obtain the equivalent per-unit circuit by transforming the voltage and impedance values to per-unit values.

On the one hand, we compute the per-unit voltage v as follows:

$$v = \frac{V}{V^{\mathrm{B}}} = \frac{220}{220} = 1 \text{ puV}.$$

In order to obtain the per-unit impedance, first we need to compute the base-impedance value, which is obtained as the base-voltage value divided by the base-current value, i.e.:

$$Z^{\mathrm{B}} = \frac{V^{\mathrm{B}}}{I^{\mathrm{B}}} = \frac{220}{2} = 110\,\Omega.$$

Then, we obtain the per-unit impedance as:

$$z = \frac{Z}{Z^{\mathrm{B}}} = \frac{55}{110} = 0.5 \text{ pu}\Omega.$$

Finally, we derive the equivalent single-phase circuit using per-unit values and depict it in Fig. 2.19.

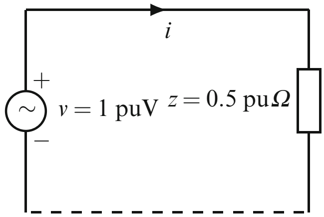


Fig. 2.19 Illustrative Example 2.5: single-phase circuit using per-unit magnitudes

Next we can compute the per-unit current i as:

$$i = \frac{v}{z} = \frac{1}{0.5} = 2 \text{ puA}.$$

Finally, we can obtain the actual current I as:

$$I = i\, I^{\mathrm{B}} = 2 \times 2 = 4 \text{ A}.$$

□

Using per-unit analysis in Illustrative Example 2.5 is not convenient since it is possible to directly compute current I from the circuit in Fig. 2.18 as $\frac{V}{Z} = \frac{220}{55} =$ 4 A. However, the use of a per-unit system to analyze power systems with multiple voltage levels is most convenient for two reasons (provided that the per-unit system is properly defined):

1. Power transformers disappear from the equivalent single-phase circuit. This is further analyzed and shown in Sect. 3.3 of Chap. 3.
2. Voltage values are close to 1 puV, which allows detecting errors.

Besides these two important advantages, there is an additional advantage of using a per-unit analysis for three-phase power systems:

3. The per-unit impedances of machines generally take values within tight bounds, independently of their nominal values, which facilitates their characterization.

2.6.3 *Definition of Base Values*

An appropriate definition of the base values is as follows:

1. We select a common single-phase base power, typically $1/3$ of the rated power of the component of highest rated power, i.e.:

$$S_{\mathrm{B}} = \frac{1}{3} S_{\mathrm{N}k}, \tag{2.76}$$

where:

- S_B is the single-phase base power and
- $S_{\mathrm{N}k}$ is the three-phase rated power of component k.

2. Recalling that power transformers separate the voltage zones of the network, we select the base voltage in one zone as the rated phase voltage of one component in that zone, i.e.:

$$U_{\mathrm{B}i} = \frac{1}{\sqrt{3}} U_{\mathrm{N}j}, \tag{2.77}$$

where:

- $U_{\mathrm{B}i}$ is the base-voltage value at zone i and
- $U_{\mathrm{N}j}$ is the rated three-phase voltage of component j in zone i.

3. The base-voltage values in other zones are determined strictly complying with the transformation ratios of the power transformers, which makes transformers disappear from equivalent single-phase circuits, i.e.:

$$U_{\mathrm{B}i} = U_{\mathrm{B}j} \frac{U_i}{U_j}, \tag{2.78}$$

where:

- $U_{\mathrm{B}i}$ is the base-voltage value in zone i,
- $U_{\mathrm{B}j}$ is the base-voltage value in zone j, and
- U_i/U_j is the three-phase transformation ratio of the power transformer coupling zones i and j.

4. We define the base-current value and the base-impedance value per zone as

$$I_{\mathrm{B}i} = \frac{S_\mathrm{B}}{U_{\mathrm{B}i}} \tag{2.79}$$

and:

$$Z_{\mathrm{B}i} = \frac{U_{\mathrm{B}i}^2}{S_\mathrm{B}}, \tag{2.80}$$

respectively, where:

- $I_{\mathrm{B}i}$ is the base-current value at zone i and
- $Z_{\mathrm{B}i}$ is the base-impedance value at zone i.

5. We specify phase shifts due to power transformers including delta and zigzag connections. We explain transformer phase shifts in Chap. 3.

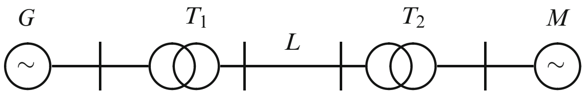


Fig. 2.20 Illustrative Example 2.6: power system

Illustrative Example 2.6 *Per-unit analysis: base values*

We consider the three-phase power system depicted in Fig. 2.20. This system comprises a generator, two power transformers, a transmission line, and a motor. The rated powers and voltages of these components are provided below:

- Generator G: 60 MVA, 11 kV.
- Transformer T_1: 60 MVA, 132/12 kV.
- Line L: 70 MVA, 132 kV.
- Transformer T_2: 60 MVA, 125/5 kV.
- Motor M: 80 MVA, 5 kV.

Using these data, we determine below the number of voltage zones, and the base values of each zone.

Each power transformer divides the system into two voltage zones. Thus, we have three voltage zones, namely, generator zone (zone 1), line zone (zone 2), and motor zone (zone 3).

Next, we follow the procedure described above to determine the base values in each zone:

1. We select the single-phase base power as 1/3 of the rated power of the motor, which has the highest rated power, i.e., $S_{\mathrm{B}} = \frac{80}{3} = 26.667$ MVA. Note that this base-power value is the same for all voltage zones.
2. We fix the base-voltage value of one of the zones as the rated phase voltage of one of the components of that zone. For example, we select the base-voltage value in the generator zone as the rated phase voltage of the generator, i.e., $U_{\mathrm{B1}} = \frac{11}{\sqrt{3}} = 6.351$ kV.
3. We compute the base-voltage values in other zones by using the transformation ratios of transformers that separate each zone, i.e., $U_{\mathrm{B2}} = \frac{132}{12}\frac{11}{\sqrt{3}} = 69.859$ kV and $U_{\mathrm{B3}} = \frac{5}{125}\frac{132}{12}\frac{11}{\sqrt{3}} = 2.794$ kV.
4. We define the base-current value and the base-impedance value per zone using the corresponding base-power and base-voltage values, as well as Eqs. (2.79) and (2.80), respectively. For example, the base-current value in the generator zone is $I_{\mathrm{B1}} = \frac{26.667 \cdot 10^6}{6.351 \cdot 10^3} = 4.199$ kA, while the base-impedance value in the line zone is $Z_{\mathrm{B2}} = \frac{\left(69.859 \cdot 10^3\right)^2}{26.667 \cdot 10^6} = 183.013\ \Omega$.

Table 2.1 summarizes the base values in each zone.

□

Table 2.1 Illustrative Example 2.6: base values

Value	Generator zone	Line zone	Motor zone
S_B [MVA]	26.667	26.667	26.667
V_B [kV]	6.351	69.859	2.794
I_B [kA]	4.199	0.382	9.544
Z_B [Ω]	1.512	183.013	0.293

Note that the base values defined in this section are single-phase base values. However, three-phase base values, equivalent to single-phase ones, are similarly defined.

If we define the three-phase base-power value as:

$$S_{B3} \triangleq 3S_B, \tag{2.81}$$

and the line base-voltage value in each zone as:

$$U_{B3i} \triangleq \sqrt{3}U_{Bi}, \tag{2.82}$$

then single-phase and three-phase base-current and single-phase and three-phase base-impedance values coincide. That is:

$$I_{B3i} = \frac{S_{B3}}{\sqrt{3}U_{B3i}} = \frac{3S_B}{\sqrt{3}\sqrt{3}U_{Bi}} = \frac{S_B}{U_{Bi}} = I_{Bi} \tag{2.83}$$

and:

$$Z_{B3i} = \frac{U_{B3}^2}{S_{B3}} = \frac{\left(\sqrt{3}U_{Bi}\right)^2}{3S_B} = \frac{U_{Bi}^2}{S_B} = Z_{Bi}, \tag{2.84}$$

respectively.

2.6.4 Per-Unit Analysis Procedure

The procedure to analyze power systems using a per-unit system comprises the five steps below:

1. Define base values as explained in Sect. 2.6.3.
2. Transform the three-phase power system into an equivalent single-phase circuit in which impedances are expressed in per unit.
3. Apply the operating conditions (in per unit).
4. Solve the circuit (in per unit).
5. Obtain actual values by multiplying per-unit values by the corresponding base values.

A number of examples to illustrate this procedure to analyze power systems are provided in Sect. 3.6 of Chap. 3.

2.7 Summary and Further Reading

This chapter provides an overview of the fundamentals of power systems. First, we define balanced sequences, balanced voltages and currents, and powers. Second, we illustrate why three-phase power systems are used instead of single-phase ones. Finally, we provide an introduction to the analysis of power systems considering a per-unit system, which is used and further analyzed in the following chapters of this book.

Basic references regarding power system analysis include Kothari and Nagrath [4] and Duncan Glover et al. [2]. Advanced references include Gómez-Expósito et al. [3] and Bergen and Vittal [1].

2.8 End-of-Chapter Exercises

2.1 Why three-phase power systems are used instead of single-phase ones?

2.2 List the advantages of analyzing power systems using a per-unit system.

2.3 Consider the three-phase circuit depicted in Fig. 2.21. Voltage sources constitute a balanced three-phase positive sequence:

$$\begin{cases} \bar{E}_R = 100\angle 0° V, \\ \bar{E}_S = 100\angle -120° V, \\ \bar{E}_T = 100\angle 120° V. \end{cases}$$

Impedances $\bar{Z}_i$ are equal to $j5\,\Omega$, impedances $\bar{Z}_\ell$ are equal to $j15\,\Omega$, and impedances $\bar{Z}_{RS} = \bar{Z}_{ST} = \bar{Z}_{TR}$ are equal to $j30\,\Omega$.

Using these data:

1. Compute currents $\bar{I}_R$, $\bar{I}_S$, and $\bar{I}_T$, as well as currents $\bar{I}_{RS}$, $\bar{I}_{ST}$, and $\bar{I}_{TR}$.
2. Compute voltages $\bar{U}_{R1}$, $\bar{U}_{S1}$, and $\bar{U}_{T1}$, as well as voltages $\bar{U}_{R2}$, $\bar{U}_{S2}$, and $\bar{U}_{T2}$.
3. Recompute the above voltages and currents if an impedance of $j20\,\Omega$ is connected between nodes N and N'.
4. Compute the above voltages and currents if impedances $\bar{Z}_{RS}$, $\bar{Z}_{ST}$, and $\bar{Z}_{TR}$ are equal to $10\,\Omega$, $j10\,\Omega$, and $-j10\,\Omega$, respectively. Is it possible in this case to use the equivalent single-phase circuit?

2.4 Consider a balanced three-phase load that is supplied at a line voltage of 400 kV and absorbs a line current of $500\angle -10°$ A:

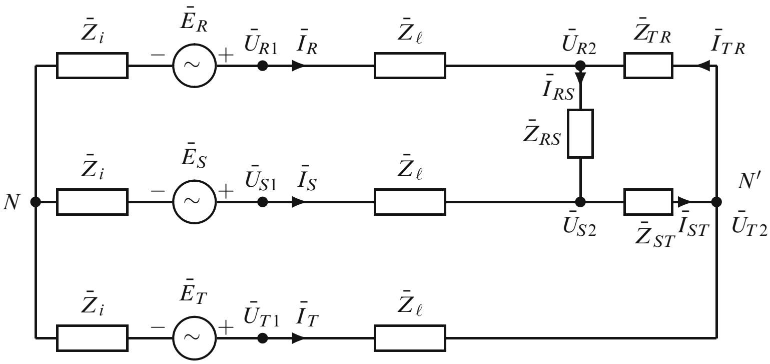


Fig. 2.21 Exercise 2.3: three-phase circuit

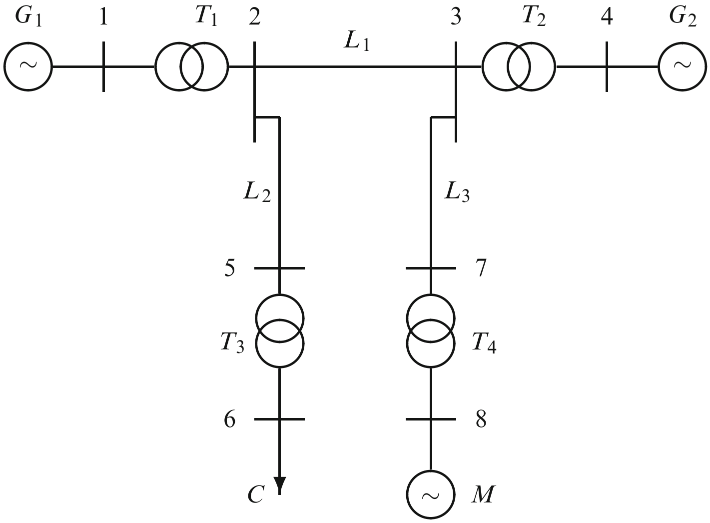


Fig. 2.22 Exercise 2.5: power system

1. Compute the instantaneous power consumed in phases R, S, and T.
2. Compute the instantaneous three-phase power consumed by the load.
3. Compute the reactive and apparent power consumed by the load.

2.5 Consider the three-phase power system depicted in Fig. 2.22. The rated powers and voltages of the system components are provided below:

- Generators G_1 and G_2: 500 MVA, 20 kV.
- Transformers T_1 and T_2: 200 MVA, 500/18 kV.
- Transformers T_3 and T_4: 150 MVA, 500/20 kV.
- Motor M: 111 MW, $\cos\phi = 0.8$ (inductive), 20 kV.

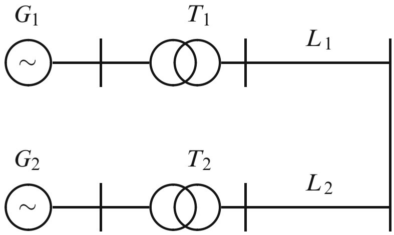


Fig. 2.23 Exercise 2.6: power system

Using these data, determine the number of voltage zones and the base values of each zone.

2.6 Consider the three-phase power system depicted in Fig. 2.23. The rated powers and voltages of the system components are provided below:

- Generator G_1: 50 MVA, 12 kV.
- Generator G_2: 100 MVA, 15 kV.
- Transformer T_1: 50 MVA, 10/138 kV.
- Transformer T_2: 100 MVA, 15/138 kV.

Using as base values the rated parameters of generator G_2, determine the number of voltage zones and the base values of each zone.

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