Chapter 2
Continuum Methods (CM): Basic Continuum Mechanics

2.1 Notation

Throughout this work, boldface symbols denote vectors or tensors. Furthermore, we exclusively employ a Cartesian basis. For the inner product of two vectors (first-order tensors), $\boldsymbol{u}$ and $\boldsymbol{v}$, we have $\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3$ in three dimensions, where a Cartesian basis and Einstein index summation notation are used. In this introduction, for clarity of presentation, *we will ignore the difference between second-order tensors and matrices.* Accordingly, if we consider the second-order tensor $\boldsymbol{A} = A_{ik}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_k$, then a first-order contraction (inner product) of two second-order tensors $\boldsymbol{A} \cdot \boldsymbol{B}$ is defined by the matrix product $[\boldsymbol{A}][\boldsymbol{B}]$, with components of $A_{ij} B_{jk} = C_{ik}$. It is clear that the range of the inner index j must be the same for $[\boldsymbol{A}]$ and $[\boldsymbol{B}]$. For three dimensions, we have $i, j = 1, 2, 3$. The inner product of a tensor (matrix) with a vector is defined as $\boldsymbol{A} \cdot \boldsymbol{v} = A_{ij} v_j$. The second-order inner (scalar) product of two tensors (matrices) is defined as $\boldsymbol{A} : \boldsymbol{B} = A_{ij} B_{ij} = tr([\boldsymbol{A}]^T [\boldsymbol{B}])$. Monograph Appendix 1 provides a basic mathematical review.

2.2 Kinematics of Deformations

In this chapter, we synopsize a more detailed discussion found in Zohdi and Wriggers [1]. The term deformation refers to a change in the shape of a continuum between a reference configuration and current configuration. In the reference configuration, a representative particle of a continuum occupies a point P in space and has the position vector (Fig. 2.1)

$$\boldsymbol{X} = X_1 \boldsymbol{e}_1 + X_2 \boldsymbol{e}_2 + X_3 \boldsymbol{e}_3, \tag{2.1}$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is a Cartesian reference triad, and X_1, X_2, X_3 (with center O) can be thought of as labels for a material point. Sometimes the coordinates or labels

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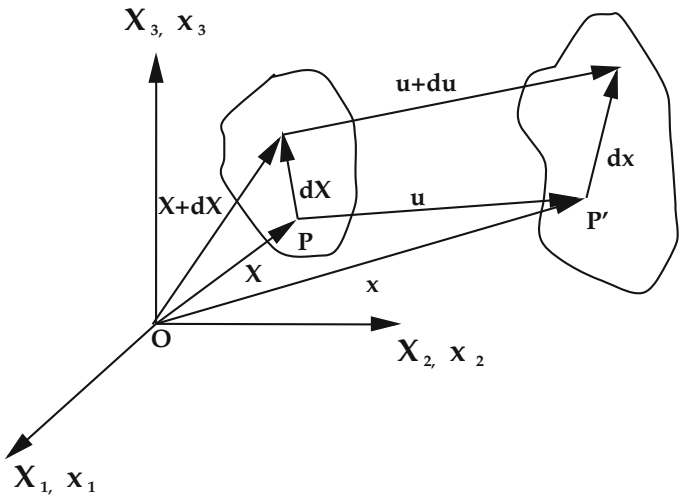

Fig. 2.1 Different descriptions of a deforming body. Ω_o is the reference configuration, and Ω is the current configuration

(X_1, X_2, X_3) are called the referential or material coordinates. In the current configuration, the particle originally located at point P (at time $t = 0$) is located at point P' and can be also expressed in terms of another position vector $\boldsymbol{x}$, with coordinates (x_1, x_2, x_3). These are called the current coordinates. In this framework, the displacement is $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ for a point originally at $\boldsymbol{X}$ and with final coordinates $\boldsymbol{x}$.

When a continuum undergoes deformation (or flow), its points move along various paths in space. This motion may be expressed as a function of $\boldsymbol{X}$ and t as

$$\boldsymbol{x}(\boldsymbol{X}, t) = \boldsymbol{u}(\boldsymbol{X}, t) + \boldsymbol{X}(t), \tag{2.2}$$

which gives the present location of a point at time t, written in terms of the referential coordinates X_1, X_2, X_3. The previous position vector may be interpreted as a mapping of the initial configuration onto the current configuration. In classical approaches, it is assumed that such a mapping is one-to-one and continuous, with continuous partial derivatives to whatever order is required. The description of motion or deformation expressed previously is known as the Lagrangian formulation. Alternatively, if the independent variables are the coordinates $\boldsymbol{x}$ and time t, then $\boldsymbol{x}(x_1, x_2, x_3, t) = \boldsymbol{u}(x_1, x_2, x_3, t) + \boldsymbol{X}(x_1, x_2, x_3, t)$, and the formulation is denoted as Eulerian (Fig. 2.1).[1]

[1] Frequently, analysts consider the referential configuration to be fixed in time; thus, in that case it is not a function of time, $\boldsymbol{X} \neq \boldsymbol{X}(t)$. We shall consider $\boldsymbol{X} \neq \boldsymbol{X}(t)$ for the remainder of the monograph.

2.2.1 Deformation of Line Elements

Partial differentiation of the displacement vector $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$, with respect to $\boldsymbol{X}$, produces the following displacement gradient:

$$\nabla_X \boldsymbol{u} = \boldsymbol{F} - \mathbf{1}, \tag{2.3}$$

where

$$\boldsymbol{F} \stackrel{\text{def}}{=} \nabla_X \boldsymbol{x} \stackrel{\text{def}}{=} \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}. \tag{2.4}$$

$\boldsymbol{F}$ is known as the material deformation gradient.

Now, consider the length of a differential element in the reference configuration $d\boldsymbol{X}$ and $d\boldsymbol{x}$ in the current configuration, $d\boldsymbol{x} = \nabla_X \boldsymbol{x} \cdot d\boldsymbol{X} = \boldsymbol{F} \cdot d\boldsymbol{X}$. Taking the difference in the squared magnitudes of these elements yields

$$\begin{aligned} d\boldsymbol{x} \cdot d\boldsymbol{x} - d\boldsymbol{X} \cdot d\boldsymbol{X} &= (\nabla_X \boldsymbol{x} \cdot d\boldsymbol{X}) \cdot (\nabla_X \boldsymbol{x} \cdot d\boldsymbol{X}) - d\boldsymbol{X} \cdot d\boldsymbol{X} \\ &= d\boldsymbol{X} \cdot (\boldsymbol{F}^T \cdot \boldsymbol{F} - \mathbf{1}) \cdot d\boldsymbol{X} \stackrel{\text{def}}{=} 2\, d\boldsymbol{X} \cdot \boldsymbol{E} \cdot d\boldsymbol{X}. \end{aligned} \tag{2.5}$$

Equation (2.5) defines the so-called *Lagrangian* strain tensor

$$\boldsymbol{E} \stackrel{\text{def}}{=} \tfrac{1}{2}(\boldsymbol{F}^T \cdot \boldsymbol{F} - \mathbf{1}) = \tfrac{1}{2}[\nabla_X \boldsymbol{u} + (\nabla_X \boldsymbol{u})^T + (\nabla_X \boldsymbol{u})^T \cdot \nabla_X \boldsymbol{u}]. \tag{2.6}$$

Remark 1 It should be clear that $d\boldsymbol{x}$ can be reinterpreted as the result of a mapping $\boldsymbol{F} \cdot d\boldsymbol{X} \rightarrow d\boldsymbol{x}$ or a change in configuration (reference to current). An important quantity is the Jacobian of the deformation gradient, $J \stackrel{\text{def}}{=} det\, \boldsymbol{F}$, which relates differential volumes in the reference configuration ($d\Omega_0$) to differential volumetric domains in the current configuration ($d\Omega$) via $d\Omega = J\, d\Omega_0$. The Jacobian of the deformation gradient must remain positive, otherwise we obtain physically impossible "negative" volumes. One way is compare the differential volume of mutually orthogonal triad of differential vectors in the reference configuration $d\boldsymbol{X}^{(1)}$, $d\boldsymbol{X}^{(2)}$, $d\boldsymbol{X}^{(3)}$ (forming the edges of a cube), given by the triple product $d\boldsymbol{X}^{(1)} \cdot (d\boldsymbol{X}^{(2)} \times d\boldsymbol{X}^{(3)})$ to the volume of the mapping of the triad $d\boldsymbol{x}^{(1)} = \boldsymbol{F} \cdot d\boldsymbol{X}^{(1)}$, $d\boldsymbol{x}^{(2)} = \boldsymbol{F} \cdot d\boldsymbol{X}^{(2)}$, $d\boldsymbol{x}^{(3)} = \boldsymbol{F} \cdot d\boldsymbol{X}^{(3)}$, given by $d\boldsymbol{x}^{(1)} \cdot (d\boldsymbol{x}^{(2)} \times d\boldsymbol{x}^{(3)})$. Another way to prove this is by formulating a conservation of mass over an arbitrary volume within the domain

$$\int_{\omega_o} \rho_o\, d\omega_o = \int_{\omega} \rho\, d\omega = \int_{\omega_o} \rho J\, d\omega_o, \tag{2.7}$$

which immediately leads to the conclusion that $\rho_o = \rho J$, since ω_o is arbitrary. For more details, we refer the reader to the texts of Malvern [2], Gurtin [3], Chandrasekharaiah and Debnath [4], and Zohdi and Wriggers [1].

Remark 2 One may develop so-called Eulerian formulations, employing the current configuration coordinates to generate Eulerian strain tensor measures (see Zohdi and Wriggers [1]).

2.3 Equilibrium/Kinetics of Continua

The balance of linear momentum in the deformed (current) configuration is

$$\underbrace{\int_{\partial\omega} \boldsymbol{t}\, da}_{\text{surface forces}} + \underbrace{\int_{\omega} \rho \boldsymbol{b}\, d\omega}_{\text{body forces}} = \underbrace{\frac{d}{dt}\int_{\omega} \rho \dot{\boldsymbol{u}}\, d\omega}_{\text{inertial forces}}, \tag{2.8}$$

where $\omega \subset \Omega$ is an arbitrary portion of the body (Fig. 2.1), with boundary $\partial\omega$, ρ is the material density, $\boldsymbol{b}$ is the body force per unit mass, and $\dot{\boldsymbol{u}}$ is the time derivative of the displacement. The surface force densities, $\boldsymbol{t}$, are commonly referred to as "tractions."

2.3.1 Postulates on Volume and Surface Quantities

Now, consider a tetrahedron (commonly referred to as a Cauchy tetrahedron) in equilibrium, as shown in Fig. 2.2, where a balance of forces yields

$$\boldsymbol{t}^{(n)}\Delta A^{(n)} + \boldsymbol{t}^{(-1)}\Delta A^{(1)} + \boldsymbol{t}^{(-2)}\Delta A^{(2)} + \boldsymbol{t}^{(-3)}\Delta A^{(3)} + \rho \boldsymbol{b}\Delta V = \rho \ddot{\boldsymbol{u}}\Delta V, \tag{2.9}$$

where $\Delta A^{(n)}$ is the surface area of the face of the tetrahedron with normal $\boldsymbol{n}$ and ΔV is the tetrahedron volume. As the distance (h) between the tetrahedron

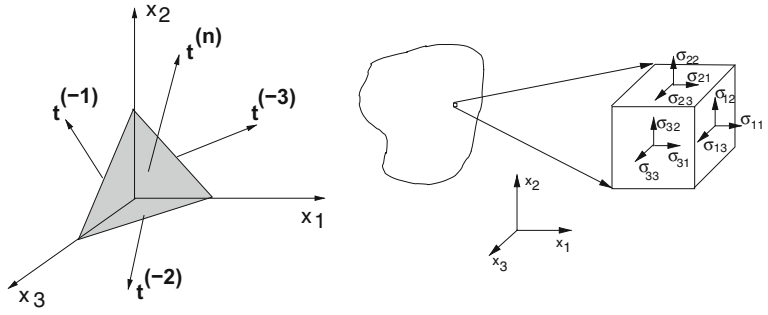

Fig. 2.2 Left: Cauchy tetrahedron: a "sectioned point" and right: stress at a point

base (located at (0, 0, 0)) and the surface center goes to zero ($h \to 0$), we have $\Delta A^{(n)} \to 0 \Rightarrow \frac{\Delta V}{\Delta A^{(n)}} \to 0$. Geometrically, we have $\frac{\Delta A^{(i)}}{\Delta A^{(n)}} = cos(x_i, x_n) \stackrel{\text{def}}{=} n_i$, and therefore $\boldsymbol{t}^{(n)} + \boldsymbol{t}^{(-1)} cos(x_1, x_n) + \boldsymbol{t}^{(-2)} cos(x_2, x_n) + \boldsymbol{t}^{(-3)} cos(x_3, x_n) = \boldsymbol{0}$, where (x_i, x_n) indicates the angle between the x_i and x_n directions. It is clear that forces on the surface areas could be decomposed into three linearly independent, mutually orthogonal, components. It is convenient to introduce the concept of stress at a point, representing the surface forces (tractions) there, pictorially represented by a cube surrounding a point. The fundamental issue that must be resolved is the characterization of these surface forces. We can represent the traction on a surface by the component representation:

$$\boldsymbol{t}^{(i)} \stackrel{\text{def}}{=} \begin{Bmatrix} \sigma_{i1} \\ \sigma_{i2} \\ \sigma_{i3} \end{Bmatrix}, \tag{2.10}$$

where the second index represents the direction of the component and the first index represents components of the normal to corresponding coordinate plane. We have $\boldsymbol{t}^{(n)} = \boldsymbol{\sigma}^T \cdot \boldsymbol{n}$, where

$$\boldsymbol{\sigma} \stackrel{\text{def}}{=} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, \tag{2.11}$$

or explicitly ($\boldsymbol{t}^{(1)} = -\boldsymbol{t}^{(-1)}$, $\boldsymbol{t}^{(2)} = -\boldsymbol{t}^{(-2)}$, $\boldsymbol{t}^{(3)} = -\boldsymbol{t}^{(-3)}$)

$$\boldsymbol{t}^{(n)} = \boldsymbol{t}^{(1)} n_1 + \boldsymbol{t}^{(2)} n_2 + \boldsymbol{t}^{(3)} n_3 = \boldsymbol{\sigma}^T \cdot \boldsymbol{n} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}^T \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}, \tag{2.12}$$

where $\boldsymbol{\sigma}$ is the so-called Cauchy stress tensor. Henceforth, we will drop the superscript notation of $\boldsymbol{t}^{(n)}$, where it is implicitly assumed that $\boldsymbol{t} \stackrel{\text{def}}{=} \boldsymbol{t}^{(n)}$.

Remark In the absence of micromoment stresses, a balance of angular momentum implies a symmetry of stress, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, and thus, the difference in notations becomes immaterial. Explicitly, starting with an angular momentum balance, under the assumptions that no infinitesimal "micromoments" or so-called couple-stresses exist, then it can be shown that the stress tensor must be symmetric,[2] i.e., $\int_{\partial\omega} \boldsymbol{x} \times \boldsymbol{t}\, da + \int_\omega \boldsymbol{x} \times \rho \boldsymbol{b}\, d\omega = \frac{d}{dt} \int_\omega \boldsymbol{x} \times \rho \dot{\boldsymbol{u}}\, d\omega$; that is, $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$.

[2]It is somewhat easier to simply consider a differential element, such as in Fig. 2.2, and to simply sum moments about the center. Doing this, one immediately obtains $\sigma_{12} = \sigma_{21}, \sigma_{23} = \sigma_{32}$ and $\sigma_{13} = \sigma_{31}$. Consequently, $\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{\sigma}^T \cdot \boldsymbol{n}$.

2.3.2 Balance Law Formulations

Substitution of Eq. 2.12 into Eq. 2.8 yields ($\omega \subset \Omega$)

$$\underbrace{\int_{\partial\omega} \boldsymbol{\sigma} \cdot \boldsymbol{n}\, da}_{\text{surface forces}} + \underbrace{\int_{\omega} \rho \boldsymbol{b}\, d\omega}_{\text{body forces}} = \underbrace{\frac{d}{dt} \int_{\omega} \rho \dot{\boldsymbol{u}}\, d\omega}_{\text{inertial forces}}. \tag{2.13}$$

A relationship can be determined between the densities in the current and reference configurations, $\int_{\omega} \rho d\omega = \int_{\omega_0} \rho J d\omega_0 = \int_{\omega_0} \rho_0 d\omega_0$. Therefore, the Jacobian can also be interpreted as the ratio of material densities at a point. Since the volume is arbitrary, we can assume that $\rho J = \rho_0$ holds at every point in the body. Therefore, we may write $\frac{d}{dt}(\rho_0) = \frac{d}{dt}(\rho J) = 0$, when the system is mass conservative over time. This leads to writing the last term in Eq. 2.13 as $\frac{d}{dt}\int_{\omega} \rho \dot{\boldsymbol{u}}\, d\omega = \int_{\omega_0} \frac{d(\rho J)}{dt} \dot{\boldsymbol{u}}\, d\omega_0 + \int_{\omega_0} \rho \ddot{\boldsymbol{u}} J\, d\omega_0 = \int_{\omega} \rho \ddot{\boldsymbol{u}}\, d\omega$. From Gauss's divergence theorem, and an implicit assumption that $\boldsymbol{\sigma}$ is differentiable, we have $\int_{\omega} (\nabla_x \cdot \boldsymbol{\sigma} + \rho \boldsymbol{b} - \rho \ddot{\boldsymbol{u}})\, d\omega = \boldsymbol{0}$. If the volume is selected as being arbitrary, then the integrand must be equal to zero at every point, yielding

$$\nabla_x \cdot \boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho \ddot{\boldsymbol{u}}. \tag{2.14}$$

2.4 The First Law of Thermodynamics/An Energy Balance

The interconversions of mechanical, thermal, and chemical energy in a system are governed by the first law of thermodynamics, which states that the time rate of change of the total energy, $\mathcal{K} + \mathcal{I}$, is equal to the rate of input of energy $\dot{W}$. Specifically, we can relate the kinetic and potential energy states at two instances of time by

$$\mathcal{K}(t) + \mathcal{I}(t) + \Delta W = \mathcal{K}(t + \Delta t) + \mathcal{I}(t + \Delta t) \tag{2.15}$$

or, as $\Delta t \to 0$,

$$\frac{d}{dt}(\mathcal{K} + \mathcal{I}) = \dot{W} = \mathcal{P} + \mathcal{H} + \mathcal{Q}, \tag{2.16}$$

where the mechanical power is $\mathcal{P}$ and the net heat supplied from sources and conduction is $\mathcal{Q} + \mathcal{H}$. Here, the kinetic energy of a subvolume of material contained in Ω, denoted ω, is

$$\mathcal{K} \overset{\text{def}}{=} \int_{\omega} \frac{1}{2} \rho \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}}\, d\omega, \tag{2.17}$$

the power (rate of work) of the external forces acting on ω is given by

$$\mathcal{P} \stackrel{\text{def}}{=} \int_{\omega} \rho \boldsymbol{b} \cdot \dot{\boldsymbol{u}}\, d\omega + \int_{\partial\omega} \boldsymbol{\sigma} \cdot \boldsymbol{n} \cdot \dot{\boldsymbol{u}}\, da, \tag{2.18}$$

the heat flow into the volume by conduction is

$$\mathcal{Q} \stackrel{\text{def}}{=} -\int_{\partial\omega} \boldsymbol{q} \cdot \boldsymbol{n}\, da = -\int_{\omega} \nabla_x \cdot \boldsymbol{q}\, d\omega, \tag{2.19}$$

$\boldsymbol{q}$ being the heat flux, the heat generated due to sources, *such as chemical reactions*, is

$$\mathcal{H} \stackrel{\text{def}}{=} \int_{\omega} \rho z\, d\omega, \tag{2.20}$$

where z is the reaction source rate per unit mass, and the internal energy is

$$\mathcal{I} \stackrel{\text{def}}{=} \int_{\omega} \rho w\, d\omega, \tag{2.21}$$

where w being the internal energy per unit mass. Differentiating the kinetic energy yields

$$\begin{aligned} \frac{d\mathcal{K}}{dt} = \frac{d}{dt}\int_{\omega} \frac{1}{2}\rho\dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}}\, d\omega &= \int_{\omega_0} \frac{d}{dt}\frac{1}{2}(\rho J \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}})\, d\omega_0 \\ &= \int_{\omega_0} (\frac{d}{dt}\rho_0)\frac{1}{2}\dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}}\, d\omega_0 + \int_{\omega} \rho \frac{d}{dt}\frac{1}{2}(\dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}})\, d\omega \\ &= \int_{\omega} \rho\dot{\boldsymbol{u}} \cdot \ddot{\boldsymbol{u}}\, d\omega, \end{aligned} \tag{2.22}$$

where we have assumed that the mass in the system is constant. We also have

$$\frac{d\mathcal{I}}{dt} = \frac{d}{dt}\int_{\omega} \rho w\, d\omega = \frac{d}{dt}\int_{\omega_0} \rho J w\, d\omega_0 = \int_{\omega_0} \underbrace{\frac{d}{dt}(\rho_0)}_{=0} w\, d\omega_0 + \int_{\omega} \rho\dot{w}\, d\omega = \int_{\omega} \rho\dot{w}\, d\omega. \tag{2.23}$$

By using the divergence theorem, we obtain

$$\int_{\partial\omega} \boldsymbol{\sigma} \cdot \boldsymbol{n} \cdot \dot{\boldsymbol{u}}\, da = \int_{\omega} \nabla_x \cdot (\boldsymbol{\sigma} \cdot \dot{\boldsymbol{u}})\, d\omega = \int_{\omega} (\nabla_x \cdot \boldsymbol{\sigma}) \cdot \dot{\boldsymbol{u}}\, d\omega + \int_{\omega} \boldsymbol{\sigma} : \nabla_x \dot{\boldsymbol{u}}\, d\omega. \tag{2.24}$$

Combining the results, and enforcing a balance of linear momentum, leads to

$$\begin{aligned} &\int_{\omega} (\rho\dot{w} + \dot{\boldsymbol{u}} \cdot (\rho\ddot{\boldsymbol{u}} - \nabla_x \cdot \boldsymbol{\sigma} - \rho\boldsymbol{b}) - \boldsymbol{\sigma} : \nabla_x \dot{\boldsymbol{u}} + \nabla_x \cdot \boldsymbol{q} - \rho z)\, d\omega = \\ &\qquad \int_{\omega} (\rho\dot{w} - \boldsymbol{\sigma} : \nabla_x \dot{\boldsymbol{u}} + \nabla_x \cdot \boldsymbol{q} - \rho z)\, d\omega = 0. \end{aligned} \tag{2.25}$$

Since the volume ω is arbitrary, the integrand must hold locally and we have

$$\rho\dot{w} - \boldsymbol{\sigma} : \nabla_x \dot{\boldsymbol{u}} + \nabla_x \cdot \boldsymbol{q} - \rho z = 0. \tag{2.26}$$

When dealing with multifield problems, this equation is used extensively.

2.5 Linearly Elastic Constitutive Equations

We now discuss relationships between the stress and the strain, so-called *material laws* or *constitutive relations* for linearly elastic cases (infinitesimal deformations).

2.5.1 The Infinitesimal Strain Case

In infinitesimal deformation theory, the displacement gradient components are considered small enough that higher-order terms like $(\nabla_X \boldsymbol{u})^T \cdot \nabla_X \boldsymbol{u}$ and $(\nabla_x \boldsymbol{u})^T \cdot \nabla_x \boldsymbol{u}$ can be neglected in the strain measure $\boldsymbol{E} = \frac{1}{2}(\nabla_X \boldsymbol{u} + (\nabla_X \boldsymbol{u})^T + (\nabla_X \boldsymbol{u})^T \cdot \nabla_X \boldsymbol{u})$, leading to $\boldsymbol{E} \approx \boldsymbol{\epsilon} \stackrel{\text{def}}{=} \frac{1}{2}[\nabla_X \boldsymbol{u} + (\nabla_X \boldsymbol{u})^T]$. If the displacement gradients are small compared with unity, $\boldsymbol{\epsilon}$ coincides closely to $\boldsymbol{E}$. If we assume $\frac{\partial}{\partial \boldsymbol{X}} \approx \frac{\partial}{\partial \boldsymbol{x}}$, we may use $\boldsymbol{E}$ or $\boldsymbol{\epsilon}$ interchangeably. Usually $\boldsymbol{\epsilon}$ is the symbol used for infinitesimal strains. Furthermore, to avoid confusion, when using models employing the geometrically linear infinitesimal strain assumption, we use the symbol of ∇ with no $\boldsymbol{X}$ or $\boldsymbol{x}$ subscript. Hence, the infinitesimal strains are defined by

$$\boldsymbol{\epsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T). \tag{2.27}$$

2.5.2 Material Response

If we neglect thermal effects, Eq. 2.26 implies $\rho\dot{w} = \boldsymbol{\sigma} : \nabla_x \dot{\boldsymbol{u}}$ which, in the infinitesimal strain linearly elastic case, is $\rho\dot{w} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}$. From the chain rule of differentiation, we have

$$\rho\dot{w} = \rho \frac{\partial w}{\partial \boldsymbol{\epsilon}} : \frac{d\boldsymbol{\epsilon}}{dt} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} \Rightarrow \boldsymbol{\sigma} = \rho \frac{\partial w}{\partial \boldsymbol{\epsilon}}. \tag{2.28}$$

The starting point to develop a constitutive theory is to assume a stored elastic energy function exists, a function denoted $W \stackrel{\text{def}}{=} \rho w$, which depends only on the mechanical deformation. The simplest function that fulfills $\boldsymbol{\sigma} = \rho \frac{\partial w}{\partial \boldsymbol{\epsilon}}$ is $W = \frac{1}{2}\boldsymbol{\epsilon} : \mathbb{E} : \boldsymbol{\epsilon}$, where $\mathbb{E}$ is the fourth rank elasticity tensor. Such a function satisfies the intuitive physical requirement that, for any small strain from an undeformed state, energy must be stored in the material. Subsequently, a small strain material law can be

derived from $\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\epsilon}}$ and $W \approx c_0 + \mathbf{c}_1 : \boldsymbol{\epsilon} + \frac{1}{2}\boldsymbol{\epsilon} : \mathbb{E} : \boldsymbol{\epsilon} + \ldots$ which implies $\boldsymbol{\sigma} \approx \mathbf{c}_1 + \mathbb{E} : \boldsymbol{\epsilon} + \ldots$. We are free to set $c_0 = 0$ (it is arbitrary) in order to have zero strain energy at zero strain, and furthermore, we assume that no stresses exist in the reference state ($\mathbf{c}_1 = \mathbf{0}$). With these assumptions, we obtain the familiar relation

$$\boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\epsilon}. \tag{2.29}$$

This is a linear relation between stresses and strains. The existence of a strictly positive stored energy function in the reference configuration implies that the linear elasticity tensor must have positive eigenvalues at every point in the body. Typically, different materials are classified according to the number of independent components in $\mathbb{E}$. In theory, $\mathbb{E}$ has 81 components, since it is a fourth-order tensor relating nine components of stress to strain. However, the number of components can be reduced to 36 since the stress and strain tensors are symmetric. This is observed from the matrix representation[3] of $\mathbb{E}$:

$$\underbrace{\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix}}_{\stackrel{\text{def}}{=}\{\boldsymbol{\sigma}\}} = \underbrace{\begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1112} & E_{1123} & E_{1113} \\ E_{2211} & E_{2222} & E_{2233} & E_{2212} & E_{2223} & E_{2213} \\ E_{3311} & E_{3322} & E_{3333} & E_{3312} & E_{3323} & E_{3313} \\ E_{1211} & E_{1222} & E_{1233} & E_{1212} & E_{1223} & E_{1213} \\ E_{2311} & E_{2322} & E_{2333} & E_{2312} & E_{2323} & E_{2313} \\ E_{1311} & E_{1322} & E_{1333} & E_{1312} & E_{1323} & E_{1313} \end{bmatrix}}_{\stackrel{\text{def}}{=}[\mathbb{E}]} \underbrace{\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \end{Bmatrix}}_{\stackrel{\text{def}}{=}\{\boldsymbol{\epsilon}\}}. \tag{2.30}$$

The existence of a scalar energy function forces $\mathbb{E}$ to be symmetric since the strains are symmetric; in other words, $W = \frac{1}{2}\boldsymbol{\epsilon} : \mathbb{E} : \boldsymbol{\epsilon} = \frac{1}{2}(\boldsymbol{\epsilon} : \mathbb{E} : \boldsymbol{\epsilon})^T = \frac{1}{2}\boldsymbol{\epsilon}^T : \mathbb{E}^T : \boldsymbol{\epsilon}^T = \frac{1}{2}\boldsymbol{\epsilon} : \mathbb{E}^T : \boldsymbol{\epsilon}$ which implies $\mathbb{E}^T = \mathbb{E}$. Consequently, $\mathbb{E}$ has only 21 independent components. The nonnegativity of W imposes the restriction that $\mathbb{E}$ remains positive definite. At this point, based on many factors that depend on the material microstructure, it can be shown that the components of $\mathbb{E}$ may be written in terms of anywhere between 21 and 2 independent parameters. Accordingly, for isotropic materials, we have two planes of symmetry and an infinite number of planes of directional independence (two free components), yielding

$$\mathbb{E} \stackrel{\text{def}}{=} \begin{bmatrix} \kappa + \frac{4}{3}\mu & \kappa - \frac{2}{3}\mu & \kappa - \frac{2}{3}\mu & 0 & 0 & 0 \\ \kappa - \frac{2}{3}\mu & \kappa + \frac{4}{3}\mu & \kappa - \frac{2}{3}\mu & 0 & 0 & 0 \\ \kappa - \frac{2}{3}\mu & \kappa - \frac{2}{3}\mu & \kappa + \frac{4}{3}\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}. \tag{2.31}$$

[3]The symbol $[\cdot]$ is used to indicate the matrix notation equivalent to a tensor form, while $\{\cdot\}$ is used to indicate the vector representation.

In this case, we have

$$\boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\epsilon} = 3\kappa \frac{tr\boldsymbol{\epsilon}}{3}\mathbf{1} + 2\mu\boldsymbol{\epsilon}' \Rightarrow W = \frac{1}{2}\boldsymbol{\epsilon} : \mathbb{E} : \boldsymbol{\epsilon} = \frac{9}{2}\kappa(\frac{tr\boldsymbol{\epsilon}}{3})^2 + \mu\boldsymbol{\epsilon}' : \boldsymbol{\epsilon}', \tag{2.32}$$

where $tr\boldsymbol{\epsilon} = \epsilon_{ii}$ and $\boldsymbol{\epsilon}' = \boldsymbol{\epsilon} - \frac{1}{3}(tr\boldsymbol{\epsilon})\mathbf{1}$ is the deviatoric strain. The eigenvalues of an isotropic elasticity tensor are $(3\kappa, 2\mu, 2\mu, \mu, \mu, \mu)$. Therefore, we must have $\kappa > 0$ and $\mu > 0$ to retain positive definiteness of $\mathbb{E}$. All of the material components of $\mathbb{E}$ may be spatially variable, as in the case of composite particulate-functionalized media.

2.5.3 Material Component Interpretation

There are a variety of ways to write isotropic constitutive laws, each time with a physically meaningful pair of material values.

Splitting the Strain

It is sometimes important to split infinitesimal strains into two physically meaningful parts

$$\boldsymbol{\epsilon} = \frac{tr\boldsymbol{\epsilon}}{3}\mathbf{1} + (\boldsymbol{\epsilon} - \frac{tr\boldsymbol{\epsilon}}{3}\mathbf{1}). \tag{2.33}$$

An expansion of the Jacobian of the deformation gradient yields $J = det(\mathbf{1} + \nabla_X \boldsymbol{u}) \approx 1 + tr\nabla_X \boldsymbol{u} + \mathcal{O}(\nabla_X \boldsymbol{u}) = 1 + tr\boldsymbol{\epsilon} + \ldots$. Therefore, with infinitesimal strains, $(1 + tr\boldsymbol{\epsilon})d\omega_0 = d\omega$ and we can write $tr\boldsymbol{\epsilon} = \frac{d\omega - d\omega_0}{d\omega_0}$. Hence, $tr\boldsymbol{\epsilon}$ is associated with the *volumetric part of the deformation*. Furthermore, since $\boldsymbol{\epsilon}' \overset{\text{def}}{=} \boldsymbol{\epsilon} - \frac{tr\boldsymbol{\epsilon}}{3}\mathbf{1}$, the so-called strain deviator describes distortion in the material.

Infinitesimal Strain Material Laws

The stress $\boldsymbol{\sigma}$ can be split into two parts (dilatational and a deviatoric):

$$\boldsymbol{\sigma} = \frac{tr\boldsymbol{\sigma}}{3}\mathbf{1} + (\boldsymbol{\sigma} - \frac{tr\boldsymbol{\sigma}}{3}\mathbf{1}) \overset{\text{def}}{=} -p\mathbf{1} + \boldsymbol{\sigma}', \tag{2.34}$$

where we call the symbol p the hydrostatic pressure and $\boldsymbol{\sigma}'$ the stress deviator. With (2.32), we write

$$p = -3\,\kappa \left(\frac{tr\boldsymbol{\epsilon}}{3}\right) \qquad \text{and} \quad \boldsymbol{\sigma}' = 2\,\mu\,\boldsymbol{\epsilon}'. \tag{2.35}$$

This is one form of Hooke's law. The resistance to change in the volume is measured by κ. We note that $(\frac{tr\boldsymbol{\sigma}}{3}\mathbf{1})' = \mathbf{0}$, which indicates that this part of the stress produces no distortion.

Another fundamental form of Hooke's law is

$$\boldsymbol{\sigma} = \frac{E^y}{1+\nu}\left(\boldsymbol{\epsilon} + \frac{\nu}{1-2\nu}(tr\boldsymbol{\epsilon})\mathbf{1}\right), \tag{2.36}$$

and the inverse form is

$$\boldsymbol{\epsilon} = \frac{1+\nu}{E^y}\boldsymbol{\sigma} - \frac{\nu}{E^y}(tr\boldsymbol{\sigma})\mathbf{1}, \tag{2.37}$$

where E^y, the Young's modulus, is the ratio of the uniaxial stress to the corresponding strain component and the Poisson ratio, ν, is the ratio of the transverse strains to the uniaxial strain. To interpret the material values, consider an idealized uniaxial tension test (pulled in the x_1-direction inducing a uniform stress state) where $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$, which implies $\epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0$. Also, we have $\sigma_{22} = \sigma_{33} = 0$. Under these conditions, we have $\sigma_{11} = E^y\epsilon_{11}$ (axial stiffness) and $\epsilon_{22} = \epsilon_{33} = -\nu\epsilon_{11}$ (the ratio of transverse to axial strain).

Another commonly used set of stress–strain forms are the Lamé relations,

$$\boldsymbol{\sigma} = \lambda(tr\boldsymbol{\epsilon})\mathbf{1} + 2\mu\boldsymbol{\epsilon} \quad \text{or} \quad \boldsymbol{\epsilon} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}(tr\boldsymbol{\sigma})\mathbf{1} + \frac{\boldsymbol{\sigma}}{2\mu}, \tag{2.38}$$

where λ is referred to a the Lame parameter. To interpret the material values, consider a homogeneous pressure test (uniform stress) where $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$ and where $\sigma_{11} = \sigma_{22} = \sigma_{33}$. Under these conditions, we have

$$\kappa = \lambda + \frac{2}{3}\mu = \frac{E^y}{3(1-2\nu)} \quad \text{and} \quad \mu = \frac{E^y}{2(1+\nu)}, \tag{2.39}$$

and consequently

$$\frac{\kappa}{\mu} = \frac{2(1+\nu)}{3(1-2\nu)}. \tag{2.40}$$

We observe that $\frac{\kappa}{\mu} \to \infty$ implies $\nu \to \frac{1}{2}$, and $\frac{\kappa}{\mu} \to 0$ implies $\Rightarrow \nu \to -1$. Therefore, since both κ and μ must be positive and finite, this implies $-1 < \nu < 1/2$ and $0 < E^y < \infty$. For example, some polymeric foams exhibit $\nu < 0$, steels $\nu \approx 0.3$, and some forms of rubber have $\nu \to 1/2$. We note that λ *can be positive or negative.* For more details, see Malvern [2], Gurtin [3], Chandrasekharaiah and Debnath [4].

Remark See Zohdi and Wriggers [1] for a variety of different finite-deformation constitutive laws.

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