Chapter 2
Review of Analytical Mechanics

Abstract This chapter treats simple structural members based on two different analytical approaches. On the one hand based on fundamental equations of continuum mechanics, i.e., the kinematics, the equilibrium and the constitutive equation, the describing partial differential equations are provided, including their general solution based on constants of integration. As an alternative approach, the total strain energy of a system is introduced and applied in Castigliano's theorems. The covered structural members are rods (tensile deformation) as well as thin and thick beams (bending deformation). The provided concepts are finally applied to the extensometer design problem.

2.1 Overview: One-Dimensional Structural Members

2.2 Partial Differential Equation-Based Approaches

2.2.1 Rods

A rod is defined as a prismatic body whose axial dimension is much larger than its transverse dimensions [2, 10, 16, 34, 36]. This structural member is only loaded in the direction of the main body axes, see Fig. 2.1. As a result of this loading, the deformation occurs only along its main axis.

Derivations are restricted many times to the following simplifications:

- only applying to straight rods,
- displacements are (infinitesimally) small,
- strains are (infinitesimally) small, and
- the material is linear-elastic.

The three basic equations of continuum mechanics, i.e. the kinematics relationship, the constitutive law and the equilibrium equation, as well as their combination to the describing partial differential equation (PDE) are summarized in Table 2.1.

Under the assumption of constant material (E = const.) and geometric (A = const.) properties, the differential equation in Table 2.1 can be easily integrated twice

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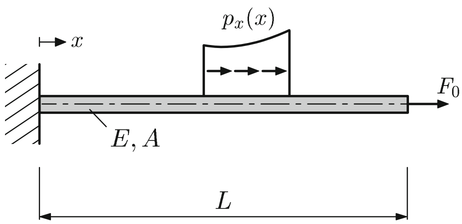


Fig. 2.1 Schematic representation of a continuum rod

Table 2.1 Different formulations of the basic equations for a rod (x-axis along the principal rod axis), with $\mathcal{L}_1(\dots) = \frac{\mathrm{d}(\dots)}{\mathrm{d}x}$

Specific formulation	General formulation
Kinematics	
$\varepsilon_x(x) = \frac{\mathrm{d}u_x(x)}{\mathrm{d}x}$	$\varepsilon_x(x) = \mathcal{L}_1\left(u_x(x)\right)$
Constitution	
$\sigma_x(x) = E\varepsilon_x(x)$	$\sigma_x(x) = C\varepsilon_x(x)$
Equilibrium	
$\frac{\mathrm{d}\sigma_x(x)}{\mathrm{d}x} + \frac{p_x(x)}{A} = 0$	$\mathcal{L}_1^{\mathrm{T}}\left(\sigma_x(x)\right) + b = 0$
PDE	
$\frac{\mathrm{d}}{\mathrm{d}x}\left(E(x)A(x)\frac{\mathrm{d}u_x}{\mathrm{d}x}\right) + p_x(x) = 0$	$\mathcal{L}_1^{\mathrm{T}}\left(EA\mathcal{L}_1\left(u_x(x)\right)\right) + p_x = 0$

for constant distributed load ($p_x = p_0 = \text{const.}$) to obtain the general solution of the problem [24]:

$$u_x(x) = \frac{1}{EA}\left(-\frac{1}{2}\,p_0 x^2 + c_1 x + c_2\right), \tag{2.1}$$

where the two constants of integration c_i $(i = 1, 2)$ must be determined based on the boundary conditions (see Table 2.2). The following equation for the internal normal force N_x was obtained based on one-time integration of the PDE and might be useful to determine some of the constants of integration:

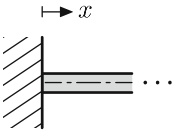
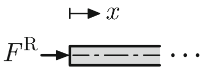
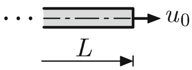
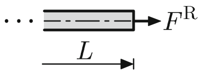
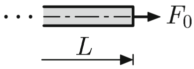
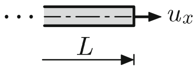
$$N_x(x) = EA\frac{\mathrm{d}u_x(x)}{\mathrm{d}x} = -p_0 x + c_1\,. \tag{2.2}$$

The internal reactions in a rod become visible if one cuts — at an arbitrary location x — the member in two parts. As a result, two opposite oriented normal forces N_x can be indicated. Summing up the internal reactions from both parts must result in zero. Their positive direction is connected with the direction of the outward surface normal vector and the orientation of the positive x-axis, see Fig. 2.2.

Once the internal normal force N_x is known, the normal stress σ_x can be calculated:

$$\sigma_x(x) = \frac{N_x(x)}{A}\,. \tag{2.3}$$

Table 2.2 Different boundary conditions and corresponding reactions for a continuum rod (deformation occurs along the x-axis)

Case	Boundary Condition	Reaction
	$u_x(x=0)=0$	
	$u_x(x=L)=u_0$	
	$EA\frac{\mathrm{d}u_x(L)}{\mathrm{d}x}=N_x(L)=F_0$	

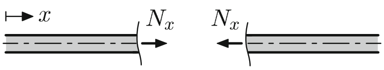


Fig. 2.2 Internal reactions for a continuum rod

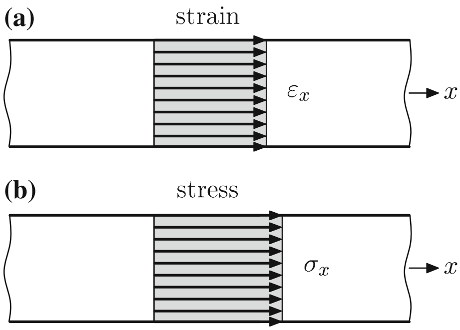


Fig. 2.3 Axially loaded rod: **a** strain and **b** stress distribution

Application of HOOKE's law (see Table 2.1) allows us to calculate the normal strain ε_x. Typical distributions of stress and strain in a rod element are shown in Fig. 2.3. It can be seen that both distributions are constant over the cross section.

To be able to realize a closed-form presentation with discontinuities (e.g. load, material, or geometry), the so-called MACAULAY bracket[1] can be used for closed-form representations. This mathematical notation has the following meaning:

$$\langle x-a\rangle^n=\begin{cases}0 & \text{for } x<a\\ (x-a)^n & \text{for } x\geq a\,.\end{cases} \tag{2.4}$$

[1] In the German literature, this approach is named after August Otto FÖPPL (1854–1942).

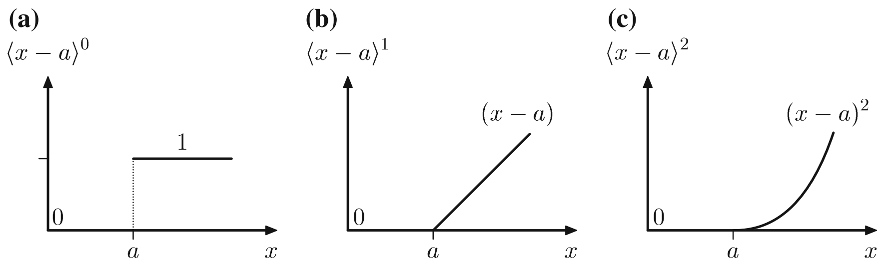


Fig. 2.4 Graphical representation of the first three discontinuous functions according to Eq. (2.4): **a** jump ($n = 0$); **b** kink ($n = 1$); **c** smooth transition ($n = 2$). Adapted from [2]

In particular with the case $n = 0$

$$\langle x-a\rangle^0 = \begin{cases} 0 \text{ for } x < a \\ 1 \text{ for } x \geq a \end{cases} \tag{2.5}$$

the closed-form presentation of jumps can be realized. The first three discontinuous functions are shown in Fig. 2.4. Furthermore, derivations and integrals are defined by regarding the triangular bracket symbol as classical round brackets:

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle x-a\rangle^n = n\langle x-a\rangle^{n-1}\,, \tag{2.6}$$

$$\int \langle x-a\rangle^n \mathrm{d}x = \frac{1}{n+1}\langle x-a\rangle^{n+1} + c\,. \tag{2.7}$$

Table 2.3 shows a few examples of discontinuous loads and their corresponding representations due to the discontinuous function given in Eq. (2.4).

In regards to the first case in Table 2.3, it should be noted that a positive singe force ($F_0 > 0$) results in a negative jump in the normal force distribution (N_x).

If no single closed-form representation is required, all the previous equations (see Table 2.2 and Eqs. (2.1)–(2.3)) can be applied to each continuous section. As a result, transmission conditions between the continuous sections must be formulated to determine the additional constants of integration, see Problem 2.3.

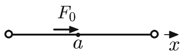
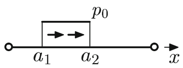
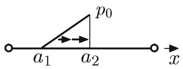
2.1 Cantilever Rod with Point Loads

Given is a rod of length L and constant axial tensile stiffness EA as shown in Fig. 2.5. At the left-hand side there is a fixed support and the right-hand side is either elongated by a displacement u_0 (case a) or loaded by a single force F_0 (case b). Determine the analytical solution for the elongation $u_x(x)$, the strain $\varepsilon_x(x)$, and the stress $\sigma_x(x)$ along the rod axis. Sketch for both cases the corresponding distributions.

2.1 Solution

Case (a): Let us start the solution procedure by sketching the free-body diagram as

Table 2.3 Discontinuous loads expressed due to discontinuous functions (deformation occurs along the x-axis). Adapted from [2]

Case	Load (Discontinuity Function)
	$N_x(x) = -F_0\langle x-a\rangle^0 + c$
	$p_x(x) = p_0\big(\langle x-a_1\rangle^0 - \langle x-a_2\rangle^0\big)$
	$p_x(x) = \dfrac{p_0}{a_2-a_1}(x-a_1)\big(\langle x-a_1\rangle^0 - \langle x-a_2\rangle^0\big)$

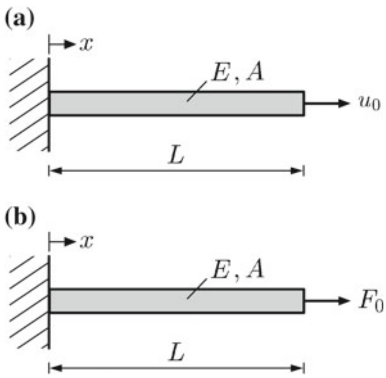


Fig. 2.5 Rod under different loading conditions: **a** displacement and **b** force

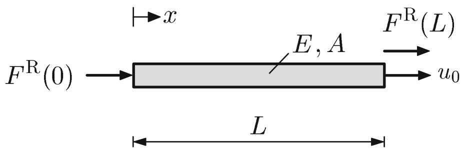


Fig. 2.6 Free-body diagram of the rod with displacement boundary condition

shown in Fig. 2.6. It should be noted here that the imposed displacement u_0 at the right-hand boundary results in a reaction force $F^{\mathrm{R}}(L)$.

The next step is to identify the boundary conditions of the problem. They can be immediately stated as:

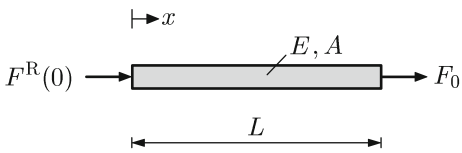


Fig. 2.7 Free-body diagram of the rod with force boundary condition

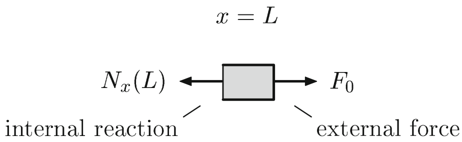


Fig. 2.8 Equilibrium between internal normal force N_x and external load F_0

$$u_x(0) = 0\,, \tag{2.8}$$

$$u_x(L) = u_0\,. \tag{2.9}$$

Consideration of the first boundary condition in Eq. (2.1) results with $p_0 = 0$ directly in $c_2 = 0$. Considering the second boundary condition in Eq. (2.1) gives then $c_1 = \frac{EAu_0}{L}$. Thus, the distributions of elongation, strain, and stress are obtained as:

$$u_x(x) = u_0 \frac{x}{L}\,, \tag{2.10}$$

$$\varepsilon_x(x) = \frac{\mathrm{d}u_x(x)}{\mathrm{d}x} = \frac{u_0}{L}\,, \tag{2.11}$$

$$\sigma_x(x) = E\varepsilon_x(x) = \frac{u_0 E}{L}\,. \tag{2.12}$$

Case (b): Let us start the solution procedure by sketching the free-body diagram as shown in Fig. 2.7.

The first boundary condition is again $u_x(0) = 0$ which results with Eq. (2.1) directly in $c_2 = 0$. The second boundary condition might be not so obvious and requires to consider of the force equilibrium for a small element at $x = L$, see Fig. 2.8.

The horizontal force equilibrium yields the second boundary condition as $N_x(L) = F_0$. Introducing this second condition into Eq. (2.2), the second constant of integration is obtained for $p_0 = 0$ as $c_1 = F_0$. Thus, the distributions of elongation, strain, and stress are obtained as:

$$u_x(x) = \frac{F_0}{EA} x\,, \tag{2.13}$$

$$\varepsilon_x(x) = \frac{\mathrm{d}u_x(x)}{\mathrm{d}x} = \frac{F_0}{EA}\,, \tag{2.14}$$

$$\sigma_x(x) = E\varepsilon_x(x) = \frac{F_0}{A}\,. \tag{2.15}$$

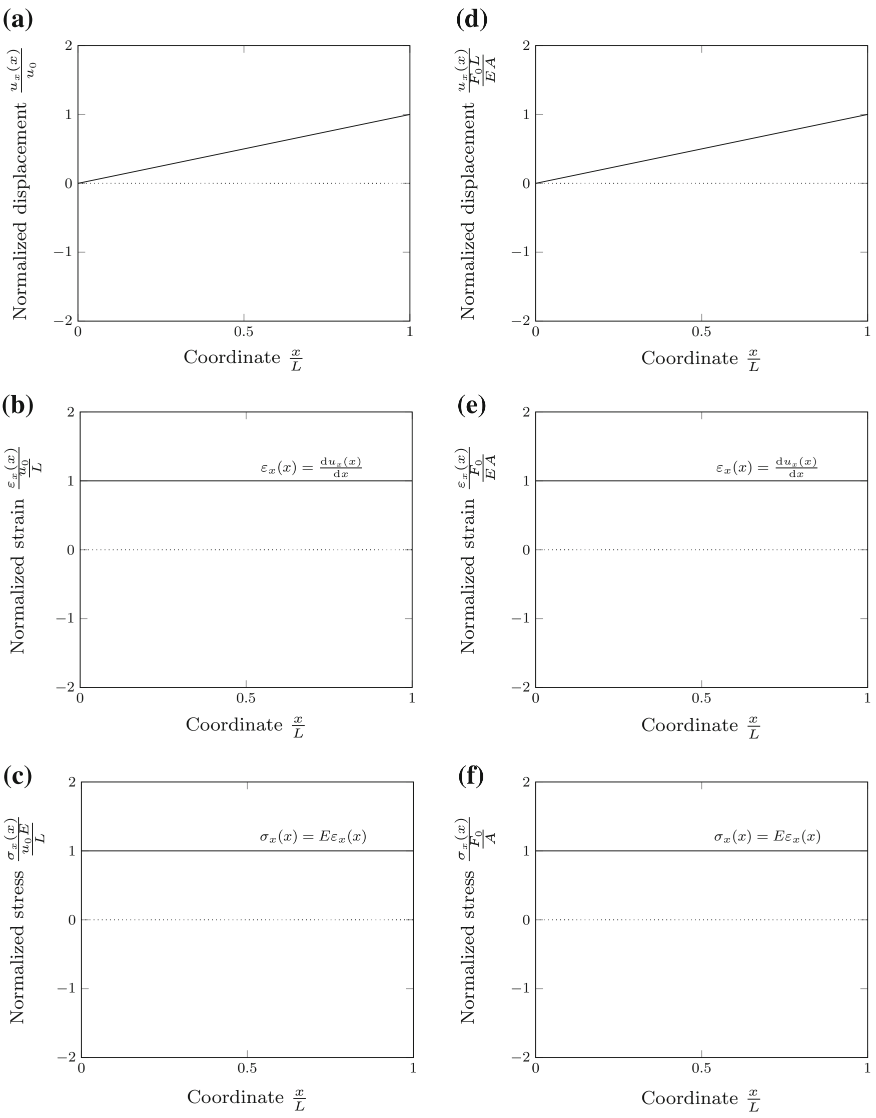


Fig. 2.9 Graphical representation of the field variables: **a**–**c** displacement boundary conditions (u_0), and **d**–**f** force boundary condition (F_0)

Equation (2.15) is the classical definition of engineering stress in the case of a uni-axial tensile test. The graphical representation of the field variables (displacement, strain, and stress) is shown in Fig. 2.9.

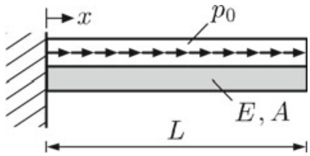


Fig. 2.10 Rod with distributed load

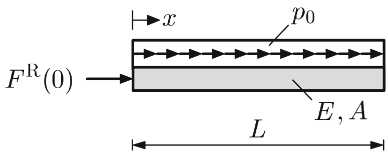


Fig. 2.11 Free-body diagram of the rod with distributed load

2.2 Cantilever Rod with Distributed Load

Given is a rod of length L and constant axial tensile stiffness EA as shown in Fig. 2.10. At the left-hand side there is a fixed support and a constant distributed load p_0 is acting along the entire rod axis. Determine the analytical solution for the elongation $u_x(x)$, the strain $\varepsilon_x(x)$, and the stress $\sigma_x(x)$ along the rod axis.

2.2 Solution

Let us start the solution procedure by sketching the free-body diagram as shown in Fig. 2.11.

As outlined in the previous example, the boundary conditions can be stated as $u_x(0) = 0$ and $N_x(L) = 0$. However, we must consider now that a constant distributed load p_0 is acting and the evaluation of Eq. (2.1) based on the first boundary condition gives $c_2 = 0$. Application of the second boundary condition in Eq. (2.2) gives now $c_1 = p_0 L$. Thus, the distributions of elongation, strain, and stress are obtained as:

$$u_x(x) = \frac{p_0 L^2}{EA}\left(-\frac{1}{2}\left[\frac{x}{L}\right]^2 + \left[\frac{x}{L}\right]\right), \tag{2.16}$$

$$\varepsilon_x(x) = \frac{\mathrm{d}u_x(x)}{\mathrm{d}x} = \frac{p_0 L}{EA}\left(-\left[\frac{x}{L}\right] + 1\right), \tag{2.17}$$

$$\sigma_x(x) = E\varepsilon_x(x) = \frac{p_0 L}{A}\left(-\left[\frac{x}{L}\right] + 1\right). \tag{2.18}$$

2.3 Cantilever Rod with Different Sections

Given is a rod of length $3L$ and constant axial tensile stiffness EA as shown in Fig. 2.12. At the left-hand side there is a fixed support and a constant distributed load $2p_0$ is acting in the range $0 \leq x \leq 2L$ whereas a load of p_0 is acting in the range $2L \leq x \leq 3L$. Determine the analytical solution for the elongation $u_x(x)$, the strain $\varepsilon_x(x)$, and the stress $\sigma_x(x)$ along the rod axis.

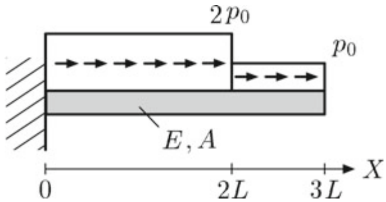


Fig. 2.12 Rod with different sections

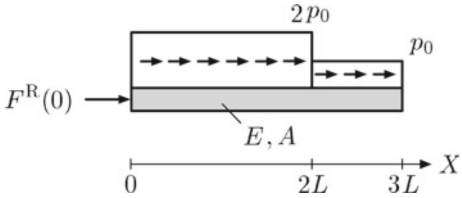


Fig. 2.13 Free-body diagram of the rod with different sections

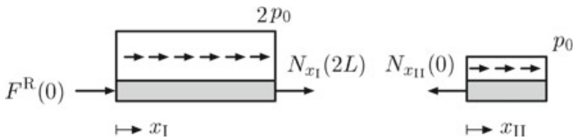


Fig. 2.14 Free-body diagram of the rod decomposed into two sections

2.3 Solution

Let us start the solution procedure by sketching the free-body diagram as shown in Fig. 2.13.

The discontinuity in the distributed load can be handled by splitting the rod at $X = 2L$ into two parts, see Fig. 2.14. The left-hand part is now described by the local coordinate x_{I} with $0 \le x_{\mathrm{I}} \le 2L$ while the right-hand part is described by the local coordinate x_{II} with $0 \le x_{\mathrm{II}} \le L$.

Consideration of two parts means that Eqs. (2.1) and (2.2) must be applied to both sections and in total four integration constants, i.e. two for each section (here c_1 and c_2 for the left-hand section while c_3 and c_4 is assigned to the right-hand section), must be determined. The following two boundary and two transmission conditions can be stated:

$$u_x(x_{\mathrm{I}} = 0) = 0\,, \qquad N_x(x_{\mathrm{II}} = L) = 0\,, \tag{2.19}$$

$$u_x(x_{\mathrm{I}} = 2L) = u_x(x_{\mathrm{II}} = 0)\,, \qquad N_x(x_{\mathrm{I}} = 2L) = N_x(x_{\mathrm{II}} = 0)\,. \tag{2.20}$$

Consideration of boundary condition $(2.19)_1$ in Eq. (2.1) gives immediately $c_2 = 0$. Consideration of the second boundary condition $(2.19)_2$ in Eq. (2.2) provides $c_3 = p_0 L$. The next step requires the application of the transmission conditions. Let us start with the transmission condition for the normal force $(2.20)_2$:

$$-(2p_0)(2L) + c_1 = c_3 \overset{2^{\text{nd}}\,\text{BC}}{=} p_0 L\,, \tag{2.21}$$

from which a further constant can be determined as $c_1 = 5p_0L$. The final constant can be obtained from the displacement transmission condition $(2.20)_1$:

$$-\frac{1}{2}(2p_0)(2L)^2 + c_1(2L) = c_4\,, \tag{2.22}$$

which can be solved for the remaining constant: $c_4 = 6p_0L^2$. Thus, the distributions of elongation, strain, and stress are obtained as for each section as:

$$u_x(x_{\mathrm{I}}) = \frac{p_0L^2}{EA}\left(-\left[\frac{x_{\mathrm{I}}}{L}\right]^2 + 5\left[\frac{x_{\mathrm{I}}}{L}\right]\right)\,, \tag{2.23}$$

$$\varepsilon_x(x_{\mathrm{I}}) = \frac{\mathrm{d}u_x(x_{\mathrm{I}})}{\mathrm{d}x} = \frac{p_0L}{EA}\left(-2\left[\frac{x_{\mathrm{I}}}{L}\right] + 5\right)\,, \tag{2.24}$$

$$\sigma_x(x_{\mathrm{I}}) = E\varepsilon_x(x_{\mathrm{I}}) = \frac{p_0L}{A}\left(-2\left[\frac{x_{\mathrm{I}}}{L}\right] + 5\right)\,, \tag{2.25}$$

and

$$u_x(x_{\mathrm{II}}) = \frac{p_0L^2}{EA}\left(-\frac{1}{2}\left[\frac{x_{\mathrm{II}}}{L}\right]^2 + \left[\frac{x_{\mathrm{II}}}{L}\right] + 6\right)\,, \tag{2.26}$$

$$\varepsilon_x(x_{\mathrm{II}}) = \frac{\mathrm{d}u_x(x_{\mathrm{II}})}{\mathrm{d}x} = \frac{p_0L}{EA}\left(-\left[\frac{x_{\mathrm{II}}}{L}\right] + 1\right)\,, \tag{2.27}$$

$$\sigma_x(x_{\mathrm{II}}) = E\varepsilon_x(x_{\mathrm{II}}) = \frac{p_0L}{A}\left(-\left[\frac{x_{\mathrm{II}}}{L}\right] + 1\right)\,. \tag{2.28}$$

An alternative solution approach can be based on the MACAULAY brackets as outlined in Eq. (2.4). Based on this particular approach to express discontinuities, we can state the distribution of the distributed load in the global coordinate X as:

$$p_X(X) = 2p_0\left(\langle X\rangle^0 - \langle X-2L\rangle^0\right) + p_0\left(\langle X-2L\rangle^0\right)\,. \tag{2.29}$$

This expression can be introduced in the second-order differential equation (see Table 2.2) as load function:

$$EA\frac{\mathrm{d}^2u_X(X)}{\mathrm{d}X^2} = -2p_0\left(\langle X\rangle^0 - \langle X-2L\rangle^0\right) - p_0\left(\langle X-2L\rangle^0\right)\,. \tag{2.30}$$

Integration twice gives:

$$EA\frac{\mathrm{d}^1u_X}{\mathrm{d}X^1} = -2p_0\left(\langle X\rangle^1 - \langle X-2L\rangle^1\right) - p_0\left(\langle X-2L\rangle^1\right) + c_1\,, \tag{2.31}$$

$$EAu_X = -2p_0\left(\frac{1}{2}\langle X\rangle^2 - \frac{1}{2}\langle X-2L\rangle^2\right) - p_0\left(\frac{1}{2}\langle X-2L\rangle^2\right) + c_1X + c_2\,. \tag{2.32}$$

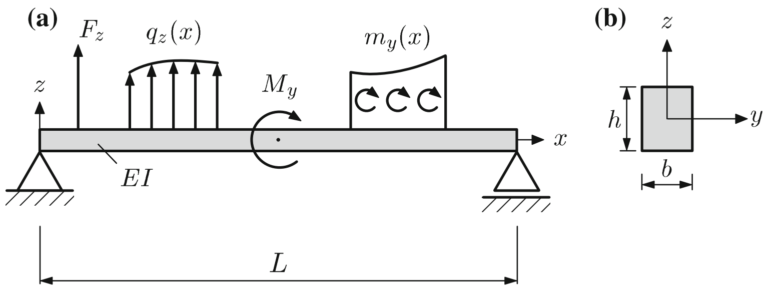


Fig. 2.15 General configuration for EULER–BERNOULLI beam problems: **a** example of boundary conditions and external loads; **b** cross-sectional area (bending occurs in the x-z plane)

The constants can be obtained based on the boundary conditions (2.19) as $c_2 = 0$ and $c_1 = 5p_0L$. Thus, the distributions of elongation, strain, and stress are obtained in closed-form representation as:

$$u_X(X) = \frac{p_0}{EA}\left\{-\langle X\rangle^2 + \langle X-2L\rangle^2 - \frac{1}{2}\langle X-2L\rangle^2 + 5LX\right\}, \tag{2.33}$$

$$\varepsilon_X(X) = \frac{p_0}{EA}\left\{-2\langle X\rangle^1 + 2\langle X-2L\rangle^1 - \langle X-2L\rangle^1 + 5L\right\}, \tag{2.34}$$

$$\sigma_X(X) = \frac{p_0}{A}\left\{-2\langle X\rangle^1 + 2\langle X-2L\rangle^1 - \langle X-2L\rangle^1 + 5L\right\}. \tag{2.35}$$

2.2.2 Euler–Bernoulli Beams

A thin or EULER–BERNOULLI beam is defined as a long prismatic body whose axial dimension is much larger than its transverse dimensions [2, 10, 16, 34, 36]. This structural member is only loaded perpendicular to its longitudinal body axis by forces (single forces F_z or distributed loads q_z) or moments (single moments M_y or distributed moments m_y). Perpendicular means that the line of application of a force or the direction of a moment vector forms a right angle with the x-axis, see Fig. 2.15. As a result of this loading, the deformation occurs only perpendicular to its main axis.

Derivations are restricted many times to the following simplifications:

- only applying to straight beams,
- no elongation along the x-axis,
- no torsion around the x-axis,
- deformations in a single plane, i.e. symmetrical bending,
- infinitesimally small deformations and strains,
- simple cross sections, and
- the material is linear-elastic.

Table 2.4 Different formulations of the basic equations for a BERNOULLI beam (bending occurs in the x-z plane), with $\mathcal{L}_2(\ldots) = \frac{\mathrm{d}^2(\ldots)}{\mathrm{d}x^2}$

Specific formulation	General formulation
Kinematics	
$\varepsilon_x(x,z) = -z\frac{\mathrm{d}^2 u_z(x)}{\mathrm{d}x^2}$	$\varepsilon_x(x,z) = -z\mathcal{L}_2\left(u_z(x)\right)$
$\kappa = -\frac{\mathrm{d}^2 u_z(x)}{\mathrm{d}x^2}$	$\kappa = -\mathcal{L}_2\left(u_z(x)\right)$
Constitution	
$\sigma_x(x,z) = E\varepsilon_x(x,z)$	$\sigma_x(x,z) = C\varepsilon_x(x,z)$
$M_y(x) = EI_y\kappa(x)$	$M_y(x) = D\kappa(x)$
Equilibrium	
force	
$\frac{\mathrm{d}Q_z(x)}{\mathrm{d}x} = -q_z(x)$	
moment	
$\frac{\mathrm{d}M_y(x)}{\mathrm{d}x} = Q_z(x)$	
combined	
$\frac{\mathrm{d}^2 M_y(x)}{\mathrm{d}x^2} + q_z(x) = 0$	$\mathcal{L}_2^{\mathrm{T}}\left(M_y(x)\right) + q_z(x) = 0$
PDE	
$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI_y\frac{\mathrm{d}^2 u_z(x)}{\mathrm{d}x^2}\right) - q_z(x) = 0$	$\mathcal{L}_2^{\mathrm{T}}\left(D\mathcal{L}_2\left(u_z(x)\right)\right) - q_z(x) = 0$
$\frac{\mathrm{d}}{\mathrm{d}x}\left(EI_y\frac{\mathrm{d}^2 u_z(x)}{\mathrm{d}x^2}\right) = -Q_z(x)$	
$EI_y\frac{\mathrm{d}^2 u_z(x)}{\mathrm{d}x^2} = -M_y(x)$	

The three basic equations of continuum mechanics, i.e. the kinematics relationship, the constitutive law and the equilibrium equation, as well as their combination to the describing partial differential equation are summarized in Table 2.4.

Under the assumption of constant material ($E = \text{const.}$) and geometric ($I_y = \text{const.}$) properties, the differential equation in Table 2.4 can be integrated four times for constant distributed load ($q_z = q_0 = \text{const.}$) to obtain the general analytical solution of the problem:

$$u_x(x) = \frac{1}{EI_y}\left(\frac{q_0 x^4}{24} + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4\right), \tag{2.36}$$

where the four constants of integration c_i $(i = 1, \ldots, 4)$ must be determined based on the boundary conditions (see Table 2.5). The following equations for the shear force Q_z, the bending moment M_y, and the rotation φ_y were obtained based on one-, two- and three-times integration and might be useful to determine some of the constants of integration:

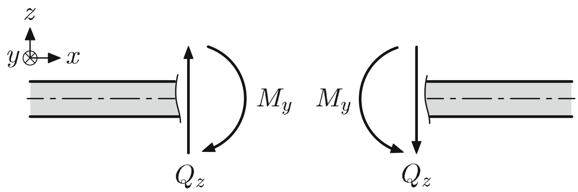


Fig. 2.16 Internal reactions for a continuum EULER–BERNOULLI beam

$$Q_z(x) = -q_0 x - c_1 \,, \tag{2.37}$$

$$M_y(x) = -\frac{q_0 x^2}{2} - c_1 x - c_2 \,, \tag{2.38}$$

$$\varphi_y(x) = -\frac{\mathrm{d}u_z(x)}{\mathrm{d}x} = -\frac{1}{EI_y}\left(\frac{q_0 x^3}{6} + \frac{c_1 x^2}{2} + c_2 x + c_3\right) . \tag{2.39}$$

The internal reactions in a beam become visible if one cuts — at an arbitrary location x — the member in two parts. As a result, two opposite oriented shear forces Q_z and bending moments M_y can be indicated. Summing up the internal reactions from both parts must result in zero. Their positive direction is connected with the positive coordinate directions at the positive face (outward surface normal vector parallel to the positive x-axis). This means that at a positive face the positive reactions have the same direction as the positive coordinate axes, see Fig. 2.16.

Once the internal bending moment M_y is known, the normal stress σ_x can be calculated:

$$\sigma_x(x, z) = \frac{M_y(x)}{I_y} z(x) \,, \tag{2.40}$$

whereas the shear force Q_z allows us to calculate the shear stress distribution. For a rectangular cross section (width b, height h, see Fig. 2.15) under the assumption that the shear stress is constant along the width, the following distribution is obtained [16]:

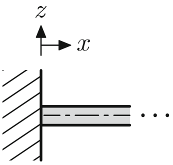
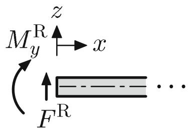
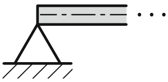
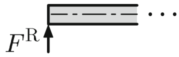
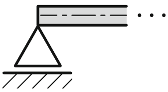
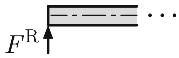
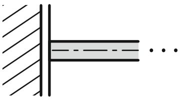
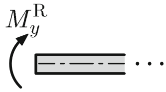
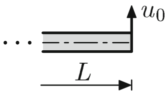
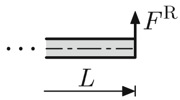
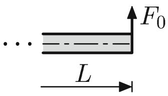
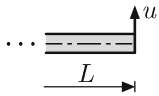
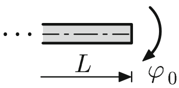
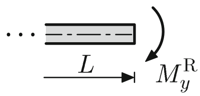
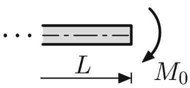
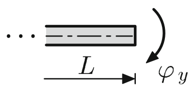
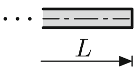
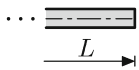
$$\tau_{xz}(x, z) = \frac{Q_z(x)}{2I_y}\left[\left(\frac{h}{2}\right)^2 - z^2\right] . \tag{2.41}$$

Application of HOOKE's law (i.e., $\sigma_x = E\varepsilon_x$ and $\tau_{xz} = G\gamma_{xz}$) allows us to calculate the normal and shear strains. Typical distributions of the two stress components in a beam element are shown in Fig. 2.17. It can be seen that normal stress distribution is linear while the shear stress distribution is parabolic over the cross section.

Finally, it should be noted here that the one-dimensional EULER–BERNOULLI beam theory has its two-dimensional analogon in the form of KIRCHHOFF plates[2] [3, 4, 6, 11, 20, 38].

[2] Also called thin or shear-rigid plates.

Table 2.5 Different boundary conditions and corresponding reactions for a continuum Euler–Bernoulli beam (bending occurs in the x-z plane)

Case	Boundary Condition	Reaction
	$u_z(0) = 0, \varphi_y(0) = 0$	
	$u_z(0) = 0, M_y(0) = 0$	
	$u_z(0) = 0, M_y(0) = 0$	
	$\varphi_y(0) = 0, Q_z(0) = 0$	
	$u_z(L) = u_0, M_y(L) = 0$	
	$Q_z(L) = F_0, M_y(L) = 0$	
	$\varphi_y(L) = \varphi_0, Q_z(L) = 0$	
	$M_y(L) = M_0, Q_z(L) = 0$	
	$M_y(L) = 0, Q_z(L) = 0$	

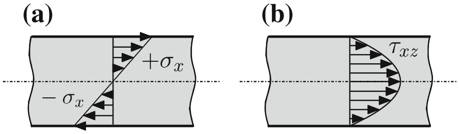


Fig. 2.17 Different stress distributions of an EULER–BERNOULLI beam with rectangular cross section and linear-elastic material behavior: **a** normal stress and **b** shear stress (bending occurs in the x-z plane)

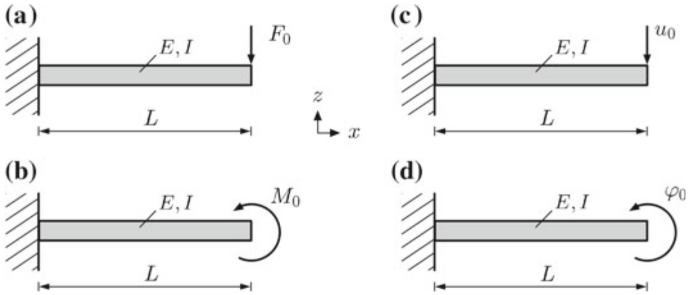


Fig. 2.18 Cantilever beam with different end loads and deformations: **a** single force; **b** single moment; **c** displacement; **d** rotation

2.4 Cantilever Beam with Different End Loads and Deformations

Calculate the analytical solutions for the deflection $u_z(x)$ and rotation $\varphi_y(x)$ of the cantilever beam shown in Fig. 2.18. Calculate in addition for all four cases the reactions at the fixed support and the distributions of the bending moment and shear force. It can be assumed for this exercise that the bending stiffness EI_y is constant.

2.4 Solution

Case (a): Let us start the solution procedure by sketching the free-body diagram as shown in Fig. 2.19a.

The consideration of the global force and moment equilibrium would allow to calculate the reactions at the fixed support, i.e., at $x = 0$:

$$\sum_i F_{z_i} = 0 \quad \Leftrightarrow \quad F_z^{\mathrm{R}}(0) - F_0 = 0 \quad \Rightarrow \quad F_z^{\mathrm{R}}(0) = F_0\,, \tag{2.42}$$

$$\sum_i M_{y_i} = 0 \quad \Leftrightarrow \quad M_y^{\mathrm{R}}(0) + F_0 L = 0 \quad \Rightarrow \quad M_y^{\mathrm{R}}(0) = -F_0 L\,. \tag{2.43}$$

The boundary conditions can be stated at the left-hand end as

$$u_z(0) = 0 \;\text{ and }\; \varphi_y(0) = 0\,, \tag{2.44}$$

while the consideration of the force and moment equilibrium at the right-hand boundary (see Fig. 2.20) requires that

$$Q_z(L) = -F_0 \quad \text{and} \quad M_y(L) = 0\,. \tag{2.45}$$

Consideration of the boundary condition $(2.44)_1$ in the general expression for the displacement distribution (2.36) gives the fourth constant of integration as: $c_4 = 0$. In a similar way, the third constant of integration can be obtained by considering the boundary condition $(2.44)_2$ in the general expression for the rotation distribution (2.39): $c_3 = 0$. Introducing the boundary conditions at the right-hand end, i.e. Eq. (2.45) in the expressions for the bending moment and shear force according to Eqs. (2.37) and (2.38), the remaining constants are obtained as: $c_1 = F_0$ and $c_2 = -F_0 L$. Thus, the distributions of deflection, rotational angle, bending moment, and shear force can be stated as:

$$u_z(x) = \frac{F_0 L^3}{EI}\left\{\frac{1}{6}\left(\frac{x}{L}\right)^3 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right\}, \tag{2.46}$$

$$\varphi_y(x) = \frac{F_0 L^2}{EI}\left\{-\frac{1}{2}\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)\right\}, \tag{2.47}$$

$$M_y(x) = F_0 L\left\{-\left(\frac{x}{L}\right) + 1\right\}, \tag{2.48}$$

$$Q_z(x) = -F_0\,. \tag{2.49}$$

The other three cases can be solved in a similar way and the final results for the distributions are summarized in the following:

Case (b): Single moment M_0 at $x = L$

$$u_z(x) = \frac{M_0 L^2}{EI}\left\{\frac{1}{2}\left(\frac{x}{L}\right)^2\right\}, \tag{2.50}$$

$$\varphi_y(x) = -\frac{M_0 L}{EI}\left(\frac{x}{L}\right), \tag{2.51}$$

$$M_y(x) = -M_0\,, \tag{2.52}$$

$$Q_z(x) = 0\,. \tag{2.53}$$

Case (c): Displacement u_0 at $x = L$

$$u_z(x) = \left\{\frac{1}{2}\left(\frac{x}{L}\right)^3 - \frac{3}{2}\left(\frac{x}{L}\right)^2\right\} u_0\,, \tag{2.54}$$

$$\varphi_y(x) = \left\{-\frac{3}{2}\left(\frac{x}{L}\right)^2 + 3\left(\frac{x}{L}\right)\right\}\frac{u_0}{L}\,, \tag{2.55}$$

$$M_y(x) = \frac{3EIu_0}{L^2}\left\{-\left(\frac{x}{L}\right) + 1\right\}, \tag{2.56}$$

$$Q_z(x) = -\frac{3EIu_0}{L^3}\,. \tag{2.57}$$

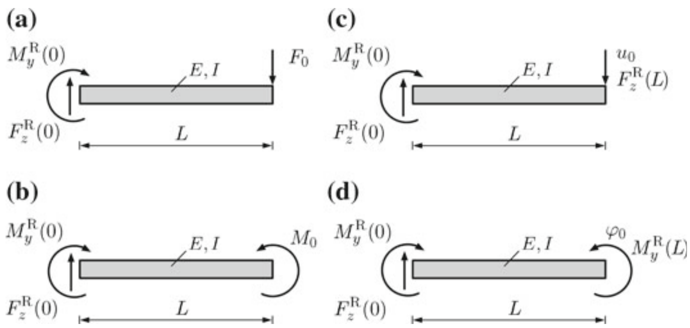


Fig. 2.19 Free-body diagrams of the cantilever beams with different end loads and deformations: **a** single force; **b** single moment; **c** displacement; **d** rotation

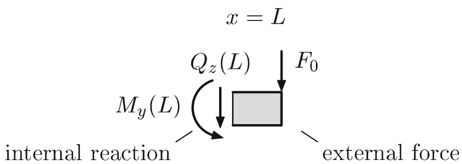


Fig. 2.20 Equilibrium between internal reactions and external load at $x = L$

Case (d): Rotation φ_0 at $x = L$

$$u_z(x) = \frac{\varphi_0 L}{2}\left(\frac{x}{L}\right)^2, \tag{2.58}$$

$$\varphi_y(x) = -\varphi_0\left(\frac{x}{L}\right), \tag{2.59}$$

$$M_y(x) = -\frac{\varphi_0 EI}{L}, \tag{2.60}$$

$$Q_z(x) = 0. \tag{2.61}$$

2.5 Cantilever Beam with Distributed Load

Given is a beam with different support conditions which is loaded by a constant distributed load q_0, see Fig. 2.21. It can be assumed for this exercise that the bending stiffness EI_y is constant. Calculate the analytical solution for the deflection $u_z(x)$, rotation $\varphi_y(x)$, the reactions at the supports as well as the distributions of the bending moment and shear force.

2.5 Solution

Case (a): Let us start the solution procedure by sketching the free-body diagram as shown in Fig. 2.22a.

The consideration of the global force and moment equilibrium would allow to calculate the reactions at the fixed support, i.e., at $x = 0$:

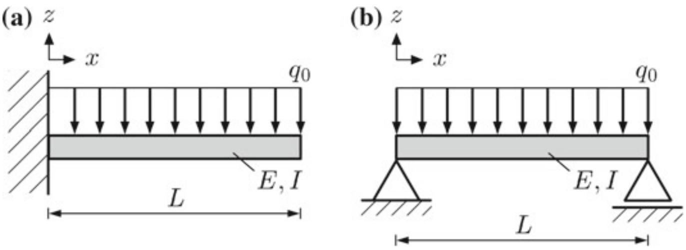


Fig. 2.21 Beam loaded under constant distributed load and different boundary conditions: **a** cantilever and **b** simply supported

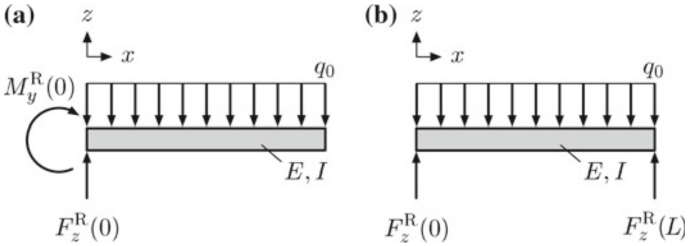


Fig. 2.22 Free-body diagrams of the beams loaded under constant distributed load and different boundary conditions: **a** cantilever and **b** simply supported

$$\sum_i F_{z_i} = 0 \quad \Leftrightarrow \quad F_z^{\mathrm{R}}(0) - q_0 L = 0 \quad \Rightarrow \quad F_z^{\mathrm{R}}(0) = q_0 L\,, \tag{2.62}$$

$$\sum_i M_{y_i} = 0 \quad \Leftrightarrow \quad M_y^{\mathrm{R}}(0) + \frac{q_0 L^2}{2} = 0 \quad \Rightarrow \quad M_y^{\mathrm{R}}(0) = -\frac{q_0 L^2}{2}\,. \tag{2.63}$$

The boundary conditions can be stated at the left-hand end as

$$u_z(0) = 0 \text{ and } \varphi_y(0) = 0\,, \tag{2.64}$$

while the consideration of the force and moment equilibrium at the right-hand boundary (see Fig. 2.20) requires that

$$Q_z(L) = 0 \text{ and } M_y(L) = 0\,. \tag{2.65}$$

Consideration of these conditions in the corresponding distributions results in the following constants of integration: $c_1 = q_0 L$, $c_2 = -\frac{1}{2}\,q_0 L^2$, $c_3 = c_4 = 0$. Thus, the distributions of deflection, rotational angle, bending moment, and shear force can be stated as:

$$u_z(x) = -\frac{q_0 L^4}{24EI}\left(\left[\frac{x}{L}\right]^4 - 4\left[\frac{x}{L}\right]^3 + 6\left[\frac{x}{L}\right]^2\right), \tag{2.66}$$

$$\varphi_y(x) = -\frac{q_0 L^3}{6EI}\left(-\left[\frac{x}{L}\right]^3 + 3\left[\frac{x}{L}\right]^2 - 3\left[\frac{x}{L}\right]\right), \tag{2.67}$$

$$M_y(x) = \frac{q_0 L^2}{2}\left(\left[\frac{x}{L}\right]^2 - 2\left[\frac{x}{L}\right] + 1\right), \tag{2.68}$$

$$Q_z(x) = q_0 L\left(\left[\frac{x}{L}\right] - 1\right). \tag{2.69}$$

Case (b): The set of boundary conditions is in this case given as

$$u_z(0) = 0, \qquad M_y(0) = 0, \tag{2.70}$$

$$u_z(L) = 0, \qquad M_y(L) = 0, \tag{2.71}$$

which results in the following constants of integration: $c_1 = \frac{q_0 L}{2}, c_2 = 0, c_3 = -\frac{q_0 L^3}{24}$, and $c_4 = 0$. Thus, the distributions of deflection, rotational angle, bending moment, and shear force can be stated as:

$$u_z(x) = -\frac{q_0 L^4}{24EI}\left(\left[\frac{x}{L}\right]^4 - 2\left[\frac{x}{L}\right]^3 + \left[\frac{x}{L}\right]\right), \tag{2.72}$$

$$\varphi_y(x) = -\frac{q_0 L^3}{24EI}\left(-4\left[\frac{x}{L}\right]^3 + 6\left[\frac{x}{L}\right]^2 - 1\right), \tag{2.73}$$

$$M_y(x) = \frac{q_0 L^2}{2}\left(\left[\frac{x}{L}\right]^2 - \left[\frac{x}{L}\right]\right), \tag{2.74}$$

$$Q_z(x) = \frac{q_0 L}{2}\left(2\left[\frac{x}{L}\right] - 1\right). \tag{2.75}$$

2.6 Cantilever Beam with Different Sections

Given is a cantilever beam of length L and constant bending stiffness EI as shown in Fig. 2.23. At the left-hand side there is a fixed support and a constant distributed load p_0 is acting in the range $a \le x \le b$. Calculate the analytical solution for the deflection $u_z(x)$, rotation $\varphi_y(x)$, the reactions at the support as well as the distributions of the bending moment and shear force.

2.6 Solution

Let us start the solution procedure by sketching the free-body diagram of the entire structure as shown in Fig. 2.24.

The two discontinuities in regards to the load at $X = a$ and $X = b$ requires to split the structure in three parts as indicated in Fig. 2.25. The left-hand part is now described by the local coordinate x_{I} with $0 \le x_{\mathrm{I}} \le a$, the middle part is described by the local coordinate x_{II} with $0 \le x_{\mathrm{II}} \le b - a$ while the right-hand part is described by the local coordinate x_{III} with $0 \le x_{\mathrm{III}} \le L - b$.

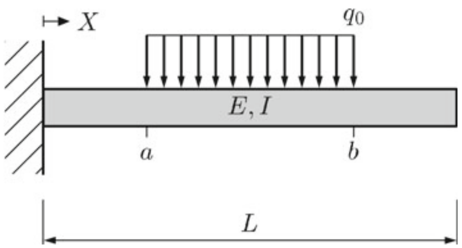


Fig. 2.23 Beam with different sections

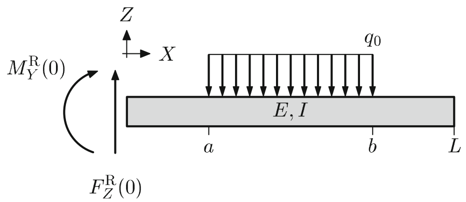


Fig. 2.24 Free-body diagram of the beam with different sections

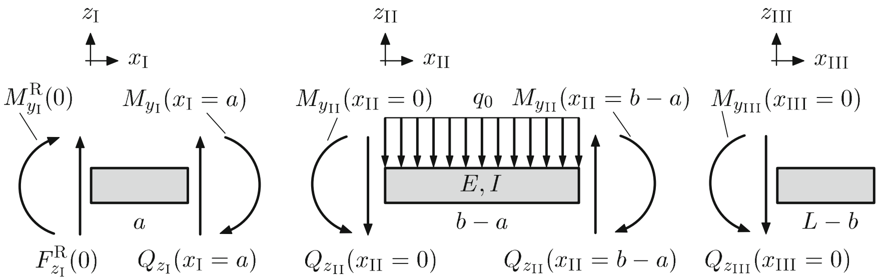


Fig. 2.25 Free-body diagrams of the different sections

Consideration of three parts means that Eqs. (2.36)–(2.39) must be applied to all sections and in total 12 integration constants, i.e. four for each section (here $c_1 \ldots c_4$ for the left-hand, $c_5 \ldots c_8$ for the middle section while c_9 and c_{12} for the right-hand section), must be determined. The following four boundary and eight transmission conditions can be stated:

$$u_z(x_I = 0) = 0\,, \qquad M_y(x_{III} = L - b) = 0\,, \tag{2.76}$$

$$\varphi_y(x_I = 0) = 0\,, \qquad Q_z(x_{III} = L - b) = 0\,, \tag{2.77}$$

and

$$u_z(x_{\mathrm{I}} = a) = u_z(x_{\mathrm{II}} = 0)\,, \qquad u_z(x_{\mathrm{II}} = b - a) = u_z(x_{\mathrm{III}} = 0)\,, \tag{2.78}$$

$$\varphi_y(x_{\mathrm{I}} = a) = \varphi_y(x_{\mathrm{II}} = 0)\,, \qquad \varphi_y(x_{\mathrm{II}} = b - a) = \varphi_y(x_{\mathrm{III}} = 0)\,, \tag{2.79}$$

$$Q_z(x_{\mathrm{I}} = a) = Q_z(x_{\mathrm{II}} = 0)\,, \qquad Q_z(x_{\mathrm{II}} = b - a) = Q_z(x_{\mathrm{III}} = 0)\,, \tag{2.80}$$

$$M_y(x_{\mathrm{I}} = a) = M_y(x_{\mathrm{II}} = 0)\,, \qquad M_y(x_{\mathrm{II}} = b - a) = M_y(x_{\mathrm{III}} = 0)\,. \tag{2.81}$$

The general solutions for the displacements, rotations, shear forces and bending moments, i.e., Eqs. (2.36)–(2.39), can be stated for the three sections as:

$$u_z(x_{\mathrm{I}}) = \frac{1}{EI_y}\left(\frac{c_1 x_{\mathrm{I}}^3}{6} + \frac{c_2 x_{\mathrm{I}}^2}{2} + c_3 x_{\mathrm{I}} + c_4\right)\,, \tag{2.82}$$

$$Q_z(x_{\mathrm{I}}) = -c_1\,, \tag{2.83}$$

$$M_y(x_{\mathrm{I}}) = -c_1 x_{\mathrm{I}} - c_2\,, \tag{2.84}$$

$$\varphi_y(x_{\mathrm{I}}) = -\frac{\mathrm{d}u_z(x_{\mathrm{I}})}{\mathrm{d}x} = -\frac{1}{EI_y}\left(\frac{c_1 x_{\mathrm{I}}^2}{2} + c_2 x_{\mathrm{I}} + c_3\right)\,, \tag{2.85}$$

and for the second section

$$u_z(x_{\mathrm{II}}) = \frac{1}{EI_y}\left(\frac{-q_0 x_{\mathrm{II}}^4}{24} + \frac{c_5 x_{\mathrm{II}}^3}{6} + \frac{c_6 x_{\mathrm{II}}^2}{2} + c_7 x_{\mathrm{II}} + c_8\right)\,, \tag{2.86}$$

$$Q_z(x_{\mathrm{II}}) = +q_0 x_{\mathrm{II}} - c_5\,, \tag{2.87}$$

$$M_y(x_{\mathrm{II}}) = +\frac{q_0 x_{\mathrm{II}}^2}{2} - c_5 x_{\mathrm{II}} - c_6\,, \tag{2.88}$$

$$\varphi_y(x_{\mathrm{II}}) = -\frac{\mathrm{d}u_z(x_{\mathrm{II}})}{\mathrm{d}x} = -\frac{1}{EI_y}\left(\frac{-q_0 x_{\mathrm{II}}^3}{6} + \frac{c_5 x_{\mathrm{II}}^2}{2} + c_6 x_{\mathrm{II}} + c_7\right)\,, \tag{2.89}$$

and for the third section

$$u_z(x_{\mathrm{III}}) = \frac{1}{EI_y}\left(\frac{c_9 x_{\mathrm{III}}^3}{6} + \frac{c_{10} x_{\mathrm{III}}^2}{2} + c_{11} x_{\mathrm{III}} + c_{12}\right)\,, \tag{2.90}$$

$$Q_z(x_{\mathrm{III}}) = -c_9\,, \tag{2.91}$$

$$M_y(x_{\mathrm{III}}) = -c_9 x_{\mathrm{III}} - c_{10}\,, \tag{2.92}$$

$$\varphi_y(x_{\mathrm{III}}) = -\frac{\mathrm{d}u_z(x_{\mathrm{III}})}{\mathrm{d}x} = -\frac{1}{EI_y}\left(\frac{c_9 x_{\mathrm{III}}^2}{2} + c_{10} x_{\mathrm{III}} + c_{11}\right)\,. \tag{2.93}$$

Consideration of the 12 boundary and transmissions conditions in this set of equations gives 12 conditions for the unknown constants of integration $c_1 \ldots c_{12}$ which can be expressed in matrix form as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{a^3}{6} & \frac{a^2}{2} & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \frac{a^2}{2} & a & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(b-a)^3}{6} & \frac{(b-a)^2}{2} & (b-a) & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \frac{(b-a)^2}{2} & (b-a) & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(b-a) & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(L-b) & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \\ c_{11} \\ c_{12} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{q_0(b-a)^4}{24} \\ \frac{q_0(b-a)^3}{6} \\ -q_0(b-a) \\ \frac{-q_0(b-a)^2}{2} \\ 0 \end{bmatrix} . \tag{2.94}$$

Multiplication of the inversed coefficient matrix with the right-hand side allows us to determine the constants as:

$$c_1 = q_0(b-a)\,, \qquad c_2 = -\frac{q_0}{2}(b^2-a^2)\,, \tag{2.95}$$

$$c_3 = 0\,, \qquad c_4 = 0\,, \tag{2.96}$$

$$c_5 = q_0(b-a)\,, \qquad c_6 = -\frac{q_0}{2}(b-a)^2\,, \tag{2.97}$$

$$c_7 = -\frac{q_0 ab}{2}(b-a)\,, \qquad c_8 = \frac{q_0 a^2}{12}(a^2+2ab-3b^2)\,, \tag{2.98}$$

$$c_9 = 0\,, \qquad c_{10} = 0\,, \tag{2.99}$$

$$c_{11} = -\frac{q_0}{6}(b^3-a^3)\,, \qquad c_{12} = -\frac{q_0}{24}(a^4+3b^4-4a^3b)\,. \tag{2.100}$$

Based on these constants of integration, the general expressions (2.82)–(2.93) for the distributions can be concretized as:

$$u_z(x_\mathrm{I}) = \frac{q_0}{EI}\left(\frac{b-a}{6}\,x_\mathrm{I}^3 - \frac{b^2-a^2}{4}\,x_\mathrm{I}^2\right), \tag{2.101}$$

$$u_z(x_\mathrm{II}) = \frac{q_0}{EI}\left(\frac{-x_\mathrm{II}^4}{24} + \frac{b-a}{6}\,x_\mathrm{II}^3 - \frac{(b-a)^2}{4}\,x_\mathrm{II}^2 - \frac{ab(b-a)}{2}\,x_\mathrm{II} + \right.$$
$$\left. + \frac{a^2(a^2+2ab-3b^2)}{12}\right), \tag{2.102}$$

$$u_z(x_\mathrm{III}) = \frac{q_0}{EI}\left(-\frac{b^3-a^3}{6}\,x_\mathrm{III} - \frac{a^4+3b^4-4a^3b}{24}\right), \tag{2.103}$$

and for the rotations

$$\varphi_y(x_\mathrm{I}) = -\frac{q_0}{2EI}\left((b-a)x_\mathrm{I}^2 - (b^2-a^2)x_\mathrm{I}\right), \tag{2.104}$$

$$\varphi_y(x_\mathrm{II}) = -\frac{q_0}{2EI}\left(-\frac{x_\mathrm{II}^3}{3} + (b-a)x_\mathrm{II}^2 - (b-a)^2x_\mathrm{II} - ab(b-a)\right), \tag{2.105}$$

$$\varphi_y(x_\mathrm{III}) = \frac{q_0}{6EI}\left(b^3-a^3\right), \tag{2.106}$$

and for the bending moments

$$M_y(x_\mathrm{I}) = q_0\left(-(b-a)x_\mathrm{I} + \frac{b^2-a^2}{2}\right), \tag{2.107}$$

$$M_y(x_\mathrm{II}) = \frac{q_0}{2}\left(x_\mathrm{II}^2 - 2(b-a)x_\mathrm{II} + (b-a)^2\right), \tag{2.108}$$

$$M_y(x_\mathrm{III}) = 0\,, \tag{2.109}$$

and for the shear forces

$$Q_z(x_\mathrm{I}) = -q_0(b-a)\,, \tag{2.110}$$

$$Q_z(x_\mathrm{II}) = q_0\left(x_\mathrm{II} - (b-a)\right), \tag{2.111}$$

$$Q_z(x_\mathrm{III}) = 0\,. \tag{2.112}$$

An alternative solution approach can be based on the MACAULAY brackets as outlined in Eq. (2.4). Based on this particular approach to express discontinuities, we can state the distribution of the distributed load in the global coordinate X as:

$$q_Z(X) = -q_0\left(\langle X-a\rangle^0 - \langle X-b\rangle^0\right). \tag{2.113}$$

This expression can be introduced in the fourth-order differential equation (see Table 2.4) as load function:

$$EI\,\frac{\mathrm{d}^4u_z(X)}{\mathrm{d}X^4} = q_Z(X) = -q_0\left(\langle X-a\rangle^0 - \langle X-b\rangle^0\right). \tag{2.114}$$

Four times integration of the last equation gives:

$$EI\frac{\mathrm{d}^3 u_z(X)}{\mathrm{d}X^3} = -Q_Z(X) = -q_0\left(\langle X-a\rangle^1 - \langle X-b\rangle^1\right) + c_1\,, \tag{2.115}$$

$$EI\frac{\mathrm{d}^2 u_z(X)}{\mathrm{d}X^2} = -M_Y(X) = -q_0\left(\frac{1}{2}\langle X-a\rangle^2 - \frac{1}{2}\langle X-b\rangle^2\right) + c_1X + c_2\,, \tag{2.116}$$

$$EI\frac{\mathrm{d}^1 u_z(X)}{\mathrm{d}X^1} = EI(-\varphi_Y(X)) = -q_0\left(\frac{1}{6}\langle X-a\rangle^3 - \frac{1}{6}\langle X-b\rangle^3\right) + \\ + \frac{c_1}{2}X^2 + c_2X + c_3\,, \tag{2.117}$$

$$EIu_z(X) = -q_0\left(\frac{1}{24}\langle X-a\rangle^4 - \frac{1}{24}\langle X-b\rangle^4\right) + \frac{c_1}{6}X^3 + \frac{c_2}{2}X^2 + \\ + c_3X + C_4\,. \tag{2.118}$$

The constants can be obtained based on the boundary conditions (2.76)– (2.77) as $c_1 = q_0(b-a)$, $c_2 = -\frac{q_0}{2}(b^2-a^2)$, $c_3 = 0$, and $c_4 = 0$. Thus, the distribution of the deflection is obtained in closed-form representation as:

$$u_Z(X) = \frac{-q_0}{EI}\left(\frac{\langle X-a\rangle^4}{24} - \frac{\langle X-b\rangle^4}{24} - \frac{(b-a)}{6}X^3 + \frac{(b^2-a^2)}{4}X^2\right)\,. \tag{2.119}$$

It should be noted that the end deflection of the beam can be obtained for $X = L$ as:

$$u_Z(L) = \frac{-q_0}{EI}\left(-\frac{La^3}{6} + \frac{a^4}{24} + \frac{Lb^3}{6} - \frac{b^4}{24}\right)\,. \tag{2.120}$$

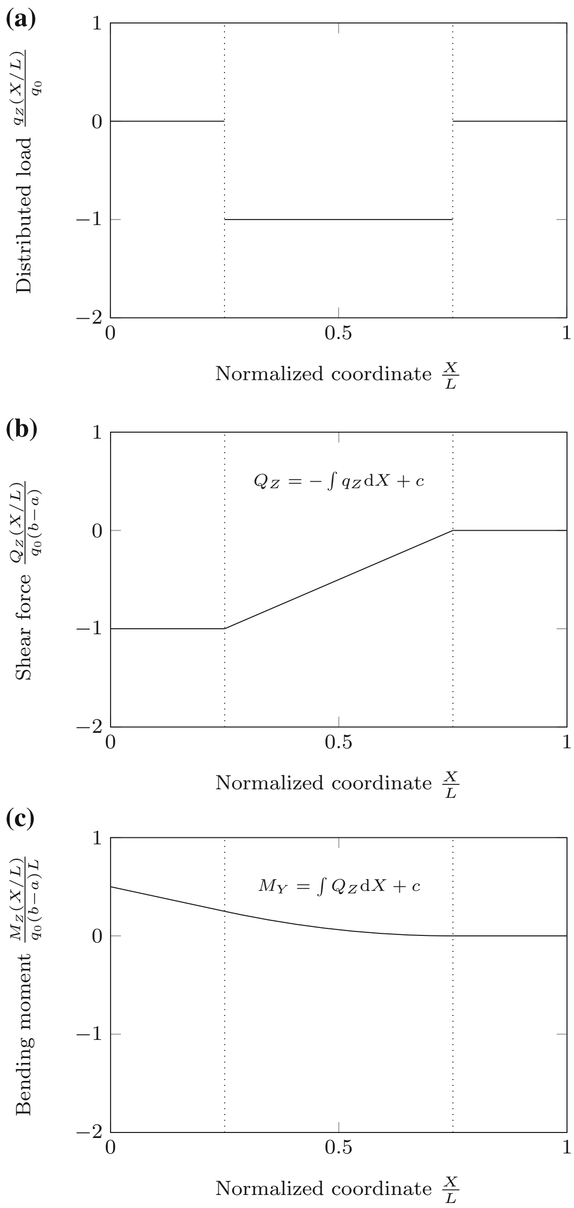
The special case that the distributed load extends over the entire beam, i.e., $a = 0$ and $b = L$, gives the classical result for the end deflection: $u_Z(L) = \frac{q_0L^4}{8EI}$.

The distributions of the load q_Z, shear force Q_Z, and bending moment M_Y are shown in Fig. 2.26 and allows us to understand the dependency of these quantities.

2.2.3 Timoshenko Beams

A thick or TIMOSHENKO beam is defined as a long prismatic body whose axial dimension is much larger than its transverse dimensions [33, 40]. This structural member is only loaded perpendicular to its longitudinal body axis by forces (single forces F_z or distributed loads q_z) or moments (single moments M_y or distributed moments m_y). Perpendicular means that the line of application of a force or the direction of a moment vector forms a right angle with the x-axis, see Fig. 2.15. As a result of this loading, the deformation occurs only perpendicular to its main axis. The

Fig. 2.26 Beam with different sections: **a** distributed load, **b** shear force, and **c** bending moment



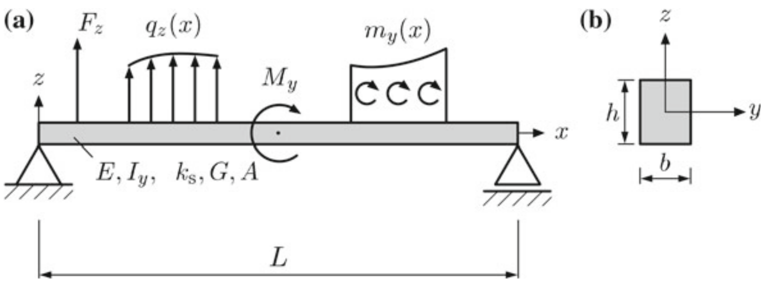


Fig. 2.27 General configuration for TIMOSHENKO beam problems: **a** example of boundary conditions and external loads; **b** cross-sectional area (bending occurs in the x-z plane)

formulation is a shear-flexible theory which means that the shear forces contribute to the bending deformation (Fig. 2.27).

Derivations are restricted many times to the following simplifications:

- only applying to straight beams,
- no elongation along the x-axis,
- no torsion around the x-axis,
- deformations in a single plane, i.e. symmetrical bending,
- infinitesimally small deformations and strains,
- simple cross sections, and
- the material is linear-elastic.

The three basic equations of continuum mechanics, i.e. the kinematics relationship, the constitutive law and the equilibrium equation, as well as their combination to the describing partial differential equations are summarized in Table 2.6. It should be noted here that the deflection u_z and the rotation ϕ_y are now independent variables and both represented in the coupled differential equations.

Under the assumption of constant material (E, G) and geometric (I_y, A, k_s) properties, the system of differential equations in Table 2.6 can be solved for constant distributed loads ($q_z = q_0 =$ const. and $m_y = 0$) to obtain the general analytical solution of the problem [39, 40]:

$$u_z(x) = \frac{1}{EI_y}\left(\frac{q_0 x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4\right), \tag{2.121}$$

$$\phi_y(x) = -\frac{1}{EI_y}\left(\frac{q_0 x^3}{6} + c_1\frac{x^2}{2} + c_2 x + c_3\right) - \frac{q_0 x}{k_s AG} - \frac{c_1}{k_s AG}, \tag{2.122}$$

$$M_y(x) = -\left(\frac{q_0 x^2}{2} + c_1 x + c_2\right) - \frac{q_0 EI_y}{k_s AG}, \tag{2.123}$$

$$Q_z(x) = -(q_0 x + c_1), \tag{2.124}$$

Table 2.6 Different formulations of the basic equations for a TIMOSHENKO beam (bending in the x-z plane). $\boldsymbol{e}$: generalized strains; $\boldsymbol{s}$: generalized stresses

Specific formulation	General formulation
Kinematics	
$\begin{bmatrix} \frac{\mathrm{d}u_z}{\mathrm{d}x} + \phi_y \\ \frac{\mathrm{d}\phi_y}{\mathrm{d}x} \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & 1 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}x} \end{bmatrix} \begin{bmatrix} u_z \\ \phi_y \end{bmatrix}$	$\boldsymbol{e} = \boldsymbol{\mathcal{L}}_1 \boldsymbol{u}$
Constitution	
$\begin{bmatrix} -Q_z \\ M_y \end{bmatrix} = \begin{bmatrix} -k_{\mathrm{s}}AG & 0 \\ 0 & EI_y \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}u_z}{\mathrm{d}x} + \phi_y \\ \frac{\mathrm{d}\phi_y}{\mathrm{d}x} \end{bmatrix}$	$\boldsymbol{s} = \boldsymbol{D}\boldsymbol{e}$
Equilibrium	
$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 1 & \frac{\mathrm{d}}{\mathrm{d}x} \end{bmatrix} \begin{bmatrix} -Q_z \\ M_y \end{bmatrix} + \begin{bmatrix} -q_z \\ +m_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\boldsymbol{\mathcal{L}}_1^{\mathrm{T}} \boldsymbol{s} + \boldsymbol{b} = \boldsymbol{0}$
PDE	
$-\frac{\mathrm{d}}{\mathrm{d}x}\left[k_{\mathrm{s}}GA\left(\frac{\mathrm{d}u_z}{\mathrm{d}x} + \phi_y\right)\right] - q_z = 0$	
$\frac{\mathrm{d}}{\mathrm{d}x}\left(EI_y \frac{\mathrm{d}\phi_y}{\mathrm{d}x}\right) - k_{\mathrm{s}}GA\left(\frac{\mathrm{d}u_z}{\mathrm{d}x} + \phi_y\right) + m_y = 0\,,$	$\boldsymbol{\mathcal{L}}_1^{\mathrm{T}} \boldsymbol{D} \boldsymbol{\mathcal{L}}_1 \boldsymbol{u} + \boldsymbol{b} = \boldsymbol{0}$

where the four constants of integration $c_i (i = 1, \ldots, 4)$ must be determined based on the boundary conditions, see Table 2.7.

The internal reactions in a beam become visible if one cuts — at an arbitrary location x — the member in two parts. As a result, two opposite oriented shear forces Q_z and bending moments M_y can be indicated. Summing up the internal reactions from both parts must result in zero. Their positive directions are connected with the positive coordinate directions at the positive face (outward surface normal vector parallel to the positive x-axis). This means that at a positive face the positive reactions have the same direction as the positive coordinate axes, see Fig. 2.28.

Once the internal bending moment M_y is known, the normal stress σ_x can be calculated:

$$\sigma_x(x, z) = \frac{M_y(x)}{I_y} z(x) = E \frac{\mathrm{d}\phi_y(x)}{\mathrm{d}x} z(x)\,, \tag{2.125}$$

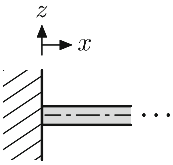
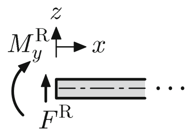
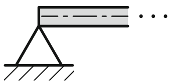
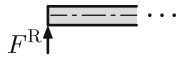
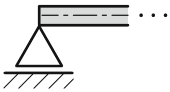
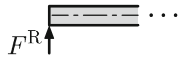
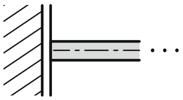
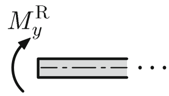
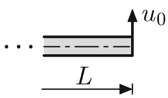
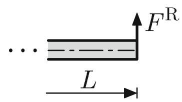
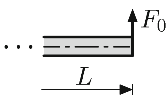
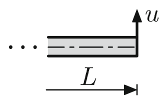
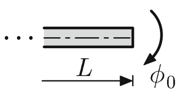
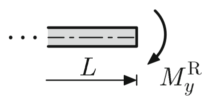
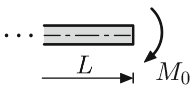
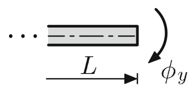
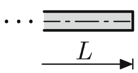
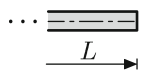
whereas the shear stress τ_{xz} is assumed constant over the cross section:

$$\tau_{xz} = \frac{Q_z(x)}{A_{\mathrm{s}}} = \frac{Q_z(x)}{k_{\mathrm{s}}A} = G\gamma_{xz}(x)\,. \tag{2.126}$$

In the above equation, the relation between the shear area A_{s} and the actual cross-sectional area A is referred to as the shear correction factor k_{s} [9, 13]:

$$k_{\mathrm{s}} = \frac{A_{\mathrm{s}}}{A}\,. \tag{2.127}$$

Table 2.7 Different boundary conditions and their corresponding reactions for a continuum TIMOSHENKO beam (bending occurs in the x-z plane)

Case	Boundary Condition	Reaction
	$u_z(0) = 0, \phi_y(0) = 0$	
	$u_z(0) = 0, M_y(0) = 0$	
	$u_z(0) = 0, M_y(0) = 0$	
	$\phi_y(0) = 0, Q_z(0) = 0$	
	$u_z(L) = u_0, M_y(L) = 0$	
	$Q_z(L) = F_0, M_y(L) = 0$	
	$\phi_y(L) = \phi_0, Q_z(L) = 0$	
	$M_y(L) = M_0, Q_z(L) = 0$	
	$M_y(L) = 0, Q_z(L) = 0$	

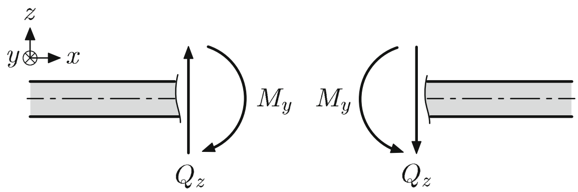


Fig. 2.28 Internal reactions for a continuum TIMOSHENKO beam (bending occurs in the x-z plane)

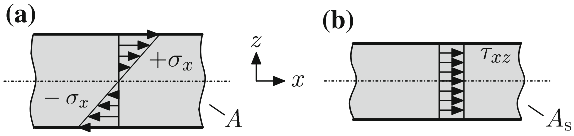


Fig. 2.29 Different stress distributions of a TIMOSHENKO beam with rectangular cross section and linear-elastic material behavior: **a** normal stress and **b** shear stress (bending occurs in the x-z plane)

The value of the shear correction factor is, for example, for a circular cross section equal to $\frac{9}{10}$ and for a square cross section equal to $\frac{5}{6}$, see [41]. The relationship between the YOUNG's and shear modulus (see Eqs. (2.125) and (2.126)) is given by [7]:

$$G = \frac{E}{2(1+\nu)}, \tag{2.128}$$

where ν is POISSON's ratio. The graphical representations of the different stress components are shown in Fig. 2.29. The normal stress is, as in the case of the EULER–BERNOULLI beam, linearly distributed whereas the shear stress is now assumed to be constant.

If more realistic shear stress distributions are considered, one reaches so-called theories of higher-order [18, 28, 29]. Finally, it should be noted here that the one-dimensional TIMOSHENKO beam theory has its two-dimensional analogon in the form of REISSNER- MINDLIN plates[3] [3, 11, 21, 31, 35].

2.7 Beam Under Pure Bending Load

The cantilever TIMOSHENKO beam shown in Fig. 2.30 is loaded by a moment M_0 at the free right-hand end. The bending stiffness EI and the shear stiffness k_sAG are constant and the total length of the beam is equal to L. Determine, based on the TIMOSHENKO beam theory, the bending line and compare the result with the EULER–BERNOULLI theory.

2.7 Solution

The set of equations for deflection, rotational angle, bending moment and shear force as given in Eqs. (2.121)–(2.124) reduces for $q_0 = 0$ to the following formulation:

[3] Also called thick plates.

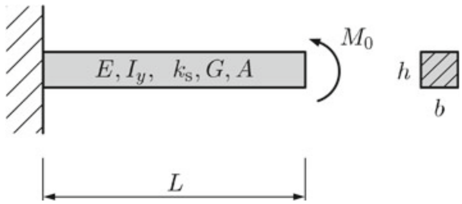


Fig. 2.30 Beam loaded under pure bending moment

$$u_z(x) = \frac{1}{EI_y}\left(c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4\right), \tag{2.129}$$

$$\phi_y(x) = -\frac{1}{EI_y}\left(+c_1\frac{x^2}{2} + c_2x + c_3\right) - \frac{c_1}{k_sAG}, \tag{2.130}$$

$$M_y(x) = -(c_1x + c_2), \tag{2.131}$$

$$Q_z(x) = -(c_1). \tag{2.132}$$

The boundary conditions for the case shown in Fig. 3.34 can be stated as

$$u_z(0) = 0, \qquad M_y(L) = -M_0, \tag{2.133}$$

$$\varphi_y(0) = 0, \qquad Q_z(L) = 0, \tag{2.134}$$

which allow to determine the constants of integration in Eqs. (2.129)–(2.132) as $c_1 = 0$, $c_2 = M_0$, $c_3 = 0$, and $c_4 = 0$. Thus, the bending line can be expressed as

$$u_z(x) = \frac{M_0x^2}{2EI}. \tag{2.135}$$

This result is identical with the solution according to the EULER–BERNOULLI beam theory.

2.8 Cantilever Beam Under the Influence of a Point or Distributed Load

The cantilever TIMOSHENKO beam shown in Fig. 2.31 is either loaded by a single force F_0 at its right-hand end or by a distributed load q_0. The bending stiffness EI and the shear stiffness k_sAG are constant, the total length of the beam is equal to L, and the circular cross section has a diameter of d. Determine the expressions of the bending lines ($u_z(x)$) and sketch the deflections of the right-hand end ($x = L$) as a function of the slenderness ratio $\frac{d}{L}$ for $\nu = 0.0, 0.3$, and 0.5.

2.8 Solution

Case (a): The set of equations for deflection, rotational angle, bending moment and shear force as given in Eqs. (2.121)–(2.124) reduces for $q_0 = 0$ to the following formulation:

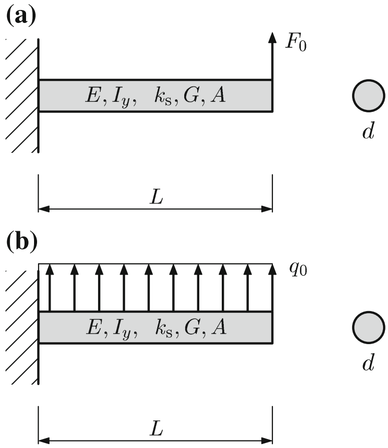


Fig. 2.31 Cantilever TIMOSHENKO beam: **a** single force case and **b** distributed load case

$$u_z(x) = \frac{1}{EI_y}\left(c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4\right), \tag{2.136}$$

$$\phi_y(x) = -\frac{1}{EI_y}\left(+c_1\frac{x^2}{2} + c_2x + c_3\right) - \frac{c_1}{k_sAG}, \tag{2.137}$$

$$M_y(x) = -(c_1x + c_2), \tag{2.138}$$

$$Q_z(x) = -(c_1). \tag{2.139}$$

The boundary conditions for the case shown in Fig. 2.31a can be stated as

$$u_z(0) = 0, \qquad M_y(L) = 0, \tag{2.140}$$

$$\varphi_y(0) = 0, \qquad Q_z(L) = F, \tag{2.141}$$

which allow to determine the constants of integration in Eqs. (2.136)–(2.139) as $c_1 = -F_0$, $c_2 = F_0L$, $c_3 = \frac{EIF_0}{k_sAG}$, and $c_4 = 0$. Thus, the bending line can be expressed as

$$u_z(x) = \frac{1}{EI}\left(-F_0\frac{x^3}{6} + F_0L\frac{x^2}{2} + \frac{EIF_0}{k_sAG}x\right), \tag{2.142}$$

or in normalized representation as:

$$\frac{u_z\left(\frac{x}{L}\right)}{\frac{F_0L^3}{EI}} = -\frac{1}{6}\left(\frac{x}{L}\right)^3 + \frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{EI}{k_sAGL^2}\left(\frac{x}{L}\right). \tag{2.143}$$

In the case of the considered circular cross section, one can use $k_s = \frac{9}{10}$, $A = \frac{\pi d^2}{4}$, and $I = \frac{\pi d^4}{64}$ to simplify Eq. (2.143):

$$\frac{u_z\left(\frac{x}{L}\right)}{\frac{F_0L^3}{EI}} = -\frac{1}{6}\left(\frac{x}{L}\right)^3 + \frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{5}{36}(1+\nu)\left(\frac{x}{L}\right)\left(\frac{d}{L}\right)^2, \tag{2.144}$$

or only at the right-hand end, i.e., $x = L$:

$$\frac{u_z\left(\frac{x}{L}=1\right)}{\frac{F_0L^3}{EI}} = \frac{1}{3} + \frac{5}{36}(1+\nu)\left(\frac{d}{L}\right)^2. \tag{2.145}$$

The graphical representation of the deflection at the right-hand end for different values of POISSON's ratio is given in Fig. 2.32.

Case (b): The set of equations for deflection, rotational angle, bending moment and shear force must be considered as given in Eqs. (2.121)–(2.124):

$$u_z(x) = \frac{1}{EI_y}\left(\frac{q_0x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4\right), \tag{2.146}$$

$$\phi_y(x) = -\frac{1}{EI_y}\left(\frac{q_0x^3}{6} + c_1\frac{x^2}{2} + c_2x + c_3\right) - \frac{q_0x}{k_sAG} - \frac{c_1}{k_sAG}, \tag{2.147}$$

$$M_y(x) = -\left(\frac{q_0x^2}{2} + c_1x + c_2\right) - \frac{q_0EI_y}{k_sAG}, \tag{2.148}$$

$$Q_z(x) = -(q_0x + c_1). \tag{2.149}$$

The boundary conditions for the case shown in Fig. 2.31b can be stated as

$$u_z(0) = 0, \qquad M_y(L) = 0, \tag{2.150}$$

$$\varphi_y(0) = 0, \qquad Q_z(L) = 0, \tag{2.151}$$

which allow to determine the constants of integration in Eqs. (2.146)–(2.149) as $c_1 = -q_0L$, $c_2 = \frac{q_0L^2}{2} - \frac{q_0EI}{k_sAG}$, $c_3 = \frac{q_0LEI}{k_sAG}$, and $c_4 = 0$. Thus, the bending line can be expressed as

$$u_z(x) = \frac{1}{EI}\left(\frac{q_0x^4}{24} - \frac{q_0Lx^3}{6} + \left[\frac{q_0L^2}{2} - \frac{q_0EI}{k_sAG}\right]\frac{x^2}{2} + \frac{q_0LEI}{k_sAG}x\right), \tag{2.152}$$

or in normalized representation as:

$$\frac{u_z\left(\frac{x}{L}\right)}{\frac{q_0L^4}{EI}} = \frac{1}{24}\left(\frac{x}{L}\right)^4 - \frac{1}{6}\left(\frac{x}{L}\right)^3 + \frac{1}{2}\left[\frac{1}{2} - \frac{EI}{k_sAGL^2}\right]\left(\frac{x}{L}\right)^2 + \frac{EI}{k_sAGL^2}\left(\frac{x}{L}\right). \tag{2.153}$$

In the case of the considered circular cross section, one can use $k_s = \frac{9}{10}$, $A = \frac{\pi d^2}{4}$, and $I = \frac{\pi d^4}{64}$ to simplify Eq. (2.153):

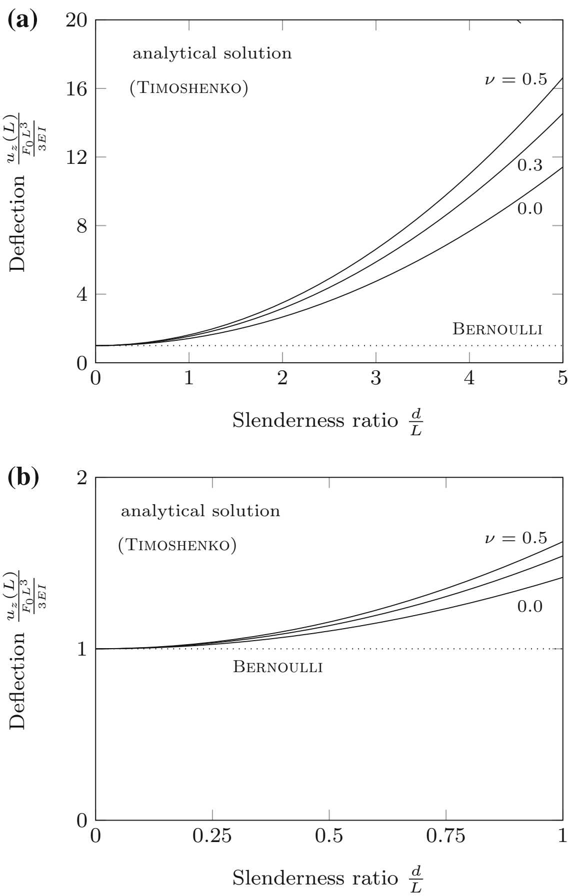


Fig. 2.32 Deflection of the right-hand-end of a TIMOSHENKO beam based on analytical solutions for single force loading **a** general view and **b** magnification for small slenderness ratios

$$\frac{u_z\left(\frac{x}{L}\right)}{\frac{q_0 L^4}{EI}} = \frac{1}{24}\left(\frac{x}{L}\right)^4 - \frac{1}{6}\left(\frac{x}{L}\right)^3 + \frac{1}{2}\left[\frac{1}{2} - \frac{5}{36}(1+\nu)\left(\frac{d}{L}\right)^2\right]\left(\frac{x}{L}\right)^2 + \frac{5}{36}(1+\nu)\left(\frac{d}{L}\right)^2\left(\frac{x}{L}\right), \tag{2.154}$$

or only at the right-hand end, i.e., $x = L$:

$$\frac{u_z\left(\frac{x}{L}\right)}{\frac{q_0 L^4}{EI}} = \frac{1}{8} + \frac{5}{72}(1+\nu)\left(\frac{d}{L}\right)^2. \tag{2.155}$$

The graphical representation of the deflection at the right-hand end for different values of POISSON's ratio is given in Fig. 2.33.

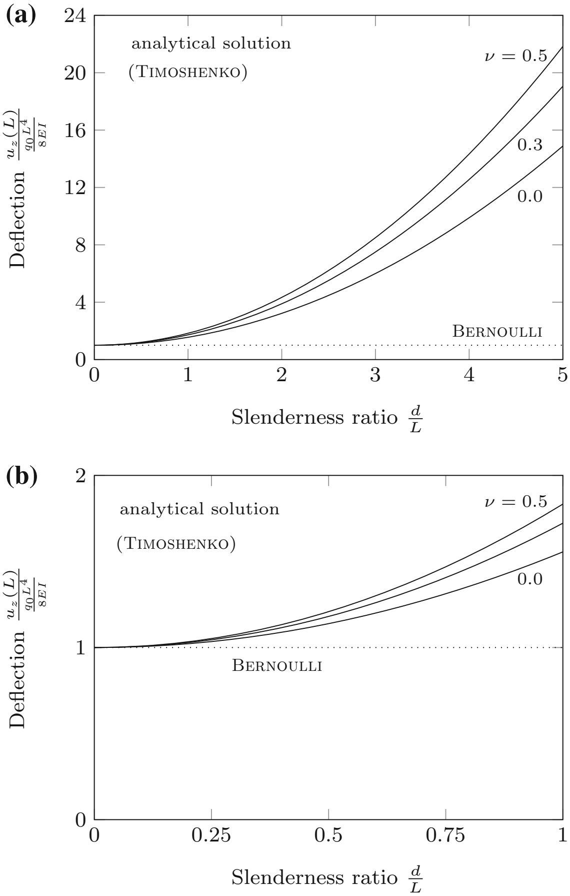


Fig. 2.33 Deflection of the right-hand-end of a TIMOSHENKO beam based on analytical solutions for distributed force loading **a** general view and **b** magnification for small slenderness ratios

2.9 Cantilever Beam with Two Different Sections

The cantilever TIMOSHENKO beam shown in Fig. 2.34 is composed of two sections, i.e., section one (I) with $0 \leq X \leq L_\mathrm{I}$ and section two (II) with $L_\mathrm{I} \leq X \leq L_\mathrm{II}$. The beam is loaded by a single force F_I at $X = L_\mathrm{I}$ and at its right-hand end by a single force F_II. The bending and the shear stiffnesses are EI_I and $k_\mathrm{s}A_\mathrm{I}G$ in section I while EI_II and $k_\mathrm{s}A_\mathrm{II}G$ holds for section II. This means that the beam is made of the same material and that the cross sections have similar shapes. Determine the expressions of the bending line.

2.9 Solution

The discontinuity in the cross section can be handled by splitting the beam at $X = L_\mathrm{I}$ into two parts. The left-hand part is now described by the local coordinate x_I with $0 \leq x_\mathrm{I} \leq L_\mathrm{I}$ while the right-hand part is described by the local coordinate x_II with $0 \leq x_\mathrm{II} \leq L_\mathrm{II}$. Consideration of two parts means that Eqs. (2.121) and (2.124) must

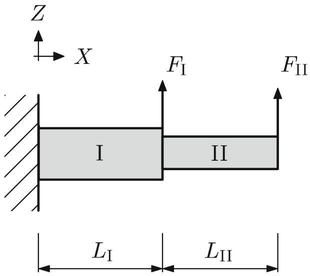


Fig. 2.34 Cantilever TIMOSHENKO beam with two different sections

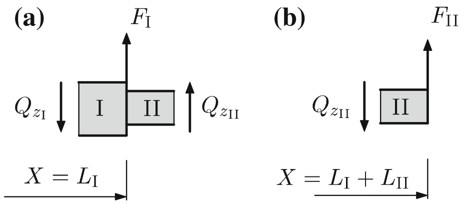


Fig. 2.35 Cantilever TIMOSHENKO beam with two different sections: **a** detail for transmission condition; **b** detail for boundary condition

be applied to both sections and in total eight integration constants, i.e. four for each section (here $c_1, \ldots, c_4$ for the left-hand section while $c_5, \ldots, c_8$ is assigned to the right-hand section), must be determined:

$$u_z(x_{\mathrm{I}}) = \frac{1}{EI_y}\left(c_1\frac{x_{\mathrm{I}}^3}{6} + c_2\frac{x_{\mathrm{I}}^2}{2} + c_3x_{\mathrm{I}} + c_4\right), \tag{2.156}$$

$$\phi_y(x_{\mathrm{I}}) = -\frac{1}{EI_y}\left(c_1\frac{x_{\mathrm{I}}^2}{2} + c_2x_{\mathrm{I}} + c_3\right) - \frac{c_1}{k_{\mathrm{s}}AG}, \tag{2.157}$$

$$M_y(x_{\mathrm{I}}) = -\left(c_1x_{\mathrm{I}} + c_2\right), \tag{2.158}$$

$$Q_z(x_{\mathrm{I}}) = -\left(c_1\right), \tag{2.159}$$

and

$$u_z(x_{\mathrm{II}}) = \frac{1}{EI_y}\left(c_5\frac{x_{\mathrm{II}}^3}{6} + c_6\frac{x_{\mathrm{II}}^2}{2} + c_7x_{\mathrm{II}} + c_8\right), \tag{2.160}$$

$$\phi_y(x_{\mathrm{II}}) = -\frac{1}{EI_y}\left(c_5\frac{x_{\mathrm{II}}^2}{2} + c_6x_{\mathrm{II}} + c_7\right) - \frac{c_5}{k_{\mathrm{s}}AG}, \tag{2.161}$$

$$M_y(x_{\mathrm{II}}) = -\left(c_5x_{\mathrm{II}} + c_6\right), \tag{2.162}$$

$$Q_z(x_{\mathrm{II}}) = -\left(c_5\right). \tag{2.163}$$

The following four boundary and four transmission conditions can be stated (see Fig. 2.35):

$$u_z(x_\mathrm{I}=0)=0\,,\qquad M_y(x_\mathrm{II}=L_\mathrm{II})=0\,,\tag{2.164}$$
$$\varphi_y(x_\mathrm{I}=0)=0\,,\qquad Q_z(x_\mathrm{II}=L_\mathrm{II})=F_\mathrm{II}\,,\tag{2.165}$$

and

$$u_z(x_\mathrm{I}=L_\mathrm{I})=u_z(x_\mathrm{II}=0)\,,\qquad \varphi_y(x_\mathrm{I}=L_\mathrm{I})=\varphi_y(x_\mathrm{II}=0)\,,\tag{2.166}$$
$$Q_z(x_\mathrm{I}=L_\mathrm{I})=F_\mathrm{I}+Q_z(x_\mathrm{II}=0)\,,\qquad M_y(x_\mathrm{I}=L_\mathrm{I})=M_y(x_\mathrm{II}=0)\,.\tag{2.167}$$

Consideration of the eight boundary and transmissions conditions in this set of equations gives eight conditions for the unknown constants of integration $c_1\ldots c_8$ which can be expressed in matrix form as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{k_\mathrm{s}A_\mathrm{I}G} & 0 & \frac{1}{EI_\mathrm{I}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -L_\mathrm{II} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{L_\mathrm{I}^3}{6} & \frac{L_\mathrm{I}^2}{2} & L_\mathrm{I} & 1 & 0 & 0 & 0 & -\frac{I_\mathrm{I}}{I_\mathrm{II}} \\ -\left(\frac{L_\mathrm{I}^2}{2}+\frac{EI_\mathrm{I}}{k_\mathrm{s}A_\mathrm{I}G}\right) & -L_\mathrm{I} & -1 & 0 & \frac{EI_\mathrm{I}}{k_\mathrm{s}A_\mathrm{II}G} & 0 & \frac{I_\mathrm{I}}{I_\mathrm{II}} & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -L_\mathrm{I} & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ -F_\mathrm{II} \\ 0 \\ 0 \\ F_\mathrm{I} \\ 0 \end{bmatrix}\,.\tag{2.168}$$

Multiplication of the inverse coefficient matrix with the right-hand side allows us to determine the constants as:

$$c_1=-(F_\mathrm{I}+F_\mathrm{II})\,,\qquad c_2=F_\mathrm{I}L_\mathrm{I}+F_\mathrm{II}(L_\mathrm{I}+L_\mathrm{LL})\,,\tag{2.169}$$
$$c_3=\frac{EI_\mathrm{I}(F_\mathrm{I}+F_\mathrm{II})}{k_\mathrm{s}A_\mathrm{I}G}\,,\qquad c_4=0\,,\tag{2.170}$$
$$c_5=-F_\mathrm{II}\,,\qquad c_6=F_\mathrm{II}L_\mathrm{II}\,,\tag{2.171}$$

$$c_7=\frac{1}{2}\frac{I_\mathrm{II}\left(k_\mathrm{s}A_\mathrm{II}G\left[F_\mathrm{I}L_\mathrm{I}^2+F_\mathrm{II}L_\mathrm{I}^2+2F_\mathrm{II}L_\mathrm{I}L_\mathrm{II}\right]+2EI_\mathrm{I}F_\mathrm{II}\right)}{I_\mathrm{I}k_\mathrm{s}A_\mathrm{II}G}\,,\tag{2.172}$$
$$c_8=\frac{1}{6}\frac{L_\mathrm{I}I_\mathrm{II}\left(k_\mathrm{s}A_\mathrm{I}G\left[2F_\mathrm{I}L_\mathrm{I}^2+2F_\mathrm{II}L_\mathrm{I}^2+3F_\mathrm{II}L_\mathrm{I}L_\mathrm{II}\right]+6EI_\mathrm{I}(F_\mathrm{I}+F_\mathrm{II})\right)}{I_\mathrm{I}k_\mathrm{s}A_\mathrm{I}G}\,.\tag{2.173}$$

Thus, based on these constants of integration, the bending lines given in Eqs. (2.156) and (2.160) are determined.

An interesting special case is obtained at the right-hand end for $A_\mathrm{I}=A_\mathrm{II}=A$, $I_\mathrm{I}=I_\mathrm{II}=I$, $L_\mathrm{I}=L_\mathrm{II}=\frac{L}{2}$, and $F_\mathrm{I}=0$ and $F_\mathrm{II}=F$:

$$u_z(X=L)=\frac{FL^3}{3EI}+\frac{FL}{k_\mathrm{s}AG}\,.\tag{2.174}$$

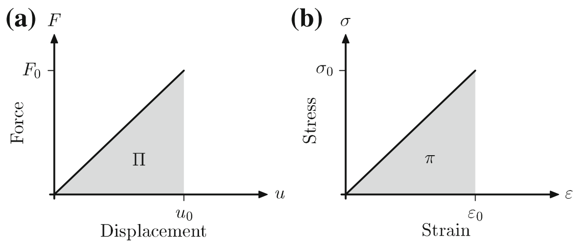


Fig. 2.36 Recorded data from a uniaxial tensile test: **a** force-displacement diagram; **b** stress-strain diagram

2.3 Energy-Based Approaches

As an alternative approach to the analytical solution procedures based on partial differential equations (see Sects. 2.2.1–2.2.3), the following section is related to energy approaches, in particular CASTIGLIANO's theorems, see [1, 12, 14, 16].

Let us first illustrate the energy which is stored in a material due to deformation, i.e. the so-called strain energy. For an ideal uniaxial tensile test with linear-elastic material behavior, Fig. 2.36 illustrates schematic force-displacement and stress-strain diagrams.

The area under the force-displacement diagram (see Fig. 2.36a) represents the total strain energy (Π) and can be calculated as[4]:

$$\Pi = \frac{1}{2} F_0 u_0 \,, \tag{2.175}$$

or in an integral approach:

$$\Pi = \int_0^{u_0} F(u)\mathrm{d}u = \int_0^{u_0} \frac{EA}{L}\, u\mathrm{d}u = \frac{EA}{2L}\, u_0^2 = \frac{F_0^2 L}{2EA} = \int_0^L \frac{N_x(x)^2}{2EA}\, \mathrm{d}x \,. \tag{2.176}$$

The transformations in the last equation used HOOKE's law and the equilibrium between the external load (F_0) and the internal reaction ($N_x(x)$). On the other hand, the area under the stress-strain diagram (see Fig. 2.36b) represents the volumetric strain energy ($\pi = \frac{\Pi}{V}$):

$$\pi = \int_0^{\varepsilon_0} \sigma(\epsilon)\mathrm{d}\epsilon = \int_0^{\varepsilon_0} E\varepsilon\mathrm{d}\varepsilon = \frac{E}{2}\,\varepsilon_0^2 = \frac{1}{2}\,\sigma_0\varepsilon_0 \,. \tag{2.177}$$

The last equation can extended to the total strain energy in the following way:

$$\mathrm{d}\Pi = \frac{1}{2}\,\sigma\varepsilon\mathrm{d}V = \frac{1}{2}\,\sigma\varepsilon\,\mathrm{d}(A\mathrm{d}x) \,. \tag{2.178}$$

[4]Confer the unit of energy: 1 J = 1 Nm = 1 Ws.

Similar derivations can be written for other simple modes of deformation and the following cases can be distinguished for linear-elastic material behavior:

- Tension or compression:

$$\Pi = \int_0^L \frac{N_x(x)^2}{2EA}\,\mathrm{d}x\,. \tag{2.179}$$

- Bending:

$$\Pi = \int_0^L \frac{M_y(x)^2}{2EI_y}\,\mathrm{d}x\,. \tag{2.180}$$

- Shear:

$$\Pi = \int_0^L \frac{Q_z(x)^2}{2GA_\mathrm{s}}\,\mathrm{d}x = \int_0^L \frac{Q_z(x)^2}{2k_\mathrm{s}GA}\,\mathrm{d}x\,. \tag{2.181}$$

- Torsion[5]:

$$\Pi = \int_0^L \frac{M_x(x)^2}{2GI_\mathrm{p}}\,\mathrm{d}x\,. \tag{2.182}$$

Thus, the total strain energy in a rod/beam-like structural member can be expressed as

$$\Pi = \int_0^L \frac{N_x(x)^2}{2EA}\,\mathrm{d}x + \int_0^L \frac{M_y(x)^2}{2EI_y}\,\mathrm{d}x + \int_0^L \frac{Q_z(x)^2}{2GA_\mathrm{s}}\,\mathrm{d}x + \int_0^L \frac{M_x(x)^2}{2GI_\mathrm{p}}\,\mathrm{d}x\,, \tag{2.183}$$

where the N_x, M_y, Q_z, M_x represent the distributions of the internal reactions. Depending on the mode of deformation, the corresponding terms in Eq. (2.183) must be considered. Based on the following theorems which make use of the strain energy, different quantities can be determined:

CASTIGLIANO's first theorem:
The partial derivative of the total strain energy with respect to the generalized displacement (displacement or rotation) gives the generalized force (force or moment). In equations, this can be expressed as:

$$\frac{\partial \Pi(x, u_i, \ldots)}{\partial u_i} = F_i\,, \tag{2.184}$$

$$\frac{\partial \Pi(x, \varphi_i, \ldots)}{\partial \varphi_i} = M_i\,. \tag{2.185}$$

[5] Only shown for completeness and not further covered here.

CASTIGLIANO's second theorem:
The partial derivative of the total strain energy with respect to the generalized force (force or moment) gives the generalized displacement (displacement or rotation) in the direction of that generalized force. In equations, this can be expressed as:

$$\frac{\partial \Pi(x, F_i, \ldots)}{\partial F_i} = u_i\,, \tag{2.186}$$

$$\frac{\partial \Pi(x, M_i, \ldots)}{\partial M_i} = \varphi_i\,. \tag{2.187}$$

The procedure also allows us to determine deformations where no external generalized forces are acting. This can be handled by introducing an auxiliary generalized force (F_{a} or M_{a}) and setting the auxiliary quantity to zero in the final equation for the generalized displacement:

$$\left(\frac{\partial \Pi(x, F_{\mathrm{a}_i}, \ldots)}{\partial F_{\mathrm{a}_i}}\right)_{F_{\mathrm{a}_i}=0} = u_i\,, \tag{2.188}$$

$$\left(\frac{\partial \Pi(x, M_{\mathrm{a}_i}, \ldots)}{\partial M_{\mathrm{a}_i}}\right)_{M_{\mathrm{a}_i}=0} = \varphi_i\,. \tag{2.189}$$

Based on this procedure, it is even possible to calculate entire distributions if the auxiliary quantity is introduced at a variable position. For practical calculations with constant material and geometrical properties (EA, EI_y, $k_{\mathrm{s}}GA$, GI_{p}), it might be useful to perform the partial derivative first and only after the integration. For example, the case of tension/compression can be written as:

$$u_{x,1} = \frac{\partial \Pi}{\partial F_1} = \frac{\partial}{\partial F_1}\left(\int_0^L \frac{N_x^2(x)}{2EA}\,\mathrm{d}x\right) = \int_0^L \frac{N_x(x)}{EA}\frac{\partial N_x(x, F_1, \ldots)}{\partial F_1}\,\mathrm{d}x\,. \tag{2.190}$$

2.10 Cantilever Rod with Point Loads (Alternative Solution Procedure of Problem 2.1)
Given is a rod of length L and constant axial tensile stiffness EA as shown in Fig. 2.37. At the left-hand side there is a fixed support and the right-hand side is either elongated by a displacement u_0 (case a) or loaded by a single force F_0 (case b). Determine based on CASTIGLIANO's theorems the solution for the elongation $u_x(x)$, the strain $\varepsilon_x(x)$, and the stress $\sigma_x(x)$ along the rod axis.

2.10 Solution
In case that only the reaction force $F^{\mathrm{R}}(L)$ at $x = L$ (case a) or the displacement $u_x(L)$ at $x = L$ (case b) would be wanted, we could simply determine the normal force distributions as, see Fig. 2.38:

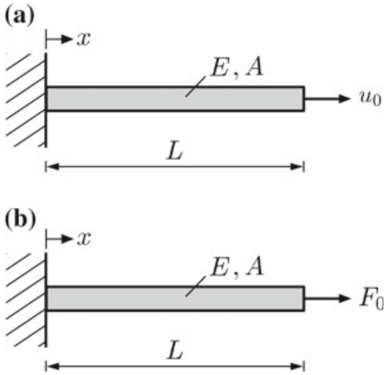


Fig. 2.37 Rod under different loading conditions: **a** displacement and **b** force

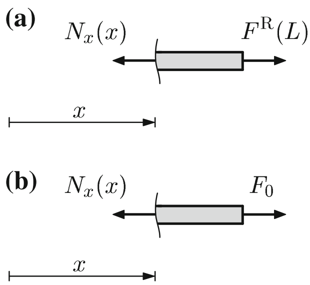


Fig. 2.38 Determination of the normal force distribution for the rod under different loading conditions: **a** displacement and **b** force

$$N_x(x) = F^{\mathrm{R}}(L) \qquad \text{(case a)}\,, \tag{2.191}$$

$$N_x(x) = F_0 \qquad \text{(case b)}\,. \tag{2.192}$$

For case (a), we can state based on CASTIGLIANO's second theorem that

$$\begin{aligned} u_0 &= \frac{\partial \Pi}{\partial F^{\mathrm{R}}(L)} = \frac{\partial}{\partial F^{\mathrm{R}}(L)} \int_0^L \frac{N_x^2(x)}{2EA}\,\mathrm{d}x = \int_0^L \frac{N_x(x)}{EA} \frac{\partial N_x(x, F^{\mathrm{R}})}{\partial F^{\mathrm{R}}(L)}\,\mathrm{d}x \\ &= \int_0^L \frac{F^{\mathrm{R}}(L)}{EA} \times 1\mathrm{d}x = \frac{F^{\mathrm{R}}(L)}{EA}\,[x]_0^L = \frac{F^{\mathrm{R}}(L)L}{EA}\,, \end{aligned} \tag{2.193}$$

or solved for the unknown reaction force at $x = L$:

$$F^{\mathrm{R}}(L) = \frac{EAu_0}{L}\,. \tag{2.194}$$

For case (b), we can state based on CASTIGLIANO's second theorem that

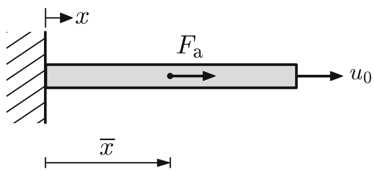


Fig. 2.39 Rod with displacement boundary condition and auxiliary force

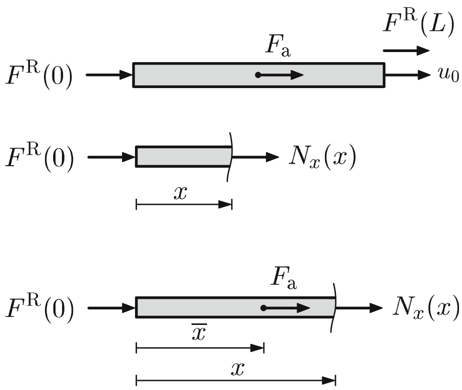


Fig. 2.40 Rod with displacement boundary condition and auxiliary force: different sections for normal force determination

$$u_x(L) = \frac{\partial \Pi}{\partial F_0} = \int_0^L \frac{N_x(x)}{EA}\frac{\partial N_x(x, F_0)}{\partial F_0}\,dx = \int_0^L \frac{F_0}{EA} \times 1dx = \frac{F_0 L}{EA}\,. \tag{2.195}$$

However, if we need to find the distributions of displacement ($u_x = u_x(x)$), stress ($\sigma_x = \sigma_x(x)$), and strain ($\varepsilon_x = \varepsilon_x(x)$), we need to follow a slightly different approach. For this purpose, an auxiliary force F_a is introduced at an arbitrary position $\overline{x}$. This is shown for case (a) in Fig. 2.39 together with the corresponding free-body diagram.

From the horizontal force equilibrium, we can conclude that

$$+F^R(0) + F_a + F^R(L) = 0 \quad \text{or} \quad F^R(0) = -F_a - F^R(L)\,. \tag{2.196}$$

Since we have now at $x = \overline{x}$ a discontinuity, we must determine the normal force distribution for two sections, see Fig. 2.40.

For the section $x < \overline{x}$, the internal normal force can be expressed as

$$N_x(x) = -F^R(0) = F_a + F^R(L)\,, \tag{2.197}$$

while the section $x \geq \overline{x}$ gives:

$$N_x(x) = -F^R(0) - F_a = F^R(L)\,. \tag{2.198}$$

Let us first apply CASTIGLIANO's second theorem to determine the unknown reaction force at the right-hand end:

$$u_0 = \int\limits_0^L \frac{N_x(x)}{EA} \frac{\partial N_x(x, F^{\mathrm{R}})}{\partial F^{\mathrm{R}}(L)} \, \mathrm{d}x = \int\limits_0^{\overline{x}} \frac{F_{\mathrm{a}} + F^{\mathrm{R}}(L)}{EA} \times 1 \mathrm{d}x + \int\limits_{\overline{x}}^L \frac{F^{\mathrm{R}}(L)}{EA} \times 1 \mathrm{d}x$$
$$= \frac{F_{\mathrm{a}} + F^{\mathrm{R}}(L)}{EA} \overline{x} + \frac{F^{\mathrm{R}}(L)}{EA}(L - \overline{x}) = \frac{F_{\mathrm{a}}\overline{x}}{EA} + \frac{F^{\mathrm{R}}(L)L}{EA} \,. \tag{2.199}$$

With $F_{\mathrm{a}} \to 0$ (and $\overline{x} \to x$), one obtains the reactions force as:

$$F^{\mathrm{R}}(L) = \frac{EAu_0}{L} \,. \tag{2.200}$$

The next application of CASTIGLIANO's second theorem allows us to determine the distribution of the displacement field:

$$u_x(x) = \int\limits_0^L \frac{N_x(x)}{EA} \frac{\partial N_x(x, F_{\mathrm{a}})}{\partial F_{\mathrm{a}}} \, \mathrm{d}x = \int\limits_0^{\overline{x}} \frac{F_{\mathrm{a}} + F^{\mathrm{R}}(L)}{EA} \times 1 \mathrm{d}x + \int\limits_{\overline{x}}^L \frac{F^{\mathrm{R}}(L)}{EA} \times 0 \mathrm{d}x$$
$$= \frac{F_{\mathrm{a}} + F^{\mathrm{R}}(L)}{EA} \overline{x} \,. \tag{2.201}$$

With $F_{\mathrm{a}} \to 0$ and $\overline{x} \to x$, one obtains the displacement field as:

$$u_x(x) = \frac{F^{\mathrm{R}}(L)x}{EA} = u_0 \frac{x}{L} \,. \tag{2.202}$$

The application of the kinematics and constitutive relationship (see Table 2.2) gives immediately the strain and stress distributions:

$$\varepsilon_x(x) = \frac{\partial u_x(x)}{\partial x} = \frac{u_0}{L} \,, \tag{2.203}$$

$$\sigma_x(x) = E\varepsilon_x(x) = \frac{u_0 E}{L} \,. \tag{2.204}$$

The configuration for case (b) and the corresponding free-body diagram is shown in Fig. 2.41.

From the horizontal force equilibrium, we can calculate the reaction force at the left-hand end:

$$+ F^{\mathrm{R}}(0) + F_{\mathrm{a}} + F_0 = 0 \quad \text{or} \quad F^{\mathrm{R}}(0) = -F_0 - F_{\mathrm{a}} \,. \tag{2.205}$$

Due to the discontinuity, the normal force distribution is required for two sections, see Fig. 2.42.

For the section $x < \overline{x}$, the internal normal force can be expressed as

$$N_x(x) = -F^{\mathrm{R}}(0) = F_0 + F_{\mathrm{a}} \,, \tag{2.206}$$

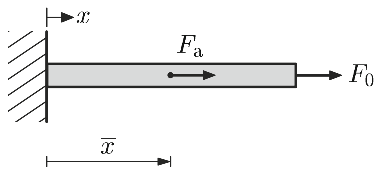


Fig. 2.41 Rod with force boundary condition and auxiliary force

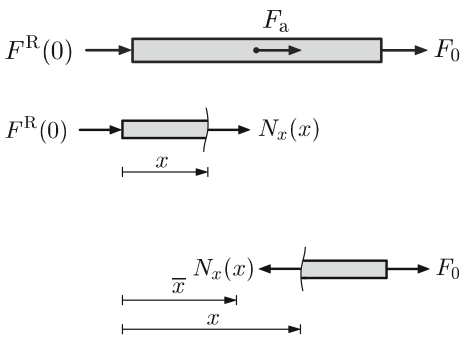


Fig. 2.42 Rod with force boundary condition and auxiliary force: different sections for normal force determination

while the section $x \geq \overline{x}$ gives:

$$N_x(x) = F_0 \,. \tag{2.207}$$

Application of CASTIGLIANO's second theorem allows us to determine the distribution of the displacement field:

$$\begin{aligned} u_x(x) &= \int_0^L \frac{N_x(x)}{EA} \frac{\partial N_x(x, F_a)}{\partial F_a} \mathrm{d}x = \int_0^{\overline{x}} \frac{F_0 + F_a}{EA} \times 1 \mathrm{d}x + \int_{\overline{x}}^L \frac{F_0}{EA} \times 0 \mathrm{d}x \\ &= \frac{F_0 + F_a}{EA} \overline{x} \,. \end{aligned} \tag{2.208}$$

With $F_a \to 0$ and $\overline{x} \to x$, one obtains the displacement field as:

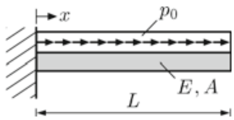
$$u_x(x) = \frac{F_0 x}{EA} \,. \tag{2.209}$$

The application of the kinematics and constitutive relationships (see Table 2.2) gives immediately the strain and stress distributions:

$$\varepsilon_x(x) = \frac{\partial u_x(x)}{\partial x} = \frac{F_0}{EA} \,, \tag{2.210}$$

$$\sigma_x(x) = E\varepsilon_x(x) = \frac{F_0}{A} \,. \tag{2.211}$$

Fig. 2.43 Rod with distributed load



2.11 Cantilever Rod with Distributed Load (Alternative Solution Procedure of Problem 2.2)

Given is a rod of length L and constant axial tensile stiffness EA as shown in Fig. 2.43. At the left-hand side there is a fixed support and a constant distributed load p_0 is acting along the entire rod axis. Determine based on CASTIGLIANO's theorems the analytical solution for the elongation $u_x(x)$, the strain $\varepsilon_x(x)$, and the stress $\sigma_x(x)$ along the rod axis.

2.11 Solution

The determination of the distributions of displacement ($u_x = u_x(x)$), stress ($\sigma_x = \sigma_x(x)$), and strain ($\varepsilon_x = \varepsilon_x(x)$) requires that an auxiliary force F_a is introduced at an arbitrary position $\overline{x}$. This is shown in Fig. 2.44 together with the corresponding free-body diagram.

From the horizontal force equilibrium, we can conclude that

$$+F^\mathrm{R}(0) + F_\mathrm{a} + p_0 L = 0 \quad \text{or} \quad F^\mathrm{R}(0) = -F_\mathrm{a} - p_0 L\,. \tag{2.212}$$

Since we have now at $x = \overline{x}$ a discontinuity, we must determine the normal force distribution for two sections, see Fig. 2.45.

For the section $x < \overline{x}$, the internal normal force can be expressed as

$$N_x(x) = F_\mathrm{a} + p_0(L - x)\,, \tag{2.213}$$

while the section $x \geq \overline{x}$ gives:

$$N_x(x) = p_0(L - x)\,. \tag{2.214}$$

Application of CASTIGLIANO's second theorem allows us to determine the distribution of the displacement field:

$$\begin{aligned} u_x(x) &= \int_0^L \frac{N_x(x)}{EA}\frac{\partial N_x(x, F_\mathrm{a})}{\partial F_\mathrm{a}}\,\mathrm{d}x \\ &= \int_0^{\overline{x}} \frac{F_\mathrm{a} + p_0(L-x)}{EA} \times 1\mathrm{d}x + \int_{\overline{x}}^L \frac{p_0(L-x)}{EA} \times 0\mathrm{d}x \\ &= \frac{1}{EA}\left(F_\mathrm{a}\overline{x} + p_0 L\overline{x} - \frac{p_0\overline{x}^2}{2}\right)\,. \end{aligned} \tag{2.215}$$

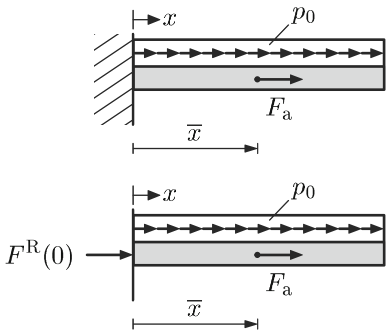


Fig. 2.44 Rod with distributed load: introduction of auxiliary force

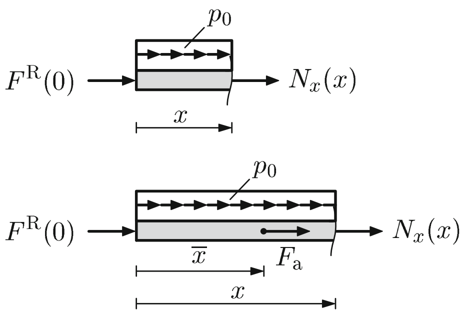


Fig. 2.45 Rod with distributed load: different sections for normal force determination

With $F_\mathrm{a} \to 0$ and $\overline{x} \to x$, one obtains the displacement field as:

$$u_x(x) = \frac{p_0}{EA}\left(Lx - \frac{x^2}{2}\right) = \frac{p_0 L^2}{EA}\left(-\frac{1}{2}\left[\frac{x}{L}\right]^2 + \left[\frac{x}{L}\right]\right). \tag{2.216}$$

The application of the kinematics and constitutive relationships (see Table 2.2) gives immediately the strain and stress distributions:

$$\varepsilon_x(x) = \frac{\partial u_x(x)}{\partial x} = \frac{p_0 L}{EA}\left(-\left[\frac{x}{L}\right] + 1\right), \tag{2.217}$$

$$\sigma_x(x) = E\varepsilon_x(x) = \frac{p_0 L}{A}\left(-\left[\frac{x}{L}\right] + 1\right). \tag{2.218}$$

2.12 Cantilever Beam with Different End Loads and Deformations (Alternative Solution Procedure of Problem 2.4)

Calculate based on CASTIGLIANO's theorems the analytical solutions for the deflection $u_z(x)$ and rotation $\varphi_y(x)$ of the cantilever beam shown in Fig. 2.46. Calculate in addition for all four cases the reactions at the fixed support and the distributions of the bending moment and shear force. It can be assumed for this exercise that the bending stiffness EI_y is constant.

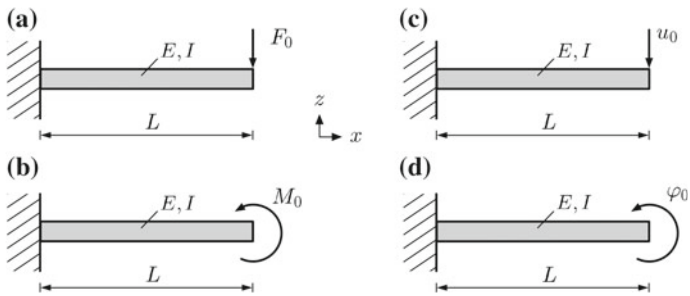


Fig. 2.46 Cantilever beam with different end loads and deformations: **a** single force; **b** single moment; **c** displacement; **d** rotation

2.12 Solution

Case (a): The determination of the distributions of deflection ($u_z = u_z(x)$) and rotation ($\varphi_y = \varphi_y(x)$) requires that an auxiliary force F_a is introduced at an arbitrary position $\overline{x}$. This is shown in Fig. 2.47 together with the corresponding free-body diagram.

From the vertical force and moment equilibrium, we can conclude that

$$+F_z^{\mathrm{R}}(0) - F_{\mathrm{a}} - F_0 = 0 \quad \text{or} \quad F_z^{\mathrm{R}}(0) = F_{\mathrm{a}} + F_0\,, \tag{2.219}$$

$$+M_y^{\mathrm{R}}(0) + F_{\mathrm{a}}\overline{x} + F_0 L = 0 \quad \text{or} \quad M_y^{\mathrm{R}}(0) = -F_{\mathrm{a}}\overline{x} - F_0 L\,. \tag{2.220}$$

Since we have now at $x = \overline{x}$ a discontinuity, we must determine the bending moment (the shear force distribution is only required if the shear contribution on the deformation should be considered) distribution for two sections, see Fig. 2.48.

For the section $x < \overline{x}$, the internal bending moment can be expressed as

$$M_y(x) = -F_z^{\mathrm{R}}(0)x - M_y^{\mathrm{R}}(0) = F_{\mathrm{a}}(\overline{x} - x) + F_0(L - x)\,, \tag{2.221}$$

while the section $x \geq \overline{x}$ gives:

$$M_y(x) = -F_z^{\mathrm{R}}(0)x - M_y^{\mathrm{R}}(0) + F_{\mathrm{a}}(x - \overline{x}) = F_0(L - x)\,. \tag{2.222}$$

Application of CASTIGLIANO's second theorem allows us to determine the distribution of the displacement field:

$$u_z(x) = \int_0^L \frac{M_y(x)}{EI}\frac{\partial M_y(x, F_{\mathrm{a}})}{\partial F_{\mathrm{a}}}\,\mathrm{d}x$$

$$= \int_0^{\overline{x}} \frac{M_y(x)}{EI}\frac{\partial M_y(x, F_{\mathrm{a}})}{\partial F_{\mathrm{a}}}\,\mathrm{d}x + \int_{\overline{x}}^L \frac{M_y(x)}{EI}\frac{\partial M_y(x, F_{\mathrm{a}})}{\partial F_{\mathrm{a}}}\,\mathrm{d}x$$

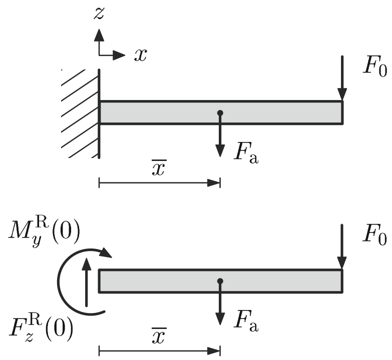


Fig. 2.47 Cantilever beam with force boundary condition: introduction of auxiliary force

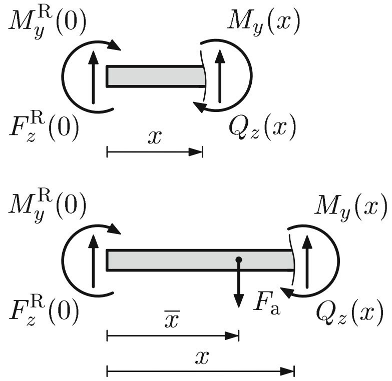


Fig. 2.48 Cantilever beam with force boundary condition: different sections for internal reactions

$$= \int_0^{\bar{x}} \frac{F_a(\bar{x} - x) + F_0(L - x)}{EI} \times (x - \bar{x})\,dx + \int_0^{\bar{x}} \frac{F_a(L - x)}{EI} \times 0\,dx\,.$$

The evaluation of these integrals gives finally under consideration of $F_a \to 0$ and $\bar{x} \to x$:

$$u_z(x) = \frac{F_0 L^3}{EI}\left(-\frac{1}{6}\left[\frac{x}{L}\right]^3 + \frac{1}{2}\left[\frac{x}{L}\right]^2\right), \tag{2.223}$$

which is the deflection in direction of F_a.

The other subproblems (b–d) can be solved in a similar manner.

2.4 Extensometer Analysis

The extensometer shown and illustrated in Figs. 1.1 and 1.2 can be modeled in a first attempt as a ⊔-shaped frame with different properties for the horizontal and vertical members (see Fig. 2.49a). Looking at this mechanical model, it is obvious that the

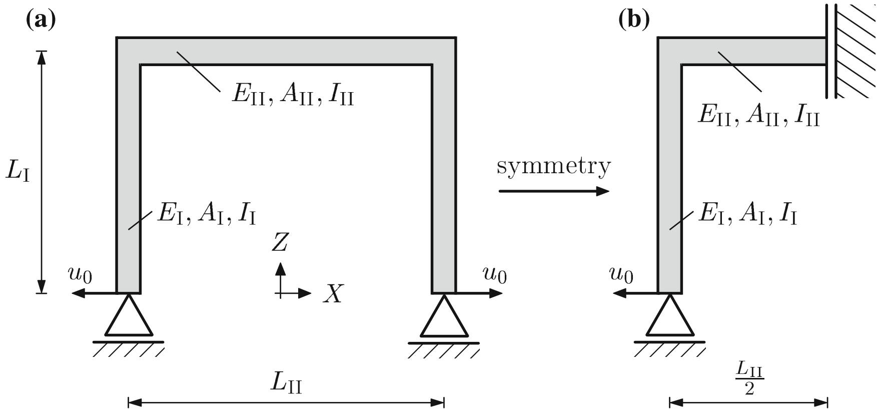


Fig. 2.49 Mechanical model of the extensometer: **a** entire sensor and **b** consideration of symmetry

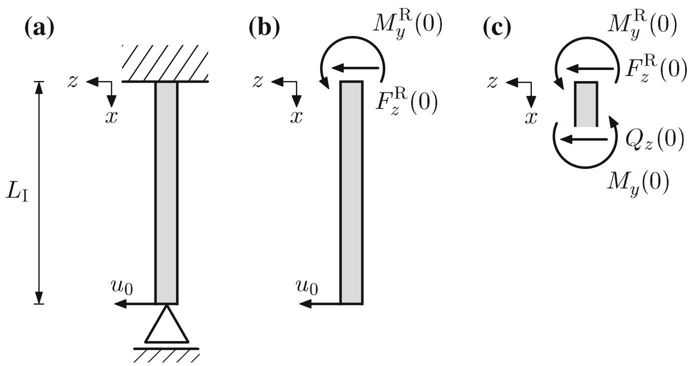


Fig. 2.50 Simplified approach for vertical members: **a** approximation as cantilever beam; **b** free-body diagram; **c** infinitesimal element at $x = 0$

structure is symmetric with respect to a vertical axis and can be reduced as indicated in Fig. 2.49b.

A rough mechanical model can be obtained by splitting the frame into a vertical and horizontal member. The vertical member (I) is assumed to be a cantilever beam (see Fig. 2.50) which perfectly transmits the reaction moment and force to the vertical member (II), see Fig. 2.51. It is obvious that the small rotation in the frame corner is not perfectly represented in this simple model. However, it allows us to derive a simple design equation based on the straight beam equations provided in Sects. 2.2.1–2.2.2. The horizontal member can be assumed to be a simply supported beam of length L_{II} as shown in Fig. 2.51a or as a cantilever of length $\frac{L_{\mathrm{II}}}{2}$ as shown in Fig. 2.51b.

Let us have a closer look at the vertical member as shown in Fig. 2.50. From Eqs. (2.55) and (2.57) we can conclude with $u_0 \to -u_0$ the bending moment and shear force distributions to be:

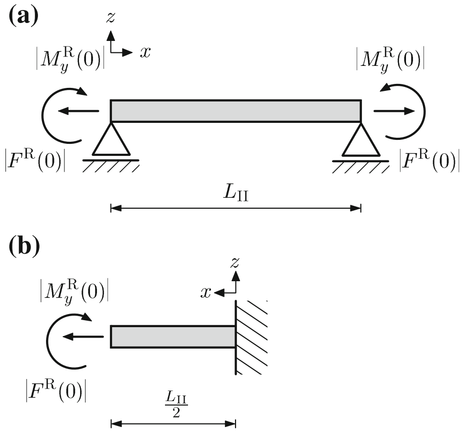


Fig. 2.51 Simplified approach for horizontal member: **a** approximation as simply supported beam and **b** consideration of symmetry

$$M_y(x) = \frac{3E_{\mathrm{I}}I_{\mathrm{I}}u_0}{L_{\mathrm{I}}^2}\left(\frac{x}{L} - 1\right) , \tag{2.224}$$

$$Q_z(x) = \frac{3E_{\mathrm{I}}I_{\mathrm{I}}u_0}{L_{\mathrm{I}}^3} , \tag{2.225}$$

or at $x = 0$:

$$M_y(0) = -\frac{3E_{\mathrm{I}}I_{\mathrm{I}}u_0}{L_{\mathrm{I}}^2} , \tag{2.226}$$

$$Q_z(0) = \frac{3E_{\mathrm{I}}I_{\mathrm{I}}u_0}{L_{\mathrm{I}}^3} . \tag{2.227}$$

These internal reactions must be balanced at $x = L$ by the reactions of the fixed support. The force and moment equilibrium at $x = L$ reads:

$$+Q_z(0) + F_z^{\mathrm{R}}(0) = 0 \quad \Rightarrow \quad F_z^{\mathrm{R}}(0) = -\frac{3E_{\mathrm{I}}I_{\mathrm{I}}u_0}{L_{\mathrm{I}}^3} , \tag{2.228}$$

$$+M_y(0) + M_y^{\mathrm{R}}(0) = 0 \quad \Rightarrow \quad M_y^{\mathrm{R}}(0) = \frac{3E_{\mathrm{I}}I_{\mathrm{I}}u_0}{L_{\mathrm{I}}^2} . \tag{2.229}$$

These reactions are now applied at the horizontal member, see Fig. 2.51. To avoid any confusion with the sign of these quantities, it is advised to simply take the absolute values and consider the correct directions as indicated in the figure. This configuration relates to the case that the base sample is under tensile load.

Let us first consider the case that only the bending moment is acting, i.e. the case of pure bending. The internal bending moment distribution for both cases shown is Fig. 2.51 is obtained as:

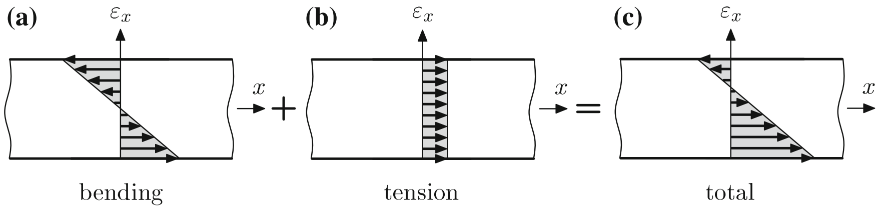


Fig. 2.52 Strain distributions in the horizontal member of the extensometer: **a** pure bending; **b** pure tension; **c** superposition of both cases

$$M_y(x) = -\frac{3E_\mathrm{I}I_\mathrm{I}u_0}{L_\mathrm{I}^2} = \text{const.} \tag{2.230}$$

Equation (2.40) together with HOOKE's law allows us to express the normal strain in the horizontal members (II) as:

$$\varepsilon_{x,\mathrm{II}}(z) = \frac{M_y(x)}{E_\mathrm{II}I_\mathrm{II}}\,z(x) = -\frac{3E_\mathrm{I}I_\mathrm{I}u_0}{E_\mathrm{II}I_\mathrm{II}L_\mathrm{I}^2}\,z(x)\,. \tag{2.231}$$

In the next step, we can express the displacement u_0 by the strain in the specimen ε_sp, i.e.,

$$\varepsilon_\mathrm{sp} = \frac{2u_0}{L_\mathrm{II}} = \frac{u_0}{\frac{L_\mathrm{II}}{2}}\,, \tag{2.232}$$

which allows us to express the strain in the horizontal member of the extensometer as:

$$\varepsilon_{x,\mathrm{II}}(z) = -\frac{3}{2} \times \frac{E_\mathrm{I}I_\mathrm{I}L_\mathrm{II}}{E_\mathrm{II}I_\mathrm{II}L_\mathrm{I}^2} \times \varepsilon_\mathrm{sp} z(x)\,. \tag{2.233}$$

The strain distribution under pure bending is sketched in Fig. 2.52a where a linear distribution can be observed. Furthermore, the distribution is symmetric with a compressive regime for $z > 0$ and a tensile regime for $z < 0$.

Reviewing again Fig. 2.50b, we can identify a shear force $F^\mathrm{y}(0)$ which acts on the horizontal member as a tensile force, see Fig. 2.51. This 'tensile' force results in the following tensile strain:

$$\varepsilon_{x,\mathrm{II}} = \frac{|F^\mathrm{R}|}{E_\mathrm{II}A_\mathrm{II}} = \frac{3E_\mathrm{I}I_\mathrm{I}u_0}{L_\mathrm{I}^3E_\mathrm{II}A_\mathrm{II}} = \frac{3}{2} \times \frac{E_\mathrm{I}I_\mathrm{I}L_\mathrm{II}}{E_\mathrm{II}A_\mathrm{II}L_\mathrm{I}^3} \times \varepsilon_\mathrm{sp}\,. \tag{2.234}$$

The strain components given in Eqs. (2.233) and (2.234) can be superposed to obtain the total axial strain (see Fig. 2.52c) in the horizontal member of the extensometer as:

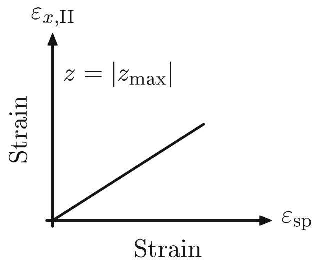


Fig. 2.53 Schematic representation of the calibration curve for the extensometer

$$\varepsilon_{x,\mathrm{II}} = \frac{3}{2}\left(-\frac{E_\mathrm{I} I_\mathrm{I} L_\mathrm{II}}{E_\mathrm{II} I_\mathrm{II} L_\mathrm{I}^2} \times z(x) + \frac{E_\mathrm{I} I_\mathrm{I} L_\mathrm{II}}{E_\mathrm{II} A_\mathrm{II} L_\mathrm{I}^3}\right)\varepsilon_{\mathrm{sp}} \,. \tag{2.235}$$

Let us assume in the following a square cross section (with width b_II and height h_II) for the horizontal member. Based on the relationship $A_\mathrm{II} = \frac{12 I_\mathrm{II}}{h_\mathrm{II}^2}$, the total strain can be expressed as:

$$\varepsilon_{x,\mathrm{II}} = \frac{3}{2} \times \frac{E_\mathrm{I} I_\mathrm{I} L_\mathrm{II}}{E_\mathrm{II} I_\mathrm{II} L_\mathrm{I}}\left(-\frac{z(x)}{L_\mathrm{I}} + \frac{1}{12}\left(\frac{h_\mathrm{II}}{L_\mathrm{I}}\right)^2\right)\varepsilon_{\mathrm{sp}} \,. \tag{2.236}$$

The extreme values at the free surfaces, i.e. $z = +\frac{h_\mathrm{II}}{2}$ and $z = -\frac{h_\mathrm{II}}{2}$, are obtained as follows:

$$\varepsilon_{x,\mathrm{II}}\Big|_{z=+\frac{h_\mathrm{II}}{2}} = \frac{3}{2} \times \frac{E_\mathrm{I} I_\mathrm{I} L_\mathrm{II}}{E_\mathrm{II} I_\mathrm{II} L_\mathrm{I}} \times \frac{h_\mathrm{II}}{L_\mathrm{I}} \times \left(-\frac{1}{2} + \frac{1}{12}\left(\frac{h_\mathrm{II}}{L_\mathrm{I}}\right)\right)\varepsilon_{\mathrm{sp}} \,, \tag{2.237}$$

$$\varepsilon_{x,\mathrm{II}}\Big|_{z=-\frac{h_\mathrm{II}}{2}} = \frac{3}{2} \times \frac{E_\mathrm{I} I_\mathrm{I} L_\mathrm{II}}{E_\mathrm{II} I_\mathrm{II} L_\mathrm{I}} \times \frac{h_\mathrm{II}}{L_\mathrm{I}} \times \left(+\frac{1}{2} + \frac{1}{12}\left(\frac{h_\mathrm{II}}{L_\mathrm{I}}\right)\right)\varepsilon_{\mathrm{sp}} \,. \tag{2.238}$$

Based on Eqs. (2.237) and/or (2.238), it is now possible to calculate and draw the calibration curve for the extensometer, i.e. the relation between the measured strain in the extensometer ($\varepsilon_{x,\mathrm{II}}$) and the strain in the specimen ($\varepsilon_{\mathrm{sp}}$), see Fig. 2.53. From a practical point of view, one could measure the strain on the top, or the bottom (larger signal since two positive strain components are summed up) of the beam or even average both signals (under consideration that the distribution is no longer symmetric).

Both Eqs. (2.237) and (2.238) can be generally written as

$$\varepsilon_{x,\mathrm{II}} = \varepsilon_{\mathrm{sp}}(E_\mathrm{I}, E_\mathrm{II}, I_\mathrm{I}, I_\mathrm{II}, L_\mathrm{I}, L_\mathrm{II}) \,, \tag{2.239}$$

which allows us to design the extensometer in the boundaries of minimum strain (sensitivity) and maximum strain (failure of the strain gage).

Let us look in the following at a solution procedure which is based on the strain energy as outlined in Sect. 2.3. This allows us to consider the entire frame (see

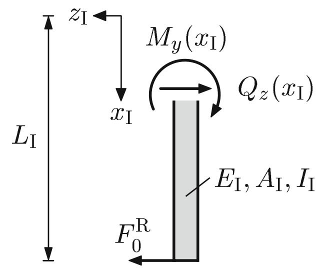


Fig. 2.54 Vertical beam section for the determination of the internal reactions

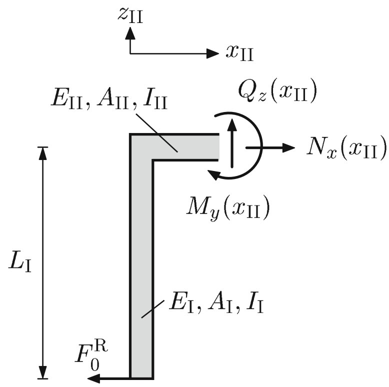


Fig. 2.55 Section of the frame structure for the determination of the internal reactions in the horizontal member

Fig. 2.49) without the strong simplification in regards to the connection of the vertical and horizontal members.

The horizontal force and moment equilibrium (see Fig. 2.54) gives the internal reactions of the vertical member as follows:

$$Q_z(x_{\mathrm{I}}) = F_0^{\mathrm{R}}\,, \tag{2.240}$$

$$M_y(x_{\mathrm{I}}) = -F_0^{\mathrm{R}}(L_{\mathrm{I}} - x_{\mathrm{I}})\,. \tag{2.241}$$

For the internal reactions of the horizontal members, it is advantageous to consider the left-hand half as shown in Fig. 2.55. Horizontal and vertical force as well as the moment equilibrium give the following internal reactions in the horizontal member:

$$N_x(x_{\mathrm{II}}) = F_0^{\mathrm{R}}\,, \tag{2.242}$$

$$Q_z(x_{\mathrm{II}}) = 0\,, \tag{2.243}$$

$$M_z(x_{\mathrm{II}}) = -F_0^{\mathrm{R}} L_{\mathrm{I}}\,. \tag{2.244}$$

It should be noted here that the reaction force F_0^{R} is still unknown. Based on CASTIGLIANO's second theorem, it is possible to express the horizontal displacement as[6]:

$$u_0 = \int\limits_0^{L_{\mathrm{I}}} \frac{M_y(x_{\mathrm{I}})}{E_{\mathrm{I}} I_{\mathrm{I}}} \frac{\partial M_y(x_{\mathrm{I}}, F_0^{\mathrm{R}})}{\partial F_0^{\mathrm{R}}} \mathrm{d}x_{\mathrm{I}} + \int\limits_0^{\frac{L_{\mathrm{II}}}{2}} \frac{M_y(x_{\mathrm{II}})}{E_{\mathrm{II}} I_{\mathrm{II}}} \frac{\partial M_y(x_{\mathrm{II}}, F_0^{\mathrm{R}})}{\partial F_0^{\mathrm{R}}} \mathrm{d}x_{\mathrm{II}} +$$
$$+ \int\limits_0^{\frac{L_{\mathrm{II}}}{2}} \frac{N_x(x_{\mathrm{II}})}{E_{\mathrm{II}} A_{\mathrm{II}}} \frac{\partial N_x(x_{\mathrm{II}}, F_0^{\mathrm{R}})}{\partial F_0^{\mathrm{R}}} \mathrm{d}x_{\mathrm{II}}$$
$$= F_0^{\mathrm{R}} \left(\frac{1}{3} \frac{L_{\mathrm{I}}^3}{E_{\mathrm{I}} I_{\mathrm{I}}} + \frac{1}{2} \frac{L_{\mathrm{I}}^2 L_{\mathrm{II}}}{E_{\mathrm{II}} I_{\mathrm{II}}} + \frac{1}{2} \frac{L_{\mathrm{II}}}{E_{\mathrm{II}} A_{\mathrm{II}}} \right) . \tag{2.245}$$

If we assume a square cross section for the horizontal member (width b_{II} and height h_{II}), we can relate the cross sectional area to the second moment of area, i.e. $A_{\mathrm{II}} = \frac{12 I_{\mathrm{II}}}{h_{\mathrm{II}}^2}$, and Eq. (2.245) can be expressed as:

$$u_0 = F_0^{\mathrm{R}} \left(\frac{1}{3} \frac{L_{\mathrm{I}}^3}{E_{\mathrm{I}} I_{\mathrm{I}}} + \frac{1}{2} \frac{L_{\mathrm{I}}^2 L_{\mathrm{II}}}{E_{\mathrm{II}} I_{\mathrm{II}}} + \frac{1}{24} \frac{L_{\mathrm{II}} h_{\mathrm{II}}^2}{E_{\mathrm{II}} I_{\mathrm{II}}} \right) , \tag{2.246}$$

or rearranged for the unknown reaction force:

$$F_0^{\mathrm{R}} = \frac{u_0}{\frac{1}{3} \frac{L_{\mathrm{I}}^3}{E_{\mathrm{I}} I_{\mathrm{I}}} + \frac{1}{2} \frac{L_{\mathrm{I}}^2 L_{\mathrm{II}}}{E_{\mathrm{II}} I_{\mathrm{II}}} + \frac{1}{24} \frac{L_{\mathrm{II}} h_{\mathrm{II}}^2}{E_{\mathrm{II}} I_{\mathrm{II}}}} . \tag{2.247}$$

Based on this reaction force, the internal reactions are known in both members, see Eqs. (2.240)–(2.244). Let us now calculate the total strain in the horizontal member of the extensometer. The axial strain due to the bending deformation can be expressed as:

$$\varepsilon_{x,\mathrm{II}} = \frac{1}{E_{\mathrm{II}}} \frac{M_y(x_{\mathrm{II}})}{I_{\mathrm{II}}} z(x_{\mathrm{II}}) = \frac{1}{E_{\mathrm{II}} I_{\mathrm{II}}} \left(- \frac{u_0 L_{\mathrm{I}}}{\frac{1}{3} \frac{L_{\mathrm{I}}^3}{E_{\mathrm{I}} I_{\mathrm{I}}} + \frac{1}{2} \frac{L_{\mathrm{I}}^2 L_{\mathrm{II}}}{E_{\mathrm{II}} I_{\mathrm{II}}} + \frac{1}{24} \frac{L_{\mathrm{II}} h_{\mathrm{II}}^2}{E_{\mathrm{II}} I_{\mathrm{II}}}} \right) z(x_{\mathrm{II}}) , \tag{2.248}$$

which can be rearranged under consideration of $u_0 = \frac{L_{\mathrm{II}}}{2} \varepsilon_{\mathrm{sp}}$ to:

$$\varepsilon_{x,\mathrm{II}} = - \frac{\varepsilon_{\mathrm{sp}}}{\frac{2}{3} \frac{E_{\mathrm{II}} I_{\mathrm{II}} L_{\mathrm{I}}}{E_{\mathrm{I}} I_{\mathrm{I}} L_{\mathrm{II}}} + 1 + \frac{1}{12} \frac{h_{\mathrm{II}}^2}{L_{\mathrm{I}}^2}} \left(\frac{z_{\mathrm{II}}}{L_{\mathrm{I}}} \right) . \tag{2.249}$$

On the other hand, the axial strain due to the tensile deformation can be expressed as:

[6]It is assumed that the beams are thin and that the shear force is not contributing to the bending deformation modes.

$$\varepsilon_{x,\mathrm{II}} = \frac{1}{E_{\mathrm{II}}}\frac{F_0^{\mathrm{R}}}{A_{\mathrm{II}}} = \frac{1}{E_{\mathrm{II}}A_{\mathrm{II}}}\left(\frac{u_0}{\frac{1}{3}\frac{L_{\mathrm{I}}^3}{E_{\mathrm{I}}I_{\mathrm{I}}} + \frac{1}{2}\frac{L_{\mathrm{I}}^2 L_{\mathrm{II}}}{E_{\mathrm{II}}I_{\mathrm{II}}} + \frac{1}{2}\frac{L_{\mathrm{II}}}{E_{\mathrm{II}}A_{\mathrm{II}}}}\right), \tag{2.250}$$

which can be rearranged under consideration of $A_{\mathrm{II}} = \frac{12 I_{\mathrm{II}}}{h_{\mathrm{II}}^2}$ and $u_0 = \frac{L_{\mathrm{II}}}{2}\varepsilon_{\mathrm{sp}}$ to:

$$\varepsilon_{x,\mathrm{II}} = \frac{1}{12}\frac{h_{\mathrm{II}}^2}{L_{\mathrm{I}}^2}\left(\frac{\varepsilon_{\mathrm{sp}}}{\frac{2}{3}\frac{E_{\mathrm{II}}I_{\mathrm{II}}L_{\mathrm{I}}}{E_{\mathrm{I}}I_{\mathrm{I}}L_{\mathrm{II}}} + 1 + \frac{1}{12}\frac{h_{\mathrm{II}}^2}{L_{\mathrm{I}}^2}}\right). \tag{2.251}$$

Thus, the total strain resulting from bending and tension is obtained as:

$$\varepsilon_{x,\mathrm{II}} = \frac{1}{\frac{2}{3}\frac{E_{\mathrm{II}}I_{\mathrm{II}}L_{\mathrm{I}}}{E_{\mathrm{I}}I_{\mathrm{I}}L_{\mathrm{II}}} + 1 + \frac{1}{12}\frac{h_{\mathrm{II}}^2}{L_{\mathrm{I}}^2}}\left(-\frac{z_{\mathrm{II}}}{L_{\mathrm{I}}} + \frac{1}{12}\left(\frac{h_{\mathrm{II}}}{L_{\mathrm{I}}}\right)^2\right)\varepsilon_{\mathrm{sp}}. \tag{2.252}$$

The extreme values at the free surfaces, i.e. $z = +\frac{h_{\mathrm{II}}}{2}$ and $z = -\frac{h_{\mathrm{II}}}{2}$, are obtained as follows:

$$\varepsilon_{x,\mathrm{II}}\Big|_{z=+\frac{h_{\mathrm{II}}}{2}} = \frac{1}{\frac{2}{3}\frac{E_{\mathrm{II}}I_{\mathrm{II}}L_{\mathrm{I}}}{E_{\mathrm{I}}I_{\mathrm{I}}L_{\mathrm{II}}} + 1 + \frac{1}{12}\frac{h_{\mathrm{II}}^2}{L_{\mathrm{I}}^2}} \times \frac{h_{\mathrm{II}}}{L_{\mathrm{I}}} \times \left(-\frac{1}{2} + \frac{1}{12}\left(\frac{h_{\mathrm{II}}}{L_{\mathrm{I}}}\right)\right)\varepsilon_{\mathrm{sp}}, \tag{2.253}$$

$$\varepsilon_{x,\mathrm{II}}\Big|_{z=-\frac{h_{\mathrm{II}}}{2}} = \frac{1}{\frac{2}{3}\frac{E_{\mathrm{II}}I_{\mathrm{II}}L_{\mathrm{I}}}{E_{\mathrm{I}}I_{\mathrm{I}}L_{\mathrm{II}}} + 1 + \frac{1}{12}\frac{h_{\mathrm{II}}^2}{L_{\mathrm{I}}^2}} \times \frac{h_{\mathrm{II}}}{L_{\mathrm{I}}} \times \left(+\frac{1}{2} + \frac{1}{12}\left(\frac{h_{\mathrm{II}}}{L_{\mathrm{I}}}\right)\right)\varepsilon_{\mathrm{sp}}. \tag{2.254}$$

Let us do a simple calculation at $z = -\frac{h_{\mathrm{II}}}{2}$ for the special case $E_{\mathrm{I}} = E_{\mathrm{II}}$, $I_{\mathrm{I}} = I_{\mathrm{II}}$, $L_{\mathrm{I}} = L_{\mathrm{II}}$, and $h_{\mathrm{II}} = \frac{L_{\mathrm{I}}}{10}$. From Eq. (2.254), we get

$$\varepsilon_{x,\mathrm{II}}\Big|_{z=-\frac{h_{\mathrm{II}}}{2}} = \frac{61}{2000}\varepsilon_{\mathrm{sp}} = 0.0305\varepsilon_{\mathrm{sp}}, \tag{2.255}$$

while the simplified model according to Eq. (2.238) gives:

$$\varepsilon_{x,\mathrm{II}}\Big|_{z=-\frac{h_{\mathrm{II}}}{2}} = \frac{61}{800}\varepsilon_{\mathrm{sp}} = 0.07625\varepsilon_{\mathrm{sp}}. \tag{2.256}$$

Obviously there is quite a significant difference between both approaches but the results have at least the same order of magnitude.

The derivation of the equation for the displacement u_0 as given in Eqs. (2.245) and (2.246) was based on the assumption that the shear force is not contributing to the deformation of the frame. The results for the shear force in Eqs. (2.240) and (2.243) indicate that only the vertical frame part is loaded by a shear force. In the case that this member is designed as a short beam, i.e. the application of the thick beam might be more appropriate, CASTIGLIANO's statement can be modified to the following expression:

$$u_0 = \int_0^{L_\mathrm{I}} \frac{M_y(x_\mathrm{I})}{E_\mathrm{I} I_\mathrm{I}} \frac{\partial M_y(x_\mathrm{I}, F_0^\mathrm{R})}{\partial F_0^\mathrm{R}} \mathrm{d}x_\mathrm{I} + \underbrace{\int_0^{L_\mathrm{I}} \frac{Q_z(x_\mathrm{I})}{k_{\mathrm{s,I}} G_\mathrm{I} A_\mathrm{I}} \frac{\partial Q_z(x_\mathrm{I}, F_0^\mathrm{R})}{\partial F_0^\mathrm{R}} \mathrm{d}x_\mathrm{I} +}_{\text{shear contribution}}$$

$$+ \int_0^{\frac{L_\mathrm{II}}{2}} \frac{M_y(x_\mathrm{II})}{E_\mathrm{II} I_\mathrm{II}} \frac{\partial M_y(x_\mathrm{II}, F_0^\mathrm{R})}{\partial F_0^\mathrm{R}} \mathrm{d}x_\mathrm{II} + \int_0^{\frac{L_\mathrm{II}}{2}} \frac{N_x(x_\mathrm{II})}{E_\mathrm{II} A_\mathrm{II}} \frac{\partial N_x(x_\mathrm{II}, F_0^\mathrm{R})}{\partial F_0^\mathrm{R}} \mathrm{d}x_\mathrm{II}$$

$$= F_0^\mathrm{R} \left(\frac{1}{3} \frac{L_\mathrm{I}^3}{E_\mathrm{I} I_\mathrm{I}} + \frac{1}{2} \frac{L_\mathrm{I}^2 L_\mathrm{II}}{E_\mathrm{II} I_\mathrm{II}} + \frac{1}{2} \frac{L_\mathrm{II}}{E_\mathrm{II} A_\mathrm{II}} + \frac{L_\mathrm{I}}{k_{\mathrm{s,I}} G_\mathrm{I} A_\mathrm{I}} \right). \tag{2.257}$$

The last equation can be rearranged for the unknown reaction force:

$$F_0^\mathrm{R} = \frac{u_0}{\frac{1}{3} \frac{L_\mathrm{I}^3}{E_\mathrm{I} I_\mathrm{I}} + \frac{1}{2} \frac{L_\mathrm{I}^2 L_\mathrm{II}}{E_\mathrm{II} I_\mathrm{II}} + \frac{1}{2} \frac{L_\mathrm{II}}{E_\mathrm{II} A_\mathrm{II}} + \frac{L_\mathrm{I}}{k_{\mathrm{s,I}} G_\mathrm{I} A_\mathrm{I}}}. \tag{2.258}$$

Under the consideration of a square cross section, i.e. $k_{\mathrm{s,I}} = \frac{5}{6}$, $G_\mathrm{I} = \frac{E_\mathrm{I}}{2(1+\nu_\mathrm{I})}$, $A_i = \frac{12 I_i}{h_i^2}$, $I_i = \frac{b_i h_i^3}{12}$, and that the width is the same, i.e. $b_\mathrm{I} = b_\mathrm{II}$, one can easily derive the following normalized expression:

$$\frac{F_0^\mathrm{R}}{\frac{3 E_\mathrm{II} I_\mathrm{II} u_0}{I_\mathrm{II}^3}} = \frac{\frac{2}{3}}{\frac{2}{3} \frac{E_\mathrm{II}}{E_\mathrm{I}} \left(\frac{h_\mathrm{II}}{L_\mathrm{II}}\right)^3 \left(\frac{L_\mathrm{I}}{h_\mathrm{I}}\right)^3 + \left(\frac{L_\mathrm{I}}{L_\mathrm{II}}\right)^3 + \frac{1}{12} \left(\frac{h_\mathrm{II}}{L_\mathrm{II}}\right)^2 + \frac{2}{5} (1+\nu_\mathrm{I}) \frac{E_\mathrm{II}}{E_\mathrm{I}} \left(\frac{h_\mathrm{II}}{L_\mathrm{II}}\right)^3 \left(\frac{L_\mathrm{I}}{h_\mathrm{I}}\right)}. \tag{2.259}$$

It should be noted that the last expression in the denominator (which contains POISSON's ratio) stems from the consideration of the shear contribution. Let us now do some simple estimates to predict the significance of the different contributions. The different fractions in the denominator are evaluated as a function of $\frac{h_\mathrm{I}}{L_\mathrm{I}}$ in Table 2.8 for the special case $E_\mathrm{II} = E_\mathrm{I}$, $L_\mathrm{II} = L_\mathrm{I}$, $h_\mathrm{II} = \frac{L_\mathrm{II}}{10}$, and $\nu_\mathrm{I} = 0.3$.

We can conclude from Table 2.8 that the dominant mode of deformation is bending in member II. Increasing the height (h_I) of member I with respect to its length (L_I) increases the shear contribution in this members compared to the bending fraction. However, both contributions reduce their share in the total deformation. Thus, we can justify from this investigation that we do not need to consider the contribution of the shear force on the deformation in this particular case.

Let us mention at the end of this section that the presented approach allows also to estimate the influence of the deadweight, see Fig. 2.56. The distributed bending loads in the horizontal frame elements are given by $q_\mathrm{g} = \frac{\mathrm{d}F_\mathrm{g}}{\mathrm{d}x} = \varrho_\mathrm{I} A_\mathrm{I} \mathrm{g}$ whereas the distributed axial load in the vertical frame element is given by $p_\mathrm{g} = \frac{\mathrm{d}F_\mathrm{g}}{\mathrm{d}x} = \varrho_\mathrm{II} A_\mathrm{II} \mathrm{g}$.

Table 2.8 Sensitivity of different deformation modes on the normalized reaction force as a function of the slenderness ratio $\frac{h_\mathrm{I}}{L_\mathrm{I}}$

	Member I		Member II	
	Bending	Shear	Bending	Tension
Eq.	$\frac{2}{3}\frac{E_\mathrm{II}}{E_\mathrm{I}}\left(\frac{h_\mathrm{II}}{L_\mathrm{II}}\right)^3\left(\frac{L_\mathrm{I}}{h_\mathrm{I}}\right)^3$	$\frac{2}{5}(1+\nu_\mathrm{I})\frac{E_\mathrm{II}}{E_\mathrm{I}}\left(\frac{h_\mathrm{II}}{L_\mathrm{II}}\right)^3\left(\frac{L_\mathrm{I}}{h_\mathrm{I}}\right)$	$\left(\frac{L_\mathrm{I}}{L_\mathrm{II}}\right)^3$	$\frac{1}{12}\left(\frac{h_\mathrm{II}}{L_\mathrm{II}}\right)^2$
$\frac{h_\mathrm{I}}{L_\mathrm{I}}$				
$\frac{1}{10}$	0.6666666667	0.0052000000	1.0	0.0008333333
$\frac{1}{9}$	0.4860000000	0.0046800000	1.0	0.0008333333
$\frac{1}{8}$	0.3413333333	0.0041600000	1.0	0.0008333333
$\frac{1}{7}$	0.2286666667	0.0036400000	1.0	0.0008333333
$\frac{1}{6}$	0.1440000000	0.0031200000	1.0	0.0008333333
$\frac{1}{5}$	0.0833333333	0.0026000000	1.0	0.0008333333
$\frac{1}{4}$	0.0426666667	0.0020800000	1.0	0.0008333333
$\frac{1}{3}$	0.0180000000	0.0015600000	1.0	0.0008333333
$\frac{1}{2}$	0.0053333333	0.0010400000	1.0	0.0008333333
1	0.0006666667	0.0005200000	1.0	0.0008333333

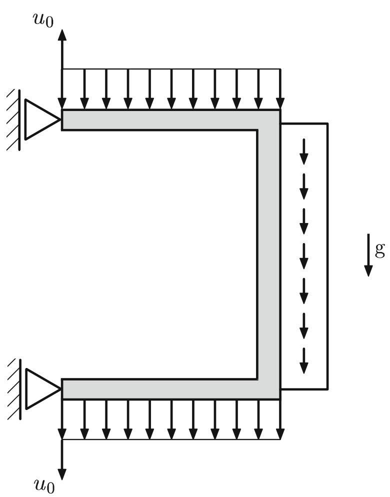


Fig. 2.56 Extensometer under consideration of the deadweight

2.5 Supplementary Problems

2.13 Rod Loaded By a Single Force in Its Middle

Given is a rod of length $2L$ and axial tensile stiffness EA which is fixed at both ends, see Fig. 2.57. A single force F_0 is acting in the middle ($X = L$) in positive direction. Determine the expression for the displacement $u_X(X)$ and the normal fore $N_X(X)$

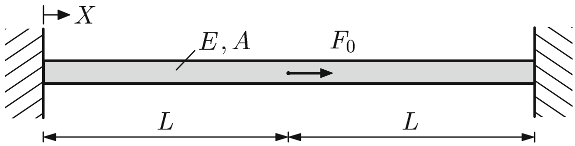


Fig. 2.57 Rod loaded by a single force in its middle

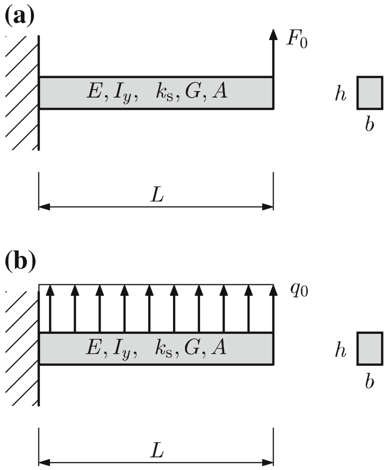


Fig. 2.58 Cantilever TIMOSHENKO beam: **a** single force case and **b** distributed load case

based on the consideration of two sections or alternatively based on the application of a discontinuous function. Sketch both distributions.

2.14 Cantilever Beam Under the Influence of a Point or Distributed Load – Rectangular Cross Section

The cantilever TIMOSHENKO beam shown in Fig. 2.58 is either loaded by a single force F_0 at its right-hand end or by a distributed load q_0. The bending stiffness EI and the shear stiffness $k_s AG$ are constant, the total length of the beam is equal to L, and the rectangular cross section has the dimensions of $b \times h$. Determine the expressions of the bending lines ($u_z(x)$) and sketch the deflections of the right-hand end ($x = L$) as a function of the slenderness ratio $\frac{h}{L}$ for $\nu = 0.0$, 0.3, and 0.5.

2.15 Cantilever Rod with Different Sections (Alternative Solution Procedure of Problem 2.3)

Given is a rod of length $3L$ and constant axial tensile stiffness EA as shown in Fig. 2.59. At the left-hand side there is a fixed support and a constant distributed load $2p_0$ is acting in the range $0 \leq x \leq 2L$ whereas a load of p_0 is acting in the range $2L \leq x \leq 3L$. Determine based on CASTIGLIANO's theorems the analytical solution for the elongation $u_x(x)$, the strain $\varepsilon_x(x)$, and the stress $\sigma_x(x)$ along the rod axis.

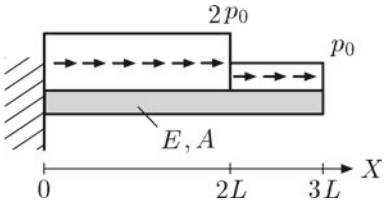


Fig. 2.59 Rod with different sections

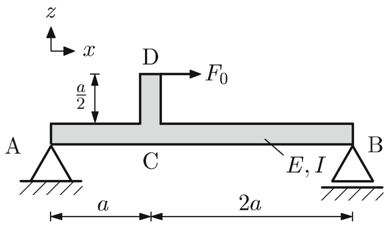


Fig. 2.60 Beam-like structure

2.16 Beam-Like Structure: Energy Approach

Given is a beam-like structure of length $3a$ and bending stiffness EI which is simply supported, see Fig. 2.60. A single force F_0 is acting at a vertical extension in positive x-direction. Determine based on CASTIGLIANO's theorems the horizontal displacement of the load application point (D) and the vertical displacement of point C.

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