Chapter 2
Convergence of Iterative Methods in Abstract Fractional Calculus

We present a semi-local convergence analysis for a class of iterative methods under generalized conditions. Some applications are suggested including Banach space valued functions of fractional calculus, where all integrals are of Bochner-type. It follows [6].

2.1 Introduction

Sections 2.1–2.3 are prerequisites for Sect. 2.4.

Let B_1, B_2 stand for Banach spaces and let Ω stand for an open subset of B_1. Let also $U(z, \rho) := \{u \in B_1 : \|u - z\| < \rho\}$ and let $\overline{U}(z, \rho)$ stand for the closure of $U(z, \rho)$.

Many problems in Computational Sciences, Engineering, Mathematical Chemistry, Mathematical Physics, Mathematical Economics and other disciplines can be brought in a form like

$$F(x) = 0 \tag{2.1.1}$$

using Mathematical Modeling [1–18], where $F : \Omega \to B_2$ is a continuous operator. The solution x^* of Eq. (2.1.1) is sought in closed form, but this is attainable only in special cases. That explains why most solution methods for such equations are usually iterative. There is a plethora of iterative methods for solving Eq. (2.1.1). We can divide these methods in two categories.

Explicit Methods [7, 8, 12, 16, 17]: Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n). \tag{2.1.2}$$

Secant method:

$$x_{n+1} = x_n - \left[x_{n-1}, x_n; F\right]^{-1} F(x_n), \tag{2.1.3}$$

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where $[\cdot, \cdot; F]$ denotes a divided difference of order one on $\Omega \times \Omega$ [8, 16, 17].

Newton-like method:

$$x_{n+1} = x_n - E_n^{-1} F(x_n), \tag{2.1.4}$$

where $E_n = E(F)(x_n)$ and $E : \Omega \to \mathcal{L}(B_1, B_2)$ the space of bounded linear operators from B_1 into B_2. Other explicit methods can be found in [8, 12, 16, 17] and the references there in.

Implicit Methods [7, 10, 12, 17]:

$$F(x_n) + A_n(x_{n+1} - x_n) = 0 \tag{2.1.5}$$

$$x_{n+1} = x_n - A_n^{-1} F(x_n), \tag{2.1.6}$$

where $A_n = A(x_{n+1}, x_n) = A(F)(x_{n+1}, x_n)$ and $A : \Omega \times \Omega \to \mathcal{L}(B_1, B_2)$.

There is a plethora on local as well as semi-local convergence results for explicit methods [1–9, 11–17]. However, the research on the convergence of implicit methods has received little attention. Authors, usually consider the fixed point problem

$$P_z(x) = x, \tag{2.1.7}$$

where

$$P_z(x) = x + F(z) + A(x, z)(x - z) \tag{2.1.8}$$

or

$$P_z(x) = z - A(x, z)^{-1} F(z) \tag{2.1.9}$$

for methods (2.1.5) and (2.1.6), respectivelly, where $z \in \Omega$ is given. If P is a contraction operator mapping a closed set into itself, then according to the contraction mapping principle [12, 16, 17], P_z has a fixed point x_z^* which can be found using the method of succesive substitutions or Picard's method [17] defined for each fixed n by

$$y_{k+1,n} = P_{x_n}\left(y_{k,n}\right), \quad y_{0,n} = x_n, x_{n+1} = \lim_{k \to +\infty} y_{k,n}. \tag{2.1.10}$$

Let us also consider the analogous explicit methods

$$F(x_n) + A(x_n, x_n)(x_{n+1} - x_n) = 0 \tag{2.1.11}$$

$$x_{n+1} = x_n - A(x_n, x_n)^{-1} F(x_n) \tag{2.1.12}$$

$$F(x_n) + A(x_n, x_{n-1})(x_{n+1} - x_n) = 0 \tag{2.1.13}$$

and

$$x_{n+1} = x_n - A(x_n, x_{n-1})^{-1} F(x_n). \tag{2.1.14}$$

In this chapter in Sect. 2.2, we present the semi-local convergence of method (2.1.5) and method (2.1.6). Section 2.3 contains the semi-local convergence of method (2.1.11), method (2.1.12), method (2.1.13) and method (2.1.14). Some applications to Abstract Fractional Calculus are suggested in Sect. 2.4 on a certain Banach space valued functions, where all the integrals are of Bochner-type [9].

2.2 Semi-local Convergence for Implicit Methods

We present the semi-local convergence analysis of method (2.1.6) using conditions (S):

(s_1) $F : \Omega \subset B_1 \to B_2$ is continuous and $A(x, y) \in \mathcal{L}(B_1, B_2)$ for each $(x, y) \in \Omega \times \Omega$.

(s_2) There exist $\beta > 0$ and $\Omega_0 \subset B_1$ such that $A(x, y)^{-1} \in \mathcal{L}(B_2, B_1)$ for each $(x, y) \in \Omega_0 \times \Omega_0$ and

$$\left\| A(x, y)^{-1} \right\| \leq \beta^{-1}.$$

Set $\Omega_1 = \Omega \cap \Omega_0$.

(s_3) There exists a continuous and nondecreasing function $\psi : [0, +\infty)^3 \to [0, +\infty)$ such that for each $x, y \in \Omega_1$

$$\|F(x) - F(y) - A(x, y)(x - y)\| \leq$$

$$\beta\psi(\|x - y\|, \|x - x_0\|, \|y - x_0\|)\|x - y\|.$$

(s_4) For each $x \in \Omega_0$ there exists $y \in \Omega_0$ such that

$$y = x - A(y, x)^{-1} F(x).$$

(s_5) For $x_0 \in \Omega_0$ and $x_1 \in \Omega_0$ satisfying (s_4) there exists $\eta \geq 0$ such that

$$\left\| A(x_1, x_0)^{-1} F(x_0) \right\| \leq \eta.$$

(s_6) Define $q(t) := \psi(\eta, t, t)$ for each $t \in [0, +\infty)$. Equation

$$t(1 - q(t)) - \eta = 0$$

has positive solutions. Denote by s the smallest such solution.

(s_7) $\overline{U}(x_0, s) \subset \Omega$, where

$$s = \frac{\eta}{1 - q_0} \text{ and } q_0 = \psi(\eta, s, s).$$

Next, we present the semi-local convergence analysis for method (2.1.6) using the conditions (S) and the preceding notation.

Theorem 2.1 *Assume that the conditions* (S) *hold. Then, sequence* $\{x_n\}$ *generated by method (2.1.6) starting at* $x_0 \in \Omega$ *is well defined in* $U(x_0, s)$*, remains in* $U(x_0, s)$ *for each* $n = 0, 1, 2, \ldots$ *and converges to a solution* $x^* \in \overline{U}(x_0, s)$ *of equation* $F(x) = 0$. *Moreover, suppose that there exists a continuous and nondecreasing function* $\psi_1 : [0, +\infty)^4 \to [0, +\infty)$ *such that for each* $x, y, z \in \Omega_1$

$$\|F(x) - F(y) - A(z, y)(x - y)\| \leq$$

$$\beta\psi_1(\|x - y\|, \|x - x_0\|, \|y - x_0\|, \|z - x_0\|)\|x - y\|$$

and $q_1 = \psi_1(\eta, s, s, s) < 1$.

Then, x^* *is the unique solution of equation* $F(x) = 0$ *in* $\overline{U}(x_0, s)$.

Proof By the definition of s and (s_5), we have $x_1 \in U(x_0, s)$. The proof is based on mathematical induction on k. Suppose that $\|x_k - x_{k-1}\| \leq q_0^{k-1}\eta$ and $\|x_k - x_0\| \leq s$.

We get by (2.1.6), $(s_2) - (s_5)$ in turn that

$$\|x_{k+1} - x_k\| = \left\|A_k^{-1}F(x_k)\right\| = \left\|A_k^{-1}(F(x_k) - F(x_{k-1}) - A_{k-1}(x_k - x_{k-1}))\right\|$$

$$\leq \left\|A_k^{-1}\right\| \|F(x_k) - F(x_{k-1}) - A_{k-1}(x_k - x_{k-1})\| \leq$$

$$\beta^{-1}\beta\psi(\|x_k - x_{k-1}\|, \|x_{k-1} - x_0\|, \|y_k - x_0\|)\|x_k - x_{k-1}\| \leq$$

$$\psi(\eta, s, s)\|x_k - x_{k-1}\| = q_0\|x_k - x_{k-1}\| \leq q_0^k\|x_1 - x_0\| \leq q_0^k\eta \tag{2.2.1}$$

and

$$\|x_{k+1} - x_0\| \leq \|x_{k+1} - x_k\| + \ldots + \|x_1 - x_0\|$$

$$\leq q_0^k\eta + \ldots + \eta = \frac{1 - q_0^{k+1}}{1 - q_0}\eta < \frac{\eta}{1 - q_0} = s.$$

The induction is completed. Moreover, we have by (2.2.1) that for $m = 0, 1, 2, \ldots$

$$\|x_{k+m} - x_k\| \leq \frac{1 - q_0^m}{1 - q_0}q_0^k\eta.$$

It follows from the preceding inequation that sequence $\{x_k\}$ is complete in a Banach space B_1 and as such it converges to some $x^* \in \overline{U}(x_0, s)$ (since $\overline{U}(x_0, s)$ is a closed ball). By letting $k \to +\infty$ in (2.2.1) we get $F(x^*) = 0$. To show the uniqueness part, let $x^{**} \in U(x_0, s)$ be a solution of equation $F(x) = 0$. By using (2.1.6) and the hypothesis on ψ_1, we obtain in turn that

$$\left\|x^{**} - x_{k+1}\right\| = \left\|x^{**} - x_k + A_k^{-1}F(x_k) - A_k^{-1}F\left(x^{**}\right)\right\| \leq$$

$$\left\|A_k^{-1}\right\| \left\|F\left(x^{**}\right) - F\left(x_k\right) - A_k\left(x^{**} - x_k\right)\right\| \leq$$

$$\beta^{-1}\beta\psi_1\left(\left\|x^{**} - x_k\right\|, \left\|x_{k-1} - x_0\right\|, \left\|x_k - x_0\right\|, \left\|x^{**} - x_0\right\|\right)\left\|x^{**} - x_k\right\| \leq$$

$$q_1\left\|x^{**} - x_k\right\| \leq q_1^{k+1}\left\|x^{**} - x_0\right\|,$$

so $\lim\limits_{k\to+\infty} x_k = x^{**}$. We have shown that $\lim\limits_{k\to+\infty} x_k = x^*$, so $x^* = x^{**}$. ■

Remark 2.2 (1) The equation in (s_6) is used to determine the smallness of η. It can be replaced by a stronger condition as follows. Choose $\mu \in (0, 1)$. Denote by s_0 the smallest positive solution of equation $q(t) = \mu$. Notice that if function q is strictly increasing, we can set $s_0 = q^{-1}(\mu)$. Then, we can suppose instead of (s_6) :

$\left(s_6'\right)$ $\eta \leq (1-\mu)s_0$

which is a stronger condition than (s_6).

However, we wanted to leave the equation in (s_6) as uncluttered and as weak as possible.

(2) Condition (s_2) can become part of condition (s_3) by considering

$(s_3)'$ There exists a continuous and nondecreasing function $\varphi : [0, +\infty)^3 \to [0, +\infty)$ such that for each $x, y \in \Omega_1$

$$\left\|A(x, y)^{-1}\left[F(x) - F(y) - A(x, y)(x, y)\right]\right\| \leq$$

$$\varphi\left(\|x - y\|, \|x - x_0\|, \|y - x_0\|\right)\|x - y\|.$$

Notice that

$$\varphi(u_1, u_2, u_3) \leq \psi(u_1, u_2, u_3)$$

for each $u_1 \geq 0$, $u_2 \geq 0$ and $u_3 \geq 0$. Similarly, a function φ_1 can replace ψ_1 for yhe uniqueness of the solution part. These replacements are of Mysovskii-type [7, 12, 16] and influence the weaking of the convergence criterion in (s_6), error bounds and the precision of s.

(3) Suppose that there exist $\beta > 0$, $\beta_1 > 0$ and $L \in \mathcal{L}(B_1, B_2)$ with $L^{-1} \in \mathcal{L}(B_2, B_1)$ such that

$$\left\|L^{-1}\right\| \leq \beta^{-1}$$

$$\|A(x, y) - L\| \leq \beta_1$$

and

$$\beta_2 := \beta^{-1}\beta_1 < 1.$$

Then, it follows from the Banach lemma on invertible operators [12], and

$$\left\|L^{-1}\right\| \|A(x, y) - L\| \leq \beta^{-1}\beta_1 = \beta_2 < 1$$

that $A(x, y)^{-1} \in \mathcal{L}(B_2, B_1)$. Let $\beta = \frac{\beta^{-1}}{1-\beta_2}$. Then, under these replacements, condition (s_2) is implied, therefore it can be dropped from the conditions (S).

(4) Clearly method (2.1.5) converges under the conditions (S), since (2.1.6) implies (2.1.5).

(5) We wanted to leave condition (s_4) as uncluttered as possible, since in practice Eqs. (2.1.6) (or (2.1.5)) may be solvable in a way avoiding the already mentioned conditions of the contraction mapping principle. However, in what follows we examine the solvability of method (2.1.5) under a stronger version of the contraction mapping principle using the conditions (V) :

$(v_1) = (s_1)$.

(v_2) There exist functions $w_1 : [0, +\infty)^4 \to [0, +\infty)$, $w_2 : [0, +\infty)^4 \to [0, +\infty)$ continuous and nondecreasing such that for each $x, y, z \in \Omega$

$$\|I + A(x, z) - A(y, z)\| \le w_1(\|x - y\|, \|x - x_0\|, \|y - x_0\|, \|z - x_0\|)$$

$$\|A(x, z) - A(y, z)\| \le w_2(\|x - y\|, \|x - x_0\|, \|y - x_0\|, \|z - x_0\|) \, ||x - y||$$

and

$$w_1(0, 0, 0, 0) = w_2(0, 0, 0, 0) = 0.$$

Set

$$h(t, t, t, t) = \begin{cases} w_1(2t, t, t, t) + w_2(2t, t, t, t)(t + \|x_0\|), z \neq x_0 \\ w_1(2t, t, t, 0) + w_2(2t, t, t, 0)\|x_0\|, z = x_0. \end{cases}$$

(v_3) There exists $\tau > 0$ satisfying

$$h(t, t, t, t) < 1$$

and

$$h(t, t, 0, t)\, t + \|F(x_0)\| \le t$$

(v_4) $\overline{U}(x_0, \tau) \subseteq D$.

Theorem 2.3 *Suppose that the conditions (V) are satisfied. Then, Eq. (2.1.5) is uniquely solvable for each $n = 0, 1, 2, \ldots$. Moreover, if $A_n^{-1} \in \mathcal{L}(B_2, B_1)$, the Eq. (2.1.6) is also uniquely solvable for each $n = 0, 1, 2, \ldots$*

Proof The result is an application of the contraction mapping principle. Let $x, y, z \in U(x_0, \tau)$. By the definition of operator P_z, (v_2) and (v_3), we get in turn that

$$\|P_z(x) - P_z(y)\| = \|(I + A(x, z) - A(y, z))(x - y) - (A(x, z) - A(y, z))z\|$$

$$\le \|I + A(x, z) - A(y, z)\| \|x - y\| + \|A(x, z) - A(y, z)\| \|z\|$$

$$\le [w_1(\|x - y\|, \|x - x_0\|, \|y - x_0\|, \|z - x_0\|) +$$

$$w_2 (\|x - y\|, \|x - x_0\|, \|y - x_0\|, \|z - x_0\|) (\|z - x_0\| + \|x_0\|)] \|x - y\|$$

$$\leq h (\tau, \tau, \tau, \tau) \|x - y\|$$

and

$$\|P_z (x) - x_0\| \leq \|P_z (x) - P_z (x_0)\| + \|P_z (x_0) - x_0\|$$

$$\leq h (\|x - x_0\|, \|x - x_0\|, 0, \|z - x_0\|) \|x - x_0\| + \|F (x_0)\|$$

$$\leq h (\tau, \tau, 0, \tau) \tau + \|F (x_0)\| \leq \tau.$$

∎

Remark 2.4 Sections 2.2 and 2.3 have an interest independent of Sect. 2.4. It is worth noticing that the results especially of Theorem 2.1 can apply in Abstract Fractional Calculus as illustrated in Sect. 2.4. By specializing function ψ, we can apply the results of say Theorem 2.1 in the examples suggested in Sect. 2.4. In particular for (2.4.1), we choose $\psi (u_1, u_2, u_3) = \frac{c_1 u_1^p}{(p+1)\beta}$ for $u_1 \geq 0$, $u_2 \geq 0$, $u_3 \geq 0$ and c_1, p are given in Sect. 2.4. Similar choices for the other examples of Sect. 2.4. It is also worth noticing that estimate (2.4.2) derived in Sect. 2.4 is of independent interest but not needed in Theorem 2.1.

2.3 Semi-local Convergence for Explicit Methods

A specialization of Theorem 2.1 can be utilized to study the semi-local convergence of the explicit methods given in the introduction of this study. In particular, for the study of method (2.1.12) (and consequently of method (2.1.11)), we use the conditions (S') :

(s_1') $F : \Omega \subset B_1 \to B_2$ is continuous and $A (x, x) \in \mathcal{L} (B_1, B_2)$ for each $x \in \Omega$.

(s_2') There exist $\beta > 0$ and $\Omega_0 \subset B_1$ such that $A (x, x)^{-1} \in \mathcal{L} (B_2, B_1)$ for each $x \in \Omega_0$ and

$$\left\|A (x, x)^{-1}\right\| \leq \beta^{-1}.$$

Set $\Omega_1 = \Omega \cap \Omega_0$.

(s_3') There exist continuous and nondecreasing functions $\psi_0 : [0, +\infty)^3 \to [0, +\infty)$, $\psi_2 : [0, +\infty)^3 \to [0, +\infty)$ with $\psi_0 (0, 0, 0) = \psi_2 (0, 0, 0) = 0$ such that for each $x, y \in \Omega_1$

$$\|F (x) - F (y) - A (y, y) (x - y)\| \leq$$

$$\beta\psi_0 (\|x - y\|, \|x - x_0\|, \|y - x_0\|) \|x - y\|$$

and

$$\|A(x,y)-A(y,y)\| \leq \beta\psi_2(\|x-y\|, \|x-x_0\|, \|y-x_0\|).$$

Set $\psi = \psi_0 + \psi_2$.

(s_4') There exist $x_0 \in \Omega_0$ and $\eta \geq 0$ such that $A(x_0,x_0)^{-1} \in \mathcal{L}(B_2, B_1)$ and

$$\left\|A(x_0,x_0)^{-1} F(x_0)\right\| \leq \eta.$$

$(s_5') = (s_6)$
$(s_6') = (s_7)$.

Next, we present the following semi-local convergence analysis of method (2.1.12) using the (S') conditions and the preceding notation.

Proposition 2.5 *Suppose that the conditions* (S') *are satisfied. Then, sequence* $\{x_n\}$ *generated by method (2.1.12) starting at* $x_0 \in \Omega$ *is well defined in* $U(x_0, s)$, *remains in* $U(x_0, s)$ *for each* $n = 0, 1, 2, \ldots$ *and converges to a unique solution* $x^* \in \overline{U}(x_0, s)$ *of equation* $F(x) = 0$.

Proof We follow the proof of Theorem 2.1 but use instead the analogous estimate

$$\|F(x_k)\| = \|F(x_k) - F(x_{k-1}) - A(x_{k-1}, x_{k-1})(x_k - x_{k-1})\| \leq$$

$$\|F(x_k) - F(x_{k-1}) - A(x_k, x_{k-1})(x_k - x_{k-1})\| +$$

$$\|(A(x_k, x_{k-1}) - A(x_{k-1}, x_{k-1}))(x_k - x_{k-1})\| \leq$$

$$\big[\psi_0(\|x_k - x_{k-1}\|, \|x_{k-1} - x_0\|, \|x_k - x_0\|) +$$

$$\psi_2(\|x_k - x_{k-1}\|, \|x_{k-1} - x_0\|, \|x_k - x_0\|)\big] \|x_k - x_{k-1}\| =$$

$$\psi(\|x_k - x_{k-1}\|, \|x_{k-1} - x_0\|, \|x_k - x_0\|) \|x_k - x_{k-1}\|.$$

The rest of the proof is identical to the one in Theorem 2.1 until the uniqueness part for which we have the corresponding estimate

$$\left\|x^{**} - x_{k+1}\right\| = \left\|x^{**} - x_k + A_k^{-1} F(x_k) - A_k^{-1} F\left(x^{**}\right)\right\| \leq$$

$$\left\|A_k^{-1}\right\| \left\|F\left(x^{**}\right) - F(x_k) - A_k\left(x^{**} - x_k\right)\right\| \leq$$

$$\beta^{-1}\beta\psi_0\left(\left\|x^{**} - x_k\right\|, \|x_{k-1} - x_0\|, \|x_k - x_0\|\right) \leq$$

$$q\left\|x^{**} - x_k\right\| \leq q^{k+1}\left\|x^{**} - x_0\right\|.$$

■

Remark 2.6 Comments similar to the ones given in Sect. 2.2 can follows but for method (2.1.13) and method (2.1.14) instead of method (2.1.5) and method (2.1.6), respectively.

2.4 Applications to Abstract Fractional Calculus

Here we deal with Banach space $(X, \|\cdot\|)$ valued functions f of real domain $[a, b]$. All integrals here are of Bochner-type, see [15]. The derivatives of f are defined similarly to numerical ones, see [18], pp. 83–86 and p. 93.

In this section we apply the earlier numerical methods to X-valued fractional calculus for solving $f(x) = 0$.

Here we would like to establish for $[a, b] \subseteq \mathbb{R}, a < b, f \in C^p([a, b], X), p \in \mathbb{N}$, that

$$\|f(y) - f(x) - A(x, y)(y - x)\| \leq c_1 \frac{|x - y|^{p+1}}{p + 1}, \tag{2.4.1}$$

$\forall\, x, y \in [a, b]$, where $c_1 > 0$, and

$$\|A(x, x) - A(y, y)\| \leq c_2 |x - y|, \tag{2.4.2}$$

with $c_2 > 0$, $\forall\, x, y \in [a, b]$.

Above A stands for a X-valued differential operator to be defined and presented per case in the next, it will be denoted as $A_+(f)$, $A_-(f)$ in the X-valued fractional cases, and $A_0(f)$ in the X-valued ordinary case.

We examine the following cases:

(I) Here see [4, 5].

Let $x, y \in [a, b]$ such that $x \geq y, \nu > 0, \nu \notin \mathbb{N}$, such that $p = [\nu]$, $[\cdot]$ the integral part, $\alpha = \nu - p$ $(0 < \alpha < 1)$.

Let $f \in C^p([a, b], X)$ and define

$$\left(J_\nu^y f\right)(x) := \frac{1}{\Gamma(\nu)} \int_y^x (x - t)^{\nu - 1} f(t)\, dt, \;\; y \leq x \leq b, \tag{2.4.3}$$

the X-valued left generalized Riemann-Liouville fractional integral.

Here Γ stands for the gamma function.

Clearly here it holds $\left(J_\nu^y f\right)(y) = 0$. We define $\left(J_\nu^y f\right)(x) = 0$ for $x < y$. By [4] $\left(J_\nu^y f\right)(x)$ is a continuous function in x, for a fixed y.

We define the subspace $C_{y+}^\nu([a, b], X)$ of $C^p([a, b], X)$:

$$C_{y+}^\nu([a, b], X) := \left\{f \in C^p([a, b], X) : J_{1-\alpha}^y f^{(p)} \in C^1([y, b], X)\right\}. \tag{2.4.4}$$

So let $f \in C_{y+}^\nu([a, b], X)$, we define the **$X$-valued generalized ν−fractional derivative of** f over $[y, b]$ as

$$D_y^{\nu} f = \left(J_{1-\alpha}^{y} f^{(p)}\right)', \tag{2.4.5}$$

that is

$$\left(D_y^{\nu} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_y^x (x-t)^{-\alpha} f^{(p)}(t)\, dt, \tag{2.4.6}$$

which exists for $f \in C_{y+}^{\nu}([a,b], X)$, for $a \leq y \leq x \leq b$.

Here we consider $f \in C^p([a,b], X)$ such that $f \in C_{y+}^{\nu}([a,b], X)$, for every $y \in [a,b]$, which means also that $f \in C_{x+}^{\nu}([a,b], X)$, for every $x \in [a,b]$ (i.e. exchange roles of x and y), we write that as $f \in C_{+}^{\nu}([a,b], X)$.

That is

$$\left(D_x^{\nu} f\right)(y) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dy} \int_x^y (y-t)^{-\alpha} f^{(p)}(t)\, dt \tag{2.4.7}$$

exists for $f \in C_{x+}^{\nu}([a,b], X)$, for $a \leq x \leq y \leq b$.

We mention the following left generalized X-valued fractional Taylor formula ($f \in C_{y+}^{\nu}([a,b], X)$, $\nu > 1$), see [5].

It holds

$$f(x) - f(y) = \sum_{k=1}^{p-1} \frac{f^{(k)}(y)}{k!}(x-y)^k + \frac{1}{\Gamma(\nu)} \int_y^x (x-t)^{\nu-1} \left(D_y^{\nu} f\right)(t)\, dt, \tag{2.4.8}$$

all $x, y \in [a,b]$ with $x \geq y$.

Similarly for $f \in C_{x+}^{\nu}([a,b], X)$ we have

$$f(y) - f(x) = \sum_{k=1}^{p-1} \frac{f^{(k)}(x)}{k!}(y-x)^k + \frac{1}{\Gamma(\nu)} \int_x^y (y-t)^{\nu-1} \left(D_x^{\nu} f\right)(t)\, dt, \tag{2.4.9}$$

all $x, y \in [a,b]$ with $y \geq x$.

So here we work with $f \in C^p([a,b], X)$, such that $f \in C_{+}^{\nu}([a,b], X)$.

We define the X-valued left linear fractional operator

$$(A_+(f))(x,y) := \begin{cases} \sum_{k=1}^{p-1} \frac{f^{(k)}(y)}{k!}(x-y)^{k-1} + \left(D_y^{\nu} f\right)(x) \frac{(x-y)^{\nu-1}}{\Gamma(\nu+1)}, & x > y, \\ \sum_{k=1}^{p-1} \frac{f^{(k)}(x)}{k!}(y-x)^{k-1} + \left(D_x^{\nu} f\right)(y) \frac{(y-x)^{\nu-1}}{\Gamma(\nu+1)}, & y > x, \\ f^{(p-1)}(x), & x = y. \end{cases} \tag{2.4.10}$$

Notice that (see [13], p. 3)

$$\|(A_+(f))(x,x) - (A_+(f))(y,y)\| = \left\| f^{(p-1)}(x) - f^{(p-1)}(y) \right\| \tag{2.4.11}$$

$$\leq \left\| f^{(p)} \right\|_{\infty} |x-y|, \; \forall\, x, y \in [a,b],$$

so that condition (2.4.2) is fulfilled.

Next we will prove condition (2.4.1). It is trivially true if $x = y$. So we examine the case of $x \neq y$.

We distinguish the subcases:

(1) $x > y$: We observe that

$$\left\| f(y) - f(x) - A_+(f)(x,y)(y-x) \right\| =$$

$$\left\| f(x) - f(y) - A_+(f)(x,y)(x-y) \right\| \overset{\text{(by (2.4.8), (2.4.10))}}{=}$$

$$\left\| \sum_{k=1}^{p-1} \frac{f^{(k)}(y)}{k!}(x-y)^k + \frac{1}{\Gamma(\nu)} \int_y^x (x-t)^{\nu-1} \left(D_y^\nu f\right)(t)\,dt - \right. \tag{2.4.12}$$

$$\left. \sum_{k=1}^{p-1} \frac{f^{(k)}(y)}{k!}(x-y)^k - \left(D_y^\nu f\right)(x) \frac{(x-y)^\nu}{\Gamma(\nu+1)} \right\| =$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_y^x (x-t)^{\nu-1} \left(D_y^\nu f\right)(t)\,dt - \frac{1}{\Gamma(\nu)} \int_y^x (x-t)^{\nu-1} \left(D_y^\nu f\right)(x)\,dt \right\|$$

by [1], p. 426, Theorem 11.43

$$= \frac{1}{\Gamma(\nu)} \left\| \int_y^x (x-t)^{\nu-1} \left(\left(D_y^\nu f\right)(t) - \left(D_y^\nu f\right)(x) \right) dt \right\| \leq \tag{2.4.13}$$

(by [9])

$$\frac{1}{\Gamma(\nu)} \int_y^x (x-t)^{\nu-1} \left\| \left(D_y^\nu f\right)(t) - \left(D_y^\nu f\right)(x) \right\| dt$$

(we assume that

$$\left\| \left(D_y^\nu f\right)(t) - \left(D_y^\nu f\right)(x) \right\| \leq \lambda_1(y) \left| t - x \right|^{p+1-\nu}, \tag{2.4.14}$$

for all $x, y, t \in [a,b]$ with $x \geq t \geq y$, with $\lambda_1(y) > 0$ and $\sup_{y \in [a,b]} \lambda_1(y) =: \lambda_1 < \infty$, also it is $0 < p + 1 - \nu < 1$)

$$\leq \frac{\lambda_1}{\Gamma(\nu)} \int_y^x (x-t)^{\nu-1} (x-t)^{p+1-\nu}\,dt = \tag{2.4.15}$$

$$\frac{\lambda_1}{\Gamma(\nu)} \int_y^x (x-t)^p\,dt = \frac{\lambda_1}{\Gamma(\nu)} \frac{(x-y)^{p+1}}{(p+1)}.$$

We have proved condition (2.4.1)

$$\|f(y)-f(x)-A_{+}(f)(x,y)(y-x)\| \leq \frac{\lambda_1}{\Gamma(\nu)}\frac{(x-y)^{p+1}}{(p+1)}, \text{ for } x>y. \tag{2.4.16}$$

(2) $x<y$: We observe that

$$\|f(y)-f(x)-(A_{+}(f))(x,y)(y-x)\| \overset{\text{(by (2.4.9), (2.4.10))}}{=}$$

$$\left\|\sum_{k=1}^{p-1}\frac{f^{(k)}(x)}{k!}(y-x)^k+\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}\left(D_x^{\nu}f\right)(t)\,dt-\right. \tag{2.4.17}$$

$$\left.\sum_{k=1}^{p-1}\frac{f^{(k)}(x)}{k!}(y-x)^k-\left(D_x^{\nu}f\right)(y)\frac{(y-x)^{\nu}}{\Gamma(\nu+1)}\right\|=$$

$$\left\|\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}\left(D_x^{\nu}f\right)(t)\,dt-\left(D_x^{\nu}f\right)(y)\frac{(y-x)^{\nu}}{\Gamma(\nu+1)}\right\|=$$

$$\left\|\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}\left(D_x^{\nu}f\right)(t)\,dt-\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}\left(D_x^{\nu}f\right)(y)\,dt\right\|= \tag{2.4.18}$$

$$\frac{1}{\Gamma(\nu)}\left\|\int_x^y (y-t)^{\nu-1}\left(\left(D_x^{\nu}f\right)(t)-\left(D_x^{\nu}f\right)(y)\right)dt\right\|\leq$$

$$\frac{1}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}\left\|\left(D_x^{\nu}f\right)(t)-\left(D_x^{\nu}f\right)(y)\right\|dt$$

(we assume here that

$$\left\|\left(D_x^{\nu}f\right)(t)-\left(D_x^{\nu}f\right)(y)\right\|\leq\lambda_2(x)\,|t-y|^{p+1-\nu}, \tag{2.4.19}$$

for all $x,y,t\in[a,b]$ with $y\geq t\geq x$, with $\lambda_2(x)>0$ and $\sup_{x\in[a,b]}\lambda_2(x)=:\lambda_2<\infty$)

$$\leq\frac{\lambda_2}{\Gamma(\nu)}\int_x^y (y-t)^{\nu-1}(y-t)^{p+1-\nu}\,dt= \tag{2.4.20}$$

$$\frac{\lambda_2}{\Gamma(\nu)}\int_x^y (y-t)^{p}\,dt=\frac{\lambda_2}{\Gamma(\nu)}\frac{(y-x)^{p+1}}{(p+1)}.$$

We have proved that

$$\left\| f(y) - f(x) - (A_+(f))(x,y)(y-x) \right\| \leq \frac{\lambda_2}{\Gamma(\nu)} \frac{(y-x)^{p+1}}{(p+1)}, \tag{2.4.21}$$

for all $x, y \in [a, b]$ such that $y > x$.

Call $\lambda := \max(\lambda_1, \lambda_2)$.

Conclusion We have proved condition (2.4.1), in detail that

$$\left\| f(y) - f(x) - (A_+(f))(x,y)(y-x) \right\| \leq \frac{\lambda}{\Gamma(\nu)} \frac{|x-y|^{p+1}}{(p+1)}, \forall\, x, y \in [a, b]. \tag{2.4.22}$$

(II) Here see [3] and [5].

Let $x, y \in [a, b]$ such that $x \leq y$, $\nu > 0$, $\nu \notin \mathbb{N}$, such that $p = [v]$, $\alpha = \nu - p$ $(0 < \alpha < 1)$.

Let $f \in C^p([a, b], X)$ and define

$$\left(J_{y-}^{\nu} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} f(z)\, dz, \; a \leq x \leq y, \tag{2.4.23}$$

the X-valued right generalized Riemann-Liouville fractional integral.

Define the subspace of functions

$$C_{y-}^{\nu}([a, b], X) := \left\{ f \in C^p([a, b], X) : J_{y-}^{1-\alpha} f^{(p)} \in C^1([a, y], X) \right\}. \tag{2.4.24}$$

Define the **X-valued right generalized $\nu-$fractional derivative of** f over $[a, y]$ as

$$D_{y-}^{\nu} f := (-1)^{p-1} \left(J_{y-}^{1-\alpha} f^{(p)} \right)'. \tag{2.4.25}$$

Notice that

$$J_{y-}^{1-\alpha} f^{(p)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^y (z-x)^{-\alpha} f^{(p)}(z)\, dz, \tag{2.4.26}$$

exists for $f \in C_{y-}^{\nu}([a, b], X)$, and

$$\left(D_{y-}^{\nu} f\right)(x) = \frac{(-1)^{p-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^y (z-x)^{-\alpha} f^{(p)}(z)\, dz. \tag{2.4.27}$$

I.e.

$$\left(D_{y-}^{\nu} f\right)(x) = \frac{(-1)^{p-1}}{\Gamma(p-\nu+1)} \frac{d}{dx} \int_x^y (z-x)^{p-\nu} f^{(p)}(z)\, dz. \tag{2.4.28}$$

Here we consider $f \in C^p([a,b], X)$ such that $f \in C^{\nu}_{y-}([a,b], X)$, for every $y \in [a,b]$, which means also that $f \in C^{\nu}_{x-}([a,b], X)$, for every $x \in [a,b]$ (i.e. exchange roles of x and y), we write that as $f \in C^{\nu}_{-}([a,b], X)$.

That is

$$\left(D^{\nu}_{x-}f\right)(y) = \frac{(-1)^{p-1}}{\Gamma(p-\nu+1)} \frac{d}{dy} \int_y^x (z-y)^{p-\nu} f^{(p)}(z)\,dz \tag{2.4.29}$$

exists for $f \in C^{\nu}_{x-}([a,b], X)$, for $a \le y \le x \le b$.

We mention the following X-valued right generalized fractional Taylor formula ($f \in C^{\nu}_{y-}([a,b], X)$, $\nu > 1$), see [5].

It holds

$$f(x) - f(y) = \sum_{k=1}^{p-1} \frac{f^{(k)}(y)}{k!}(x-y)^k + \frac{1}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} \left(D^{\nu}_{y-}f\right)(z)\,dz, \tag{2.4.30}$$

all $x, y \in [a,b]$ with $x \le y$.

Similarly for $f \in C^{\nu}_{x-}([a,b], X)$ we have

$$f(y) - f(x) = \sum_{k=1}^{p-1} \frac{f^{(k)}(x)}{k!}(y-x)^k + \frac{1}{\Gamma(\nu)} \int_y^x (z-y)^{\nu-1} \left(D^{\nu}_{x-}f\right)(z)\,dz, \tag{2.4.31}$$

all $x, y \in [a,b]$ with $x \ge y$.

So here we work with $f \in C^p([a,b], X)$, such that $f \in C^{\nu}_{-}([a,b], X)$.

We define the X-valued right linear fractional operator

$$A_-(f)(x,y) := \begin{cases} \sum_{k=1}^{p-1} \frac{f^{(k)}(x)}{k!}(y-x)^{k-1} - \left(D^{\nu}_{x-}f\right)(y) \frac{(x-y)^{\nu-1}}{\Gamma(\nu+1)}, \; x > y, \\ \sum_{k=1}^{p-1} \frac{f^{(k)}(y)}{k!}(x-y)^{k-1} - \left(D^{\nu}_{y-}f\right)(x) \frac{(y-x)^{\nu-1}}{\Gamma(\nu+1)}, \; y > x, \\ f^{(p-1)}(x), \; x = y. \end{cases} \tag{2.4.32}$$

Condition (2.4.2) is fulfilled, the same as in (2.4.11), now for $A_-(f)(x,x)$.

We would like to prove that

$$\|f(x) - f(y) - (A_-(f))(x,y)(x-y)\| \le c \cdot \frac{|x-y|^{p+1}}{p+1}, \tag{2.4.33}$$

for any $x, y \in [a,b]$, where $c > 0$.

When $x = y$ the last condition (2.4.33) is trivial. We assume $x \ne y$.

We distinguish the subcases:

(1) $x > y$: We observe that

$$\|(f(x) - f(y)) - (A_-(f))(x,y)(x-y)\| = \tag{2.4.34}$$

$$\|(f(y) - f(x)) - (A_-(f))(x,y)(y-x)\| =$$

$$\left\| \left(\sum_{k=1}^{p-1} \frac{f^{(k)}(x)}{k!} (y-x)^k + \frac{1}{\Gamma(\nu)} \int_y^x (z-y)^{\nu-1} \left(D_{x-}^{\nu} f \right)(z)\, dz \right) - \right.$$

$$\left. \left(\sum_{k=1}^{p-1} \frac{f^{(k)}(x)}{k!} (y-x)^{k-1} - \left(D_{x-}^{\nu} f \right)(y) \frac{(x-y)^{\nu-1}}{\Gamma(\nu+1)} \right) (y-x) \right\| = \tag{2.4.35}$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_y^x (z-y)^{\nu-1} \left(D_{x-}^{\nu} f \right)(z)\, dz + \left(D_{x-}^{\nu} f \right)(y) \frac{(x-y)^{\nu-1}}{\Gamma(\nu+1)} (y-x) \right\| =$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_y^x (z-y)^{\nu-1} \left(D_{x-}^{\nu} f \right)(z)\, dz - \left(D_{x-}^{\nu} f \right)(y) \frac{(x-y)^{\nu}}{\Gamma(\nu+1)} \right\| =$$

$$\frac{1}{\Gamma(\nu)} \left\| \int_y^x (z-y)^{\nu-1} \left(D_{x-}^{\nu} f \right)(z)\, dz - \int_y^x (z-y)^{\nu-1} \left(D_{x-}^{\nu} f \right)(y)\, dz \right\| =$$

$$\frac{1}{\Gamma(\nu)} \left\| \int_y^x (z-y)^{\nu-1} \left(\left(D_{x-}^{\nu} f \right)(z) - \left(D_{x-}^{\nu} f \right)(y) \right) dz \right\| \le \tag{2.4.36}$$

$$\frac{1}{\Gamma(\nu)} \int_y^x (z-y)^{\nu-1} \left\| \left(D_{x-}^{\nu} f \right)(z) - \left(D_{x-}^{\nu} f \right)(y) \right\| dz$$

(we assume that

$$\left\| \left(D_{x-}^{\nu} f \right)(z) - \left(D_{x-}^{\nu} f \right)(y) \right\| \le \lambda_1 |z-y|^{p+1-\nu}, \tag{2.4.37}$$

$\lambda_1 > 0$, for all $x, z, y \in [a, b]$ with $x \ge z \ge y$)

$$\le \frac{\lambda_1}{\Gamma(\nu)} \int_y^x (z-y)^{\nu-1} (z-y)^{p+1-\nu}\, dz = \tag{2.4.38}$$

$$\frac{\lambda_1}{\Gamma(\nu)} \int_y^x (z-y)^p\, dz = \frac{\lambda_1}{\Gamma(\nu)} \frac{(x-y)^{p+1}}{p+1} = \rho_1 \frac{(x-y)^{p+1}}{p+1},$$

where $\rho_1 := \frac{\lambda_1}{\Gamma(\nu)} > 0$.

We have proved, when $x > y$, that

$$\| f(x) - f(y) - (A_-(f))(x, y)(x-y) \| \le \rho_1 \frac{(x-y)^{p+1}}{p+1}. \tag{2.4.39}$$

(2) $y > x$: We observe that

$$\| f(x) - f(y) - (A_-(f))(x, y)(x-y) \| =$$

$$\left\| \left(\sum_{k=1}^{p-1} \frac{f^{(k)}(y)}{k!} (x-y)^k + \frac{1}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} \left(D_{y-}^{\nu} f \right)(z)\, dz \right) - \right.$$

$$\left. \left(\sum_{k=1}^{p-1} \frac{f^{(k)}(y)}{k!} (x-y)^{k-1} - \left(D_{y-}^{\nu} f \right)(x) \frac{(y-x)^{\nu-1}}{\Gamma(\nu+1)} \right)(x-y) \right\| = \tag{2.4.40}$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} \left(D_{y-}^{\nu} f \right)(z)\, dz - \left(D_{y-}^{\nu} f \right)(x) \frac{(y-x)^{\nu}}{\Gamma(\nu+1)} \right\| =$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} \left(D_{y-}^{\nu} f \right)(z)\, dz - \frac{1}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} \left(D_{y-}^{\nu} f \right)(x)\, dz \right\| = \tag{2.4.41}$$

$$\frac{1}{\Gamma(\nu)} \left\| \int_x^y (z-x)^{\nu-1} \left(\left(D_{y-}^{\nu} f \right)(z) - \left(D_{y-}^{\nu} f \right)(x) \right) dz \right\| \leq \tag{2.4.42}$$

$$\frac{1}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} \left\| \left(D_{y-}^{\nu} f \right)(z) - \left(D_{y-}^{\nu} f \right)(x) \right\| dz$$

(we assume that

$$\left\| \left(D_{y-}^{\nu} f \right)(z) - \left(D_{y-}^{\nu} f \right)(x) \right\| \leq \lambda_2 |z-x|^{p+1-\nu}, \tag{2.4.43}$$

$\lambda_2 > 0$, for all $y, z, x \in [a, b]$ with $y \geq z \geq x$)

$$\leq \frac{\lambda_2}{\Gamma(\nu)} \int_x^y (z-x)^{\nu-1} (z-x)^{p+1-\nu}\, dz = \tag{2.4.44}$$

$$\frac{\lambda_2}{\Gamma(\nu)} \int_x^y (z-x)^{p}\, dz = \frac{\lambda_2}{\Gamma(\nu)} \frac{(y-x)^{p+1}}{p+1}.$$

We have proved, for $y > x$, that

$$\| f(x) - f(y) - (A_{-}(f))(x, y)(x-y) \| \leq \rho_2 \frac{(y-x)^{p+1}}{p+1}, \tag{2.4.45}$$

where $\rho_2 := \frac{\lambda_2}{\Gamma(\nu)} > 0$.

Set $\lambda := \max(\lambda_1, \lambda_2)$ and $\rho := \frac{\lambda}{\Gamma(\nu)} > 0$.

Conclusion We have proved (2.4.1) that

$$\| f(x) - f(y) - (A_{-}(f))(x, y)(x-y) \| \leq \rho \frac{|x-y|^{p+1}}{p+1}, \tag{2.4.46}$$

for any $x, y \in [a, b]$.

(III) Let again $f \in C^p([a,b], X)$, $p \in \mathbb{N}$, $x, y \in [a,b]$.
By vector X-valued Taylor's formula we have, see [3, 4, 18],

$$f(x) - f(y) = \sum_{k=1}^{p} \frac{f^{(k)}(y)}{k!}(x-y)^k + \frac{1}{(p-1)!}\int_y^x (x-t)^{p-1}\left(f^{(p)}(t) - f^{(p)}(y)\right) dt, \tag{2.4.47}$$

$\forall\, x, y \in [a,b]$.

We define the X-valued function

$$(A_0(f))(x,y) := \begin{cases} \sum_{k=1}^{p} \frac{f^{(k)}(y)}{k!}(x-y)^{k-1}, & x \neq y, \\ f^{(p-1)}(x), & x = y. \end{cases} \tag{2.4.48}$$

Then it holds, by [13], p. 3,

$$\|(A_0(f))(x,x) - (A_0(f))(y,y)\| = \left\| f^{(p-1)}(x) - f^{(p-1)}(y) \right\| \tag{2.4.49}$$

$$\leq \left\| f^{(p)} \right\|_{\infty} |x-y|, \ \forall\, x, y \in [a,b],$$

so that condition (2.4.2) is fulfilled.

Next we observe that

$$\|f(x) - f(y) - (A_0(f))(x,y)(x-y)\| =$$

$$\left\| \sum_{k=1}^{p} \frac{f^{(k)}(y)}{k!}(x-y)^k + \frac{1}{(p-1)!}\int_y^x (x-t)^{p-1}\left(f^{(p)}(t) - f^{(p)}(y)\right) dt \right. \tag{2.4.50}$$

$$\left. - \sum_{k=1}^{p} \frac{f^{(k)}(y)}{k!}(x-y)^k \right\| =$$

$$\frac{1}{(p-1)!}\left\| \int_y^x (x-t)^{p-1}\left(f^{(p)}(t) - f^{(p)}(y)\right) dt \right\| =: (\xi). \tag{2.4.51}$$

Here we assume that

$$\left\| f^{(p)}(t) - f^{(p)}(y) \right\| \leq c\,|t-y|, \ \forall\, t, y \in [a,b], \ c > 0. \tag{2.4.52}$$

(1) Subcase of $x > y$: We have that (by [9])

$$(\xi) \leq \frac{1}{(p-1)!}\int_y^x (x-t)^{p-1}\left\| f^{(p)}(t) - f^{(p)}(y) \right\| dt \leq$$

$$\frac{c}{(p-1)!}\int_y^x (x-t)^{p-1}(t-y)^{2-1}\, dt = \tag{2.4.53}$$

$$c\frac{\Gamma(p)\Gamma(2)}{(p-1)!\Gamma(p+2)}(x-y)^{p+1} = c\frac{(p-1)!}{(p-1)!(p+1)!}(x-y)^{p+1}$$

$$= \frac{c(x-y)^{p+1}}{(p+1)!}.$$

Hence

$$(\xi) \leq c\frac{(x-y)^{p+1}}{(p+1)!}, \; x > y. \tag{2.4.54}$$

(2) Subcase of $y > x$.
We have that

$$(\xi) = \frac{1}{(p-1)!}\left\| \int_x^y (t-x)^{p-1}\left(f^{(p)}(y) - f^{(p)}(t)\right)dt \right\| \leq \tag{2.4.55}$$

$$\frac{1}{(p-1)!}\int_x^y (t-x)^{p-1}\left\| f^{(p)}(y) - f^{(p)}(t)\right\| dt \leq$$

$$\frac{c}{(p-1)!}\int_x^y (t-x)^{p-1}(y-t)\,dt =$$

$$\frac{c}{(p-1)!}\int_x^y (y-t)^{2-1}(t-x)^{p-1}\,dt = \tag{2.4.56}$$

$$\frac{c}{(p-1)!}\frac{\Gamma(2)\Gamma(p)}{\Gamma(p+2)}(y-x)^{p+1} = \frac{c}{(p-1)!}\frac{(p-1)!}{(p+1)!}(y-x)^{p+1}$$

$$= c\frac{(y-x)^{p+1}}{(p+1)!}.$$

That is

$$(\xi) \leq c\frac{(y-x)^{p+1}}{(p+1)!}, \; y > x. \tag{2.4.57}$$

Therefore it holds

$$(\xi) \leq c\frac{|x-y|^{p+1}}{(p+1)!}, \text{ all } x, y \in [a,b] \text{ such that } x \neq y. \tag{2.4.58}$$

We have found that

$$\|f(x) - f(y) - (A_0(f))(x,y)(x-y)\| \leq c\frac{|x-y|^{p+1}}{(p+1)!}, \; c > 0, \tag{2.4.59}$$

for all $x \neq y$.

When $x = y$ inequality (2.4.59) holds trivially, so (2.4.1) it is true for any $x, y \in [a, b]$.

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