Chapter 2
Stability Analysis of Time-Delay Systems

This chapter deals with the stability analysis of linear time-delay systems without and with parametric uncertainties. The stability analysis for both constant and time-varying delay in the states is considered. The focus of this chapter is to review the existing methods on delay-dependent stability analysis in an LMI framework based on Lyapunov-Krasovskii approach and consequently the improved results on delay-dependent stability analysis are presented. The results of the proposed techniques are validated by considering numerical examples and compared with existing results.

2.1 Introduction

Time-delays are often observed in many areas of engineering systems such as networked control systems, chemical processes, neural networks, milling process, nuclear reactors and long transmission lines in power systems and their presence can have effect on system stability and performance [1], so ignoring them can lead to design flaws and incorrect stability analysis. In particular the effect of delays become more pronounced in interconnected and distributed system where multiple sensors, actuators and controller introduce multiple delays. Thus stability analysis becomes the prime objective in a control system design. The stability analysis of time-delay systems using Lyapunov's second method are broadly classified into three major categories

- Delay-independent stability analysis
- Delay-dependent stability analysis
- Delay-range-dependent stability analysis

Delay-independent stability analysis considers the size of the delay to be arbitrarily large (delay value$\rightarrow \infty$) and hence the obtained stability conditions are independent of the delay value. Delay-independent stability results are, in general,

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more conservative for many important applications and especially for engineering (or physical) systems [2]. So, in early 1990s increasing attention has been devoted to delay-dependent stability analysis, which considers finite (bounded) delay value, thus in this case derived stability conditions depends on the size of the delay. For both the delay-independent and delay-dependent analysis the lower bound of the delay is assumed to be zero, whereas for the first case the delay upper bound is unbounded and for the second case, it is bounded to some finite value. Very recently [3–5], another variant of delay-dependent stability analysis has been proposed where the information of the delay ranges are available i.e., the lower bound of the delay is not assumed to be explicitly zero but can possess some finite value and the delay upper value is bounded as in the case of delay-dependent stability analysis, such stability analysis is referred as delay-range-dependent stability analysis. In this chapter, we discuss delay-dependent as well as delay-range-dependent stability analysis as they are of physical significance.

The stability analysis has been carried out for (i) nominal time-delay systems i.e., systems without parametric uncertainties and (ii) uncertain time-delay systems i.e., systems possessing uncertainties in the system matrices, the stability analysis of such time-delay systems is referred as robust stability analysis. The structure of the uncertainty is assumed to be of norm-bounded type. The stability analysis of time-delay systems can be carried out using either

- Lyapunov-Razumikhin theorem
- Lyapunov-Krasovskii (LK) theorem

All the recent literature on stability analysis of time-delay systems adopts latter method as the former method yields conservative estimate of delay upper bound compared to LK theorem [6–9] because of the following reasons:

1. The use of the condition $V(t+\theta, x(t+\theta)) \leq pV(t, x(t)), \forall\theta \in [-d, 0]$ in Lyapunov-Razumikhin theorem.
2. More number of bounding inequalities are used while deriving delay-dependent stability conditions, as in this case Lyapunov function is assumed to be very simple one i.e., $V(t) = x^T(t)Px(t)$.
3. Lyapunov-Razumikhin theorem is incapable of handling slow time-varying delay (i.e., bounded differentiable time-varying delay which implies delay-derivative <1), it can treat fast time-varying delays (i.e., non-differentiable time-varying delay that implies delay derivative ≥ 1) and constant delays.

In this thesis the attention is focused on the delay-dependent stability analysis of time-delay systems using Lyapunov-Krasovskii theorem.

2.2 Description of Time-Delay Systems

In this section, description of time-delay systems for carrying out stability as well as robust stability analysis is presented.

2.2.1 Nominal Time-Delay System for Stability Analysis

1. **System with single, constant delay**

$$\Sigma_1 : \dot{x}(t) = Ax(t) + A_d x(t-d) \tag{2.1}$$

$$x(t) = \phi(t), \forall t \in [-d_u, 0] \tag{2.2}$$

2. **System with single, time-varying delay**

$$\Sigma_2 : \dot{x}(t) = Ax(t) + A_d x(t-d(t)) \tag{2.3}$$

$$x(t) = \phi(t), \forall t \in [-d_u, 0] \tag{2.4}$$

where, $d(t)$ is time-varying delay satisfying following conditions

$$0 \le d(t) \le d_u \tag{2.5}$$

$$0 \le d_l \ \le \ d(t) \ \le \ d_u \tag{2.6}$$

Note: The condition (2.5) refers to delay-dependent stability (DDS) notion and (2.6) refers to delay-range-dependent stability (DRDS) notion.
the delay derivative satisfies the condition

$$\dot{d}(t) \le \ \mu \ < \ 1 \tag{2.7}$$

$$1 \le \dot{d}(t) \ \le \ \mu \ \le \ \infty \tag{2.8}$$

Note: The condition (2.7) refers to slowly varying time-delay and (2.8) refers to fast varying time-delay [9] and [10].

Notations: $x(t) \in \mathcal{R}^n$ is the state vector, $\phi(t)$ is the initial function in the banach (norm linear space) space, $d(t)$ is the time-varying delay, d_u is the delay upper bound and d_l is the delay lower bound, $A \in \mathcal{R}^{n\times n}$ and $A_d \in \mathcal{R}^{n\times n}$ are known constant matrices.

3. **System with multiple time-varying delays**

$$\Sigma_3 : \dot{x}(t) = Ax(t) + \sum_{i=1}^{n} A_{di} x(t-d_i(t)) \tag{2.9}$$

$$x(t) = \phi(t), \forall t \in \max[-d_{ui}, 0], i = 1, 2 \ldots n \tag{2.10}$$

Note: Here, $d_i(t)$ indicates the time-varying delay in the states and A_{di} is the associated delayed system matrices. The stability analysis of the system Σ_2 can be extended to multiple time-delay case of Σ_3 in a straight forward manner [11] and [8].

4. **System with two additive time-varying delays**

$$\Sigma_4 : \dot{x}(t) = Ax(t) + A_d x(t - d_1(t) - d_2(t)) \tag{2.11}$$

$$x(t) = \phi(t), \forall t \in \max[-d_u, 0], \tag{2.12}$$

where, $d_1(t)$ and $d_2(t)$ are the two additive delay component in the state, they satisfy following conditions

$$\begin{aligned} 0 \leq d_1(t) &\leq d_{1u} < \infty \\ 0 \leq d_2(t) &\leq d_{2u} < \infty \\ d_u &= d_{1u} + d_{2u} \end{aligned} \tag{2.13}$$

and delay derivatives satisfies the following conditions,

$$\begin{aligned} \dot{d}_1(t) &\leq \mu_1 < \infty \\ \dot{d}_2(t) &\leq \mu_2 < \infty \end{aligned} \tag{2.14}$$

2.2.2 *Uncertain Time-Delay Systems for Robust Stability Analysis*

The uncertain time-delay systems with norm-bounded parametric uncertainties for robust stability analysis is described as

$$\dot{x}(t) = A(t)x(t) + A_d(t)x(t - d(t)) \tag{2.15}$$

$$x(t) = \phi(t), \forall t \in [-d_u, 0] \tag{2.16}$$

The matrices $A(t)$ and $A_d(t)$ are uncertain system matrices and are assumed to be of the form:

$$A(t) = A + \Delta A(t) \tag{2.17}$$

$$A_d(t) = A_d + \Delta A_d(t) \tag{2.18}$$

where, A and A_d are nominal system matrices and $\Delta A(t)$ and $\Delta A_d(t)$ are time-varying matrices, which models the parametric uncertainties present in the system that are Lebesgue measurable and are norm bounded [7, 12–15]. Further these may possibly be decomposed by exploiting their structural description as

$$\Delta A(t) = D_a F_a(t) E_a \tag{2.19}$$

$$\Delta A_d(t) = D_d F_d(t) E_d \tag{2.20}$$

where the time-varying uncertain matrices $F_a(t) \in \mathcal{R}^{n_a \times n_a}$ and $F_d(t) \in \mathcal{R}^{n_d \times n_d}$ are norm bounded and satisfies $\forall\ t$ that,

$$F_a^T F_a(t) \leq I \tag{2.21}$$
$$F_d^T F_d(t) \leq I \tag{2.22}$$

The matrices D_a, E_a, D_d and E_d in (2.19) and (2.20) are constant known matrices and possibly characterizes how the matrices $F_a(t)$ and $F_d(t)$ influence the system dynamics.

2.3 Delay-Dependent Stability Condition

This section brings out the review of some significant existing LMI techniques in deriving delay-dependent stability conditions based on Lyapunov-Krasovskii functional approach. This review is useful to understand the evolution and development of improved techniques while attempting to achieve less conservative estimate of delay upper bound.

Assumption 2.1 The necessary condition for delay-dependent stability of time-delay systems in Sect. 2.2.1 is that, the matrix $[A + A_d]$ (when $d = 0$, $d(t) = 0$) must be Hurwitz.

2.3.1 Model Transformation Approach (Based on Newton-Leibniz Formula)

In this section, we review the delay-dependent stability conditions that are derived using fixed model transformation. The systems Σ_1 and Σ_2 with time-delays are transformed into systems with distributed delays using Newton-Leibniz formula for the analysis. The Newton-Leibniz formula is expressed as,

$$\int_{t-d}^{t} \dot{x}(s)ds = x(t) - x(t-d)$$
$$x(t-d) = x(t) - \int_{t-d}^{t} \dot{x}(s)ds \tag{2.23}$$

A. First Model Transformation

Using (2.23) in Σ_1 one can write

$$\dot{x}(t) = (A + A_d)x(t) - A_d \int_{t-d}^{t} [Ax(s) + A_d x(s-d)]ds \tag{2.24}$$

The transformed system obtained in (2.24) is called first model transformation, the asymptotic stability of (2.24) also guarantees the stability of the system Σ_1 [8]. Based on the transformed systems, a lot of delay-dependent stability results have been obtained.

Theorem 2.1 (Theorem 7 [8], Cor. 1 [16]) *The transformed system (2.24) is asymptotically stable for any delay satisfying* $0 \leq d \leq d_u$ *if there exist matrices* $P > 0$, $Q_1 > 0$ *and* $Q_2 > 0$ *such that,*

$$0 > \begin{bmatrix} \Delta & d_u\ PA_d & d_u\ PA_d \\ \star & -Q_1 & 0 \\ \star & 0 & -Q_2 \end{bmatrix} \tag{2.25}$$

where, $\Delta = (A + A_d)P + P(A + A_d)^T + d_u\ (A^T Q_1 A + d\ A_d^T Q_2 A_d)$.

Proof Lyapunov-Krasovskii functional chosen is given by

$$\begin{aligned} V(t) = x^T(t)Px(t) &+ \int_{-d_u}^{0}\int_{t+\theta}^{t} x^T(\beta)A^T Q_1 Ax(\beta)d\beta d\theta \\ &+ \int_{-d_u}^{0}\int_{t-d+\theta}^{t} x^T(\beta)A_d^T Q_2 A_d x(\beta)d\beta d\theta \end{aligned} \tag{2.26}$$

Finding the time-derivative of (2.26) one can obtain

$$\begin{aligned} \dot{V}(t) = {} & x^T[(A + A_d)^T P + P(A + A_d) + d_u\ (A_d^T Q_2 A_d + A^T Q_1 A)]x(t) \\ & -2\int_{t-d_u}^{t} x^T(t)PA_d Ax(\beta)d\beta - 2\int_{t-d_u}^{t} x^T(t)PA_d A_d x(\beta - d)d\beta \\ & -\int_{t-d_u}^{t} x^T(\beta)A^T Q_1 Ax(\beta)d\beta - \int_{t-d_u}^{t} x^T(\beta - d)A_d^T Q_2 A_d x(\beta - d)d\beta \end{aligned} \tag{2.27}$$

In (2.27) two cross terms $-2\int_{t-d_u}^{t} x^T(t)PA_d Ax(\beta)d\beta$ and $-2\int_{t-d_u}^{t} x^T(t)$ $PA_d Ax(\beta - d)d\beta$ appears that are approximated using the bounding Lemma stated below,

Lemma 2.1 ([2, 8, 16]) *For any* $z, y \in \mathcal{R}^n$ *and any positive definite matrix* $X \in \mathcal{R}^{n\times n}$

$$-2z^T y \leq z^T X^{-1} z + y^T X y \tag{2.28}$$

Now, using Lemma 2.1 in (2.27) one can obtain

$$\begin{aligned}\dot{V}(t) \leq\ & x^T[(A+A_d)^T P + P(A+A_d) + d_u\,(A_d^T Q_2 A_d + A^T Q_1 A)]x(t) \\ & + \int_{t-d_u}^{t} x^T(t)PA_d Q_1^{-1} A_d^T Px(t)d\beta + \int_{t-d_u}^{t} x^T(\beta)A^T Q_1 Ax(\beta)d\beta \\ & - \int_{t-d_u}^{t} x^T(\beta)A^T Q_1 Ax(\beta)d\beta - \int_{t-d_u}^{t} x^T(\beta-d)A_d^T Q_2 A_d x(\beta-d)d\beta \\ & + \int_{t-d_u}^{t} x^T(t)PA_d Q_2^{-1} A_d^T Px(t)d\beta + \int_{t-d_u}^{t} x^T(\beta-d)A_d^T Q_2 A_d x(\beta-d)d\beta \end{aligned} \quad (2.29)$$

One can observe in (2.29) that, the terms arising out after using bounding lemma will compensate for the last two integral terms in (2.27), thus yielding quadratic Lyapunov inequality in the form of LMI given in (2.25).

Remark 2.1 It is obvious from the above derivation that, the choice of LK functional for this method leads to two cross bounding terms which are approximated using bounding Lemma 2.1. If this theorem has to be extended for multiple delay case (say m delays) then the number of times bounding lemma have to be used will be '2 m'. Hence, more will be the cross bounding terms present in the LK functional derivative, the use of bounding lemma for its approximation will be more, which is a major source of conservativism in the estimate of delay bound results.

Remark 2.2 One can find the similar choice of LK functional for delay-dependent stability analysis using first model transformation for the systems Σ_1 and Σ_2 in [17] and [18] respectively.

The choice of LK functional in [17] is found to be,

$$V(t) = x^T(t)Px(t) + V_2(t) + V_3(t) \quad (2.30)$$

where,

$$\begin{aligned} V_2(t) &= \int_{-d_u}^{0}\int_{t+\theta}^{t} x^T(s)M_1 x(s)dsd\theta \\ V_3(t) &= \int_{-2d_u}^{-d}\int_{t+\theta}^{t} x^T(s)M_2 x(s)dsd\theta,\ M_1 > 0,\ M_2 > 0 \end{aligned}$$

the time-derivative of (2.30) is found to be

$$\dot{V}(t) = x^T(t)[(A+A_d)^T P + P(A+A_d)]x(t) + \dot{V}_2(t) + \dot{V}_3(t) + \xi_1(t) + \xi_2(t) \quad (2.31)$$

where,

$$\xi_1(t) \triangleq -2\int_{t-d_u}^{t} x^T(t)PA_d Ax(\alpha)d\alpha \quad (2.32)$$

and

$$\xi_2(t) \triangleq -2 \int_{t-d_u}^{t} x^T(t) P A_d A_d x(\alpha - d) d\alpha \tag{2.33}$$

To approximate the cross terms $\xi_1(t)$ and $\xi_2(t)$ bounding Lemma 2.1 is used, remaining integral terms arising out of $\dot{V_2}(t)$ and $\dot{V_3}(t)$ is canceled by the integral terms that appears after the use of Lemma 2.1 with the assumption that, $M_1 = A^T X_1 A$ and $M_2 = A_d^T X_2 A_d$, where X_1 and X_2 are positive definite matrices thus giving a quadratic LMI formulation.

In case of the system Σ_2 (time-varying delay) satisfying the conditions (2.5), the stability condition in [18] is obtained using first model transformation with the similar choice of LK functional given as

$$\begin{aligned} V(t) &= x^T(t) P x(t) + d_u \int_{-d_u}^{0} \int_{t+\theta}^{t} x^T(s) M_1 x(s) ds d\theta \\ &+ \frac{d_u}{(1-\mu)^2} \int_{-d(t)-d_u}^{-d(t)} \int_{t+\theta}^{t} x^T(s) M_2 x(s) ds d\theta, \ M_1 > 0, M_2 > 0 \end{aligned} \tag{2.34}$$

The factor $\frac{d_u}{(1-\mu)^2}$ associated in the second term of (2.34) is used to compensate for the derivative of the delay term (i.e., $\dot{d}(t)$) which arises upon differentiation of $V(t)$ due to presence of time-varying delay term in the limit of integration in (2.34).

Remark 2.3 While deriving stability condition using first model transformation, the number of cross terms to be approximated using bounding lemma is twice the number of delays present in the system Σ_1 or Σ_2, secondly, it is proved in [9, 19, 20] that the first model transformation introduces some additional eigenvalues in the transformed system, hence the characteristics of the transformed system is not equivalent to the original one (i.e., Σ_1 or Σ_2), thus the stability condition derived using this transformation yields conservative result of delay upper bound. In other words, the drawbacks associated with this approach is that all of the transformed system is not equivalent to (2.1) or (2.3).

B. Second model transformation [9, 17, 21]

The rearrangement of first model transformation in (2.24) yields the second model transformation (or neutral type transformation) and it is expressed as

$$\frac{d}{dt}[x(t) + A_d \int_{t-d}^{t} x(s) ds] = (A + A_d) x(t) \tag{2.35}$$

Using second model transformation the delay-dependent stability condition obtained in [21] for the system Σ_1 is presented in the form of following theorem.

Theorem 2.2 ([21]) *The system Σ_1 is asymptotically stable for any delay d_u, if the operator $\mathcal{D}(x_t)$ is stable and there exists symmetric and positive-definite matrices P and Q such that following LMI holds:*

$$\begin{bmatrix} (A + A_d)^T P + P(A + A_d) + d_u Q & d_u(A + A_d)^T P A_d \\ \star & -d_u Q \end{bmatrix} < 0 \tag{2.36}$$

The choice of LK functional candidate for this transformed model (2.35) is of the form (as in [21]),

$$V(t) = \mathcal{D}^T(x_t)P\mathcal{D}(x_t) + \int_{-d_u}^{0}\int_{t+\theta}^{t} x^T(s)Qx(s)dsd\theta \tag{2.37}$$

where, $P = P^T > 0$, $Q = Q^T > 0$ and $\mathcal{D}(x_t) = x(t) + A_d \int_{t-d_u}^{t} x(s)ds$. Finding the time-derivative of (2.37) one can obtain,

$$\begin{aligned} \dot{V}(t) &= \dot{\mathcal{D}}^T(x_t)P\mathcal{D}(x_t) + \mathcal{D}^T(x_t)P\dot{\mathcal{D}}(x_t) \\ &\quad + d_u x^T(t)Qx(t) - \int_{t-d_u}^{t} x^T(s)Qx(s)ds \end{aligned} \tag{2.38}$$

where,

$$\dot{\mathcal{D}}^T(x_t) = \dot{x}(t) + A_d \int_{t-d_u}^{t} \dot{x}(s)ds$$

Substituting the value of $\dot{\mathcal{D}}^T(x_t)$ and $\mathcal{D}^T(x_t)$ in (2.38) and carrying out algebraic manipulations one can get,

$$\begin{aligned} \dot{V}(t) &= x^T(t)(A + A_d)^T Px(t) + x^T(t)P(A + A_d)x(t) + d_u x^T(t)Qx(t) \\ &\quad + 2\int_{t-d_u}^{t} x^T(t)(A + A_d)^T PA_d x(s) - \int_{t-d_u}^{t} x^T(s)Qs(s)ds \end{aligned} \tag{2.39}$$

Applying bounding Lemma 2.1 on the cross term of (2.39) one can get,

$$\begin{aligned} \dot{V}(t) &\le x^T(t)(A + A_d)^T Px(t) + x^T(t)P(A + A_d)x(t) + d_u x^T(t)Qx(t) \\ &\quad + \int_{t-d_u}^{t} x^T(s)Qs(s)ds - \int_{t-d_u}^{t} x^T(s)Qs(s)ds \\ &\quad + \int_{t-d_u}^{t} x^T(t)(A + A_d)^T PA_d Q^{-1} PA_d(A + A_d)x(t)ds \end{aligned} \tag{2.40}$$

After algebraic simplification and using Schur-complement [22] on (2.40) one can get,

$$\dot{V}(t) \le x^T(t)\begin{bmatrix} (A + A_d)^T P + P(A + A_d) + d_u Q & d_u(A + A_d)^T P A_d \\ \star & -d_u Q \end{bmatrix} x(t) \tag{2.41}$$

The negativity of $\dot{V}(t)$ in (2.41) is not sufficient to guarantee the stability of the transformed system, further it is required to assure the stability of the $\mathcal{D}(x_t)$ also. The stability of $\mathcal{D}(x_t)$ is carried out using frequency domain analysis (refer Remark 14 in [21]), thus this stability analysis yields one more additional sufficient condition on $\mathcal{D}(x_t)$ which is given as,

$$d_u \parallel A_d \parallel < 1$$

Remark 2.4 Applying second transformation on Σ_1 and choosing the LK function in (2.37) for the transformed system, it is found that the derivative of this functional yields only one cross term for system Σ_1 thus it has an advantage of approximating half the number of cross terms using bounding Lemma 2.1 compared to the first model transformation. Whereas the additional constraint introduced to guarantee the stability of $\mathcal{D}(x_t)$ results into conservative estimate of the delay bound. Furthermore, this transformation is not suitable for time-varying delay, as frequency domain stability analysis is adopted for $\mathcal{D}(x_t)$ which is a complicated task for systems with differentiable time-varying delays.

C. Third model transformation

Replacing the value of $x(t-d)$ for Σ_1 and $x(t-d(t))$ for Σ_2 using Newton-Liebniz formula (2.23) into (2.1) and (2.3) respectively, one can get,

$$\dot{x}(t) = (A + A_d)x(t) - A_d \int_{t-d}^{t} \dot{x}(s)ds \tag{2.42}$$

The model expressed in (2.42) is called third model transformation. To derive the sufficient delay-dependent stability condition using (2.42) the choice of Lyapunov-Krasovskii functional is of the following form [10, 23],

$$\begin{aligned} V(t) = x^T(t)Px(t) &+ \int_{t-d}^{t} x^T(s)Nx(s)ds \\ &+ \int_{-d_u}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)A_d^T M A_d \dot{x}(s)dsd\theta \end{aligned} \tag{2.43}$$

The present author have considered stability analysis for system Σ_2 using third model transformation [6] and bounding Lemma 2.1 by selecting LK functional of the type (2.43) (assumed in [23]) satisfying the condition (2.5) to investigate the conservatism of the different model transformations. Similar results are also available in the literature and the results of delay upper bound for different model transformations are presented in Tables 2.1 and 2.2. The stability condition derived in [6] is presented in the form of following theorem.

Theorem 2.3 ([6]) *If there exist $P = P^T > 0$, $Q_1 = Q_1^T > 0$ and $Q_2 = Q_2^T > 0$, such that the following LMI holds,*

$$\phi = \begin{bmatrix} \phi_{11} & 0 & d_u A^T Q_2 & d_u P A_d \\ 0 & -(1-\mu) Q_1 & d_u A_d^T Q_2 & 0 \\ \star & \star & -d_u Q_2 & 0 \\ \star & 0 & 0 & -d_u Q_2 \end{bmatrix} < 0 \tag{2.44}$$

where, $\phi_{11} = P(A + A_d) + (A + A_d)^T P + Q_1$, then the system Σ_2 is asymptotically stable.

Proof We choose LK functional candidate as

$$V(t) = V_1(t) + V_2(t) \tag{2.45}$$

where,

$$\begin{aligned} V_1(t) &= x^T(t) P x(t) \\ V_2(t) &= \int_{t-d(t)}^{t} x^T(s) Q_1 x(s) ds + \int_{-d_u}^{0} \int_{t+\alpha}^{t} \dot{x}(s)^T Q_2 \dot{x}(s) ds d\alpha \end{aligned}$$

Taking time-derivative of (2.45), substituting $\dot{x}(t)$ from (2.42) in $\dot{V}_1(t)$ and approximating the quadratic integral term as,

$$-\int_{t-d_u}^{t} \dot{x}^T(s) Q_2 \dot{x}(s) ds \le -\int_{t-d(t)}^{t} \dot{x}^T(s) Q_2 \dot{x}(s) ds$$

one can obtain $\dot{V}(t)$ as,

$$\begin{aligned} \dot{V}(t) \le\ & 2x^T(t) P(A + A_d) x(t) - 2x^T(t) P A_d \int_{t-d(t)}^{t} \dot{x}(s) ds \\ & + x^T(t) Q_1 x(t) - (1-\mu) x^T(t - d(t)) Q_1 x(t - d(t)) + d_u \dot{x}^T(t) Q_2 \dot{x}(t) \\ & - \int_{t-d(t)}^{t} \dot{x}^T(s) Q_2 \dot{x}(s) ds \end{aligned} \tag{2.46}$$

Applying Lemma 2.1 in (2.46) the cross terms are approximated, and $\dot{x}(t)$ term in $\dot{V}_2(t)$ is substituted with (2.3). Further, algebraic manipulations and use of Schur-complement [24] will lead to,

$$\dot{V}(t) \le \xi^T(t) \phi \xi(t) \tag{2.47}$$

where, $\xi(t)$ is an augmented state vector, i.e., $\xi(t) = \left[\, x^T(t)\ x^T(t - d(t)) \,\right]^T$

Now, to guarantee the asymptotic stability of the system Σ_2, the matrix $\phi < 0$ and ϕ is an LMI defined in (2.44).

Remark 2.5 The third model transformation introduced in [10] and [23] transforms the original system Σ_1 or Σ_2 into system with distributed delay which is equivalent to the original system owing to the fact that, the integration of state dynamics is retained unlike in first model transformation [9]. This transformation is suitable to treat the systems Σ_1 and Σ_2. The additional conservatism arises due to quadratic stability condition formed by considering the state space vector $x(t)$ and $x(t-d)$ independently in the augmented state vector $\zeta(t)$. The substitution of $\dot{x}(t)$ term in (2.46) is carried out using (2.3) and not by (2.42) which is also the source of conservativeness in the stability analysis.

Numerical Example 2.1 ([19]) *Consider the system Σ_1 or Σ_2 with the following constant matrices*

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

The eigenvalues of the matrix $[A + A_d]$ are Hurwitz and the eigenvalues of the matrix $[A - A_d]$ are unstable, thus the given system is delay-dependently stable (i.e., the system is asymptotically stable for certain finite delay value, refer Sect. 1.1.4). The analytical delay upper bound for this system is $d_u = 6.1726$ for $\mu = 0$ [14, 21].

Numerical Example 2.2 ([19]) *Consider the system Σ_2 with the following constant matrices*

$$A = \begin{bmatrix} -6 & 0 \\ 0.2 & -5.8 \end{bmatrix}, A_d = \begin{bmatrix} 0 & 4 \\ -8 & -8 \end{bmatrix}$$

The eigenvalues of the matrix $[A + A_d]$ and $[A - A_d]$ are Hurwitz, the given system is delay-independently stable (i.e., the system is asymptotically stable for arbitrary large delay value, refer Sect. 1.1.4).

Remark 2.6 In [19], it has been shown that the system Σ_1 after being transformed using first model transformation has the characteristic equation of the form $\Delta_t(s) = \Delta_{add}(s)\Delta_{or}(s)$, where $\Delta_{add}(s) = det(I - \frac{1-e^{-ds}}{s} A_d)$, thus indicating that the transformed system contains additional eigenvalues which depends on the delayed matrix and delay size. The presence of these additional eigenvalues makes the stability of the transformed system different from original system when the eigenvalues of the A_d matrix are (i) complex conjugate and (ii) positive real, as in both the cases for a small positive delay the additional eigenvalues will reach the imaginary axis before the original system, but this is not the case for the eigenvalues of A_d matrix being negative real. The degree of conservatism could be better understood by observing the result for Numerical Example 2.2 presented in Table 2.2, as the original system is delay-independently stable but the transformed system is found to be stable up to certain finite value only.

The delay value results obtained using different model transformations and bounding Lemma 2.1 depicts that the third model transformation has the advantage for obtaining better estimate of delay value for different delay derivatives ($0 < \mu < 1$) over the other two transformations.

Table 2.1 Delay upper bound (d_u) results of Example 2.1

Stability methods	$\mu = 0$	$\mu = 0.5$	$\mu = 0.9$	Model Transformation
[16]	0.9999	–	–	First model transformation
[17]	0.9999	–	–	First model transformation
[18]	0.9999	0.6551	0.1743	First model transformation
[21]	0.9999	NA	NA	Second model transformation
[6]	0.9999	0.8210	0.4677	Third model transformation

Table 2.2 Delay upper bound (d_u) results of Example 2.2

Stability methods	$\mu = 0$	$\mu = 0.5$	$\mu = 0.9$	Model Transformation
[16]	0.1514	–	–	First model transformation
[18]	0.1514	0.1016	0.0280	First model transformation
[21]	0.1639	NA	NA	Second model transformation
[6]	0.3891	0.3548	0.2712	Third model transformation

It can be concluded from the above discussion and results (presented in Tables 2.1 and 2.2) that, all the transformed systems (first, second and third) discussed above are not equivalent to the original system Σ_1 and/or Σ_2 as all the transformations possesses additional eigenvalues due to the distributed delayed term and this becomes the main reason for the conservatism in estimating delay bound when the eigenvalues of A_d matrix are present on imaginary or positive real axis. Next, the conservatism in the estimate of delay value due to the adopted bounding Lemma is discussed.

2.3.2 Bounding Techniques

The main purpose of the delay-dependent stability studies of time-delay systems is to find sufficient LMI conditions that can estimate less conservative delay upper bound compared to the existing stability methods using bounding techniques [8]. The stability methods discussed so far utilized bounding Lemma 2.1 for approximating the cross terms arising out of the LK functional derivative. It is validated in [23] that the use of better tighter bounding inequality to represent the cross term arising out of LK functional derivative in the stability analysis can play a key role in reducing conservatism. An improved bounding inequality lemma proposed in [23] is presented below.

Lemma 2.2 (Park's Bounding Lemma [23]) *Assume that $a(\alpha) \in \mathcal{R}^{n_a}$, and $b(\alpha) \in \mathcal{R}^{n_b}$, are given for $\alpha \in \Omega$. Then, for any positive definite matrix $X \in \mathcal{R}^{n_a \times n_a}$ and any matrix $M \in \mathcal{R}^{n_b \times n_b}$, the following inequality holds*

$$-2\int_{\Omega} b^T(\alpha)a(\alpha)d\alpha \leq \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \begin{bmatrix} X & XM \\ \star & (2,2) \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} \tag{2.48}$$

where, (2, 2)= $(M^T X + I)X^{-1}(XM + I)$.
Then by using this inequality, an improved delay-dependent stability has been reported in [23]. The derived sufficient condition in [23] is restated as:

Theorem 2.4 ([23]) *If there exist $P > 0$, $Q > 0$, $V > 0$ and $W > 0$ then the system defined in Σ_1, satisfying the condition (2.2) is asymptotically stable if the following LMI holds,*

$$\begin{bmatrix} (1,1) & -W^T A_d & A^T A_d^T V & (1,4) \\ \star & Q & A_d^T A_d^T V & 0 \\ \star & \star & -V & 0 \\ \star & 0 & 0 & -V \end{bmatrix} < 0 \tag{2.49}$$

where, (1, 1)=$(A + A_d)^T P + P(A + A_d)+W^T A_d + A_d^T W + Q$
(1, 4)= $d_u(W^T + P)$, $W = XMP$, and $V = d_u X$

The stability Theorem 2.4 has been derived using third model transformation. The selection of LK functional is same as (2.45) except that the positive definite matrix corresponding to delay-dependent LK functional term is taken as $A_d^T X A_d$ instead of $Q_2 = Q_2^T > 0$. The cross terms that evolve from LK functional derivative is approximated using bounding Lemma 2.2.

Remark 2.7 The use of bounding Lemma 2.2 in deriving the delay-dependent stability condition in Theorem 2.4 resulted into significant increase in the delay upper bound estimate in an LMI framework. The result obtained for the system considered in Example 2.1 using Theorem 2.4 is $d_u = 4.3588$ for $\mu = 0$ which is a significant improvement in comparison to all the previous delay upper bound results of $d_u = 0.9999$ for $\mu = 0$ and also much closer to actual delay upper bound value of $d_u = 6.172$ for the considered system. This validates the fact that, approximation of cross terms by the bounding technique is one of the major source of conservatism in the delay-dependent stability analysis of TDS.

Another significance of delay-dependent stability Theorem 2.4 is that, for the system considered in Numerical Example 2.2 it could establish that the system is delay-independently stable as the delay upper bound estimate for this system turns out to be arbitrarily large. Thus one can be concluded that, the use of bounding Lemma 2.2 not only enhanced the delay upper bound estimate of delay-dependently stable system, but the delay-dependent stability condition (2.49) derived using this bounding lemma could even establish the delay-independent stability of the system in Numerical Example 2.2, this was not possible using third model transformation and bounding Lemma 2.1 as clear from the result presented in Table 2.2.

The generalization of bounding inequality Lemma 2.1 and Lemma 2.2 was proposed in [25] with an idea to provide a simple LMI structure of delay-dependent stability condition such that it can be easily extended to synthesis problems. The generalized bounding Lemma and delay-dependent stability condition obtained for Σ_1 are discussed below.

Lemma 2.3 (Moon's Bounding Lemma, [25]) *Assume that $a(\alpha) \in \mathcal{R}^{n_a}$, and $b(\alpha) \in \mathcal{R}^{n_b}$, and $\mathcal{N}(.) \in \mathcal{R}^{n_a \times n_b}$ are given for $\alpha \in \Omega$. Then, for any positive definite matrix $X \in \mathcal{R}^{n_a \times n_a}$, $Y \in \mathcal{R}^{n_a \times n_b}$ and any matrix $Z \in \mathcal{R}^{n_b \times n_b}$, the following inequality holds*

$$-2\int_{\Omega} a^T(\alpha)\mathcal{N}b(\alpha)d\alpha \leq \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \begin{bmatrix} X & Y-\mathcal{N} \\ \star & Z \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} \quad (2.50)$$

where,

$$\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \geq 0$$

Remark 2.8 The bounding Lemma 2.3 is more generalized bounding Lemma and one can obtain bounding Lemma 2.1 and bounding Lemma 2.2 from it with proper selection of matrices Y, Z and $\mathcal{N}$,

Case I Selecting $\mathcal{N} = I$, $Y = I$, and $Z = X^{-1}$ in (2.50) one can get bounding Lemma 2.1.

Case II Selecting $\mathcal{N} = I$, $Y = I + XM$, and $Z = (M^T X + I)X^{-1}(XM + I)$ in (2.50) one can get bounding Lemma 2.2.

The delay-dependent stability condition for the system Σ_1 is derived in [25] using the third model transformation selecting the LK functional as:

$$\begin{aligned} V(t) = x^T(t)Px(t) + \int_{t-d}^{t} x^T(s)Qx(s)ds \\ + \int_{-d_u}^{0}\int_{t+\alpha}^{t} \dot{x}^T(s)Z\dot{x}(s)dsd\alpha \end{aligned} \quad (2.51)$$

Taking the derivative of (2.51) along the trajectory of the transformed system and using the bounding Lemma 2.3, one can get the stability condition in an LMI framework [25]. The delay-dependent stability theorem is restated below.

Theorem 2.5 ([25]) *The system Σ_1 is asymptotically stable if there exist symmetric matrices $P > 0$, $Q > 0$, matrices $X > 0$, $Y > 0$ and any matrix Z for time-delay $d \in [-d_u, 0]$ such that following LMIs hold,*

$$\begin{bmatrix} A^T P + PA + d_u X + Y + Y^T + Q & -Y + PA_d & d_u A^T Z \\ \star & -Q & d_u A_d^T Z \\ \star & \star & -dZ \end{bmatrix} < 0 \quad (2.52)$$

$$\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \geq 0 \quad (2.53)$$

Remark 2.9 The estimate of the delay upper bound result using Theorem 2.5 is found to be $d_u = 4.3588$ for the system considered in Numerical Example 2.1, which is same as the result obtained through Theorem 2.4. The advantage of using bounding Lemma 2.3 in comparison to the use bounding Lemma 2.2 in deriving stability condition is that, the former bounding lemma results into simple LMI structure. This can easily be extended for solution of stabilization and robust stabilization problems.

From the above discussions, it is now clear that the conservatism in the delay-dependent stability analysis are due to (i) the presence of distributed delay term in the model transformation which in turn introduces additional dynamics into the transformed system and (ii) the use of bounding inequalities to approximate the cross terms. In an attempt to reduce the conservatism arising out of the model transformations discussed in Sect. 2.3.1 a new model transformation called descriptor system approach was introduced in [26].

2.3.3 Descriptor System Approach

This section discusses briefly the development and further modification of this method which are available in literature [9, 27–29]. In [26] and [9] the delay-dependent study was done on system Σ_1 (constant time-delay), whereas the method was extended to system Σ_2 (time-varying delays) in [28] and [9].

The model transformation of the original system into descriptor system with distributed delay for the system Σ_1 in (2.1) is discussed briefly below:

$$\begin{aligned} \dot{x}(t) &= y(t) \\ y(t) &= Ax(t) + A_d x(t-d) \end{aligned} \tag{2.54}$$

using Newton-Leibniz formula (2.23), the above equation (2.54) can be rewritten as

$$\begin{aligned} \dot{x}(t) &= y(t) \\ y(t) &= (A + A_d)x(t) - A_d \int_{t-d}^{t} y(s)ds \\ 0 &= -y(t) + (A + A_d)x(t) - A_d \int_{t-d}^{t} y(s)ds \end{aligned}$$

$$E\dot{\xi}(t) = \tilde{A}\xi(t) + \tilde{A}_d \int_{t-d}^{t} y(s)ds \tag{2.55}$$

where, $\xi(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ $\tilde{A} = \begin{bmatrix} 0 & I \\ A + A_d & -I \end{bmatrix}$ and $\tilde{A}_d = \begin{bmatrix} 0 \\ -A_d \end{bmatrix}$

The delay-dependent stability conditions for the transformed system (2.55) in [26] has been derived selecting the LK functional candidate as

$$V(t) = \xi^T(t)EP\xi(t) + \int_{-d_u}^{0}\int_{t+\theta}^{t} y^T(s)Ry(s)dsd\theta \tag{2.56}$$

where, $P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} \geq 0$ with $P_1 = P_1^T > 0$.

One can find from (2.56) that the LK functional corresponding to the delay-independent term (single integral term) is not present. Finding the time-derivative of the (2.56) and applying the bounding Lemma 2.1 to approximate the cross terms arising out of the LK functional derivative, one can obtain following LMI stability condition for the descriptor system (2.55) as

$$\begin{bmatrix} P_2^T(A + A_d) + (A + A_d)^T P_2 & P_1 - P_2^T + (A + A_d)^T P_3 & d_u P_2^T A_d \\ \star & -P_3 - P_3^T + d_u R & d_u P_3^T A_d \\ \star & \star & -d_u R \end{bmatrix} < 0 \tag{2.57}$$

In [9] the stability condition of the descriptor system (2.55) for time-varying delay satisfying conditions (2.5) and (2.7) was derived. The selection of the LK functional in this case is considered as,

$$V(t) = \xi^T(t)EP\xi(t) + \int_{t-d(t)}^{t} x^T(s)Sx(s)ds + \int_{-d_u}^{0}\int_{t+\theta}^{t} y^T(s)A_d^T RA_d y(s)ds \tag{2.58}$$

where, $\xi(t)$ and P are defined earlier.

The time-derivative of the (2.58) results into cross terms which is approximated using bounding Lemma 2.2. The introduction of functional corresponding to delay-independent term (single integral term) is for the application of bounding lemma. The resulting LMI stability condition is stated below in the form of theorem.

Theorem 2.6 ([9]) *The time-delay system Σ_2 satisfying (2.7) is asymptotically stable for any delay $d(t) \in [-d_u, 0]$ if there exist $P_1 = P_1^T > 0$, P_2, P_3, $R = R^T > 0$, $S = S^T > 0$, W_1 and W_2, such that the following LMI holds,*

$$\begin{bmatrix} (1,1) & (1,2) & d_u(W_1^T + P_2^T) & W_1^T A_d \\ \star & (2,2) & d_u(W_2^T + P_3^T) & W_2^T A_d \\ \star & \star & -d_u R & 0 \\ \star & \star & 0 & -S(1-\mu) \end{bmatrix} < 0 \tag{2.59}$$

where, $(1,1)$=$P_2^T(A + A_d) + (A + A_d)^T P_2 + A_d^T W_1 + W_1 A_d + S$,
$(1,2)$=$P_1 - P_2^T + (A + A_d)^T P_3 + W_2^T A_d$, $(2,2)$=$-P_3 - P_3^T + d_u A_d^T RA_d$,
W_1=RMP_2, and W_2=RMP_3

A modified version of Theorem 2.6 can be found in [28] by introducing the following modifications, (i) cross bounding Lemma 2.3 was used instead of Lemma 2.2, as this lemma leads to simple structure of LMI condition and (ii) the positive definite matrix associated with delay-dependent term in (2.58) is replaced by a symmetric positive-definite matrix R such that it is suitable for the use of bounding Lemma 2.3. The stability conditions derived is restated in the form of theorem.

Theorem 2.7 ([28]) *The time-delay system Σ_2 satisfying (2.7) is asymptotically stable for any delay $d(t) \in [-d_u, 0]$ if there exist $P_1 > 0$, P_2, P_3, $R = R^T > 0$, $S = S^T > 0$, Y_1, Y_2, Z_1, Z_2 and Z_3 such that the following LMIs hold:*

$$\begin{bmatrix} (1,1) & (1,2) & P_2^T A_d - Y_1^T \\ \star & -P_3 - P_3^T + d_u Z_3 + d_u R & P_3^T A_d - Y_2^T \\ \star & \star & -S(1-\mu) \end{bmatrix} < 0 \tag{2.60}$$

$$\begin{bmatrix} R & Y_1 & Y_2 \\ \star & Z_1 & Z_2 \\ \star & \star & Z_3 \end{bmatrix} \geq 0 \tag{2.61}$$

where, $(1,1) = P_2^T A + A^T P_2 + Y_1 + Y_1^T + S + d_u Z_1$, $(1,2) = P_1 - P_2^T + A^T P_3 + Y_2 + d_u Z_2$

Remark 2.10 Comparing the stability conditions in (2.59) and (2.60)–(2.61) it can be observed that (i) the dimension of main LMI in (2.60) is less compared to that in (2.59) and (ii) the structure of LMI is simpler in terms of product of the Lyapunov matrix variables with the system matrices. Due to these reasons, the extension of stability conditions in (2.60) could be extended easily for synthesis problem in [28].

Remark 2.11 The stability condition in (2.57) (derived using descriptor system and bounding Lemma 2.1) when tested on Example 2.1 gives delay upper bound of $d_u = 0.9999$ which is same as the result obtained using the condition derived in [6] (using third model transformation and bounding Lemma 2.1). But the stability conditions in Theorem 2.6 and Theorem 2.7 when tested on Example 2.1 yielded delay upper bound estimate of $d_u = 4.47$ which is less conservative result compared to the delay value of $d_u = 4.3588$ obtained in [23].

Thus from above discussions and results it can be concluded that, a delay-dependent stability conditions using LK functional requires (i) appropriate choice of LK functional [26] followed by appropriate bounding technique to bind the cross terms arising out of the LK functional derivative, in order to reduce the conservatism in the delay bound results.

The advantage of descriptor method is that, it can be easily extended for state feedback synthesis of controller as in [28] and [27], but the observation reveals that the dimension and structure of the LMI conditions are larger and complicated respectively due to the use of descriptor system instead of original system.

Recently in [29] a new generalized delay-dependent stability condition has been proposed for neutral time-delay system using Finsler's Lemma and the bounding

Lemma 2.3. First we present Finsler's Lemma and then the stability condition for time-delay systems Σ_2 using the method proposed in [29] is presented.

Lemma 2.4 (Finsler's Lemma [24, 29]) *The following statements hold* $x^T Qx + f(x) < 0, \forall \bar{B}x = 0, x \neq 0$, *where* $Q = Q^T$, $\bar{B} \in \mathcal{R}^{m\times n}$ *(such that rank* $(\bar{B}) = m < n$ *and* $f(x)$ *is a scalar function, if there exists matrix* $X \in \mathcal{R}^{n\times m}$, *such that*

$$x^T[Q + X\bar{B} + \bar{B}^T X^T]x + f(x) < 0, \forall x \neq 0$$

Theorem 2.8 ([29]) *System* Σ_2 *satisfying (2.7) is asymptotically stable for the delay* $d(t) \in [-d_u, 0]$ *if there exist* $P_1 = P_1^T > 0, S > 0, P_i, i = 2, 3, 4, Y_1, Y_2, Z_1, Z_2, Z_3$ *and* $R > 0$ *such that following LMIs hold:*

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) \\ \star & (2,2) & (2,3) \\ \star & \star & (3,3) \end{bmatrix} < 0 \tag{2.62}$$

$$\begin{bmatrix} R & Y_1 & Y_2 \\ \star & Z_1 & Z_2 \\ \star & \star & Z_3 \end{bmatrix} \geq 0 \tag{2.63}$$

where,(1, 1)=$P_2^T A + A^T P_2 + S + Y_1 + Y_1^T + d_u Z_1$,

(1, 2)=$P_1 - P_2^T + A^T P_3 + Y_2 + d_u Z_2$

(1, 3)=$A^T P_4 - Y_1^T + P_2^T A_d$, (2, 2)=$-P_3 - P_3^T + d_u R + d_u Z_3$

(2, 3)=$-P_4 - Y_2^T + P_3^T A_d$, and (3, 3)=$-(1-\mu)S + A_d^T P_4 + P_4^T A_d$

Proof The Lyapunov-Krasovskii functional candidate chosen is

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{2.64}$$

where, $V_1(t) = x^T(t)P_1x(t)$, $V_2(t) = \int_{-d_u}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta$, $V_3(t) = \int_{t-d(t)}^{t} x^T(s)Sx(s)ds$

Time-derivative of the (2.64) is

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) \tag{2.65}$$

$$\dot{V}_1(t) = \begin{bmatrix} x^T(t) & \dot{x}^T(t) \end{bmatrix} \begin{bmatrix} 0 & P_1 \\ P_1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \tag{2.66}$$

$$\dot{V}_2(t) \leq d_u \dot{x}^T(t) R \dot{x}(t) - \int_{t-d_u}^{t} \dot{x}^T(s) R \dot{x}(s) ds \tag{2.67}$$

$$\dot{V}_3(t) \leq x^T(t) S x(t) - (1-\mu) x^T(t-d(t)) S x(t-d(t)) \tag{2.68}$$

Defining, augmented state space vector as

$$\xi(t) \triangleq \begin{bmatrix} x(t) \\ \dot{x}(t) \\ x(t-d(t)) \end{bmatrix}$$

In terms of $\xi(t)$ we can express (2.65) as

$$\dot{V}(t) = \xi^T(t) \begin{bmatrix} 0 & P_1 & 0 \\ P_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi(t) + \dot{V}_2(t) + \dot{V}_3(t) < 0 \tag{2.69}$$

for $\forall\, \xi(t) \in \mathcal{R}^{3n}$ s.t. $[A, -I, A_d]\, \xi(t) = 0$, this is obtained from (2.3).

Applying Lemma 2.4 in (2.69) yields

$$\begin{aligned} 0 > \xi^T(t) \Bigg\{ & \begin{bmatrix} 0 & P_1 & 0 \\ P_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} P_2 & P_3 & P_4 \end{bmatrix}^T \begin{bmatrix} A & -I & A_d \end{bmatrix} \\ & \begin{bmatrix} A & -I & A_d \end{bmatrix}^T \begin{bmatrix} P_2 & P_3 & P_4 \end{bmatrix} \Bigg\} \xi(t) + \dot{V}_2(t) + \dot{V}_3(t) \end{aligned} \tag{2.70}$$

Substituting the values of $\dot{V}_2(t)$ and $\dot{V}_3(t)$ from (2.67) and (2.68) respectively into (2.70) one can write

$$\begin{aligned} 0 > \xi^T(t) \Bigg\{ & \begin{bmatrix} P_2^T A + A^T P_2 + S & P_1 - P_2^T + A^T P_3 & P_2^T A_d + A^T P_4 \\ \star & -P_3^T - P_3 + d_u R & P_3^T A_d - P_4 \\ \star & \star & -(1-\mu)S + P_4^T A_d + A_d^T P_4 \end{bmatrix} \\ & + \begin{bmatrix} 0 & 0 & P_2^T A_d \\ 0 & 0 & P_3^T A_d \\ \star & \star & 0 \end{bmatrix} \Bigg\} \xi(t) - \int_{t-d_u}^{t} \dot{x}^T(s) R \dot{x}(s) ds \end{aligned} \tag{2.71}$$

Define the last two terms of (2.71) as,

$$\mu(t) \triangleq \xi^T(t) \begin{bmatrix} 0 & 0 & P_2^T A_d \\ 0 & 0 & P_3^T A_d \\ \star & \star & 0 \end{bmatrix} \xi(t) - \int_{t-d_u}^{t} \dot{x}^T(s) R \dot{x}(s) ds$$

and carrying out the algebraic manipulations with Lemma 2.3,

$$\begin{aligned}\mu(t) = \xi^T(t)\begin{bmatrix} P_2 & P_3 & 0\end{bmatrix}^T\begin{bmatrix} 0 & 0 & A_d\end{bmatrix}\begin{bmatrix} x^T(t) & \dot{x}^T(t) & x^T(t-d(t))\end{bmatrix}^T \\ +\begin{bmatrix} x^T(t) & \dot{x}^T(t) & x^T(t-d(t))\end{bmatrix}\begin{bmatrix} 0 & 0 & A_d\end{bmatrix}^T\begin{bmatrix} P_2 & P_3 & 0\end{bmatrix}\xi(t) \\ -\int_{t-d_u}^{t}\dot{x}^T(s)R\dot{x}(s)ds\end{aligned}$$

$$\begin{aligned}\mu(t) = \xi^T(t)\begin{bmatrix} P_2 & P_3 & 0\end{bmatrix}^T A_d x(t-d(t)) + x^T(t-d(t))A_d^T\begin{bmatrix} P_2 & P_3 & 0\end{bmatrix}\xi(t) \\ -\int_{t-d_u}^{t}\dot{x}^T(s)R\dot{x}(s)ds\end{aligned} \tag{2.72}$$

Using Newton-Leibniz formula defined in (2.23) on (2.72), one can obtain

$$\begin{aligned}\mu(t) = 2\xi^T(t)\begin{bmatrix} P_2 & P_3 & 0\end{bmatrix}^T A_d x(t) - 2\int_{t-d(t)}^{t}\xi^T(t)\begin{bmatrix} P_2 & P_3 & 0\end{bmatrix}^T A_d\dot{x}(s)ds \\ -\int_{t-d_u}^{t}\dot{x}^T(s)R\dot{x}(s)ds\end{aligned} \tag{2.73}$$

Applying bounding Lemma 2.3 on (2.73) one can get,

$$\begin{aligned}\mu(t) \le 2x^T(t)Y\begin{bmatrix} x(t) \\ \dot{x}(t)\end{bmatrix} + d_u\begin{bmatrix} x^T(t) & \dot{x}^T(t)\end{bmatrix}Z\begin{bmatrix} x(t) \\ \dot{x}(t)\end{bmatrix} \\ -2x^T(t-d(t))Y\begin{bmatrix} x(t) \\ \dot{x}(t)\end{bmatrix} + 2x^T(t-d(t))A_d^T\begin{bmatrix} P_2 & P_3\end{bmatrix}\begin{bmatrix} x(t) \\ \dot{x}(t)\end{bmatrix}\end{aligned} \tag{2.74}$$

where, $Y = \begin{bmatrix} Y_1 & Y_2\end{bmatrix}$ and $Z = \begin{bmatrix} Z_1 & Z_2 \\ \star & Z_3\end{bmatrix}$.
Substituting $\mu(t)$ from (2.74) into (2.71) and carrying out further algebraic manipulations, one can easily obtain

$$\dot{V}(t) \le \xi^T(t)\Xi\xi(t) \tag{2.75}$$

where matrix Ξ is defined in (2.62). For asymptotic stability of the system Σ_2, the matrix $\Xi < 0$, an additional LMI (2.63) appears in the stability formulation due to adoption of bounding Lemma 2.3.

Remark 2.12 One can observe that the LMI structure in (2.62) and (2.60) are similar except that in the former condition few additional terms consisting of a free matrix P_4 are involved. Now, if one chooses $P_4 = 0$ in (2.62) then one can obtain the condition (2.60) and thus the condition (2.62) is a generalized one. This generalization of LMI was possible due to the use of Finsler's lemma, but in turn introduces additional free matrix variable (in this case P_4).

When stability conditions in (2.62)–(2.63) are tested on Numerical Example 2.2 the delay upper bound obtained is $d_u = 4.4721$ which is same as the result obtained with Theorem 2.7, thus indicating that the use of additional matrix variable P_4 is redundant here.

2.3.4 Free Weighting Matrix Approach

Recently it was pointed out in [13] and [30] that, the derivative of the LK functional that contained $x(t - d(t))$ term was replaced with $x(t) - \int_{t-d_u}^{t} \dot{x}(s)ds$ (due to third model transformation in [23] and [25]) for obtaining the quadratic stability condition whereas the term $d_u\dot{x}^T(t)Z\dot{x}(t)$ in the LK derivative was not replaced by Newton-Leibniz formula rather $\dot{x}(t)$ was substituted with (2.3) (as in [25]), thus the replacements are not done uniformly everywhere in the formulation. In free weighting matrix method, the term $\dot{x}(t)$ is treated as one of the state in augmented state space vector and the relationship among the terms $x(t)$, $x(t - d(t))$ and $\dot{x}(t)$ are expressed using Newton-Leibniz formula by introducing some free matrices. Free weighting matrix method proposed in [30] plays an important role in deriving delay-dependent stability conditions that is restated below.

Theorem 2.9 ([30]) *The system Σ_2 satisfying the conditions (2.5) and (2.7) is asymptotically stable for any delay $d(t) \in [-d_u, 0]$, if there exist $P = P^T > 0$, $Q = Q^T > 0$ and $Z = Z^T > 0$ along with appropriately dimensioned matrices N_i and T_i for $(i = 1, 2, 3)$ such that following LMIs hold:*

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & d_u N_1 \\ \star & (2,2) & (2,3) & d_u N_2 \\ \star & \star & (3,3) & d_u N_3 \\ \star & \star & \star & -d_u Z \end{bmatrix} < 0 \tag{2.76}$$

where, $(1,1) = -T_1A - A^T T_1^T + N_1 + N_1^T + Q$, $(1,2) = -N_1 + N_2^T - A^T T_2^T - T_1 A_d$

$(1,3) = P + N_3^T + T_1 - A^T T_3^T$, $(2,2) = -N_2 - N_2^T - (1-\mu)Q - T_2A_d - A_d^T T_2^T$

$(2,3) = T_2 - N_3^T - A_d^T T_3^T$, and $(3,3) = d_u Z + T_3 + T_3^T$

Proof The LK functional candidate chosen is

$$V(t) = x^T(t)Px(t) + \int_{t-d(t)}^{t} x^T(s)Qx(s)ds + \int_{-d_u}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z\dot{x}(s)dsd\theta \tag{2.77}$$

One can write the time-derivative of (2.77) as

$$\dot{V}(t) \leq 2x^T(t)P\dot{x}(t) + x^T(t)Qx(t) - (1-\mu)x^T(t-d(t))Qx(t-d(t)) \\ +d_u\dot{x}^T(t)Z\dot{x}(t) - \int_{t-d(t)}^{t} \dot{x}^T(s)Z\dot{x}(s)ds \tag{2.78}$$

The inequality sign in (2.78) is due to fact that, $\dot{d}(t)$ has been approximated as μ and the integral $-\int_{t-d_u}^{t}(\dot{x}^T(s)Z\dot{x}(s)ds)$ arising out of the derivative of double integral term in (2.77) is approximated as $-\int_{t-d_u}^{t}(\dot{x}^T(s)Z\dot{x}(s)ds) \leq -\int_{t-d(t)}^{t}(\dot{x}^T(s)$ $Z\dot{x}(s)ds)$.

Earlier methods that are based on model transformations, replace $\dot{x}(t)$ in LK derivative by (2.3), whereas in this method an appropriately dimensioned free weighting matrices N_i for $i = 1, 2, 3$ have been introduced to express the relationship between the terms $x(t),\ x(t-d(t))$, and $\dot{x}(t)$ using Newton-Leibniz formula as shown below,

$$0 = 2[x^T(t)N_1 + x^T(t-d(t))N_2 + \dot{x}^T(t)N_3] \\ \times \left[x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)ds\right] \tag{2.79}$$

Another set of free weighting matrices T_i for $i = 1, 2, 3$ are introduced using the following relation,

$$0 = 2[x^T(t)T_1 + x^T(t-d(t))T_2 + \dot{x}^T(t)T_3] \\ \times[\dot{x}(t) - Ax(t) - A_dx(t-d(t))] \tag{2.80}$$

A semi-positive definite matrix $X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ \star & X_{22} & X_{23} \\ \star & \star & X_{33} \end{bmatrix} \geq 0$ is introduced and the following holds

$$d_u\xi^T(t)X\xi(t) - \int_{t-d(t)}^{t} \xi^T(t)X\xi(t)ds \geq 0 \tag{2.81}$$

where, $\xi(t) = \left[x^T(t)\ x^T(t-d(t)\ \dot{x}^T(t)\right]^T$.

Adding the terms (2.79)–(2.81) into $\dot{V}(t)$, one can express $\dot{V}(t)$ as

$$\dot{V}(t) \leq \xi^T(t)\Upsilon\xi(t) - \int_{t-d(t)}^{t} \eta^T(t,s)\Theta\eta(t,s)ds \tag{2.82}$$

where, $\eta(t,s)$=$\left[\xi^T(t)\ \dot{x}^T(s)\right]^T$

$$\Upsilon = \begin{bmatrix} (1,1) + d_uX_{11} & (1,2) + d_uX_{12} & (1,3) + d_uX_{13} \\ \star & (2,2) + d_uX_{22} & (2,3) + d_uX_{23} \\ \star & \star & (3,3) + d_uX_{33} \end{bmatrix}$$

$$\Theta = \begin{bmatrix} X_{11} & X_{12} & X_{13} & N_1 \\ \star & X_{22} & X_{23} & N_2 \\ \star & \star & X_{33} & N_3 \\ \star & \star & \star & Z \end{bmatrix}$$

Selection of $Z > 0$ and $X = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} Z^{-1} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}^T$ ensures that $X \geq 0$ and $\Theta \geq 0$.

Thus, one can write $\Upsilon + d_u \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} Z^{-1} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}^T$ using Schur-complement equivalent to LMI (2.76).

The asymptotic stability of the system Σ_2 is guaranteed if the LMI in (2.76) is negative definite.

Remark 2.13 In the Theorem 2.9 as $\dot{x}(t)$ is retained in the formulation of stability condition so (2.80) is introduced such that system matrices A and A_d appear in the stability condition. In [13] similar kind of stability condition as in [30] is proposed, $\dot{x}(t)$ which appears in the derivative of LK functional is now replaced by the RHS of (2.3). The stability condition in [13] is presented next in the form of following theorem.

Theorem 2.10 ([13]) *The system Σ_2 is asymptotically stable for any delay $d(t) \in [-d_u, 0]$, if there exist $P = P^T > 0$, $Q = Q^T > 0$ and $Z = Z^T > 0$, a symmetric semi-positive-definite matrix $X = \begin{bmatrix} X_{11} & X_{12} \\ \star & X_{22} \end{bmatrix} \geq 0$, and appropriately dimensioned matrices Y and T such that following LMIs hold:*

$$\begin{bmatrix} (1,1) & (1,2) & d_u A^T Z \\ \star & (2,2) & d_u A_d^T Z \\ \star & \star & -d_u Z \end{bmatrix} < 0 \tag{2.83}$$

$$\begin{bmatrix} X_{11} & X_{12} & Y \\ \star & X_{22} & T \\ \star & \star & Z \end{bmatrix} \geq 0 \tag{2.84}$$

where, $(1,1) = PA + A^T P + Y + Y^T + Q + d_u X_{11}$, $(1,2) = PA_d - Y + Y^T + d_u X_{12}$

$$(2,2) = -T - T^T - (1-\mu)Q + d_u X_{22}$$

This theorem can be proved in a similar manner as in Theorem 2.9, except that (2.80) need not to be considered now as $\dot{x}(t)$ is substituted by RHS of (2.3).

The present author has recently investigated a stability condition for system Σ_2 using free weighting matrix approach and introduced the following modifications

(i) without using inequality (2.81) (i.e., avoiding the use of semi-positive definite matrix in the formulation) and (ii) without retaining the $\dot{x}(t)$ term and hence not using condition (2.80) over Theorems 2.9 and 2.10 in [31]. The theorem is stated below.

Theorem 2.11 ([31]) *The system* Σ_2 *is asymptotically stable for any delay* $d(t) \in [-d_u, 0]$, *satisfying the conditions (2.7) if there exist* $P = P^T > 0$, $Q_1 = Q_1^T > 0$ *and* $Q_2 = Q_2^T > 0$, *with appropriately dimensioned free matrices* $T_i (i = 1, 2)$ *such that following LMIs holds:*

$$\Omega = \begin{bmatrix} (1,1) & (1,2) & T_1 \\ \star & (2,2) & T_2 \\ \star & \star & -d_u^{-1} Q_2 \end{bmatrix} < 0 \tag{2.85}$$

where, $(1,1) = d_u A^T Q_2 A + A^T P + PA + T_1 + T_1^T + Q_1$, $(1,2) = PA_d + d_u A^T Q_2 A_d - T_1 + T_2^T$ $(2,2) = d_u A_d^T Q_2 A_d - T_2 - T_2^T - (1-\mu) Q_1$

Proof Consider LK functional candidate chosen as

$$V(t) = x^T(t) P x(t) + \int_{t-d(t)}^{t} x^T(s) Q_1 x(s) ds + \int_{-d_u}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) Q_2 \dot{x}(s) ds d\theta \tag{2.86}$$

One can write the time-derivative of (2.86) as

$$\begin{aligned} \dot{V}(t) \leq\ & 2x^T(t) P \dot{x}(t) + x^T(t) Q_1 x(t) - (1-\mu) x^T(t-d(t)) Q_1 x(t-d(t)) \\ & + d_u \dot{x}^T(t) Q_2 \dot{x}(t) - \int_{t-d(t)}^{t} \dot{x}^T(s) Q_2 \dot{x}(s) ds \end{aligned} \tag{2.87}$$

For delay-dependent condition, one can use the following expression based on Newton-Leibniz formula in the derivative of LK functional.

$$\begin{aligned} 0 = 2 & \left[x^T(t)\ x^T(t-d(t)) \right] \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \\ & \times \left[x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s) ds \right] \end{aligned} \tag{2.88}$$

where, T_1 and T_2 are free matrices. Expanding (2.88), one can get

$$\xi^T(t) \begin{bmatrix} T_1 + T_1^T & -T_1 + T_2^T \\ \star & -T_2 - T_2^T \end{bmatrix} \xi(t) - \int_{t-d(t)}^{t} 2\xi^T(t) \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \dot{x}(s) ds = 0 \tag{2.89}$$

where, $\xi(t) = \left[x^T(t)\ x^T(t-d(t)) \right]^T$. Applying bounding Lemma 2.1 in the last term of (2.89) one can obtain

$$-\int_{t-d(t)}^{t} \dot{x}^T(s)Q_2\dot{x}(s)ds \leq \xi^T(t)\begin{bmatrix} T_1+T_1^T & -T_1+T_2^T \\ \star & -T_2-T_2^T \end{bmatrix}\xi(t)$$
$$+\xi^T(t)d_u\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} Q_2^{-1} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^T \xi(t) \qquad (2.90)$$

Substituting the value of $\dot{x}(t) = Ax(t) + A_dx(t-d(t))$ and RHS of (2.90) in (2.87) one can obtain

$$\dot{V}(t) \leq \xi^T(t)\Omega\xi(t) \qquad (2.91)$$

where, matrix Ω is defined in (2.85), if $\Omega < 0$ then it ensures the asymptotic stability of the system under consideration.

This stability analysis can be extended for systems with delay-derivative ($\mu > 1$) i.e., fast time-varying delay satisfying the condition (2.8), which is stated in the following corollary.

Corollary 2.1 *For $\mu > 1$, the system Σ_2 is asymptotically stable if there exist matrices $P = P^T > 0$, $Q_2 = Q_2^T > 0$, any free matrices T_1 and T_2 of appropriate dimensions, such that the following LMI holds:*

$$\begin{bmatrix} (1,1) & (1,2) & T_1 \\ \star & (2,2) & T_2 \\ \star & \star & -d_u^{-1}Q_2 \end{bmatrix} < 0 \qquad (2.92)$$

where,$(1,1)$=$d_uA^TQ_2A + A^TP + PA + T_1 + T_1^T$, $(1,2)$=$PA_d + d_uA^TQ_2A_d - T_1 + T_2^T$
$(1,3)$=$d_uA_d^TQ_2A_d - T_2 - T_2^T$

Proof The proof of this corollary is straight forward following the proof of Theorem 2.11, the stability result is obtained by considering $Q_1 = 0$ in (2.86), this assumption makes the Lyapunov functional candidate corresponding to delay-independent term zero.

Remark 2.14 Theorem 2.11 provides a generalized framework for stability analysis as it can treat systems Σ_1 (constant delay case), Σ_2 (time-varying delay case) for both the types of time-varying delays-slow ($\mu < 1$) and fast ($\mu > 1$). Advantage of the stability condition obtained in Theorem 2.11 compared to Theorem 2.10 (condition (2.83)–(2.84)) are (i) it consists of lesser matrix variables and (ii) lesser number of LMIs need to be solved, whereas compared to Theorem 2.9 the LMI dimension in (2.85) is smaller. Furthermore, Theorem 2.11 has also been extended for system Σ_3 (system with multiple state delays) in [11].

A stability condition has been proposed recently in [14] for the system Σ_2 satisfying condition (2.7). The stability condition is derived by using (i) augmented

Lyapunov-Krasovskii functional candidate, (ii) Jensen's integral inequality (for eliminating the integral terms arising out of the derivative of LK functional) and (iii) free weighting matrices by utilizing Newton-Leibniz formula.

The delay-dependent stability condition of [14] is presented after stating the Jensens integral inequality Lemma [7] as it is significant in establishing this stability condition.

Lemma 2.5 (Jensens inequality [19]) *For any symmetric positive definite matrix $M > 0$, scalar $\gamma > 0$ and vector function $\omega : [0, \gamma] \to R^n$ such that the integrations concerned are well defined, the following inequality holds:*

$$\left(\int_0^{\gamma} \omega(s)ds\right)^T M \left(\int_0^{\gamma} \omega(s)ds\right) \leq \gamma \left(\int_0^{\gamma} \omega(s)^T M\omega(s)ds\right) \quad (2.93)$$

Theorem 2.12 ([14]) *The system Σ_2 is asymptotically stable for any time-delay $d(t) \in [-d_u, 0]$ satisfying (2.7), if there exist symmetric positive definite matrices, P, Q, R, T and any matrices $S_i (i = 1, \ldots 4)$ with the appropriate dimensions satisfying following LMIs:*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ \star & P_{22} \end{bmatrix} \geq 0 \quad with\ P_{11} > 0 \quad (2.94)$$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ \star & Q_{22} \end{bmatrix} \geq 0 \quad (2.95)$$

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \mu P_{12} \\ \star & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & 0 \\ \star & \star & -Q_{11} & \Gamma_{34} & \mu P_{22} \\ \star & \star & \star & \Gamma_{44} & 0 \\ \star & \star & \star & \star & -\mu T \end{bmatrix} < 0 \quad (2.96)$$

where, $\Gamma_{11}= A^T P_{11} + P_{11}A + d_u^2(Q_{11} + A^T Q_{12}^T + Q_{12}A + A^T Q_{22}A) + R + S_1 + S_1^T$
$\Gamma_{12}=P_{11}A_d - S_1^T + S_2 + d_u^2(Q_{12}A_d + A^T Q_{22}A_d)$, $\Gamma_{13}=A^T P_{12} + S_3$
$\Gamma_{14}=P_{12} - S_1^T + S_4$, $\Gamma_{22}=-(1-\mu)R + \mu T + d_u^2 A_d^T Q_{22}A_d - S_2^T - S_2$
$\Gamma_{23}=A_d^T P_{12} - S_3$, $\Gamma_{24}=-S_2^T - S_4$, $\Gamma_{34}=P_{22} - Q_{12} - S_3^T$ and
$\Gamma_{44}=-Q_{22} - S_4^T - S_4$

Proof The augmented LK functional candidate chosen here is

$$V(t) = V_1(t) + V_2(t) + V_3(t) \quad (2.97)$$

where, $V_1(t)=\eta^T(t)P\eta(t),\ V_2(t)=d_u\int_{-d_u}^{0}\int_{t+\theta}^{t}\xi^T(s)Q\xi(s)dsd\theta,\ V_3(t)=\int_{t-d(t)}^{t}x^T(s)Rx(s)ds$
$\eta(t)=\begin{bmatrix} x^T(t) & \left(\int_{t-d(t)}^{t}x(s)ds\right)^T \end{bmatrix}^T,\ \xi(s)=\begin{bmatrix} x^T(s) & \dot{x}^T(s)\end{bmatrix}^T$

Finding time-derivative of (2.97), one can get the following

$$\dot{V}_1(t) = 2\eta^T(t)P\dot{\eta}(t)$$

$$\dot{V}_1(t) = 2\eta^T(t)P[\eta_1(t) + \dot{d}(t)\eta_2(t)] \tag{2.98}$$

where, $\dot{\eta}(t) = \begin{bmatrix} \dot{x}^T(t) & \left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)^T \end{bmatrix}^T$, $\eta_1(t)=\begin{bmatrix} Ax(t) + A_d x(t-d(t)) \\ \int_{t-d(t)}^{t}\dot{x}(s)ds \end{bmatrix}$ and
$\eta_2(t)=\begin{bmatrix} 0 \\ I \end{bmatrix} x(t-d(t))$

Defining the augmented vector as

$$\tau(t) = \begin{bmatrix} x^T(t) \ x^T(t-d(t)) \ \left(\int_{t-d(t)}^{t}x(s)ds\right)^T \ \left(\int_{t-d(t)}^{t}\dot{x}(s)ds\right)^T \end{bmatrix}^T$$

Now, following vectors can be expressed in terms of $\tau(t)$ as given below
$\eta(t)=\Theta_1\tau(t)$, $\eta_1(t)=\Theta_2\tau(t)$, $[0, I]P\eta(t)=\Theta_3\tau(t)$ and $x(t-d(t))=\Theta_4\tau(t)$

where, $\Theta_1=\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}$, $\Theta_2=\begin{bmatrix} A & A_d & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$, $\Theta_3=\begin{bmatrix} P_{12}^T & 0 & P_{22} & 0 \end{bmatrix}$ and
$\Theta_4=\begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix}$

The term $2\eta^T(t)P\dot{d}(t)\eta_2(t)$ in (2.98) can be rewritten as,

$$2\eta^T(t)P\dot{d}(t)\eta_2(t) = 2\dot{d}(t)\eta^T(t)P\begin{bmatrix} 0 \\ I \end{bmatrix} x(t-d(t))$$

thus in view of above notations, one can equivalently write the above equation as,

$$\dot{d}(t)2\eta^T(t)P\begin{bmatrix} 0 \\ I \end{bmatrix} x(t-d(t)) = \dot{d}(t)2\tau^T(t)\Theta_3^T\Theta_4^T\tau(t)$$

Using the bounding inequality Lemma 2.1 one can write,

$$\dot{d}(t)2\tau^T(t)\Theta_3^T\Theta_4^T\tau(t) \le \mu\tau^T(t)\Theta_3^T T^{-1}\Theta_3\eta(t) + \mu\tau^T(t)\Theta_4^T T\Theta_4\eta(t),\ T = T^T > 0 \tag{2.99}$$

Substituting (2.99) into (2.98) one can obtain

$$\dot{V}_1(t) \leq \tau^T(t)(\Gamma_1 + \mu\Theta_3^T T^{-1}\Theta_3)\tau(t) \tag{2.100}$$

where, $\Gamma_1 = \Theta_1^T P\Theta_2 + \Theta_2^T P\Theta_1 + \mu\Theta_4^T T\Theta_4$

Now, the time-derivative of $V_2(t)$ can be written as,

$$\dot{V}_2(t) = d_u^2\xi^T(t)Q\xi(t) - d_u\int_{t-d_u}^{t}\xi^T(s)Q\xi(s)ds$$

using Jensens integral inequality Lemma 2.5 one can write,

$$\dot{V}_2(t) \leq \tau^T(t)\Gamma_2\tau(t) \tag{2.101}$$

where,$\Gamma_2 = d_u^2\Theta_5^T Q\Theta_5 - \Theta_6^T Q\Theta_6$, Θ_5= $\begin{bmatrix} I & 0 & 0 & 0 \\ A & A_d & 0 & 0 \end{bmatrix}$, and Θ_6= $\begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$

Finally $\dot{V}_3(t)$ can be written as

$$\dot{V}_3(t) \leq \tau^T(t)\Gamma_3\tau(t) \tag{2.102}$$

where,$\Gamma_3 = \Theta_7^T R\Theta_7 - (1-\mu)\Theta_4^T R\Theta_4$, Θ_7= $\begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}$

Using Newton-Leibniz formula relationship among various states of an augmented state vector $\tau(t)$ is expressed by introducing free matrices S such that following equality is satisfied,

$$2\tau^T(t)S^T\Theta_8\tau(t) = 0 \tag{2.103}$$

where, $\Theta_8 = \begin{bmatrix} I & -I & 0 & -I \end{bmatrix}$ and $S = \begin{bmatrix} S_1 & S_2 & S_3 & S_4 \end{bmatrix}$

One can write (2.103) in terms of $\eta(t)$ as

$$\tau^T(t)S^T\Gamma_4\tau(t) = 0 \tag{2.104}$$

where, $\Gamma_4 = S^T\Theta_8 + \Theta_8^T S$
Adding (2.100), (2.101), (2.102) and (2.104), which yields the expression of $\dot{V}(t)$

$$\dot{V}(t) \leq \tau^T(\Gamma_0 + \mu\Theta_3^T T^{-1}\Theta_3)\tau(t) \tag{2.105}$$

where, $\Gamma_0 = \Sigma_{i=1}^3\Gamma_i$

Taking Schur-complement of (2.105), one can obtain the stability condition in (2.95).

Remark 2.15 The LK functional in [13] can be obtained by setting $P_{11} = P_{22} = 0$ and $Q_{11} = Q_{12} = 0$ in functional (2.97), thus this LK functional is more generalized than in [13]. The construction of new LK functional was attempted such that there is proper distribution of delay information in the obtained LMI structure leading to less conservative estimate of delay upper bound.

An improved method of delay-dependent stability analysis has recently appeared in [32] and [3] where it is pointed out that the approximation of the integral term $-\int_{t-d_u}^{t} f(.)d(.) \leq -\int_{t-d(t)}^{t} f(.)d(.)$ is conservative, in sequel exact replacement of the above mentioned integral was proposed which is expressed as,

$$-\int_{t-d_u}^{t} f(.)d(.) = -\int_{t-d(t)}^{t} f(.)d(.) - \int_{t-d(t)}^{t-d_u} f(.)d(.)$$

In all earlier delay-dependent stability methods the integral term $-\int_{t-d(t)}^{t-d_u} f(.)d(.)$ was ignored leading to conservative estimate of the delay upper bound. In [32] and [3] an improved delay-dependent stability condition was derived considering new LK functional so as to accommodate this integral term. The stability theorem of [32] is presented below:

Theorem 2.13 ([32]) *The system* Σ_2 *is asymptotically stable for any time-delay* $d(t) \in [-d_u, 0]$ *satisfying (2.7), if there exist symmetric positive definite matrices,* $P,\ Q,\ R,\ Z_i\ (i = 1, 2)$ *and free matrices* N_i, M_i *and* S_i $(i = 1, 2, 3)$ *with appropriate dimensions such that following LMI holds:*

$$\begin{bmatrix} \Phi & d_u N & d_u S & d_u M & d_u A_{c1}^T(Z_1 + Z_2) \\ \star & -d_u Z_1 & 0 & 0 & 0 \\ \star & \star & -d_u Z_1 & 0 & 0 \\ \star & \star & \star & -d_u Z_2 & 0 \\ \star & \star & \star & \star & -d_u(Z_1 + Z_2) \end{bmatrix} < 0 \tag{2.106}$$

where, $\Phi = \Phi_1 + \Phi_2 + \Phi_2^T$,

$$\Phi_1 = \begin{bmatrix} PA + A^T P + Q + R & PA_d & 0 \\ \star & -(1-\mu)Q & 0 \\ \star & \star & -R \end{bmatrix},$$

$\Phi_2 = \begin{bmatrix} N + M & -N + S & -M - S \end{bmatrix}$, $A_{c1} = \begin{bmatrix} A & A_d & 0 \end{bmatrix}$,

$N = \begin{bmatrix} N_1^T & N_2^T & N_3^T \end{bmatrix}^T$, $S = \begin{bmatrix} S_1^T & S_2^T & S_3^T \end{bmatrix}^T$, and $M = \begin{bmatrix} M_1^T & M_2^T & M_3^T \end{bmatrix}^T$

Proof The LK functional chosen here is

$$V(t) = x^T(t)Px(t) + \int_{-d_u}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s)dsd\theta$$

$$+ \int_{t-d(t)}^{t} x^T(s)Qx(s)ds + \int_{t-d_u}^{t} x^T(s)Rx(s)ds \tag{2.107}$$

The time-derivative of (2.107) along with the exact substitution of quadratic integral term arising out of the LK functional derivative of the double integral (associated with Z_1 matrix) as $-\int_{t-d_u}^{t} f(.)d(.) = -\int_{t-d(t)}^{t} f(.)d(.) - \int_{t-d(t)}^{t-d_u} f(.)d(.)$ one can write,

$$\begin{aligned}\dot{V}(t) \leq\ & 2x^T(t)P\dot{x}(t) + x^T(Q+R)x(t) - (1-\mu)x^T(t-d(t))Qx(t-d(t)) \\ & -x^T(t-d_u)Rx(t-d_u) + d_u\dot{x}^T(t)(Z_1+Z_2)\dot{x}(t) - \int_{t-d(t)}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds \\ & - \int_{t-d_u}^{t-d(t)} \dot{x}^T(s)Z_1\dot{x}(s)ds - \int_{t-d_u}^{t} \dot{x}^T(s)Z_2\dot{x}(s)ds \end{aligned} \tag{2.108}$$

For delay-dependent condition Newton-Leibniz formula is used, which satisfies following equations involving free matrices

$$2\xi^T(t)N\left[x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)ds\right] = 0$$

$$2\xi^T(t)S\left[x(t-d(t)) - x(t-d_u) - \int_{t-d_u}^{t-d(t)} \dot{x}(s)ds\right] = 0$$

$$2\xi^T(t)M\left[x(t) - x(t-d_u) - \int_{t-d_u}^{t} \dot{x}(s)ds\right] = 0 \tag{2.109}$$

where, $N=\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$, $S=\begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}$, $M=\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}$ and
$\xi(t)=\begin{bmatrix} x^T(t) & x^T(t-d(t)) & x^T(t-d_u) \end{bmatrix}^T$

Adding all the terms of (2.109) to (2.108), and with further rearrangement of terms one can get,

$$\begin{aligned}\dot{V}(t) \leq\ & \xi^T(t)\{\Phi + d_u A_{c1}^T(Z_1+Z_2)A_{c1} + d_u N Z_1^{-1} N^T \\ & + d_u S Z_1^{-1} S^T + d_u M Z_2^{-1} M^T\}\xi(t) - \int_{t-d(t)}^{t} [\xi^T(t)N + \dot{x}^T(s)Z_1] \\ & \times Z_1^{-1}[N^T\xi(t) + Z_1\dot{x}(s)]ds - \int_{t-d_u}^{t-d(t)} [\xi^T(t)S + \dot{x}^T(s)Z_1] \\ & \times Z_1^{-1}[S^T\xi(t) + Z_1\dot{x}(s)]ds - \int_{t-d_u}^{t} [\xi^T(t)M + \dot{x}^T(s)Z_2] \\ & \times Z_2^{-1}[M^T\xi(t) + Z_2\dot{x}(s)]ds \end{aligned} \tag{2.110}$$

It may be noted that, the last three integral terms of (2.110) are all less than zero, so if $\xi^T(t)[\Phi+d_u A_{c1}^T(Z_1+Z_2)A_{c1}+d_u N Z_1^{-1}N^T+d_u S Z_1^{-1}S^T+d_u M Z_2^{-1}M^T]\xi(t) < 0$ than by using Schur complement one can obtain the stability condition in LMI form as in (2.106).

The free weighting matrix method has been widely used in the stability analysis of continuous systems with two additive time-varying delay in the states. The closed-loop operation of a networked controlled systems is an example of systems with two additive time-varying delays [33]. The stability condition of system Σ_4 while considering as a single time delay term yields conservative results of delay upper bound compared to the case when the two additive time-varying delays are treated separately in the formulation, because the delays may have different properties as they occur in different places of the network [33].

The delay-dependent stability condition derived in [33] for system Σ_4 based on free-weighting matrix method is presented below.

Theorem 2.14 ([33]) *System Σ_4 in (2.11) with delays $d_1(t)$ and $d_2(t)$ satisfying (2.13) is asymptotically stable if there exist matrices $P > 0$, $Q_1 \geq Q_2 > 0$, $Q_3 \geq Q_4 > 0$, $M_1 \geq M_2 > 0$, $M_3 \geq M_4 > 0$, $N_i, i = 1, \ldots 8$, such that following LMI holds,*

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & PA_d & A^T\Pi_{55} & N_1 & 0 & N_5 & 0 \\ \star & \Pi_{22} & 0 & \Pi_{24} & 0 & N_2 & N_3 & 0 & 0 \\ \star & \star & \Pi_{33} & \Pi_{34} & 0 & 0 & 0 & N_6 & N_7 \\ \star & \star & \star & \Pi_{44} & A_d^T\Pi_{55} & 0 & N_4 & 0 & N_8 \\ \star & \star & \star & \star & -\Pi_{55} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -d_{1u}^{-1}M_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -d_{2u}^{-1}M_2 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -d_{2u}^{-1}M_3 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & -d_{1u}^{-1}M_4 \end{bmatrix} < 0 \quad (2.111)$$

where, $\Pi_{11}=A^TP + PA + Q_1 + Q_3 + N_1 + N_1^T + N_5 + N_5^T$,
$\Pi_{12}=-N_1 + N_2^T$, $\Pi_{13}=-N_5 + N_6^T$
$\Pi_{22}=-(1-\mu_1)(Q_1 - Q_2) - N_2 - N_2^T + N_3 + N_3^T$, $\Pi_{24}=-N_3 + N_4^T$,
$\Pi_{33}=-(1-\mu_2)(Q_3 - Q_4) - N_6 - N_6^T + N_7 + N_7^T$, $\Pi_{34}=-N_7 + N_8^T$,
$\Pi_{44}=-(1-\mu_1-\mu_2)(Q_2 + Q_4) - N_4 - N_4^T - N_8 - N_8^T$
$\Pi_{55}=(d_{1u}M_1 + d_{2u}M_2 + d_{2u}M_3 + d_{1u}M_4)$

Remark 2.16 The LK functional candidate selected in [33] for Theorem 2.14) is,

$$V(t) = V_1 + V_2 + V_3 + V_4 + V_5$$

$$V_1(t) = x^T(t)Px(t)$$

$$V_2(t) = \int_{t-d_1(t)}^{t} x^T(s)Q_1x(s)ds + \int_{t-d_1(t)-d_2(t)}^{t-d_1(t)} x^T(s)Q_2x(s)ds$$

$$V_3(t) = \int_{-d_{1u}}^{0}\int_{\beta}^{0} \dot{x}^T(t+\alpha)M_1\dot{x}(t+\alpha)d\alpha d\beta + \int_{-d_{1u}-d_{2u}}^{-d_{1u}}\int_{\beta}^{0} \dot{x}^T(t+\alpha)M_2\dot{x}(t+\alpha)d\alpha d\beta$$

$$V_4(t) = \int_{t-d_2(t)}^{t} x^T(s)Q_3x(s)ds + \int_{t-d_1(t)-d_2(t)}^{t-d_2(t)} x^T(s)Q_4x(s)ds$$

$$\begin{aligned} V_5(t) &= \int_{-d_{2u}}^{0}\int_{\beta}^{0} \dot{x}^T(t+\alpha)M_3\dot{x}(t+\alpha)d\alpha d\beta \\ &+ \int_{-d_{1u}-d_{2u}}^{-d_{2u}}\int_{\beta}^{0} \dot{x}^T(t+\alpha)M_4\dot{x}(t+\alpha)d\alpha d\beta \end{aligned}$$

The selection of the LK functional considered above contains repetitive delay information in some region that can lead to a conservative estimate of the delay upper bound. Moreover, the dimension of the LMI obtained by this method is more due to the introduction of free weighting matrices for approximating quadratic integral terms using Newton-Leibniz formula.

Also, introduction of semi-positive definite matrices to satisfy the inequalities (24)–(27) in [33] and consequently replacing it with inequalities (32) in [33] are not equivalent. This is turn, leads to conservative delay upper bound estimate.

2.4 Delay-Range-Dependent Stability Condition

It was pointed out in [3] that, in practice the delay lower bound cannot necessarily be always restricted to 0 as in many engineering (or physical) systems, delay may vary in a ranges (or intervals) unlike for the system considered in Σ_1, Σ_2, Σ_3 and Σ_4 satisfying (2.5). The stability conditions derived by restricting the lower delay bound to 0 are referred in literature as delay-dependent stability conditions.

The stability condition in an LMI framework for systems with time delay varying in ranges have been reported in [3–5, 34, 35]. In [3] stability condition has been proposed for the system Σ_2 satisfying (2.6) (i.e., delay lower bound is not restricted to 0) by proposing a new LK functional suitable for the condition (2.6), such stability condition are referred as delay-range-dependent stability condition in the literature. The stability condition derived in [3] is presented below.

Theorem 2.15 ([3]) *The system Σ_2 is asymptotically stable for any time-delay $d(t) \in [-d_u, 0]$ satisfying (2.6), (2.7) and (2.8), if there exist symmetric positive* *definite matrices,* P, T, Q, R, Z_i $(j = 1, 2)$ *such that following LMI holds:*

$$\Phi < 0 \tag{2.112}$$

where, $\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & M_1 & -S_1 & d_u N_1 & d_{lu} S_1 & d_{lu} M_1 & A^T U \\ \star & \Phi_{22} & M_2 & -S_2 & d_u N_2 & d_{lu} S_2 & d_{lu} M_2 & A_d^T U \\ \star & \star & -T & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -R & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -d_u Z_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -d_{lu}(Z_1 + Z_2) & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -d_{lu} Z_2 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -U \end{bmatrix} < 0$

$\Phi_{11}{=}PA + A^T P + Q + T + R + N_1 + N_1^T$, $\Phi_{12}{=}PA_d + N_2^T - N_1 + S_1 - M_1$
$\Phi_{22}{=}-(1-\mu)Q + S_2 + S_2^T - N_2 - N_2^T - M_2 - M_2^T$, $U{=}d_u Z_1 + d_{lu} Z_2$ and $d_{lu}{=}d_u - d_l$

Proof The LK functional is chosen here as

$$\begin{aligned} V(t) = {} & x^T(t)Px(t) + \int_{-d_u}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z_1\dot{x}(s)dsd\theta \\ & + \int_{-d_u}^{-d_l}\int_{t+\theta}^{t} \dot{x}^T(s)Z_2\dot{x}(s)dsd\theta + \int_{t-d(t)}^{t} x^T(s)Qx(s)ds \\ & + \int_{t-d_u}^{t} x^T(s)Rx(s)ds + \int_{t-d_l}^{t} x^T(s)Tx(s)ds \end{aligned} \tag{2.113}$$

The time-derivative of (2.113) is given by

$$\begin{aligned} \dot{V}(t) \le {} & 2x^T(t)P\dot{x}(t) + x^T(t)(Q + R + T)x(t) - (1-\mu)x^T(t - d(t))Qx(t - d(t)) \\ & -x^T(t - d_u)Rx(t - d_u) - x^T(t - d_l)Tx(t - d_l) + \dot{x}^T(t)(d_u Z_1 + d_{lu} Z_2)\dot{x}(t) \\ & - \int_{t-d_u}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds - \int_{t-d_u}^{t-d_l} \dot{x}^T(s)Z_2\dot{x}(s)ds \end{aligned} \tag{2.114}$$

As stated in [32] that the conservative estimate of the delay bound is obtained as the term $-\int_{t-du}^{t-d(t)}(.)$ was ignored while approximating the term $-\int_{t-du}^{t}(.)$. Hence, the exact expression for the last two integral terms in (2.114) is considered as,

$$\begin{aligned} -\int_{t-d_u}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds = {} & -\int_{t-d(t)}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds \\ & -\int_{t-d_u}^{t-d(t)} \dot{x}^T(s)Z_1\dot{x}(s)ds \end{aligned} \tag{2.115}$$

$$-\int_{t-d_u}^{t-d_l} \dot{x}^T(s)Z_2\dot{x}(s)ds = -\int_{t-d_u}^{t-d(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds - \int_{t-d(t)}^{t-d_l} \dot{x}^T(s)Z_2\dot{x}(s)ds \tag{2.116}$$

Thus in view of (2.115) and (2.116), one can write (2.114) as

$$\begin{aligned}\dot{V}(t) \leq\ & 2x^T(t)P\dot{x}(t) + x^T(Q+R+T)x(t) - (1-\mu)x^T(t-d(t))Qx(t-d(t)) \\ & -x^T(t-d_u)Rx(t-d_u) - x^T(t-d_l)Tx(t-d_l) + \dot{x}^T(t)(d_u Z_1 + d_{lu}Z_2)\dot{x}(t) \\ & -\int_{t-d_u}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds - \int_{t-d_u}^{t-d(t)} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds \\ & -\int_{t-d(t)}^{t-d_l} \dot{x}^T(s)Z_2\dot{x}(s)ds \end{aligned} \tag{2.117}$$

For obtaining delay-dependent condition one can use Newton-Leibniz formula such that it satisfies following equations involving free matrices

$$\begin{aligned} 2\xi^T(t)N\left[x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)ds\right] &= 0 \\ 2\xi^T(t)S\left[x(t-d(t)) - x(t-d_u) - \int_{t-d_u}^{t-d(t)} \dot{x}(s)ds\right] &= 0 \\ 2\xi^T(t)M\left[x(t-d_l) - x(t-d(t)) - \int_{t-d(t)}^{t-d_l} \dot{x}(s)ds\right] &= 0 \end{aligned} \tag{2.118}$$

where, $N=\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$, $S=\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$, $M=\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ and $\xi(t) = \begin{bmatrix} x^T(t) & x^T(t-d(t)) \end{bmatrix}^T$

Adding (2.118) into (2.117), then carrying out algebraic manipulations, using bounding techniques (discussed in Theorem 2.11 and applying Schur-complement one can obtain,

$$\dot{V}(t) = \zeta^T(t)\Phi\zeta(t)$$

where $\zeta(t) = [x(t)^T, x(t-d(t)^T, x(t-d_l)^T, x(t-d_u)^T]^T$. The LMI Φ is already defined above. If $\Phi < 0$ then the system Σ_2 is guaranteed to be asymptotically stable.

Remark 2.17 It is possible to obtain delay-dependent stability condition from this theorem by setting $d_l = 0$ in the LK functional (2.113) thus reducing the double integral term to $\int_{-d_u}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)(Z_1 + Z_2)\dot{x}(s)dsd\theta$ and setting $T = 0$ the single integral term becomes zero while rest of the term appears as in (2.113).

Considering $N = [N_1^T, N_2^T, 0]^T$, $S = [S_1^T, S_2^T, 0]^T$ and $M = [0]$ in (2.109) of Theorem 2.13 one can obtain the delay-dependent condition of Corollary 3 of [3].

Theorem 2.13 and corollary 3 of [3] both are applicable for unknown $\mu \geq 1$ due to the presence of free matrices in the (2,2) element of the LMI conditions, so separate condition need not to be derived for this case.

Further improvements of delay-range-dependent stability criteria as well as delay-dependent stability criteria with less number of matrix variables applicable for both slow and fast varying time-delay (i.e., satisfying the conditions (2.7) and (2.8)) have been proposed in [4] and [5]. Both the stability criteria are presented below.

Theorem 2.16 ([4]) *The system Σ_2 is asymptotically stable for any time-delay $d(t) \in [-d_u, 0]$ satisfying (2.6), (2.7) and (2.8), if there exist symmetric positive definite matrices, P, T, Q, R, Z_i $(j = 1, 2)$ such that following LMI holds:*

$$\begin{bmatrix} \gamma_{11} & PA_d & Z_1 & 0 & d_l A^T Z_1 & d_{lu} A^T Z_2 \\ \star & \gamma_{22} & Z_2 & Z_2 & d_l A_d^T Z_1 & d_{lu} A_d^T Z_2 \\ \star & \star & \gamma_{33} & 0 & 0 & 0 \\ \star & \star & \star & -R - Z_2 & 0 & 0 \\ \star & \star & \star & \star & -Z_1 & 0 \\ \star & \star & \star & \star & \star & -Z_2 \end{bmatrix} < 0 \tag{2.119}$$

where, γ_{11}=$PA + A^T P + Q + T + R - Z_1$, γ_{22}=$-(1 - \mu)Q - 2Z_2$ and γ_{33}=$-T - Z_1 - Z_2$

Proof The LK functional candidate is selected as,

$$\begin{aligned} V(t) = x^T(t)Px(t) &+ \int_{-d_l}^{0} \int_{t+\theta}^{t} d_l \dot{x}^T(s) Z_1 \dot{x}(s) ds d\theta \\ &+ \int_{-d_u}^{-d_l} \int_{t+\theta}^{t} d_{lu} \dot{x}^T(s) Z_2 \dot{x}(s) ds d\theta + \int_{t-d(t)}^{t} x^T(s) Q x(s) ds \\ &+ \int_{t-d_u}^{t} x^T(s) R x(s) ds + \int_{t-d_l}^{t} x^T(s) T x(s) ds \end{aligned} \tag{2.120}$$

Considering time-derivative of (2.120) and substituting the value of $\dot{x}(t) = Ax(t) + A_d x(t - d(t))$ one can obtain

$$\begin{aligned} \dot{V}(t) \leq\ & 2x^T(t)P(Ax(t) + A_d x(t - d(t))) + x^T(Q + R + T)x(t) \\ & -(1 - \mu)x^T(t - d(t))Qx(t - d(t)) \\ & -x^T(t - d_u)Rx(t - d_u) - x^T(t - d_l)Tx(t - d_l) \\ & +(Ax(t) + A_d x(t - d(t)))^T (d_l^2 Z_1 + d_{lu}^2 Z_2)(Ax(t) + A_d x(t - d(t))) \\ & - \int_{t-d_l}^{t} d_l \dot{x}^T(s) Z_1 \dot{x}(s) ds - \int_{t-d_u}^{t-d_l} d_{lu} \dot{x}^T(s) Z_2 \dot{x}(s) ds \end{aligned} \tag{2.121}$$

Applying Lemma 2.5 (Jensen's integral inequality), the integral terms in (2.121) are approximated as,

$$-\int_{t-d_l}^{t} d_l \dot{x}(s)^T Z_1 \dot{x}(s) ds \leq -(x(t) - x(t-d_l))^T Z_1 (x(t) - x(t-d_l)) \tag{2.122}$$

and,

$$\begin{aligned} -\int_{t-d_u}^{t-d_l} d_{lu} \dot{x}(s)^T Z_2 \dot{x}(s) ds \leq & -\int_{t-d_u}^{t-d(t)} (d_u - d(t)) \dot{x}(s)^T Z_2 \dot{x}(s) ds \\ & -\int_{t-d(t)}^{t-d_l} (d(t) - d_l) \dot{x}(s)^T Z_2 \dot{x}(s) ds \end{aligned} \tag{2.123}$$

further one can write (2.123) as,

$$\begin{aligned} -\int_{t-d_u}^{t-d_l} d_{lu} \dot{x}(s)^T Z_2 \dot{x}(s) ds \leq & -(x(t-d(t)) - x(t-d_u))^T Z_2 (x(t-d(t)) - x(t-d_u)) \\ & -(x(t-d_l) - x(t-d(t)))^T Z_2 (x(t-d_l) - x(t-d(t))) \end{aligned} \tag{2.124}$$

Using (2.121)–(2.124) and with algebraic manipulations, one can easily obtain the following expression,

$$\dot{V}(t) \leq \zeta^T \Gamma \zeta(t) \tag{2.125}$$

where, $\zeta(t) = \begin{bmatrix} x^T(t) & x^T(t-d(t)) & x^T(t-d_l) & x^T(t-d_u) \end{bmatrix}^T$

$$\Gamma = \begin{bmatrix} \gamma_{11} & PA_d & Z_1 & 0 \\ \star & \gamma_{22} & Z_2 & Z_2 \\ \star & \star & \gamma_{33} & 0 \\ \star & \star & \star & -R - Z_2 \end{bmatrix} + \begin{bmatrix} A & A_d & 0 & 0 \end{bmatrix}^T (d_l^2 Z_1 + d_{lu}^2 Z_2) \begin{bmatrix} A & A_d & 0 & 0 \end{bmatrix}$$

γ_{11}, γ_{22} and γ_{33} are defined in (2.119), if $\Gamma < 0$ in (2.125) then the system is asymptotically stable.

Remark 2.18 When $d_l = 0$, the above Theorem reduces to delay-dependent stability condition, which is given in corollary 1 of [4] and it is valid for $\mu < 1$.

When $\mu > 1$ then Theorem 2.16 is not applicable, thus setting $Q = 0$ in Theorem 2.16, one can easily obtain corollary 2 of [4].

The modifications made in Theorem 2.16 compared to Theorem 2.15 are (i) selection of different LK functional (ii) use of Jensens integral to approximate the quadratic integral terms arising in the LK functional derivative unlike introducing free matrices via Newton-Leibniz formula in [3]. It is clear from the above theorem that, even without using free matrices it is possible to derive an LMI condition which can provide feasible solution for both $\mu < 1$ and $\mu > 1$, with lower matrix variables due to the use of Jensen integral inequality.

As only weighting matrices of LK functional is involved in the LMI of Theorem 2.16 thus the computational burden of this theorem is much lesser then that of [3], as the latter method involves lot of free weighting matrices.

Recently in [5] another improved delay-range-dependent stability analysis has been reported with a tight bounding of the following integral terms,

$$\begin{aligned} &-\int_{t-d_u}^{t-d(t)} d_{lu}\dot{x}(s)^T Z_2 \dot{x}(s)ds \\ and\ &-\int_{t-d(t)}^{t-d_l} d_{lu}\dot{x}(s)^T Z_2 \dot{x}(s)ds \end{aligned}$$

The bounding of the above integral terms carried out in [5] and it is expressed as,

$$\begin{aligned} -\int_{t-d_u}^{t-d_l} d_{lu}\dot{x}(s)^T Z_2 \dot{x}(s)ds &= -\int_{t-d_u}^{t-d(t)} d_{lu}\dot{x}(s)^T Z_2 \dot{x}(s)ds \\ &\quad -\int_{t-d(t)}^{t-d_l} d_{lu}\dot{x}(s)^T Z_2 \dot{x}(s)ds \\ -\int_{t-d_u}^{t-d_l} d_{lu}\dot{x}(s)^T Z_2 \dot{x}(s)ds &= -\int_{t-d_u}^{t-d(t)} (d_u - d(t))\dot{x}(s)^T Z_2 \dot{x}(s)ds \\ &\quad -\int_{t-d_u}^{t-d(t)} (d(t) - d_l)\dot{x}(s)^T Z_2 \dot{x}(s)ds \\ &\quad -\int_{t-d(t)}^{t-d_l} (d(t) - d_l)\dot{x}(s)^T Z_2 \dot{x}(s)ds \\ &\quad -\int_{t-d(t)}^{t-d_l} (d_u - d(t))\dot{x}(s)^T Z_2 \dot{x}(s)ds \end{aligned} \quad (2.126)$$

Defining $\beta = (d(t) - d_l)/d_{lu}$, so $1 - \beta = (d_u - d(t))/d_{lu}$, thus following will be true

$$\begin{aligned} -\int_{t-d_u}^{t-d(t)} (d(t) - d_l)\dot{x}(s)^T Z_2 \dot{x}(s)ds &= -\beta \int_{t-d_u}^{t-d(t)} d_{lu}\dot{x}(s)^T Z_2 \dot{x}(s)ds \\ &\leq -\beta \int_{t-d_u}^{t-d(t)} (d_u - d(t))\dot{x}(s)^T Z_2 \dot{x}(s)ds \\ and, \quad -\int_{t-d(t)}^{t-d_l} (d_u - d(t))\dot{x}(s)^T Z_2 \dot{x}(s)ds &= (1-\beta) \int_{t-d(t)}^{t-d_l} d_{lu}\dot{x}(s)^T Z_2 \dot{x}(s)ds \\ &\leq -(1-\beta) \int_{t-d(t)}^{t-d_l} (d(t) - d_l)\dot{x}(s)^T Z_2 \dot{x}(s)ds \end{aligned}$$

Incorporating the above modifications, the quadratic integral term $-\int_{t-d_u}^{t-d_l} d_{lu}\dot{x}(s)^T Z_2\dot{x}(s)ds$ is approximated using Lemma 2.5 (Jensens Integral Inequality) as

$$\begin{aligned}-\int_{t-d_u}^{t-d_l} d_{lu}\dot{x}(s)^T Z_2\dot{x}(s)ds \leq & -(x(t-d(t))-x(t-d_u))^T Z_2(x(t-d(t))-x(t-d_u)) \\ & -(x(t-d_l)-x(t-d(t)))^T Z_2(x(t-d_l)-x(t-d(t))) \\ & -\beta(x(t-d(t))-x(t-d_u))^T Z_2(x(t-d(t))-x(t-d_u)) \\ & -(1-\beta)(x(t-d_l)-x(t-d(t)))^T Z_2(x(t-d_l)-x(t-d(t)))\end{aligned} \tag{2.127}$$

Considering similar LK functional as in [4] and applying bounding technique of (2.127) results a stability condition which is stated below.

Theorem 2.17 ([5]) *The system Σ_2 is asymptotically stable for any time-delay $d(t) \in [-d_u, 0]$ satisfying (2.6), (2.7), if there exist symmetric positive definite matrices, $P, T, Q, R, Z_i(j = 1, 2)$ such that following LMI holds:*

$$\Phi_1 = \Phi - \begin{bmatrix} 0 & -I & I & 0 \end{bmatrix}^T Z_2 \begin{bmatrix} 0 & -I & I & 0 \end{bmatrix} < 0 \tag{2.128}$$

and

$$\Phi_2 = \Phi - \begin{bmatrix} 0 & I & 0 & -I \end{bmatrix}^T Z_2 \begin{bmatrix} 0 & I & 0 & -I \end{bmatrix} < 0 \tag{2.129}$$

where, $\Phi = \begin{bmatrix} \gamma_{11} & PA_d & Z_1 & 0 \\ \star & \gamma_{22} & Z_2 & Z_2 \\ \star & \star & \gamma_{33} & 0 \\ \star & \star & \star & -R-Z_2 \end{bmatrix} + \begin{bmatrix} A & A_d & 0 & 0 \end{bmatrix}^T (d_l^2 Z_1 + d_{lu}^2 Z_2) \begin{bmatrix} A & A_d & 0 & 0 \end{bmatrix}$

$\gamma_{11}=PA + A^T P + Q + T + R - Z_1$, $\gamma_{22}=-(1-\mu)Q - 2Z_2$ and

$\gamma_{33}=-T - Z_1 - Z_2$

Proof Considering the similar LK functional candidate as in Theorem 2.15 and incorporating the integral term approximation in (2.127) one can easily obtain,

$$\dot{V}(t) \leq \zeta(t)^T[(1-\beta)\Phi_1 + \beta\Phi_2]\zeta(t)$$

where, $\zeta(t) = \begin{bmatrix} x(t)^T & x(t-d(t))^T & x(t-d_l)^T & x(t-d_u)^T \end{bmatrix}^T$. One can observe that the above $\dot{V}(t)$ expression is a convex combination of the matrices Φ_1 and Φ_2. The negativity of $\dot{V}(t)$ is ensured if both Φ_1 and Φ_2 are negative definite, which in turn guarantees the asymptotic stability of the system Σ_2. The details derivation can be found in [5].

Remark 2.19 The improvement on delay upper bound result compared to the Theorem 2.17 is expected as the over bounding of the integrals (2.122)–(2.124) has been avoided in the present theorem.

Delay-dependent stability condition is a special case of delay-range-dependent stability condition when $d_l = 0$.

Delay-dependent stability conditions in LMI framework have been reviewed extensively that are directly relevant to the present work of the thesis. Initially, some basic delay-independent stability condition are recalled. Next, four basic approaches have been discussed in length for developing delay-dependent stability conditions in LMI framework. they are namely: Model transformation approach, Bounding techniques, Descriptor system approach and Free-Weighting matrix approach. Finally the recent results on additive delay and delay-range-dependent stability conditions have been discussed in previous sections.

For convenience of the discussion of the main results of this chapter, some preliminaries including few definitions, basic theorems on stability of time-delay systems which are related to the main results are presented in previous sections.

The main and improved results on delay-dependent stability analysis are proposed in this chapter and presented below.

2.5 Main Results on Stability Analysis of Time-Delay System

In this section, delay-dependent-stability analysis of nominal time-delay systems are presented with an objective to (i) obtain less conservative estimate of the delay upper bound result and (ii) obtain a stability condition in an LMI framework (without sacrificing the conservatism) that can be further extended to derive improved robust stability and/(or) stabilization conditions. The necessary condition for the delay-dependent stability of the time-delay systems is that the matrix $(A + A_d)$ must be Hurwitz.

The results of the proposed stability conditions are compared with existing methods by considering several numerical examples.

2.5.1 Stability Analysis of TDS with Single Time Delay

Consider the system Σ_2 in (2.3), satisfying the condition (2.5) and (2.7), the stability condition is presented by considering a new LK functional and replacing the quadratic integral inequalities arising in the LK derivative with more exact expressions as suggested in [3]. The proposed method is described in the form of theorem below.

Theorem 2.18 *The system Σ_2 satisfying the condition (2.5) for $0 < \mu < 1$, is asymptotically stable if there exist matrices $R_i = R_i^T > 0$, (i=1,2) and any free* *matrices* L_i, $(i = 1, ..3)$, M_i $(i = 1, ..3)$, G_i, $(i = 1, 2)$ *such that following LMI's are satisfied*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0, \tag{2.130}$$

$$Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} > 0 \tag{2.131}$$

and,

$$\begin{bmatrix} \Omega & \mu\tilde{P} & d_u M \\ \star & -\mu T & 0 \\ \star & 0 & -Q_{22} \end{bmatrix} < 0 \tag{2.132}$$

$$\begin{bmatrix} \Omega & \mu\tilde{P} & d_u L \\ \star & -\mu T & 0 \\ \star & 0 & -Q_{22} \end{bmatrix} < 0 \tag{2.133}$$

where,

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & 0 & \Omega_{16} \\ \star & \Omega_{22} & \Omega_{23} & \Omega_{24} & 0 & \Omega_{26} \\ \star & \star & \Omega_{33} & 0 & 0 & 0 \\ \star & \star & 0 & -Q_{11} & 0 & \Omega_{46} \\ 0 & 0 & 0 & 0 & -Q_{11} & 0 \\ \star & \star & 0 & \star & 0 & \Omega_{66} \end{bmatrix} \tag{2.134}$$

and, $L = \begin{bmatrix} L_1^T & L_2^T & L_3^T & 0 & 0 & 0 \end{bmatrix}^T$, $M = \begin{bmatrix} M_1^T & M_2^T & M_3^T & 0 & 0 & 0 \end{bmatrix}^T$

$\tilde{P} = \begin{bmatrix} P_{12}^T & 0 & 0 & P_{22} & 0 & 0 \end{bmatrix}^T$

where, $\Omega_{11} = d_u^2 Q_{11} + G_1^T A + A^T G_1 + P_{12} + P_{12}^T + R_1 + R_2 + d_u(L_1 + L_1^T)$

$\Omega_{12} = G_1^T A_d - P_{12} + d_u(-L_1 + L_2^T + M_1)$, $\Omega_{13} = d_u(L_3^T - M_1)$, $\Omega_{14} = P_{22}$,

$\Omega_{16} = P_{11} - G_1^T + A^T G_2$ $\Omega_{22} = -(1-\mu)R_1 + \mu T + d_u(-L_2 - L_2^T + M_2 + M_2^T)$

$\Omega_{23} = d_u(-M_2 + M_3^T - L_3^T)$, $\Omega_{24} = -P_{22}$, $\Omega_{26} = A_d^T G_2$, $\Omega_{33} = d_u(-M_3 - M_3^T) - R_2$,

$\Omega_{66} = -G_2 - G_2^T + d_u^2 Q_{22}$

Proof The LK functional is selected as,

$$V(t) = V_1(t) + V_2(t) + V_3(t)$$

where,

$$V_1(t) = \varphi^T(t) P \varphi(t) \tag{2.135}$$

with, $\varphi(t) = \left[\, x^T(t)\ \left(\int_{t-d(t)}^{t} x(s)ds\right)^T \,\right]^T$

$$V_2(t) = \int_{t-d(t)}^{t} x^T(s) R_1 x(s)ds + \int_{t-d_u}^{t} x^T(s) R_2 x(s)ds \tag{2.136}$$

$$V_3(t) = \int_{-d_u}^{0} \int_{t+\theta}^{t} d_u \gamma^T(s) Q \gamma(s) ds d\theta \tag{2.137}$$

where, $\gamma(s) = \left[\, x^T(s)\ \dot{x}^T(s) \,\right]^T$. The time-derivative of (2.135) along the solution of (2.3), one can obtain

$$\dot{V}_1(t) = 2\varphi^T(t) P \left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & -I \end{bmatrix} \varphi_1(t) + \dot{d}(t) \begin{bmatrix} 0 \\ I \end{bmatrix} x(t-d(t)) \right) \tag{2.138}$$

where,

$$\varphi_1(t) = \left[\, \dot{x}^T(t)\ x^T(t)\ x^T(t-d(t)) \,\right]^T$$

Defining the augmented state vector as, $\eta(t) = \left[\, x^T(t)\ x^T(t-d(t))\ x^T(t-d_u)\ \left(\int_{t-d(t)}^{t} x(s)ds\right)^T\ \left(\int_{t-d_u}^{t-d(t)} x(s)ds\right)^T\ \dot{x}^T(t) \,\right]^T$
One can rewrite the second term of (2.138) as,

$$2\dot{d}(t)\eta^T \tilde{P} \left[\, 0\ I\ 0\ 0\ 0\ 0 \,\right] \eta(t) = 2\dot{d}(t) a^T(.) b(.) \tag{2.139}$$

where, $a(.) = \tilde{P}^T \eta(t)$, $b(.) = \left[\, 0\ I\ 0\ 0\ 0\ 0 \,\right] \eta(t)$ and $\tilde{P} = \left[\, P_{12}^T\ 0\ 0\ P_{22}\ 0\ 0 \,\right]^T$. Using bounding lemma (Lemma 2.1) on (2.139) one can have,

$$\begin{aligned} 2\dot{d}(t) a^T(.) b(.) &\le \mu \{ a^T(.) T^{-1} a(.) + b^T(.) T b(.) \} \\ &\le \mu \eta^T(t) \left(\tilde{P} T^{-1} \tilde{P}^T + \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} T \left[\, 0\ I\ 0\ 0\ 0\ 0 \,\right] \right) \eta(t) \end{aligned} \tag{2.140}$$

Substituting (2.140) on (2.138) and upon expansion of (2.138) one can have,

$$\dot{V}_1(t) \leq \eta^T(t)\left(\begin{bmatrix} P_{12}+P_{12}^T & -P_{12} & 0 & P_{22} & 0 & P_{11} \\ \star & \mu T & 0 & -P_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 & P_{12}^T \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & \star & 0 & 0 \end{bmatrix} + \tilde{P}\mu T^{-1}\tilde{P}^T\right)\eta(t) \quad (2.141)$$

Finding the time-derivative of (2.136) and (2.137), one can obtain

$$\begin{aligned}\dot{V}_2(t) \leq\ & x^T(t)(R_1+R_2)x(t) - (1-\mu)x^T(t-d(t))R_1x(t-d(t)) \\ & -x^T(t-d_u)R_2x(t-d_u)\end{aligned} \quad (2.142)$$

$$\dot{V}_3(t) \leq d_u^2\gamma^T(t)Q\gamma(t) - d_u\int_{t-d_u}^{t}\gamma^T(s)Q\gamma(s)ds \quad (2.143)$$

Express the first quadratic term of (2.143) into the following form,

$$d_u^2\gamma^T(t)Q\gamma(t) = d_u^2\eta^T(t)\Theta\ \eta(t) \quad (2.144)$$

where,

$$\Theta = \begin{bmatrix} d_u^2Q_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_u^2Q_{22} \end{bmatrix}$$

The integral term in (2.143) can be written as,

$$\begin{aligned}-d_u\int_{t-d_u}^{t}\gamma^T(s)Q\gamma(s)ds = & -d_u\int_{t-d(t)}^{t}\gamma^T(s)Q\gamma(s)ds \\ & -d_u\int_{t-d_u}^{t-d(t)}\gamma^T(s)Q\gamma(s)ds\end{aligned} \quad (2.145)$$

Expanding (2.145), one can write

$$
\begin{aligned}
-d_u\int_{t-d_u}^{t}\gamma^T(s)Q\gamma(s)ds = &-\int_{t-d(t)}^{t}d_u x^T(s)Q_{11}x(s)ds-\int_{t-d_u}^{t-d(t)}d_u x^T(s)Q_{11}x(s)ds\\
&-\int_{t-d(t)}^{t}d_u\dot{x}^T(s)Q_{22}\dot{x}(s)ds-\int_{t-d_u}^{t-d(t)}d_u\dot{x}^T(s)Q_{22}\dot{x}(s)ds
\end{aligned}
\tag{2.146}
$$

Treating first two integral terms of (2.146) using Jensen's integral inequality (Lemma 2.5), one can write

$$
\begin{aligned}
-d_u\int_{t-d_u}^{t}\gamma^T(s)Q\gamma(s)ds \le &-\left(\int_{t-d(t)}^{t}x(s)ds\right)^T Q_{11}\int_{t-d(t)}^{t}x(s)ds\\
&-\left(\int_{t-d_u}^{t-d(t)}x(s)ds\right)^T Q_{11}\int_{t-d_u}^{t-d(t)}x(s)ds\\
&-\int_{t-d_u}^{t-d(t)}d_u\dot{x}^T(s)Q_{22}\dot{x}(s)ds-\int_{t-d(t)}^{t}d_u\dot{x}^T(s)Q_{22}\dot{x}(s)ds
\end{aligned}
\tag{2.147}
$$

The last two integral terms in (2.147) are eliminated using Newton-Leibniz formula by introducing free matrices L and M and they have the following forms,

$$
2\eta^T(t)\left[L_1^T\ L_2^T\ L_3^T\ 0\ 0\ 0\right]^T[x(t)-x(t-d(t)-\int_{t-d(t)}^{t}\dot{x}(s)ds]=0
\tag{2.148}
$$

$$
2\eta^T(t)\left[M_1^T\ M_2^T\ M_3^T\ 0\ 0\ 0\right]^T[x(t-d(t))-x(t-d_u)-\int_{t-d_u}^{t-d(t)}\dot{x}(s)ds]=0
\tag{2.149}
$$

Expanding (2.148) one can obtain,

$$
\begin{aligned}
0=\eta^T(t)&\begin{bmatrix}
L_1+L_1^T & -L_1+L_2^T & L_3^T & 0\ 0\ 0\\
\star & -L_2-L_2^T & -L_3^T & 0\ 0\ 0\\
\star & \star & 0 & 0\ 0\ 0\\
0 & 0 & 0 & 0\ 0\ 0\\
0 & 0 & 0 & 0\ 0\ 0\\
0 & 0 & 0 & 0\ 0\ 0
\end{bmatrix}\eta(t)\\
&-2\int_{t-d(t)}^{t}\eta^T(t)\left[L_1^T\ L_2^T\ L_3^T\ 0\ 0\ 0^T\right]\dot{x}(s)ds
\end{aligned}
\tag{2.150}
$$

Now applying bounding lemma (Lemma 2.1) and with algebraic manipulations one can obtain,

$$-\int_{t-d(t)}^{t} d_u \dot{x}^T(s) Q_{22} \dot{x}(s) ds \le \eta^T(t) \left\{ d_u \begin{bmatrix} L_1 + L_1^T & -L_1 + L_2^T & L_3^T & 0 & 0 & 0 \\ \star & -L_2 - L_2^T & -L_3^T & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + d(t) L d_u Q_{22}^{-1} L^T \right\} \eta(t) \tag{2.151}$$

Similarly the simplification of (2.149) will yield,

$$-\int_{t-d_u}^{t-d(t)} d_u \dot{x}^T(s) Q_{22} \dot{x}(s) ds \le \eta^T(t) \left\{ d_u \begin{bmatrix} 0 & M_1 & -M_1 & 0 & 0 & 0 \\ \star & M_2 + M_2^T & -M_2 + M_3^T & 0 & 0 & 0 \\ \star & \star & -M_3 - M_3^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + (d_u - d(t)) M d_u Q_{22}^{-1} M^T \right\} \eta(t) \tag{2.152}$$

Substituting the values of integrals from (2.151) and (2.152) in (2.147) and rearranging the terms one can write

$$-d_u \int_{t-d_u}^{t} \gamma^T(s) Q \gamma(s) ds \le \eta^T(t) \{\Delta + d(t) L (d_u Q_{22}^{-1}) L^T + (d_u - d(t)) M (d_u Q_{22}^{-1}) M^T\} \eta(t) \tag{2.153}$$

where,

$$\Delta = \begin{bmatrix} d_u(L_1 + L_1^T) & d_u(-L_1 + L_2^T + M_1) & d_u(L_3^T - M_1) & 0 & 0 & 0 \\ \star & d_u(-L_2 - L_2^T + M_2 + M_2^T) & d_u(-L_3^T - M_2 + M_3^T) & 0 & 0 & 0 \\ \star & \star & d_u(-M_3 - M_3^T) & 0 & 0 & 0 \\ 0 & 0 & 0 & -Q_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & -Q_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

Substituting (2.144) and (2.153) in (2.143) one can obtain,

$$\dot{V}_3(t) \quad \le \eta^T(t)\left\{\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & 0 & 0 & 0 \\ \star & \Upsilon_{22} & \Upsilon_{23} & 0 & 0 & 0 \\ \star & \star & \Upsilon_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & -Q_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & -Q_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_u^2 Q_{22} \end{bmatrix}\right.$$
$$\left. +d(t)L(d_u Q_{22}^{-1})L^T + (d_u - d(t))M(d_u Q_{22}^{-1})M^T\right\}\eta(t) \quad (2.154)$$

where, $\Upsilon_{11}{=}d_u^2 Q_{11} + d_u(L_1 + L_1^T)$, $\Upsilon_{12}{=}d_u(-L_1 + L_2^T + M_1)$

$\Upsilon_{13}{=}d_u(L_3^T - M_1)$, $\Upsilon_{22}{=}d_u(-L_2 - L_2^T + M_2 + M_2^T)$

$\Upsilon_{23}{=}d_u(-L_3^T - M_2 + M_3^T)$, $\Upsilon_{33}{=}d_u(-M_3 - M_3^T)$

Now for any matrices G_1 and G_2 the equation shown below is satisfied,

$$2[x^T(t)G_1^T + \dot{x}^T(t)G_2^T]\,[Ax(t) + A_d x(t - d(t)) - \dot{x}(t)] = 0 \quad (2.155)$$

Expanding (2.155) one can get,

$$\eta^T(t)\begin{bmatrix} G_1^T A + A^T G_1 & G_1^T A_d & 0 & 0 & 0 & -G_1^T + A^T G_2 \\ \star & 0 & 0 & 0 & 0 & A_d^T G_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 & -G_2 - G_2^T \end{bmatrix}\eta(t) = 0 \quad (2.156)$$

Now, adding all the three derivative terms in (2.141), (2.142) and (2.154), along with (2.156) one can obtain;

$$\dot{V}(t) \le \eta^T(t)\left\{\Omega + \tilde{P}\mu T^{-1}\tilde{P}^T + d(t)L(d_u Q_{22}^{-1})L^T + (d_u - d(t))M(d_u Q_{22}^{-1})M^T\right\}\eta(t) \quad (2.157)$$

where, the matrix Ω is defined in (2.134). Since in (2.157) the term $d(t)L(d_u Q_{22}^{-1})L^T + (d_u - d(t))M(d_u Q_{22}^{-1})M^T$ is a convex combination of the matrices $L(d_u Q_{22}^{-1})L^T$ and $M(d_u Q_{22}^{-1})M^T$ on $d(t)$ so one can express (2.157) by two equivalent LMI conditions, one for $d(t) = 0$ and another for $d(t) = d_u$ that are described in (2.132) and (2.133) respectively.

Hence, to guarantee the asymptotic stability of the time-delay system Σ_2 in (2.3) the LMI conditions (2.132) and (2.133) need to be satisfied.

When the time-varying delay is not differentiable (or equivalently called fast time-varying delay), then by setting $P_{12} = 0$, $P_{22} = 0$, $Q_{11} = 0$ and $R_1 = 0$ in Theorem 2.18 one can obtain the delay-dependent stability condition for $\mu \ge 1$ which is presented in the following corollary.

Corollary 2.2 *The system Σ_2 for $\mu \geq 1$, satisfying the condition (2.5), is asymptotically stable if there exist matrices $R_2 = R_2^T > 0$, $P_{11} = P_{11}^T > 0$, $Q_{22} = Q_{22}^T > 0$ and any free matrices L_i, $(i = 1, ..3)$, $M_i(i = 1, ..3)$ such that following LMI's are satisfied:*

$$\begin{bmatrix} \tilde{\Omega} & d_u M \\ \star & -Q_{22} \end{bmatrix} < 0 \tag{2.158}$$

$$\begin{bmatrix} \tilde{\Omega} & d_u L \\ \star & -Q_{22} \end{bmatrix} < 0 \tag{2.159}$$

$$\tilde{\Omega} = \begin{bmatrix} \tilde{\Omega}_{11} & \tilde{\Omega}_{12} & \tilde{\Omega}_{13} & \Omega_{14} \\ \star & \tilde{\Omega}_{22} & \tilde{\Omega}_{23} & \tilde{\Omega}_{24} \\ \star & \star & \tilde{\Omega}_{33} & 0 \\ \star & \star & 0 & \tilde{\Omega}_{44} \end{bmatrix} < 0 \tag{2.160}$$

where, $\tilde{\Omega}_{11}$=$G_1^T A + A^T G_1 + R_2 + d_u(L_1 + L_1^T)$; $\tilde{\Omega}_{12}$=$G_1^T A_d + d_u(-L_1 + L_2^T + M_1)$

$\tilde{\Omega}_{13}$=$d_u(L_3^T - M_1)$, Ω_{14}=$P_{11} - G_1^T + A^T G_2$

$\tilde{\Omega}_{22}$=$d_u(-L_2 - L_2^T + M_2 + M_2^T)$ $\tilde{\Omega}_{23}$=$d_u(-M_2 + M_3^T - L_3^T)$, Ω_{24}=$A_d^T G_2$

$\tilde{\Omega}_{33}$=$d_u(-M_3 - M_3^T) - R_2$, Ω_{44}=$-G_2^T - G_2 + d_u^2 Q_{22}$

Corollary 2.3 *The system Σ_2 satisfying the condition (2.5) for $\mu = 0$ (i.e., constant delay), is asymptotically stable if there exist matrices $R_i = R_i^T > 0$, (i = 1,2) and any free matrices L_i, $(i = 1, ..3)$, $M_i(i = 1, ..3)$ such that following LMIs are satisfied*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0 \tag{2.161}$$

$$Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} > 0 \tag{2.162}$$

and,

$$\begin{bmatrix} \Omega & d_u M \\ \star & -Q_{22} \end{bmatrix} < 0 \tag{2.163}$$

$$\begin{bmatrix} \Omega & d_u L \\ \star & -Q_{22} \end{bmatrix} < 0 \tag{2.164}$$

where, $L = \begin{bmatrix} L_1^T & L_2^T & L_3^T & 0 & 0 & 0 \end{bmatrix}^T$, $M = \begin{bmatrix} M_1^T & M_2^T & M_3^T & 0 & 0 & 0 \end{bmatrix}^T$ *and*

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & 0 & \Omega_{16} \\ \star & \Omega_{22}|_{\mu=0} & \Omega_{23} & \Omega_{24} & 0 & \Omega_{26} \\ \star & \star & \Omega_{33} & 0 & 0 & 0 \\ \star & \star & 0 & -Q_{11} & 0 & 0 \\ \star & \star & 0 & 0 & -Q_{11} & 0 \\ \star & \star & 0 & 0 & 0 & \Omega_{66} \end{bmatrix} < 0 \tag{2.165}$$

where, $\Omega_{11}{=}d_u^2 Q_{11} + G_1^T A + A^T G_1 + P_{12} + P_{12}^T + R_1 + R_2 + d_u(L_1 + L_1^T)$
$\Omega_{12}{=}G_1^T A_d - P_{12} + d_u(-L_1 + L_2^T + M_1)$
$\Omega_{13}{=}d_u(L_3^T - M_1)$, $\Omega_{14}{=}P_{22}$, $\Omega_{16}{=}P_{11} - G_1^T + A^T G_2$
$\Omega_{22}{=}-R_1 + d_u(-L_2 - L_2^T + M_2 + M_2^T)$
$\Omega_{23}{=}d_u(-M_2 + M_3^T - L_3^T)$, $\Omega_{24}{=}-P_{22}$, $\Omega_{26}{=}A_d^T G_2$
$\Omega_{33}{=}d_u(-M_3 - M_3^T) - R_2$, $\Omega_{66}{=}-G_2 - G_2^T + d_u^2 Q_{22}$

Remark 2.20 The proposed delay-dependent stability method has been obtained in an LMI framework by combining the method of [14] and [36]. The proposed method selects augmented LK functional of the type in [14] by introducing a new LK functional term ($\int_{t-d_u}^{t} x^T(s) R_2 x(s) ds$) and finally obtaining a convex combination of LMIs as in [36].

The proposed method differs with the method in [14] and its extension in [37] withe the fact that, the augmented state vector does not include the term $\int_{(s)} \dot{x}(s) ds$ and in sequel the matrix Q_{12} in $V_3(t)$ is not considered. Further due to the introduction of new delay-independent functional, the term $\int_{t-d_u}^{t-d(t)} x(s) ds$ is included in the augmented state vector and subsequently with the use of Jensen's integral inequality in the derivative of LK functional concerning integral of $x(s)$ can be taken inside the LMI condition, whereas the quadratic integral terms associated with the vector $\dot{x}(s)$ is eliminated using Newton-Leibniz formula. Unlike the stability condition in [14], the proposed method can estimate the delay upper bound for $\mu \geq 1$ which is due to the modification as suggested in the selection of LK functional followed by different techniques adopted for bounding the quadratic integral terms appearing in the LK functional derivative.

The proposed formulation can assess the stability of (i) TDS with constant time-delay, (ii) TDS with slow varying time-delay ($\mu < 1$) and (iii) TDS with fast varying time-delay ($\mu \geq 1$).

The proposed delay-dependent stability theorem and corollaries for different delay-derivatives by considering the numerical example 2.1 have been tested. The results of the delay upper bound are tabulated in Table 2.3 and 2.4.

Table 2.3 (d_u) results of Example 2.1 for $\mu = 0$ and $\mu < 1$

Stability methods	$\mu = 0$	$\mu = 0.5$	$\mu = 0.9$
[9, 27]	4.4721	2.00	1.18
[38]	4.4721	2.00	1.18
[39]	4.472	2.00	1.18
[40]	4.472	2.00	1.18
Cor. 3 [32]	4.4721	2.04	1.37
Cor. 1 [4]	4.4721	2.04	1.3789
Cor. 1 [5]	4.4721	2.07	1.5304
Cor. 3 [36]	4.472	2.337	1.873
Cor. 1 [14]	4.472	2.0100	1.1801
[31]	4.4721	2.0083	1.1801
Proposed method	4.4721 (Cor. 2.3)	2.3372 (§ 2.18)	1.8731 (§ 2.18)

Table 2.4 (d_u) results of Example 2.1 for $\mu \geq 1$

Stability methods	Unknown μ
Cor. 1 [4]	1.3454
Cor. 1 [5]	1.3454
Cor. 3 [36]	1.868
[40]	Not applicable
[14]	Not applicable
[39]	0.999
Cor. 2.2	1.868

Numerical Example 2.3 ([41]) *Consider the system Σ_2 with the following constant matrices*

$$A = \begin{bmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{bmatrix}, A_d = \begin{bmatrix} -5.9 & 7.1 & -70.3 \\ 2 & -1 & 5 \\ 2 & 0 & 6 \end{bmatrix}$$

The eigenvalues of the matrix $[A + A_d]$ is Hurwitz and the eigenvalues of the matrix $[A - A_d]$ is unstable, thus the given system is delay-dependently stable, i.e., the system is asymptotically stable for certain finite delay value. The exact delay upper bound value for this system is $d_u = 0.1624$ for $\mu = 0$ ([19] and [41]).

The delay upper bound estimate of Numerical Example 2.3 for $\mu = 0$ and $\mu < 1$ using proposed DDS condition are presented in Table 2.6.

Remark 2.21 The stability results presented in Tables 2.3 and 2.4 shows that the proposed stability method provides better estimate of delay upper bound compared to [14] for increasing delay derivatives, whereas the results of the proposed method is

Table 2.5 Comparison of decision variables and LMIs for Example 2.1

Stability methods	Decision variables	No. of LMIs
Cor. 3 [36]	$17n^2 + 9n$	6
Cor. 2.2	$15n^2 + 5n$	4

Table 2.6 (d_u) results of Example 2.3 for $\mu = 0$ and $\mu < 1$

Stability methods	$\mu = 0$	$\mu = 0.5$
Cor. 3 [32]	0.0751	–
Cor. 1 [4]	0.0751	–
Cor. 1 [5]	0.0751	–
Cor. 1 [14]	0.1091	0.0723
[36]	–	0.0736
Present method	0.0803 (§ 2.18)	0.0736 (Cor. 2.3)

found to be same when compared with the results obtained in [4, 5, 32, 36] for $\mu = 0$. Note that the stability condition in [36] is not in a conventional LMI framework. The advantage of the proposed method over [36] is indicated in Table 2.5.

Remark 2.22 The LK functional considered in Theorem 2.18 can be treated as generalized one as other choices of functional can be obtained from (2.135)–(2.137) with following choices of the matrices described below:

1. setting $P_{12} = P_{22} = 0$, $P_{11} = P$, $Q_{11} = Q_{12} = 0$ and $Q_{22} = \frac{Q}{d_u}$ in proposed LK functional yields the LK function of [32].
2. $P_{12} = P_{22} = 0$, $P_{11} = P$, $Q_{11} = Q_{12} = 0$ and $Q_{22} = Z_2$, $Q_3 = R_1$ and $Q_2 = R_2$ in proposed LK functional yields LK functional of [5] and [4].

2.5.2 *Stability Analysis of TDS with Two Additive Time-Varying Delays*

Consider the system Σ_4 described in (2.11) satisfying the conditions (2.13), the stability condition for this system is presented in the following theorem.

Theorem 2.19 ([42]) *The system Σ_4 described in (2.11) satisfying the conditions (2.13)–(2.14) is asymptotically stable if there exist $P = P^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, $Q_3 = Q_3^T > 0$, $R_1 = R_1^T > 0$, $R_2 = R_2^T > 0$, $R_3 = R_3^T > 0$ and $G_1, G_2, G_3, G_4, M_1, M_2, M_3, M_4, N_1, N_2, N_3$ and N_4 are free matrices with $Q_2 \geq Q_3$. satisfying following LMI:*

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & L_1 & M_1 & N_1 \\ \star & \Omega_{22} & \Omega_{23} & \Omega_{24} & L_2 & M_2 & N_2 \\ \star & \star & \Omega_{33} & \Omega_{34} & L_3 & M_3 & N_3 \\ \star & \star & \star & \Omega_{44} & L_4 & M_4 & N_4 \\ \star & \star & \star & \star & -\frac{1}{d_u}R_1 & 0 & 0 \\ \star & \star & \star & \star & \star & -\frac{1}{d_{u1}}R_2 & 0 \\ \star & \star & \star & \star & \star & \star & -\frac{1}{d_{u2}}R_3 \end{bmatrix} < 0 \tag{2.166}$$

where, Ω_{11}=$Q_1 + Q_2 + G_1A + A^TG_1^T + L_1 + L1^T + M_1 + M_1^T$
Ω_{12}=$A^TG_2^T + L_2^T - M_1 + M_2^T + N_1$, Ω_{13}=$G_1A_d + A_TG_3^T - L_1 + L_3^T + M_3^T - N_1$
Ω_{14}=$P - G_1 + A^TG_4^T + L_4^T + M_4^T$, Ω_{22}=$-(1 - d_{u1})(Q_2 - Q_3) - M_2 - M_2^T + N_2 + N_2^T$
Ω_{23}=$G_2A_d - L_2 - M_3^T - N_2 + N_3^T$, Ω_{24}=$-G_2 - M_4^T + N_4^T$
Ω_{33}=$-(1 - d_{u1} - d_{u2})(Q_1 + Q_3) + G_3A_d + A_d^TG_3^T - L_3 - L_3^T - N_3 - N_3^T$
Ω_{34}=$-G_3 + A_d^TG_4^T - L_4 - N_4^T$, and Ω_{44}=$d_uR_1 + d_{u1}R_2 + d_{u2}R_3 - G_4 - G_4^T$

Proof Considering the LK functional candidate as

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{2.167}$$

$$V_1(t) = x^T(t)Px(t) \tag{2.168}$$

$$\begin{aligned} V_2(t) = &\int_{t-d_1(t)-d_2(t)}^{t} x^T(s)Q_1x(s)ds + \int_{t-d_1(t)}^{t} x^T(s)Q_2x(s)ds \\ &+ \int_{t-d_1(t)-d_2(t)}^{t-d_1(t)} x^T(s)Q_3x(s)ds \end{aligned} \tag{2.169}$$

$$\begin{aligned} V_3(t) = &\int_{t-d_{u1}-d_{u2}}^{t}\int_{\theta}^{t} \dot{x}^T(s)R_1\dot{x}(s)dsd\theta + \int_{t-d_{u1}}^{t}\int_{\theta}^{t} \dot{x}^T(s)Q_2\dot{x}(s)dsd\theta \\ &+ \int_{t-d_{u1}-d_{u2}}^{t-d_{u1}}\int_{\theta}^{t} \dot{x}^T(s)Q_3\dot{x}(s)dsd\theta \end{aligned} \tag{2.170}$$

Finding the time-derivative of (2.168)–(2.170). The time derivative of $\dot{V}_3(t)$ will be,

$$\begin{aligned}
\dot{V}_3(t) = d_u \dot{x}^T(t) R_1 \dot{x}(t) - \int_{t-d_u}^{t} \dot{x}^T(s) R_1 \dot{x}(s) ds \\
+ d_{u1} \dot{x}^T(t) R_2 \dot{x}(t) - \int_{t-d_{u1}}^{t} \dot{x}^T(s) R_2 \dot{x}(s) ds \\
+ d_{u2} \dot{x}^T(t) R_3 \dot{x}(t) - \int_{t-d_u}^{t-d_{u1}} \dot{x}^T(s) R_3 \dot{x}(s) ds
\end{aligned} \tag{2.171}$$

As $\dot{V}_3(t)$ contains integral terms so to formulate quadratic conditions we need to replace them. For any symmetric positive definite matrices R_1, R_2 and R_3, the following inequalities are satisfied,

$$\begin{aligned}
-\int_{t-d_u}^{t} \dot{x}^T(s) R_1 \dot{x}(s) ds \le -\int_{t-d(t)}^{t} \dot{x}^T(s) R_1 \dot{x}(s) ds \\
-\int_{t-d_{u1}}^{t} \dot{x}^T(s) R_2 \dot{x}(s) ds \le -\int_{t-d_1(t)}^{t} \dot{x}^T(s) R_2 \dot{x}(s) ds \\
-\int_{t-d_u}^{t-d_{u1}} \dot{x}^T(s) R_3 \dot{x}(s) ds \le -\int_{t-d(t)}^{t-d_1(t)} \dot{x}^T(s) R_3 \dot{x}(s) ds
\end{aligned} \tag{2.172}$$

thus (2.171) can be written as,

$$\begin{aligned}
\dot{V}_3(t) \le \dot{x}^T(t)(d_u R_1 + d_{u1} R_2 + d_{u2} R_3)\dot{x}(t) - \int_{t-d(t)}^{t} \dot{x}^T(s) R_1 \dot{x}(s) ds \\
- \int_{t-d_1(t)}^{t} \dot{x}^T(s) R_2 \dot{x}(s) ds - \int_{t-d(t)}^{t-d_1(t)} \dot{x}^T(s) R_3 \dot{x}(s) ds
\end{aligned} \tag{2.173}$$

Removing the integral terms from (2.173) with the help of Newton-Leibniz formula and introduction of free matrices. Free matrices $L_i, i = 1, 2, 3, 4$, $M_i, i = 1, 2, 3, 4$ and $N_i, i = 1, 2, 3, 4$ are introduced in the second, third and fourth integral terms respectively in (2.173). The following identities with free matrices will satisfy

$$\begin{aligned}
\Sigma = 2[x^T L_1 + x^T(t - d_1(t)) L_2 + x^T(t - d(t)) L_3 + \dot{x}(t) L_4] \\
\times [x(t) - x(t - d(t)) - \int_{t-d(t)}^{t} \dot{x}(s) ds] = 0
\end{aligned} \tag{2.174}$$

$$\begin{aligned}
0 = 2[x^T M_1 + x^T(t - d_1(t)) M_2 + x^T(t - d(t)) M_3 + \dot{x}^T(t) M_4] \\
\times [x(t) - x(t - d_1(t)) - \int_{t-d_1(t)}^{t} \dot{x}(s) ds]
\end{aligned} \tag{2.175}$$

$$0 = 2[x^T N_1 + x^T(t - d_1(t))N_2 + x^T(t - d(t))N_3 + \dot{x}(t)N_4] \\ \times[x(t - d_1(t)) - x(t - d(t)) - \int_{t-d(t)}^{t-d_1(t)} \dot{x}(s)ds] \tag{2.176}$$

Simple algebraic manipulations along with the application of Lemma 2.1 one can obtain following,

$$-\int_{t-d(t)}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds \le \xi^T \begin{bmatrix} L_1 + L_1^T & L_2^T & -L_1 + L_3^T & L_4^T \\ \star & 0 & -L_2 & 0 \\ \star & \star & -L_3 - L_3^T & -L_4^T \\ \star & \star & \star & 0 \end{bmatrix} \xi(t) \\ +\xi^T(t)d_u \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix} R_1^{-1} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix}^T \xi(t) \tag{2.177}$$

$$-\int_{t-d_1(t)}^{t} \dot{x}^T(s)R_2\dot{x}(s)ds \le \xi^T \begin{bmatrix} M_1 + M_1^T & -M_1 + M_2^T & M_3^T & M_4^T \\ \star & -M_2 - M_2^T & -M_3 & -M_4^T \\ \star & \star & 0 & 0 \\ \star & \star & \star & 0 \end{bmatrix} \xi(t) \\ +\xi^T d_{u1} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} R_2^{-1} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix}^T \xi(t) \tag{2.178}$$

$$-\int_{t-d(t)}^{t-d_1(t)} \dot{x}^T(s)R_2\dot{x}(s)ds \le \xi^T \begin{bmatrix} 0 & N_1 & -N_1^T & 0 \\ \star & N_2 + N_2^T & -N_2 + N_3^T & N_4^T \\ \star & \star & -N_3 - N_3^T & -N_4^T \\ \star & \star & \star & 0 \end{bmatrix} \xi(t) \\ +\xi^T d_{u2} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} R_3^{-1} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}^T \xi(t) \tag{2.179}$$

In this formulation as system dynamics $\dot{x}(t)$ is retained in the formulation, so introducing free matrices $G_i, i = 1, 2, 3, 4$ system matrices are introduced in the LMI expression as,

$$\Sigma = 2[x^T G_1 + x^T(t - d_1(t))G_2 + x^T(t - d(t))G_3 + \dot{x}(t)G_4] \\ \times[-\dot{x}(t) + Ax(t) + A_d x(t - d(t))] = 0 \tag{2.180}$$

Simplifying (2.180) one can write

$$\xi^T(t)\begin{bmatrix} G_1A + A^TG_1^T & A^TG_2^T & G_1A_d + A^TG_3^T & -G_1 + A^TG_4^T \\ \star & 0 & G_2A_d & -G_2 \\ \star & \star & G_3A_d + A_d^TG_3^T & -G_3 \\ \star & \star & \star & -G_4 - G_4^T \end{bmatrix}\xi(t) = 0 \quad (2.181)$$

where, $\xi(t) = \left[\, x(t)^T \;\; x(t-d_1(t))^T \;\; x(t-d(t))^T \;\; \dot{x}(t)^T \,\right]^T$

Substituting (2.177), (2.178) and (2.179) into (2.173) and then finally adding all the derivative terms one obtains

$$\dot{V}(t) \le \xi^T(t)\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & L_1 & M_1 & N_1 \\ \star & \Omega_{22} & \Omega_{23} & \Omega_{24} & L_2 & M_2 & N_2 \\ \star & \star & \Omega_{33} & \Omega_{34} & L_3 & M_3 & N_3 \\ \star & \star & \star & \Omega_{44} & L_4 & M_4 & N_4 \\ \star & \star & \star & \star & -\frac{1}{d_u}R_1 & 0 & 0 \\ \star & \star & \star & \star & \star & -\frac{1}{d_{u1}}R_2 & 0 \\ \star & \star & \star & \star & \star & \star & -\frac{1}{d_{u2}}R_3 \end{bmatrix}\xi(t) \quad (2.182)$$

The LMI (2.182) is defined in (2.166). This completes the proof of the theorem. □

Remark 2.23 The numerical example 2.1 is considered for validating the result of this theorem. The delay-derivatives are assumed to be $\dot{d}_1(t) \le 0.1$ and $\dot{d}_2(t) \le 0.8$. Here the delay upper bound d_{u1} or d_{u2} is calculated, when the value of either one is known. By combining the two delay factors the results of some existing stability theorems have been provided in the Tables 2.7 and 2.8 respectively. The results validate the fact that the formulation of the stability conditions in LMI framework by single delay approach for ascertaining the stability of time-delay systems provides conservative results compared to independent treatment of the delays.

Table 2.7 Computed delay bound d_{u2} for a given d_{u1} with $\mu_1 = 0.1$ and $\mu_2 = 0.8$

Stability methods	$d_{u1} = 1$	$d_{u1} = 1.2$	$d_{u1} = 1.5$	Remarks
Theorem 2.19	0.5188	0.4528	0.3777	†
[43]	0.512	0.406	0.283	†
[33]	0.415	0.376	0.283	†
[9, 13, 44]	0.180	0.080	Infeasible	‡
[45]	Infeasible	Infeasible	Infeasible	‡
[18]	Infeasible	Infeasible	Infeasible	‡

'†' – $d_1(t)$ and $d_2(t)$ treated separately
'‡'- $d_1(t)$ and $d_2(t)$ treated combinedly

Table 2.8 Computed delay bound d_{u1} for a given d_{u2} with $\mu_1 = 0.1$ and $\mu_2 = 0.1$

Stability methods	$d_{u2} = 0.1$	$d_{u2} = 0.2$	$d_{u3} = 0.3$	Remarks
Theorem 2.19	2.9182	2.3304	1.8324	†
[43]	2.300	1.779	1.453	†
[33]	2.263	1.696	1.324	†
[9, 13, 44]	1.080	0.980	0.880	‡
[45]	0.098	Infeasible	Infeasible	‡
[18]	0.074	Infeasible	Infeasible	‡

'†' - $d_1(t)$ and $d_2(t)$ treated separately
'‡'- $d_1(t)$ and $d_2(t)$ treated combinedly

2.5.3 *Stability Analysis of TDS with Interval Time Varying Delay*

Consider the system Σ_2 described in (2.3) satisfying following conditions

$$0 \le d_l \le d(t) \le d_u \tag{2.183}$$

$$d_{lu} = d_u - d_l \tag{2.184}$$

$$0 \le \dot{d}(t) \le \mu \tag{2.185}$$

To establish the stability condition using LK functional approach, the integral inequalities arising in the LK functional derivatives need to be approximated by tight bounding inequality to achieve less conservative delay upper bound estimate. To do so, two different types of bounding inequalities have been proposed depending upon the nature of the limits of integration in the integral inequalities and they are discussed below.

- If the limit of integral inequality is certain, then the approximation can be done by following bounding inequality,

$$-\int_{t-\beta}^{t-\alpha} \dot{x}^T(\theta)R\dot{x}(\theta)d\theta \le \gamma^{-1}\begin{bmatrix} x(t-\alpha) \\ x(t-\beta) \end{bmatrix}^T \begin{bmatrix} -R & R \\ \star & -R \end{bmatrix}\begin{bmatrix} x(t-\alpha) \\ x(t-\beta) \end{bmatrix} \tag{2.186}$$

- If the limit of integral inequality is uncertain, then the following bounding inequality that can be used is,

$$-\int_{t-\beta}^{t-\alpha} \dot{x}^T(\theta)R\dot{x}(\theta)d\theta \le \begin{bmatrix} x(t-\alpha) \\ x(t-\beta) \end{bmatrix}^T \begin{bmatrix} M+M^T & -M+N^T \\ \star & -N-N^T \end{bmatrix} \begin{bmatrix} x(t-\alpha) \\ x(t-\beta) \end{bmatrix} + \gamma \begin{bmatrix} x(t-\alpha) \\ x(t-\beta) \end{bmatrix}^T \begin{bmatrix} M \\ N \end{bmatrix} R^{-1} \begin{bmatrix} M \\ N \end{bmatrix}^T \begin{bmatrix} x(t-\alpha) \\ x(t-\beta) \end{bmatrix} \quad (2.187)$$

where, $\gamma = \beta - \alpha$. The main result of the delay-range-dependent stability of the time-delay system (Σ_2) is presented below in the form of theorem.

Theorem 2.20 *System Σ_2 is asymptotically stable satisfying the conditions (2.183)–(2.185) if there exist symmetric matrices $P > 0$, $Q_i > 0, i = 1, 2, \ldots 4$, $R_j > 0$, and any arbitrary matrices of appropriate dimensions $M_j, N_j, j = 1, 2$ satisfying the following LMIs:*

$$\begin{bmatrix} \Theta & \Phi_1 \\ \star & -R_2 \end{bmatrix} < 0, \quad (2.188)$$

$$\begin{bmatrix} \Theta & \Phi_2 \\ \star & -R_2 \end{bmatrix} < 0, \quad (2.189)$$

where, $\Phi_1 = \begin{bmatrix} 0 \\ M_1 \\ N_1 \\ 0 \end{bmatrix}$, $\Phi_2 = \begin{bmatrix} 0 \\ 0 \\ M_2 \\ N_2 \end{bmatrix}$, $\Theta = \begin{bmatrix} \Theta_{11} & R_1 & \Theta_{13} & 0 \\ \star & \Theta_{22} & \Theta_{23} & 0 \\ \star & \star & \Theta_{33} & \Theta_{34} \\ \star & \star & \star & \Theta_{44} \end{bmatrix}$

$\Theta_{11}{=}PA + A^TP + \sum_{i=1}^{3} Q_i + A^T(d_l^2R_1 + R_2)A - R_1$,

$\Theta_{13}{=}PA_d + A^T(d_l^2R_1 + R_2)A_d + R_1$, $\Theta_{22}{=}Q_4 - Q_1 - R_1 + d_{lu}^{-1}(M_1 + M_1^T)$,

$\Theta_{23}{=}d_{lu}^{-1}(-M_1 + N_1^T)$,

$\Theta_{33}{=}-(1-\mu)(Q_3 + Q4) + A_d^T(d_l^2R_1 + R_2)A_d + d_{lu}^{-1}(M_2 + M_2^T - N_1 - N_1^T)$,

$\Theta_{34}{=}d_{lu}^{-1}(-M_2 + N_2^T)$, $\Theta_{44}{=}-Q_2 - d_{lu}^{-1}(N_2 + N_2^T)$

Proof Selecting the LK functional candidate as

$$V(t) = x^T(t)Px(t) + \int_{t-d_l}^{t} x^T(\theta)Q_1x(\theta)d\theta + \int_{t-d_u}^{t} x^T(\theta)Q_2x(\theta)d\theta + \int_{t-d(t)}^{t} x^T(\theta)Q_3x(\theta)d\theta + \int_{t-d(t)}^{t-d_l} x^T(\theta)Q_4x(\theta)d\theta + d_l\int_{t-d_l}^{t}\int_{\theta}^{t} \dot{x}^T(s)R_1\dot{x}(s)dsd\theta$$

$$+d_{lu}^{-1}\int_{t-d_u}^{t-d_l}\int_{\theta}^{t}\dot{x}^T(s)R_2\dot{x}(s)d\theta \tag{2.190}$$

Finding the time-derivative of (2.190) one can obtain

$$\begin{aligned}\dot{V}(t) = {} & 2x^T(t)P\dot{x}(t) + x^T(t)(\sum_{i=1}^{3}Q_i)x(t)\\ & -(1-\dot{d}(t))x^T(t-d(t))(\sum_{i=3}^{4}Q_i)x(t-d(t))\\ & -x^T(t-d_l)(Q_1-Q_4)x(t-d_l) - x^T(t-d_u)Q_2x(t-d_u)\\ & +\dot{x}^T(t)(d_l^2R_1+R_2)\dot{x}(t) - d_l\int_{t-d_l}^{t}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta\\ & -d_{lu}^{-1}\int_{t-d_u}^{t-d_l}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta\end{aligned} \tag{2.191}$$

Replacing $\dot{x}(t)$ using (2.3) in (2.191) one gets

$$\begin{aligned}\dot{V}(t) = {} & 2x^T(t)PAx(t) + 2x^T(t)PA_dx(t-d(t)) + x^T(t)(\sum_{i=1}^{3}Q_i)x(t)\\ & -(1-\dot{d}(t))x^T(t-d(t))(\sum_{i=3}^{4}Q_i)x(t-d(t))\\ & -x^T(t-d_l)(Q_1-Q_4)x(t-d_l) - x^T(t-d_u)Q_2x(t-d_u)\\ & +(Ax(t)+A_dx(t-d(t)))^T\{d_l^2R_1+R_2\}(Ax(t)+A_dx(t-d(t)))\\ & -d_l\int_{t-d_l}^{t}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta - d_{lu}^{-1}\int_{t-d_u}^{t-d_l}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta\end{aligned} \tag{2.192}$$

Now, using (2.186) the first integral term in (2.192) is approximated as,

$$-d_l\int_{t-d_l}^{t}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta \le \begin{bmatrix} x(t)\\ x(t-d_l)\end{bmatrix}^T\begin{bmatrix} -R_1 & R_1\\ \star & -R_1\end{bmatrix}\begin{bmatrix} x(t)\\ x(t-d_l)\end{bmatrix} \tag{2.193}$$

Next, the last integral term in (2.192) can be written as,

$$
\begin{aligned}
-d_{lu}^{-1}\int_{t-d_u}^{t-d_l}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta = & -d_{lu}^{-1}\int_{t-d(t)}^{t-d_l}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta \\
& -d_{lu}^{-1}\int_{t-d_u}^{t-d(t)}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta
\end{aligned}
\tag{2.194}
$$

Now, using (2.187), one may approximate the two integral terms of (2.194) as,

$$
\begin{aligned}
-d_{lu}^{-1}\int_{t-d(t)}^{t-d_l}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta \leq & \begin{bmatrix} x(t-d_l) \\ x(t-d(t)) \end{bmatrix}^T \Big\{ d_{lu}^{-1} \\
& \times \begin{bmatrix} M_1^T + M_1^T & -M_1 + N_1^T \\ \star & -N_1 - N_1^T \end{bmatrix} + \varrho \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} R_2^{-1} \begin{bmatrix} M_1 \\ N_1 \end{bmatrix}^T \Big\} \\
& \times \begin{bmatrix} x(t-d_l) \\ x(t-d(t)) \end{bmatrix}
\end{aligned}
\tag{2.195}
$$

and,

$$
\begin{aligned}
-d_{lu}^{-1}\int_{t-d_u}^{t-d(t)}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta \leq & \begin{bmatrix} x(t-d(t)) \\ x(t-d_u) \end{bmatrix}^T \Big\{ d_{lu}^{-1} \\
& \times \begin{bmatrix} M_2^T + M_2^T & -M_2 + N_2^T \\ \star & -N_2 - N_2^T \end{bmatrix} \\
& + (1-\varrho) \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} R_2^{-1} \begin{bmatrix} M_2 \\ N_2 \end{bmatrix}^T \Big\} \\
& \times \begin{bmatrix} x(t-d(t)) \\ x(t-d_u) \end{bmatrix}
\end{aligned}
\tag{2.196}
$$

where,

$$
\varrho = \frac{d(t)-d_l}{d_{lu}}, 0 \leq \varrho \leq 1 \tag{2.197}
$$

In view of (2.185) one may replace the uncertain $\dot{d}(t)$ by μ in (2.192). Substituting (2.195) and (2.196) into (2.194) and substituting the integral term (2.193) into (2.192) one can get

$$
\dot{V}(t) \leq \xi^T(t)(\Theta + \varrho\Phi_1 R_2^{-1}\Phi_1^T + (1-\varrho)\Phi_2 R_2^{-1}\Phi_2^T)\xi(t) \tag{2.198}
$$

where, Θ, Φ_1 and Φ_2 are already defined above and
$\xi(t) = \left[x^T(t)\ x^T(t-d_l)\ x^T(t-d(t))\ x^T(t-d_u) \right]^T$

To ensure the asymptotic stability of (2.3) the matrix $\Theta + \varrho\Phi_1 R_2^{-1}\Phi_1^T + (1-\varrho)\Phi_2 R_2^{-1}\Phi_2^T)$ must be negative definite which further can be simplified in the following form,

$$\varrho(\Theta + \Phi_1 R_2^{-1}\Phi_1^T) + (1-\varrho)(\Theta + \Phi_2 R_2^{-1}\Phi_2^T) < 0 \tag{2.199}$$

The above matrix inequality holds equivalently can be written as the following two LMIs.

$$\Theta + \Phi_1 R_2^{-1}\phi_1^T < 0 \tag{2.200}$$

$$\Theta + \Phi_2 R_2^{-1}\phi_2^T < 0 \tag{2.201}$$

Finally using Schur-complement on (2.200) and (2.201) one can obtain the stability condition stated in (2.188) and (2.189).

For the case of $d_l = 0$, one may set $Q_1 = Q_4 = R_1 = 0$ in (2.190). Then Theorem 2.20 reduces to following corollary.

Corollary 2.4 *System Σ_2 with $d_l = 0$ is asymptotically stable satisfying the conditions (2.183)–(2.185) if there exist symmetric matrices $P > 0$, $Q_i > 0, i = 2, 3$ $R_2 > 0$, and any arbitrary matrices of appropriate dimensions $M_j, N_j, j = 1, 2$ satisfying the following LMIs:*

$$\begin{bmatrix} \Sigma & \Xi_1 \\ \star & -R_2 \end{bmatrix} < 0, \tag{2.202}$$

$$\begin{bmatrix} \Sigma & \Xi_2 \\ \star & -R_2 \end{bmatrix} < 0, \tag{2.203}$$

where, $\Xi_1 = \begin{bmatrix} M_1 \\ N_1 \\ 0 \end{bmatrix}$, $\Xi_2 = \begin{bmatrix} 0 \\ M_2 \\ N_2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & 0 \\ \star & \Sigma_{22} & \Sigma_{23} \\ \star & \star & \Sigma_{33} \end{bmatrix}$

$\Sigma_{11}=PA + A^TP + \sum_{i=2}^{3} Q_i + A^TR_2A + d_u^{-1}(M_1 + M_1^T)$,
$\Sigma_{12}=PA_d + A^TR_2A_d + d_u^{-1}(-M_1 + N_1^T)$,
$\Sigma_{22}=-(1-\mu)Q_3 + A_d^TR_2A_d + d_u^{-1}(M_2 + M_2^T - N_1 - N_1^T)$
$\Sigma_{23}=d_u^{-1}(-M_2 + N_2^T)$, $\Sigma_{33}=-Q_2 - d_u^{-1}(N_2 + N_2^T)$

Numerical Example 2.4 *Consider the system Σ_2 with the following constant matrices*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, A_d = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$$

For a given delay lower bound (for different μ) the delay upper bound estimates are computed using the obtained LMI condition for Numerical Example 2.1 and Numerical Example 2.4 that are tabulated in Tables 2.9 and 2.10 respectively. The computed results are compared with the existing results.

Remark 2.24 The proposed delay-range-dependent stability analysis is a modification over the work in [5]. It is observed in [5] that, to reduce the conservatism in the delay upper bound estimate, a convex combination of LMIs are derived by approximating the uncertain factor $\gamma = \beta - \alpha$. However, approximation of this γ could not be fully avoided in [5] as the terms $d(t) - d_l$ and $d_u - d(t)$ are assigned to zero.

To avoid the approximation on uncertain factor γ stability analysis is derived using both the integral inequalities (2.186) and (2.187), instead of using single bounding inequality as in Lemma 1 of [5]. To implement the bounding inequality in (2.187) along with (2.186) a new LK functional is proposed. The inequality in (2.187) is used when the limit of integral is uncertain, else (2.186). This is an important feature and in addition helps to improve the delay upper bound estimate.

Remark 2.25 Note that the present method emphasizes the less conservativeness of the derived stability criterion based on the LK functional that belongs to the class of (2.190). However, there exists LK functional that is based on delay partitioning approach (where the delay intervals are further divided into sub intervals) as in [35] and [47]. It appears that the use of such LK functional along with the present bounding inequalities proposed in (2.186) and (2.187) may leads to further improvement in the estimate of the delay upper bound estimate.

Remark 2.26 In order to derive less conservative stability condition a separate class of LK functional was considered that consists of triple integral as well as single-integral terms with $t - d_l$ as the upper limit of integration in the latter term [46]. But it is observed from the results presented in Tables 2.9, 2.10 and 2.11 that the suggested modifications cannot effectively and consistently reduce the conservatism in the delay bound estimate compared to the results obtained by the proposed method.

Table 2.9 Computed delay bound d_u for different μ (Example 2.1)

d_l	[35]	[3]	[5]	[46]	Present result
$\mu = 0.5$					
0	–	2.04	2.0723	–	2.2594
1	–	2.07	2.1277	–	2.3303
2	–	2.43	2.5049	–	2.6127
3	–	3.22	3.2591	–	3.3147
4	–	4.07	4.0744	–	4.0900
$\mu = 0.9$					
0	–	1.37	1.5305	–	1.8502
1	–	1.74	1.8736	–	2.0550
2	–	2.43	2.5049	–	2.6127
3	–	3.22	3.2591	–	3.3147
4	–	4.07	4.0744	–	4.0900
$\mu \geq 1$					
0	1.01	1.34	1.5295	–	1.8497
1	1.64	1.7424	1.8737	–	2.0550
2	2.39	2.4328	2.5049	2.5663	2.6127
3	3.20	3.2234	3.2591	3.3408	3.3147
4	4.06	4.0644	4.0744	4.1690	4.0900

Table 2.10 Computed delay bound d_u for $\mu = 0.3$ (Example 2.4)

d_l	[3]	[4]	[5]	[46]	Present result
0	2.19	2.1959	2.2161	–	2.3369
1	2.2125	2.2128	2.2474	2.3167 - Theorem 2	2.4043
2	2.4091	2.4179	2.4798	–	2.5871
3	3.3342	3.3382	3.3893	–	3.4766
4	4.2799	4.2819	4.3250	–	4.3978
5	5.2393	5.2403	5.2773	–	5.3394

Table 2.11 Computed delay bound d_u for $\mu \geq 1$ (Example 2.4)

d_l	[35]	[3]	[4]	[5]	[46]	Present result
0	0.67	0.77	0.7744	0.8714	–	1.0420
0.3	0.91	0.9431	0.9860	1.0715	–	1.2301
0.5	1.07	1.0991	1.1325	1.2191	–	1.3713
0.8	1.33	1.3476	1.3733	1.4539	–	1.5960
1	1.50	1.5187	1.5401	1.6169	1.6198	1.7523
2	2.39	2.4000	2.4100	2.4798	2.4884	2.5871

2.6 Robust Stability Analysis of Time-Delay System

The robust stability problem of time-delay systems considers parametric uncertainties in the system matrices. The uncertainty arises in the system model due to following reasons:

1. System parameters are often not known accurately while modeling, rather the ranges are known.
2. Due to the limitation on the part of availability of the mathematical tools, one tends to create simple mathematical models that approximate a practical systems, thus some aspects of system dynamics are ignored, which is known as un-modeled dynamics.
3. Some control strategies are required to operate the systems under different operating conditions.

2.6.1 Characteristic of Structured Uncertainties [7]

To include these parametric uncertainties, a bounding set containing all possible uncertainties in the set is considered that makes the mathematical analysis and synthesis simpler. In the present discussion, the structured uncertainties are considered for robust stability analysis. Consider a single time-delay system,

$$\dot{x}(t) = A(t)x(t) + A_d(t)x(t - d(t)) \tag{2.204}$$

where, $A(t)$ and $A_d(t)$ are uncertain system matrices, and defined within a compact uncertain set Π as,

$$(A(t), A_d(t)) \in \Pi, \forall t \geq 0 \tag{2.205}$$

The various characterization of the structured uncertainties [7] are as follows,

(i) **Polytopic uncertainty**: In practice the parameters of the system are not completely known and may vary between lower and upper limits, and these uncertain parameters are found to vary linearly in the system matrices. Thus, the collection of all possible system matrices form a polytopic uncertainty set. Say there exist n_p uncertain parameters, then the number of uncertain elements in the set Π is $n_m = 2^{n_p}$, as the parameters vary between upper and lower limits,

$$\pi^k = [A^k, A_d^k], k = 1, 2, \ldots n_m$$

n_m is known as vertices. The uncertainty set Π is expressed as the convex hull of these vertices as given by,

$$\Pi = \left\{ \sum_{k=1}^{n_m} \alpha_k \pi^k | \alpha_k \geq 0, k = 1, 2, \ldots n_m; \sum_{k=1}^{n_m} \alpha_k = 1 \right\}$$

(ii) **Sub-polytopic uncertainty**: The subpolytopic uncertainty is more general than polytopic uncertainty. In this case the uncertainty set π possesses n_m vertices and the uncertainty set Π is contained in the convex hull of the vertices

$$\Pi \subset co \left\{ \pi^i, i = 1, 2, \ldots n_m \right\}$$

where, $\pi = \sum_{i=1}^{n_m} \beta_i \pi_i$, for some scalar $\beta_i \geq 0$ and $\sum_{i=1}^{n_m} \beta_i = 1$

(iii) **Norm-bounded uncertainty**: Here the uncertain system matrices $\pi = (A(t), A_d(t))$ in (2.204) is decomposed into two parts, nominal part $\pi_n = (A, A_d)$ and the uncertain part $\Delta\pi = (\Delta A, \Delta A_d)$, thus $\pi = \pi_n + \Delta\pi$. The uncertain part can be further decomposed as [12],

$$\Delta A = D_a F_a(t) E_a$$
$$\Delta A_d = D_d F_d(t) E_d$$

where, $F_a(t)$ and $F_d(t)$ are unknown real time-varying matrices with Lebesgue measurable elements satisfying

$$\| F_a(t) \| \leq 1$$
$$\| F_d(t) \| \leq 1$$

and, D_a, D_d, E_a and E_d are known real constant matrices that characterizes how the uncertain parameters in $F_a(t)$ and $F_d(t)$ enter the nominal system matrices (A, A_d).

The delay-dependent robust stability conditions are generally obtained by directly extending the stability conditions of nominal time-delay systems. To do so, the nominal system A and A_d are replaced by $A(t)$ and $A_d(t)$ in the stability conditions derived for nominal systems. The uncertain matrices/parameters that now appear in the formulation of stability analysis are eliminated with the help of appropriate bounding inequality lemma along with the condition $\| F_a(t) \| \leq 1$ and $\| F_d(t) \| \leq 1$ to get the robust stability condition.

Note: Here, the robust stability analysis of an uncertain time-delay systems is discussed for norm-bounded type uncertainty structure only. The time-delay system with norm-bounded uncertainty structure is presented in details in Sect. 2.2.2 from (2.15)–(2.22).

Following lemmas are useful for deriving robust stability condition of an uncertain time-delay systems with norm-bounded uncertainties.

Lemma 2.6 ([12, 13]) *Let D, E and F be real matrices of appropriate dimensions with $\| F \| \leq 1$, then for any scalar $\epsilon > 0$,*

$$DFE + E^T F^T D^T \leq \frac{1}{\epsilon} DD^T + \epsilon E^T E \tag{2.206}$$

Lemma 2.7 ($\mathcal{S}$-Procedure for quadratic forms and strict inequalities [24]). *Let $T_0, \ldots, T_p \in \mathcal{R}^{n \times n}$ be symmetric matrices. Considering the following conditions on $T_0, \ldots, T_p$*

$$\zeta^T T_0 \zeta > 0, \forall \zeta \neq 0 \tag{2.207}$$

such that,

$$\zeta^T T_i \zeta \geq 0, i = 1, \ldots p \tag{2.208}$$

If there exists some scalars, $\tau_1 \geq 0, \ldots \tau_p \geq 0$, such that

$$T_0 - \sum_{i=1}^{i=p} \tau_i T_i > 0 \tag{2.209}$$

then, (2.207) and (2.208) holds.

2.6.2 Delay-Dependent Robust Stability Analysis

In this subsection, some relevant existing results on delay-dependent robust stability analysis of time-delay systems considering norm-bounded uncertainties in the system matrices are discussed to understand the application of lemmas 2.6 and 2.7 for eliminating the uncertain matrix from the derived stability condition in order to obtain desired robust stability condition.

Several literature on delay-dependent robust stability analysis considering polytopic type model uncertainties can be found in [9, 28, 29, 48], and references cited therein and are not discussed here as the present work focuses on norm-bounded uncertainty structure only.

Theorem 2.21 ([25]) *If there exist matrices $P = P^T > 0, Q = Q^T > 0, X > 0, Y > 0$, any matrix Z and scalars $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that following LMIs hold:*

$$\begin{bmatrix} Y_{11} & -Y + PA_d & d_u A^T Z & PD_a & PD_d \\ \star & -Q + \epsilon_2 E_d^T E_d & d_u A_d^T Z & 0 & 0 \\ \star & \star & -d_u Z & d_u Z D_a & d_u Z D_d \\ \star & \star & \star & -\epsilon_1 I & 0 \\ \star & \star & \star & \star & -\epsilon_2 I \end{bmatrix} < 0 \tag{2.210}$$

$$\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \geq 0 \tag{2.211}$$

where, $Y_{11} = A^T P + PA + d_u X + Y + Y^T + Q + \epsilon_1 E_a^T E_a$, *then the system (2.15)–(2.16) is asymptotically stable for any time-delay satisfying* $0 \leq d \leq d_u$ *and all admissible uncertainties as defined in (2.19)–(2.20).*

Proof The robust stability condition in (2.210) can be obtained as a direct extension of the stability results in Theorem 2.5. Note that, the stability condition in Theorem 2.5 was obtained for constant delay d.

Replace A and A_d in (2.52) with $A(t)$[1] and $A_d(t)$[2] respectively, which gives,

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) \\ \star & -Q & (2,3) \\ \star & \star & -d_u Z \end{bmatrix} \tag{2.212}$$

where, $(1,1){=}A^T P + PA + d_u X + Y + Y^T + Q + E_a^T F_a^T D_a^T P + P D_a F_a E_a$,

$$(1,2){=}-Y + PA_d + P D_d F_d E_d,\ (1,3){=}d_u(E_a^T F_a^T D_a^T Z + A^T Z),$$

$$(2,3){=}d_u(E_d^T F_d^T D_d^T Z + A_d^T Z)$$

Now, multiplying both the sides of (2.212) by any vector y_i $(i = 1,2,3)$ and its transpose one gets,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}^T \begin{bmatrix} (1,1) & (1,2) & (1,3) \\ \star & -Q & (2,3) \\ \star & \star & -d_u Z \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{2.213}$$

Expanding (2.213) one can get following cross terms involving the product of vectors y_i and the uncertain matrices,

$$y_1^T(E_a^T F_a^T D_a^T P + P D_a F_a E_a) y_1,\ y_1^T P D_d F_d E_d y_2$$
$$d_u y_1^T E_a^T F_a^T D_a^T Z y_3 \textit{ and } d_u y_2^T E_d^T F_d^T D_d^T Z y_3$$

Defining,

$$p = F_a(t) E_a y_1,\ q = F_d(t) E_d E_d y_2$$

Thus one can rewrite (2.213) in view of above as,

[1] $A(t) = A + \Delta A(t)$, where, $\Delta A(t) = D_a F_a(t) E_a$.

[2] $A_d(t) = A + \Delta A_d(t)$, where, $\Delta A_d(t) = D_d F_d(t) E_d$.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ p \\ q \end{bmatrix}^T \begin{bmatrix} X_{11} & -Y + PA_d & d_u A^T Z & PD_a & PD_d \\ \star & -Q & d_u A_d^T Z & 0 & 0 \\ \star & \star & -d_u Z & d_u Z D_a & d_u Z D_d \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ p \\ q \end{bmatrix} \tag{2.214}$$

where, $X_{11} = A^T P + PA + d_u X + Y + Y^T + Q$

It may be noted that as the elements (5,5) and (6,6) in (2.214) are zero, so it cannot be solved for negative definiteness. To overcome this problem one can have,

$$\begin{aligned} \epsilon_1 p^T p &\leq \epsilon_1 y_1^T E_a^T E_a y_1, \ with \parallel F_a(t) \parallel \leq 1 \\ \epsilon_2 q^T q &\leq \epsilon_2 y_2^T E_d^T E_d y_2, \ with \parallel F_d(t) \parallel \leq 1 \end{aligned} \tag{2.215}$$

where, $\epsilon_1 > 0$ and $\epsilon_2 > 0$.
Now, applying Lemma 2.7 ($\mathcal{S} - procedure$), one can combine (2.214) and (2.215) to obtain the LMI condition of (2.210). As, the LMI condition in (2.211) do not contain any uncertain terms so it remains unchanged as in (2.53) (see Theorem 2.5).

Theorem 2.22 ([13]) *Given scalars $d_u > 0$ and $\mu < 1$, satisfying the conditions $0 \leq d(t) \leq d_u$ and $\dot{d}(t) \leq \mu < 1$, the uncertain system (2.15) is robustly stable if there exist symmetric positive definite matrices $P = P^T > 0$, $Q = Q^T > 0$ and $Z = Z^T > 0$, a symmetric-semi-positive-definite matrix $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \geq 0$, any matrices Y and T such that following LMIs hold:*

$$\phi = \begin{bmatrix} \phi_{11} + \epsilon E_a^T E_a & \phi_{12} + \epsilon E_a^T E_d & d_u A^T Z & PD \\ \star & \phi_{22} + \epsilon E_d^T E_d & d_u A_d^T Z & 0 \\ \star & \star & -d_u Z & d_u Z D \\ \star & \star & \star & -\epsilon I \end{bmatrix} < 0 \tag{2.216}$$

$$\begin{bmatrix} X_{11} & X_{12} & Y \\ \star & X_{22} & T \\ \star & \star & Z \end{bmatrix} \geq 0 \tag{2.217}$$

where,ϕ_{11}=$PA + A^T P + Y + Y^T + Q + d_u X_{11}$
ϕ_{12}=$PA_d - Y + Y^T + d_u X_{12}$ and ϕ_{22}=$-T - T^T - (1-\mu)Q + d_u X_{22}$

Remark 2.27 The proof of this theorem follows directly from the stability condition derived in Theorem 2 of [13] by replacing the A and A_d matrices with $A + DF(t)E_a$ and $A_d + DF(t)E_d$. In this method to establish the the robust stability condition in an LMI framework the uncertain matrix $F(t)$ has been eliminated using Lemma 2.6 instead of using ($\mathcal{S} - procedure$) Lemma 2.7.

Theorem 2.23 ([14]) *Consider the uncertain systems (2.15). Given the scalars $d_u > 0$ and $\mu > 0$, the system described in (2.15) is robustly asymptotically stable for any time-delay satisfying the conditions $0 \le d(t) \le d_u$ and $\dot{d}(t) \le \mu < 1$, with the admissible uncertainties (2.19)–(2.20) satisfying (2.21)–(2.22), if there exist symmetric positive definite matrices, P, Q, R, T, matrices S_i $(i = 1, 2 \ldots, 4)$ with appropriate dimensions and scalars ϵ_i $(i = 1, 2)$, then following LMIs hold:*

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} & \Sigma_{16} & P_{11}D_a & P_{11}D_d & \mu P_{12} \\ \star & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} & \Sigma_{25} & \Sigma_{26} & 0 & 0 & 0 \\ \star & \star & -Q_{11} & \Sigma_{34} & 0 & 0 & P_{12}^T D_a & P_{12}^T D_d & \mu P_{22} \\ \star & \star & \star & \Sigma_{44} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -Q_{11} & -Q_{12} & \Sigma_{57} & \Sigma_{58} & 0 \\ \star & \star & \star & \star & \star & -Q_{22} & \Sigma_{67} & \Sigma_{68} & 0 \\ \star & \star & \star & \star & \star & \star & -\epsilon_1 I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\epsilon_2 I & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & -\mu T \end{bmatrix} < 0 \quad (2.218)$$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ \star & P_{22} \end{bmatrix} \ge 0, with \quad P_{11} > 0 \quad (2.219)$$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ \star & Q_{22} \end{bmatrix} \ge 0 \quad (2.220)$$

where, $\Sigma_{11}= A^T P_{11} + P_{11}A + R + S_1 + S_1^T + \epsilon_1 E_a^T E_a,\ \Sigma_{12}=P_{11}A_d - S_1^T + S_2$
$\Sigma_{13}=A^T P_{12} + S_3,\ \Sigma_{14}=P_{12} - S_1^T + S_4,\ \Sigma_{15}=d_u(Q_{11} + A^T Q_{12}^T)$
$\Sigma_{16}=d_u(Q_{12} + A_T Q_{22}),\ \Sigma_{22}=-(1-\mu)R + \mu T - S_2^T - S_2 + \epsilon_2 E_d^T E_d$
$\Sigma_{23}=A_d^T P_{12} - S_3,\ \Sigma_{24}=-S_2^T - S_4,\ \Sigma_{25}=d_u A_d^T Q_{12}^T,\ \Sigma_{26}=d_u A_d^T Q_{22}$
$\Sigma_{34}=P_{22} - Q_{12} - S_3^T,\ \Sigma_{44}=-Q_{22} - S_4^T - S_4$
$\Sigma_{57}=d_u Q_{12} D_a,\ \Sigma_{58}=d_u Q_{12} D_d,\ \Sigma_{67}=d_u Q_{22} D_a$, and $\Sigma_{68}=d_u Q_{22} D_d$

Proof This theorem can be proved by directly extending the stability condition (2.95) (see Theorem 2.12). To accomplish this one can replace A and A_d by $A(t)$ and $A_d(t)$ in (2.95) and subsequently the resulting matrix inequality is decomposed into nominal and uncertain matrices as,

$$\Sigma_n + \Sigma_{un}^T + \Sigma_{un} < 0 \quad (2.221)$$

where the uncertain matrices are defined as,

$$\Sigma_{un} = D_1 F_a(t) E_1 + D_2 F_d(t) E_2$$

and, $D_1 = [D_a^T P_{11}\ 0\ D_a^T P_{12}\ 0\ d_u D_a^T Q_{12}^T\ d_u D_a^T Q_{22}\ 0]^T$, $E_1 = [E_a\ 0\ 0\ 0\ 0\ 0\ 0]$, $D_2 = [D_d^T P_{11}\ 0\ D_d^T P_{12}\ 0\ d_u D_d^T Q_{12}^T\ d_u D_d^T Q_{22}\ 0]^T$ and $E_2 = [0\ E_d\ 0\ 0\ 0\ 0\ 0]$.

Employing Lemma 2.6 and Schur's complement on (2.221) one can obtain the LMI condition (2.218). If the uncertainties in (2.19) and (2.20) are given as

$$D_a = D_d = D \qquad F_a(t) = F_d(t) = F(t)$$

then the LMI condition in (2.218) reduces to the following LMI condition,

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} & \Sigma_{16} & P_{11}D & \mu P_{12} \\ \star & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} & \Sigma_{25} & \Sigma_{26} & 0 & 0 \\ \star & \star & -Q_{11} & \Sigma_{34} & 0 & 0 & P_{12}^T D & \mu P_{22} \\ \star & \star & \star & \Sigma_{44} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -Q_{11} & -Q_{12} & \Sigma_{57} & 0 \\ \star & \star & \star & \star & \star & -Q_{22} & \Sigma_{67} & 0 \\ \star & \star & \star & \star & \star & \star & -\epsilon I & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\mu T \end{bmatrix} < 0 \tag{2.222}$$

where, $\Sigma_{11}= A^T P_{11} + P_{11}A + R + S_1 + S_1^T + \epsilon E_a^T E_a$, $\Sigma_{12}=P_{11}A_d - S_1^T + S_2 + \epsilon E_a^T E_d$
$\Sigma_{13}=A^T P_{12} + S_3$, $\Sigma_{14}=P_{12} - S_1^T + S_4$, $\Sigma_{15}=d_u(Q_{11} + A^T Q_{12}^T)$
$\Sigma_{16}=d_u(Q_{12} + A_T Q_{22})$, $\Sigma_{22}=-(1-\mu)R + \mu T - S_2^T - S_2 + \epsilon E_d^T E_d$
$\Sigma_{23}=A_d^T P_{12} - S_3$, $\Sigma_{24}=-S_2^T - S_4$, $\Sigma_{25}=d_u A_d^T Q_{12}^T$, $\Sigma_{26}=d_u A_d^T Q_{22}$
$\Sigma_{34}=P_{22} - Q_{12} - S_3^T$, $\Sigma_{44}=-Q_{22} - S_4^T - S_4$ $\Sigma_{57}=d_u Q_{12}D$, $\Sigma_{67}=d_u Q_{22}D$

In [36] a delay-dependent robust stability condition has been proposed using new type of LK functional and integral inequality lemma of [25]. The uncertain time-delay system in (2.15) for $t \geq 0$ with an assumption that $D_a = D_d = D$ and $F_a(t) = F_d(t) = F(t)$ is expressed as,

$$\begin{aligned} \dot{x}(t) &= (A + DF(t)E_a)x(t) + (A_d + DF(t)E_d)x(t-d(t)) \\ &= Ax(t) + A_d x(t-d(t)) + D[F(t)E_a x(t) + F(t)E_d x(t-d(t))] \\ &= Ax(t) + A_d x(t-d(t)) + Dp(t),\ \forall\, t\ \in\ [-d_u,\ 0] \end{aligned} \tag{2.223}$$

where, $p(t) = F(t)q(t)$ and $q(t) = E_a x(t) + E_d x(t-d(t))$ and $F(t)^T F(t) \leq \gamma^{-2} I$. The LMI stability condition for the uncertain system described in (2.223) is presented below.

Theorem 2.24 ([36]) *For a given γ, the delayed uncertain system (2.223) with $\dot{d}(t) \leq \mu$ is asymptotically stable if there exist symmetric matrices $P > 0$, $Q_0 > 0$, $Q_1 > 0$, $S_0 > 0$, $S_1 > 0$, Y_{11}, Y_{12}, Y_{22}, Z_{11}, Z_{12}, Z_{22} and Σ such that the following conditions hold,*

$$
\begin{aligned}
0 > {} & \Sigma(Ae_1^T + A_d e_2^T - e_4^T + De_5^T) + Y_{12}(e_1 - e_2)^T + d_u Y_{11} \\
& + (e_1 A^T + e_2 A_d^T - e_4 + e_5 D^T)\Sigma^T + (e_1 - e_2)Y_{12}^T \\
& + (e_1 E_a^T + e_2 E_d^T)(E_a e_1^T + E_d e_2^T) + Z_{12}(e_2 - e_3)^T \\
& + (e_2 - e_3)Z_{12}^T + e_4 P e_1^T + e_1 P e_4^T - (1-\mu)e_2 Q_1 e_2^T \\
& + d_u e_4 (S_0 + S_1) e_4^T + e_1 (Q_0 + Q_1) e_1^T - e_3 Q_0 e_3^T - \gamma^2 e_5 e_5^T
\end{aligned} \tag{2.224}
$$

$$
\begin{aligned}
0 > {} & \Sigma(Ae_1^T + A_d e_2^T - e_4^T + De_5^T) + Y_{12}(e_1 - e_2)^T + d_u Z_{11} \\
& + (e_1 A^T + e_2 A_d^T - e_4 + e_5 D^T)\Sigma^T + (e_1 - e_2)Y_{12}^T \\
& + (e_1 E_a^T + e_2 E_d^T)(E_a e_1^T + E_d e_2^T) + Z_{12}(e_2 - e_3)^T \\
& + (e_2 - e_3)Z_{12}^T + e_4 P e_1^T + e_1 P e_4^T - (1-\mu)e_2 Q_1 e_2^T \\
& + d_u e_4 S_0 e_4^T + e_1 (Q_0 + Q_1) e_1^T - e_3 Q_0 e_3^T - \gamma^2 e_5 e_5^T
\end{aligned} \tag{2.225}
$$

$$
\begin{bmatrix} Y_{11} & Y_{12} \\ \star & Y_{22} \end{bmatrix} \geq 0, \; \begin{bmatrix} Z_{11} & Z_{12} \\ \star & Z_{22} \end{bmatrix} \geq 0, \; S_0 \geq Z_{22}, \; S_0 + (1-\mu)S_1 \geq Y_{22} \tag{2.226}
$$

Remark 2.28 The stability method in Theorem 2.24 uses a new LK functional of the form,

$$
\begin{aligned}
V(t) = {} & x^T(t)Px(t) + \int_{t-d_u}^{t} x^T(s)Q_0 x(s)ds + \int_{t-d(t)}^{t} x^T(s)Q_1 x(s)ds \\
& + \int_{-d_u}^{0}\int_{t+s}^{t} \dot{x}^T(\alpha)S_0\dot{x}(\alpha)d\alpha ds + \int_{-d(t)}^{0}\int_{t+s}^{t} \dot{x}^T(\alpha)S_1\dot{x}(\alpha)d\alpha ds
\end{aligned} \tag{2.227}
$$

with Q_0, Q_1, S_0 and S_1 are symmetric positive-definite matrices. One may note that, the delay-dependent functional (double integral) term in (2.227) contains uncertain limit of integration which is usually not used in any LK functional, in this regard the chosen LK functional is new. The time-derivative of the functional in (2.227) yields two quadratic integral terms of the form,

$$
-\int_{t-d_u}^{t} \dot{x}^T(s)S_0\dot{x}(s)ds, \quad and \quad (1-\dot{d}(t))\int_{t-d(t)}^{t} \dot{x}^T(s)S_1\dot{x}(s)ds
$$

The information of the system matrices are incorporated into the formulation by introducing one more matrix variables to satisfy the following,

$$
0 = 2\chi^T(t)\Sigma(Ae_1^T + A_d e_2^T - e_4^T + De_5^T)\chi(t) \tag{2.228}
$$

where, $\chi(t) = [x^T(t)\; x^T(t-d(t))\; x^T(t-d_u)\; \dot{x}(t)\; p^T(t)]^T$ and e_i $(i = 1, ..5)$ are corresponding block identity matrix. For handling the uncertainty, one more constraint is introduced which is of the form,

$$0 \leq q^T(t)q(t) - \gamma^2 p^t(t)p(t) \tag{2.229}$$

this can further be written as,

$$0 \leq \chi^T(t)\{(e_1 E_a^T + e_2 E_d^T)(E_a e_1^T + E_d e_2^T) - \gamma^2 p^t(t)p(t)\}\chi(t) \tag{2.230}$$

The integral terms shown above are approximated using integral bounding lemma (Lemma 2.3) which is further added with the other derivative terms of LK functional along with the terms of (2.228), the resulting quadratic expression is finally combined with (2.230) using $\mathcal{S} - Procedure$ lemma to get $\dot{V}(t)$ expression of the form,

$$\begin{aligned}\dot{V}(t) \leq \chi^T(t)\{d(t)\Gamma + (d_u - d(t))\Pi + \Omega\}\chi(t) - \int_{s_1}(.)^T\Phi_1(.)ds \\ - \int_{s_2}(.)^T\Phi_2(.)ds\end{aligned} \tag{2.231}$$

where, $\Gamma, \Pi, \Omega, \Phi_1$ and Φ_2 are matrices of compatible dimensions. The detailed structure of the matrices can be found in [36]. The above expression is a convex combination of matrices Γ and Π on $d(t)$ that can be further expressed by two matrix inequality conditions as,

1. When $d(t) = 0$ one can write

$$d_u\Pi + \Omega - \int_{s_1}(.)^T\Phi_1(.)ds \; - \int_{s_2}(.)^T\Phi_2(.)ds < 0$$

2. When $d(t) = d_u$ one can write

$$d_u\Gamma + \Omega - \int_{s_1}(.)^T\Phi_1(.)ds \; - \int_{s_2}(.)^T\Phi_2(.)ds < 0$$

In order to guarantee the negativity of $\dot{V}(t)$ in (2.231), first one has to impose two matrix inequality constraints, $\Phi_1 \geq 0$ and $\Phi_2 \geq 0$ such that the quadratic integral terms in the above expression remains semi-positive definite. In addition to these constraints two more LMI constraints due to the use of bounding lemma (Lemma 2.3) need to be satisfied (refer (2.226)).

Note that, the number of LMIs used are six and number of matrix variables involved are 12, so this theorem has still room for improvement in terms of reducing the number of LMI constraints and matrix variables by using efficient bounding inequalities as proposed in the stability condition in Sect. 2.5.3.

For detailed proof of Theorem 2.24 one can refer [36].

For convenience of the discussion of the main results of this chapter, some preliminaries including few definitions, basic theorems on robust stability of time-delay systems which are related to the main results on delay-dependent robust stability analysis are presented in preceding sections.

2.7 Main Results on Delay-Dependent Robust Stability Analysis of TDS

In this section, delay-dependent robust stability conditions for uncertain time-delay systems in an LMI framework are presented. The additive uncertainties are assumed to be of norm-bounded type (explained in Sect. 2.6.1). For robust stability analysis, an uncertain time-delay system described in (2.15)–(2.20) satisfying the conditions (2.21) and (2.22) (refer Sect. 2.2.2 of the chapter) is considered.

The delay-dependent robust stability conditions are direct extension of the stability analysis of TDS as discussed in Sect. 2.5.1. The nominal matrices A and A_d in the stability condition are replaced by the uncertain matrices $A(t)$ and $A_d(t)$. With the proper choice of lemma (i.e., 2.6 or 2.7) the uncertain matrices are eliminated.

2.7.1 Delay-Dependent Robust Stability Analysis of TDS with Single Time Delay

Considering an uncertain time-delay system (2.15) with the structure as described in (2.17)–(2.20), satisfying the condition (2.21)–(2.22). The robust stability condition is presented in the following theorem by constructing a new LK functional and subsequently using improved bounding inequalities for the integral quadratic terms arising in the LK functional derivative which, in turn, yields convex combination of LMIs.

The main contribution of the proposed method have already been discussed in Sect. 2.5.1 (see Remarks 2.20–2.22).

Theorem 2.25 *For given scalars $d_u > 0$ and ϵ_i $(i = 1, 2) > 0$, the system (2.15) is robustly asymptotically stable satisfying the conditions (2.5) and $0 < \mu < 1$, for the admissible uncertainties (2.19)–(2.20) satisfying (2.21)–(2.22), if there exist symmetric positive definite matrices $P,\ Q,\ R_i$ $(i = 1, 2)$, any free matrices L_i $(i = 1, 2, 3)$, M_i $(i = 1, 2, 3)$ and $G_i (i = 1, 2)$ such that, $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0$, $Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} > 0$, then the following LMIs hold:*

$$\begin{bmatrix} \Xi & \mu\tilde{P} & d_u M & D_1 & D_2 \\ \star & -\mu T & 0 & 0 & 0 \\ \star & 0 & -Q_{22} & 0 & 0 \\ \star & 0 & 0 & -\epsilon_1 I & 0 \\ \star & 0 & 0 & 0 & -\epsilon_2 I \end{bmatrix} < 0 \tag{2.232}$$

$$\begin{bmatrix} \Xi & \mu\tilde{P} & d_u L & D_1 & D_2 \\ \star & -\mu T & 0 & 0 & 0 \\ \star & 0 & -Q_{22} & 0 & 0 \\ \star & 0 & 0 & -\epsilon_1 I & 0 \\ \star & 0 & 0 & 0 & -\epsilon_2 I \end{bmatrix} < 0 \tag{2.233}$$

where, $L = \begin{bmatrix} L_1^T & L_2^T & L_3^T & 0 & 0 & 0 \end{bmatrix}^T$, $M = \begin{bmatrix} M_1^T & M_2^T & M_3^T & 0 & 0 & 0 \end{bmatrix}^T$, $D_1 = \begin{bmatrix} D_a^T G_1 & 0 & 0 & 0 & 0 & D_a^T G_2 \end{bmatrix}^T$, $D_2 = \begin{bmatrix} D_d^T G_1 & 0 & 0 & 0 & 0 & D_d^T G_2 \end{bmatrix}^T$, $\tilde{P}$ *is defined in (2.132) and*

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & 0 & \Xi_{16} \\ \star & \Xi_{22} & \Xi_{23} & \Xi_{24} & 0 & \Xi_{16} \\ \star & \star & \Xi_{33} & 0 & 0 & 0 \\ \star & \star & 0 & -Q_{11} & 0 & \Xi_{46} \\ 0 & 0 & 0 & 0 & -Q_{11} & 0 \\ \star & \star & 0 & \star & 0 & \Xi_{66} \end{bmatrix} \tag{2.234}$$

where,

$\Xi_{11} = \Omega_{11} + \epsilon_1 E_a^T E_a$, $\Xi_{12} = \Omega_{12}$, $\Xi_{13} = \Omega_{13}$, $\Xi_{14} = \Omega_{14}$, $\Xi_{16} = \Omega_{16}$
$\Xi_{22} = \Omega_{22} + \epsilon_2 E_d^T E_d$, $\Xi_{23} = \Omega_{23}$, $\Xi_{24} = \Omega_{24}$, $\Xi_{26} = \Omega_{26}$
$\Xi_{33} = \Omega_{23}$, $\Xi_{46} = \Omega_{46}$, $\Xi_{66} = \Omega_{66}$
the elements of matrix Ω ($\Omega_{i,j}$, $i = 1, ..6;\ j = 1, ..6$) are already defined in stability Theorem 2.18.

Proof The proof of this theorem follows directly from the stability condition in (2.132)–(2.134). The nominal matrices A and A_d appearing in Ω matrix in (2.132) are replaced by time-varying matrices $A(t)$ and $A_d(t)$ which are defined as,

$$A(t) = A + \Delta_a(t)$$

$$A_d(t) = A_d + \Delta_d(t)$$

where $\Delta_a(t) = D_a F_a(t) E_a$ and $\Delta_d(t) = D_d F_d(t) E_d$. With these replacement in Ω matrix one can get resulting matrix as,

$$\Xi = \begin{bmatrix} \Omega_{11}|_\Delta & \Omega_{12}|_\Delta & \Omega_{13} & \Omega_{14} & 0 & \Omega_{16}|_\Delta \\ \star & \Omega_{22} & \Omega_{23} & \Omega_{24} & 0 & \Omega_{26}|_\Delta \\ \star & \star & \Omega_{33} & 0 & 0 & 0 \\ \star & \star & 0 & -Q_{11} & 0 & \Omega_{46} \\ 0 & 0 & 0 & 0 & -Q_{11} & 0 \\ \star & \star & 0 & \star & 0 & \Omega_{66} \end{bmatrix} \tag{2.235}$$

where, $\Omega_{11}|_\Delta = d_u^2 Q_{11} + G_1^T A(t) + A^T(t) G_1 + P_{12} + P_{12}^T + R_1 + R_2 + d_u(L_1 + L_1^T)$
$\Omega_{12}|_\Delta = G_1^T A_d(t) - P_{12} + d_u(-L_1 + L_2^T + M_1)$
$\Omega_{13} = d_u(L_3^T - M_1)$, $\Omega_{14} = P_{22}$, $\Omega_{16}|_\Delta = P_{11} - G_1^T + A^T(t) G_2$
$\Omega_{22} = -(1-\mu)R_1 - \mu T + d_u(-L_2 - L_2^T + M_2 + M_2^T)$
$\Omega_{23} = d_u(-M_2 + M_3^T - L_3^T)$, $\Omega_{24} = -P_{22}$, $\Omega_{26} = A_d^T G_2$
$\Omega_{33} = d_u(-M_3 - M_3^T) - R_2$, $\Omega_{66} = -G_2 - G_2^T + d_u^2 Q_{22}$

The matrix in (2.235) can be further decomposed in the form,

$$\Xi = \Omega_{nom} + \Omega_{unc} \tag{2.236}$$

where, $\Omega_{nom} = \Omega$ and Ω_{unc} can be represented as,

$$\Omega_{unc} = \Sigma + \Sigma^T \tag{2.237}$$

where,

$$\Sigma = D_1 F_a(t) E_1 + D_2 F_d(t) E_d \tag{2.238}$$

$$D_1 = \begin{bmatrix} D_a^T G_1 & 0 & 0 & 0 & 0 & D_a^T G_2 \end{bmatrix}^T,$$

$$D_2 = \begin{bmatrix} D_d^T G_1 & 0 & 0 & 0 & 0 & D_d^T G_2 \end{bmatrix}^T,$$

$$E_1 = \begin{bmatrix} E_a & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & E_d & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using Lemma 2.6 one can write (2.237) as,

$$\Sigma + \Sigma^T \le \epsilon_1 E_1^T E_1 + \epsilon_2 E_2^T E_2 + \epsilon_1^{-1} D_1 D_1^T + \epsilon_2^{-1} D_2 D_2^T \tag{2.239}$$

So, in view of (2.239) and with the application of Schur-complement one can obtain the LMI conditions (2.232)–(2.233), while remaining conditions $P > 0$ and $Q > 0$ remains same as in stability Theorem 2.18 as they do not contain A and A_d matrices. This completes the proof. □

If the uncertainties in (2.19) and (2.20) are defined as,

$$\begin{aligned} \Delta A(t) &= DF(t)E_a \\ \Delta A_d(t) &= DF(t)E_d, \ F^T(t)F(t) \le I \end{aligned} \tag{2.240}$$

then with $D_a = D_d = D$, Theorem 2.25 can be modified in the form of following corollary.

Corollary 2.5 *Let $d_u > 0$ and $\epsilon > 0$ be given scalars, the system (2.15) with (2.5) and $0 \le \mu < 1$ is robustly asymptotically stable for the admissible uncertainties (2.19)–(2.20) satisfying (2.240), if there exist symmetric positive definite matrices $P,\ Q,\ R_i\ (i = 1, 2)$, and any free matrices $L_i\ (i = 1, 2, 3)$, $M_i\ (i = 1, 2, 3)$ and $G_i\ (i = 1, 2)$ such that, $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0$, $Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} > 0$, the following LMIs hold:*

$$\begin{bmatrix} \tilde{\Xi} & \mu\tilde{P} & d_u M & \bar{D} \\ \star & -\mu T & 0 & 0 \\ \star & 0 & -Q_{22} & 0 \\ \star & 0 & 0 & -\epsilon I \end{bmatrix} < 0 \tag{2.241}$$

$$\begin{bmatrix} \tilde{\Xi} & \mu\tilde{P} & d_u L & \bar{D} \\ \star & -\mu T & 0 & 0 \\ \star & 0 & -Q_{22} & 0 \\ \star & 0 & 0 & -\epsilon I \end{bmatrix} < 0 \tag{2.242}$$

where, $\bar{D} = \begin{bmatrix} D^T G_1 \ 0\ 0\ 0\ 0\ D^T G_2 \end{bmatrix}^T$, *and*

$$\tilde{\Xi} = \begin{bmatrix} \tilde{\Xi}_{11} & \tilde{\Xi}_{12} & \tilde{\Xi}_{13} & \tilde{\Xi}_{14} & 0 & \tilde{\Xi}_{16} \\ \star & \tilde{\Xi}_{22} & \tilde{\Xi}_{23} & \tilde{\Xi}_{24} & 0 & \tilde{\Xi}_{16} \\ \star & \star & \tilde{\Xi}_{33} & 0 & 0 & 0 \\ \star & \star & 0 & -Q_{11} & 0 & \tilde{\Xi}_{46} \\ 0 & 0 & 0 & 0 & -Q_{11} & 0 \\ \star & \star & 0 & \star & 0 & \tilde{\Xi}_{66} \end{bmatrix} < 0 \tag{2.243}$$

where, $\tilde{\Xi}_{11}{=}\Omega_{11} + \epsilon E_a^T E_a$, $\tilde{\Xi}_{12}{=}\Omega_{12} + \epsilon E_a^T E_d$,

$$\tilde{\Xi}_{13}=\Omega_{13},\ \ \tilde{\Xi}_{14}=\Omega_{14},\ \tilde{\Xi}_{16}=\Omega_{16}$$
$$\tilde{\Xi}_{22}=\Omega_{22}+\epsilon_2 E_d^T E_d,\ \tilde{\Xi}_{23}=\Omega_{23},\ \ \tilde{\Xi}_{24}=\Omega_{24},\ \ \tilde{\Xi}_{26}=\Omega_{26}$$
$$\tilde{\Xi}_{33}=\Omega_{23},\ \ \tilde{\Xi}_{46}=\Omega_{46},\ \ \tilde{\Xi}_{66}=\Omega_{66}$$

The the elements of matrix Ω ($\Omega_{i,j},\ i=1,..6;\ \ j=1,..6$) are defined in stability Theorem 2.18.

If $\mu=0$ (i.e., constant time-delay case) and the uncertainty structure is as defined in 2.240) then the robust stability condition for an uncertain system satisfying the condition $F^T(t)F(t)\leq I$, will follow directly from Corollary 2.5 by substituting $\mu=0$.

Corollary 2.6 *Let $d_u>0$ and $\epsilon>0$ be given scalars, an uncertain time-delay systems is robustly asymptotically stable for $\mu=0$ and admissible uncertainties described in (2.19)–(2.20), if there exist symmetric positive definite matrices $P,\ Q,\ R_i\ (i=1,2)$, and any free matrices $L_i\ (i=1,2,3)$, $M_i\ (i=1,2,3)$ and $G_i\ (i=1,2)$ such that, $P=\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22}\end{bmatrix}>0\ \ Q=\begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22}\end{bmatrix}>0$, the following LMIs hold:*

$$\begin{bmatrix} \tilde{\Xi} & d_u M & \bar{D} \\ \star & -Q_{22} & 0 \\ \star & 0 & -\epsilon I \end{bmatrix} < 0 \tag{2.244}$$

$$\begin{bmatrix} \tilde{\Xi} & d_u L & \bar{D} \\ \star & -Q_{22} & 0 \\ \star & 0 & -\epsilon I \end{bmatrix} < 0 \tag{2.245}$$

where, $\bar{D}$ is defined in corollary 2.5. and

$$\tilde{\Xi}=\begin{bmatrix} \tilde{\Xi}_{11} & \tilde{\Xi}_{12} & \tilde{\Xi}_{13} & \tilde{\Xi}_{14} & 0 & \tilde{\Xi}_{16} \\ \star & \tilde{\Xi}_{22}|_{\mu=0} & \tilde{\Xi}_{23} & \tilde{\Xi}_{24} & 0 & \tilde{\Xi}_{16} \\ \star & \star & \tilde{\Xi}_{33} & 0 & 0 & 0 \\ \star & \star & 0 & -Q_{11} & 0 & \tilde{\Xi}_{46} \\ 0 & 0 & 0 & 0 & -Q_{11} & 0 \\ \star & \star & 0 & \star & 0 & \tilde{\Xi}_{66} \end{bmatrix} < 0 \tag{2.246}$$

where, $\tilde{\Xi}_{22}=-R_1+d_u(-L_2-L_2^T+M_2+M_2^T)+\epsilon E_d^T E_d$ and rest of the elements of $\tilde{\Xi}$ are same as in Corollary 2.5.

To illustrate the effectiveness of the proposed theorem three numerical examples are considered and the results are compared in tabular form with the existing robust stability methods.

Numerical Example 2.5 ([14, 36]) *Consider the uncertain time-delay system (2.15) with the following constant matrices*

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$
$$E_a = \begin{bmatrix} 1.6 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad E_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad D_a = D_d = D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The analytical value of the delay upper bound d_u considering $F(t) = I$ is found to be 1.3771 [14] for $\mu = 0$, so it cannot be more than 1.3771 for any given uncertainties defined in (2.240) as $F^T(t)F(t) \leq I$. The results obtained for different delay-derivatives μ are presented in Table 2.12.

Numerical Example 2.6 ([14]) *Consider the uncertain time-delay system (2.15) with the following constant matrices*

$$A = \begin{bmatrix} -0.5 & -2 \\ 1 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.5 & -1 \\ -1 & 0.6 \end{bmatrix}$$
$$E_a = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad = E_d, \quad and\ D_a = D_d = D = I$$

The results obtained for this Example using Theorem 2.25 for different delay-derivatives (μ) are presented in Table 2.13.

Numerical Example 2.7 ([14]) *Consider the uncertain time-delay system (2.15) with the following constant matrices*

Table 2.12 (d_u) results of Example 2.5 for $0 \leq \mu < 1$, $d_l = 0$

Stability methods	$\mu = 0$	$\mu = 0.4$	$\mu = 0.5$	$\mu = 0.6$	$\mu = 0.8$	$\mu = 0.9$
[12]	0.2086	*	*	*	*	*
[18]	0.2299	–	0.1758	–	–	0.0557
[25]	Infeas.	*	*	*	*	*
[27]	1.1490	–	0.9247	–	–	0.6710
[40]	1.1490	0.973	0.9247	0.873	0.760	0.6954
[38]	1.03	0.61	0.40		0.18	–
[14]	1.1623	–	0.9273	–	–	0.6954
[36]	1.149	1.077	–	1.070	1.068	–
Cor. 2.6 and Cor. 2.5	1.1606	1.0778	1.0733	1.0708	1.0686	1.0686

'–' means result is not available in reference, '*' means improper

Table 2.13 (d_u) results of Example 2.6 for $0 \leq \mu < 1$, $d_l = 0$

Stability methods	$\mu = 0$	$\mu = 0.5$	$\mu = 0.9$
[12]	0.3010	*	*
[18]	0.3513	0.2587	0.0825
[25]	0.5799	*	*
[27]	0.6812	0.1820	0.1820
[40]	0.8435	0.2433	0.2420
[14]	1.8542	0.3459	0.2542
Cor. 2.6 and Cor. 2.5	0.8435	0.3972	0.3972

'–' means result is not available in reference, '*' means improper

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$
$$D_a = D_d = D = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad \text{and } E_a = E_d = E = I$$

The delay upper bound estimate obtained (i) using Corollary 2.6 is $d_u = 2.4317$ *for* $\mu = 0$ *(ii) using Corollary 2.5 are* $d_u = 1.5276$ *and* $d_u = 1.2658$ *for* $\mu = 0.5$ *and* 0.9 *respectively, whereas the corresponding results obtained in [14] for* $\mu = 0$ *is* $d_u = 2.4390$*, and* $d_u = 1.3214$ *and* 0.7938 *for* $\mu = 0.5$ *and* 0.9 *respectively.*

Remark 2.29 It may be noted from Tables 2.12 and 2.13 that the proposed method gives less conservative estimate of delay upper bound compared to the [14] for non zero delay derivatives for all the systems. It may be emphasized here that the stability method adopted in [14] leads to better delay upper bound only for $\mu = 0$.

2.7.2 Robust Stability Analysis of TDS with Delay Varying in Ranges

Theorem 2.26 *Given a system (2.15) for* $\mu \geq 0$ *with the admissible uncertainties (2.19)–(2.20), satisfying the conditions (2.21)–(2.22) and (2.183)–(2.184) is robustly asymptotically stable, if there exist symmetric matrices* P, $Q_i > 0, i = 1, 2, \ldots 4$, $R_j > 0$*, any free matrices of appropriate dimensions* M_j, N_j, $j = 1, 2$ *and the scalars* $\epsilon_i > 0$ $(i = 1, 2)$ *such that the following LMIs hold:*

$$\begin{bmatrix} \Psi & \Phi_1 & D_1 & D_2 \\ \star & -R_2 & 0 & 0 \\ \star & 0 & -\epsilon_1 I & 0 \\ \star & 0 & 0 & -\epsilon_2 I \end{bmatrix} < 0, \tag{2.247}$$

$$\begin{bmatrix} \Psi & \Phi_2 & D_1 & D_2 \\ \star & -R_2 & 0 & 0 \\ \star & 0 & -\epsilon_1 I & 0 \\ \star & 0 & 0 & -\epsilon_2 I \end{bmatrix} < 0, \tag{2.248}$$

where, $\Phi_1 = \begin{bmatrix} 0 & M_1^T & N_1^T & 0 & 0 & 0 \end{bmatrix}^T$, $\Phi_2 = \begin{bmatrix} 0 & 0 & M_2^T & N_2^T & 0 & 0 \end{bmatrix}$,

$$D_1 = \begin{bmatrix} D_a^T P & 0 & 0 & 0 & d_l D_a^T R_1 & D_a^T R_2 \end{bmatrix}^T, \; D_2 = \begin{bmatrix} D_d^T P & 0 & 0 & 0 & d_l D_d^T R_1 & D_d^T R_2 \end{bmatrix}^T$$

and

$$\Psi = \begin{bmatrix} \Theta_{11}|_{\Delta=0} + \epsilon_1 E_a^T E_a & R_1 & \Theta_{13}|_{\Delta=0} & 0 & \Theta_{15}|_{\Delta=0} & \Theta_{16}|_{\Delta=0} \\ \star & \Theta_{22} & \Theta_{23} & 0 & 0 & 0 \\ \star & \star & \Theta_{33}|_{\Delta=0} + \epsilon_2 E_d^T E_d & \Theta_{34} & \Theta_{35}|_{\Delta=0} & \Theta_{36}|_{\Delta=0} \\ \star & \star & \star & \Theta_{44} & 0 & 0 \\ \star & 0 & \star & 0 & -R_1 & 0 \\ \star & 0 & \star & 0 & 0 & -R_2 \end{bmatrix}$$

$\Theta_{11}|_{\Delta=0}$=$PA + A^T P + \sum_{i=1}^{3} Q_i - R_1$, $\Theta_{13}|_{\Delta=0}$=$PA_d + R_1$, $\Theta_{15}|_{\Delta=0}$=$d_l A^T R_1$,

$\Theta_{16}|_{\Delta=0}$=$A^T R_2$, Θ_{22}=$Q_4 - Q_1 - R_1 + d_{lu}^{-1}(M_1 + M_1^T)$, Θ_{23}=$d_{lu}^{-1}(-M_1 + N_1^T)$,

Θ_{33}=$-(1-\mu)(Q_3 + Q4) + d_{lu}^{-1}(M_2 + M_2^T - N_1 - N_1^T)$,

Θ_{34}=$d_{lu}^{-1}(-M_2 + N_2^T)$, $\Theta_{35}|_{\Delta=0}$=$d_l A_d^T R_1$, $\Theta_{36}|_{\Delta=0}$=$A_d^T R_2$, Θ_{44}=$-Q_2 - d_{lu}^{-1}(N_2 + N_2^T)$

Proof The proof of this theorem follows directly from the stability conditions derived in Theorem 2.20 for nominal time-delay systems. The nominal matrices A and A_d appearing in Θ matrix of (2.188) and (2.189) are replaced by time-varying matrices $A(t)$ and $A_d(t)$ respectively, and then using Schur-complement one can rewrite matrix Θ as,

$$\Theta(t) = \begin{bmatrix} \Theta_{11} & R_1 & \Theta_{13} & 0 & \Theta_{15} & \Theta_{16} \\ \star & \Theta_{22} & \Theta_{23} & 0 & 0 & 0 \\ \star & \star & \Theta_{33} & \Theta_{34} & \Theta_{35} & \Theta_{36} \\ \star & \star & \star & \Theta_{44} & 0 & 0 \\ \star & 0 & \star & 0 & -R_1 & 0 \\ \star & 0 & \star & 0 & 0 & -R_2 \end{bmatrix} \tag{2.249}$$

Θ_{11}=$PA(t) + A^T(t)P + \sum_{i=1}^{3} Q_i - R_1$, Θ_{13}=$PA_d(t) + R_1$, Θ_{15}=$d_l A^T(t) R_1$,

Θ_{16}=$A^T(t) R_2$, Θ_{22}=$Q_4 - Q_1 - R_1 + d_{lu}^{-1}(M_1 + M_1^T)$, Θ_{23}=$dlu^{-1}(-M_1 + N_1^T)$,

Θ_{33}=$-(1-\mu)(Q_3 + Q4) + d_{lu}^{-1}(M_2 + M_2^T - N_1 - N_1^T)$,

Θ_{34}=$d_{lu}^{-1}(-M_2 + N_2^T)$, Θ_{35}=$d_l A_d(t)^T R_1$, Θ_{36}=$A_d(t)^T R_2$, Θ_{44}=$-Q_2 - d_{lu}^{-1}(N_2 + N_2^T)$

Now, $A(t)$ and $A_d(t)$ are substituted in (2.249) with the values as defined in (2.17) and (2.18) respectively and subsequently separating nominal and uncertain matrices, one can rewrite (2.249) as,

$$\Theta(t) = \Theta|_{\Delta=0} + \Delta \tag{2.250}$$

where,

$$\Delta = \Pi + \Pi^T \tag{2.251}$$

and

$$\Pi = D_1 F_a(t) E_1 + D_2 F_d(t) E_d \tag{2.252}$$

where,
$D_1 = \begin{bmatrix} D_a^T P & 0 & 0 & 0 & d_l D_a^T R_1 & D_a^T R_2 \end{bmatrix}^T$, $D_2 = \begin{bmatrix} D_d^T P & 0 & 0 & 0 & d_l D_d^T R_1 & D_d^T R_2 \end{bmatrix}^T$,

$E_1 = \begin{bmatrix} E_a & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 0 & E_d & 0 & 0 & 0 \end{bmatrix}$.

Using Lemma 2.6 for eliminating the uncertain matrix, one can write (2.252) as,

$$\Pi + \Pi^T \leq \epsilon_1 E_1^T E_1 + \epsilon_2 E_2^T E_2 + \epsilon_1^{-1} D_1 D_1^T + \epsilon_2^{-1} D_2 D_2^T \tag{2.253}$$

So, using (2.253) in (2.250) and applying Schur-complement one can obtain the LMI conditions (2.243) and (2.248).

If $F_a(t) = F_d(t) = F(t)$ and $D_a = D_d = D$, then the above theorem is stated in the following corollary.

Corollary 2.7 *System (2.15) for $\mu \geq 0$ is robustly asymptotically stable with the admissible uncertainties defined in (2.240) satisfying the conditions $F(t)^T F(t) \leq I$, if there exist symmetric matrices $P > 0$, $Q_i > 0$, $i = 1, 2, \ldots 4$, $R_j > 0$, any matrices of appropriate dimensions M_j, N_j, $j = 1, 2$ and the scalars $\epsilon_i > 0$ $(i = 1, 2)$ such that following LMIs hold:*

$$\begin{bmatrix} \Psi & \Phi_1 & D \\ \star & -R_2 & 0 \\ \star & 0 & -\epsilon I \end{bmatrix} < 0, \tag{2.254}$$

$$\begin{bmatrix} \Psi & \Phi_2 & D \\ \star & -R_2 & 0 \\ \star & 0 & -\epsilon I \end{bmatrix} < 0, \tag{2.255}$$

where,

$$D = \begin{bmatrix} D^T P & 0 & 0 & 0 & d_l D^T R_1 & D^T R_2 \end{bmatrix}^T,$$

and,

$$\Psi = \begin{bmatrix} \Theta_{11}|_{\Delta=0} + \epsilon E_a^T E_a & R_1 & \Theta_{13}|_{\Delta=0} + \epsilon E_a^T E_d & 0 & \Theta_{15}|_{\Delta=0} & \Theta_{16}|_{\Delta=0} \\ \star & \Theta_{22} & \Theta_{23} & 0 & 0 & 0 \\ \star & \star & \Theta_{33}|_{\Delta=0} + \epsilon E_d^T E_d & \Theta_{34} & \Theta_{35}|_{\Delta=0} & \Theta_{36}|_{\Delta=0} \\ \star & \star & \star & \Theta_{44} & 0 & 0 \\ \star & 0 & \star & 0 & -R_1 & 0 \\ \star & 0 & \star & 0 & 0 & -R_2 \end{bmatrix}$$

Note: Ψ, Φ_1 *and* Φ_2 *are defined in Theorem 2.26.*

If $d_l = 0$ and the uncertainties are as defined in (2.240) then the Corollary 2.4 (stability condition of nominal TDS) can be extended for obtaining the corresponding delay-range-dependent robust stability condition which is presented below.

Corollary 2.8 *Given a system (2.15) with* $d_l = 0$ *for* $\mu \geq 0$ *is robustly asymptotically stable with the admissible uncertainties defined in (2.240), if there exist symmetric matrices* $P > 0$, $Q_i > 0$, $i = 2, 3$ $R_2 > 0$, *and any free matrices of appropriate dimensions* M_j, N_j, $j = 1, 2$ *and the scalar* $\epsilon > 0$ *such that the following LMIs hold:*

$$\begin{bmatrix} \Gamma & \Xi_1 & D \\ \star & -R_2 & 0 \\ \star & 0 & -\epsilon I \end{bmatrix} < 0, \tag{2.256}$$

$$\begin{bmatrix} \Gamma & \Xi_2 & D \\ \star & -R_2 & 0 \\ \star & 0 & -\epsilon I \end{bmatrix} < 0, \tag{2.257}$$

where, $\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & A^T R_2 \\ \star & \Gamma_{22} & \Gamma_{23} & A_d^T R_2 \\ \star & \star & \Gamma_{33} & 0 \\ \star & \star & \star & -R_2 \end{bmatrix}$, and

$\Gamma_{11} = PA + A^T P + \sum_{i=2}^{3} Q_i + d_u^{-1}(M_1 + M_1^T) + \epsilon E_a^T E_a$,
$\Gamma_{12} = PA_d + d_u^{-1}(-M_1 + N_1^T) + \epsilon E_a^T E_d$,
$\Gamma_{22} = -(1 - \mu)Q_3 + d_u^{-1}(M_2 + M_2^T - N_1 - N_1^T) + \epsilon E_d^T E_d$, $\Gamma_{23} = d_u^{-1}(-M_2 + N_2^T)$,
$\Gamma_{33} = -Q_2 - d_u^{-1}(N_2 + N_2^T)$

The matrices Ξ_1 *and* Ξ_2 are already defined in Corollary 2.4 while matrix D is defined as $D = \begin{bmatrix} D^T P & 0 & 0 & D^T R_2 & 0 \end{bmatrix}$.

The delay upper bound estimates using Corollary 2.7 and 2.8 for the numerical examples 2.6 and 2.7 are computed when (i) $d_l < d(t) < d_u$ (where d_l is specified) and (ii) $0 < d(t) < d_u$, for different μ (delay-derivative) values.

Table 2.14 d_u results of Example 2.6 for $0 \leq \mu < 1$ for specified d_l

Stability methods	μ	d_l	d_u
[37]	0.5	0	0.31
		0.5	–
[37]	0.9	0	0.31
		0.5	–
Corollary 2.8	0.5	0	0.3972
Corollary 2.7		0.1	0.2783
Corollary 2.7		0.2	0.3688
Corollary 2.7		0.5	0.6076
Corollary 2.8	0.9	0	0.3972
Corollary 2.7		0.1	0.2783
Corollary 2.7		0.2	0.3688
Corollary 2.7		0.5	0.6076

Table 2.15 d_u results of Example 2.7 for $0 \leq \mu < 1$ for specified d_l

Stability methods	μ	d_l	d_u
Corollary 2.8	0	0	2.3970
	0.5	0	1.4818
Corollary 2.7	0.5	0.1	1.4952
		0.5	1.5234
		1	1.5458
		2	2.1277
		2.2	2.2612
		2.38	2.3851
Corollary 2.7	0.9	0.1	1.2526
		0.5	1.3199
		1	1.5391
		2	2.1279
		2.2	2.2612
		2.38	2.3851

The numerical results presented above suggests that the proposed method improves the results over the published paper [37] mainly due to the (i) new LK functional (ii) tighter bounding conditions (2.193) and (2.194) used in the robust stability analysis.

2.8 Delay-Range-Dependent Stability Analysis of Uncertain TDS by Delay Partitioning Approach

In this section, a robust delay-range-dependent stability method in the frame work of delay partitioning approach is considered by adopting the LK functional in [47] (here the delay range between d_l to d_u is divided into two equal intervals $\delta = \frac{d_l+d_u}{2}$) and using the proposed bounding inequality discussed in Theorem 2.20 (see Sect. 2.5.3). The stability analysis is presented below in the form of theorem.

Theorem 2.27 *Given a system (2.15) for* $0 \le d_l \le d(t) \le d_u$, $0 \le \mu < 1$ *is robustly asymptotically stable with the admissible uncertainties defined in (2.240) satisfying the conditions* $F(t)^T F(t) \le I$, *if there exist symmetric matrices* $P > 0$, $Q_i > 0$, $i = 1, 2, 3$, $R_j > 0$, *and any free matrices of appropriate dimensions* M_j, N_j, $j = 1, 2$ *and the scalar* $\epsilon > 0$ *such that the following LMIs hold:*

$$\begin{bmatrix} \Phi_{11}(a) & \Phi_{12} \\ \star & -\epsilon I \end{bmatrix} < 0 \tag{2.258}$$

$$\begin{bmatrix} \Phi_{11}(b) & \Phi_{12} \\ \star & -\epsilon I \end{bmatrix} < 0 \tag{2.259}$$

where,

$$\Phi_{11}(a) = \begin{bmatrix} \Theta_{11}(a) & \Theta_{12} \\ \star & \Theta_{22} \end{bmatrix}, \ \Phi_{11}(b) = \begin{bmatrix} \Theta_{11}(b) & \Theta_{12} \\ \star & \Theta_{22} \end{bmatrix}, \tag{2.260}$$

$$\Phi_{12} = \begin{bmatrix} D^T P \ 0 \ 0 \ 0 \ 0 \ \delta D^T R_1 \ D^T R_2 \end{bmatrix}^T, \textit{and}$$

$$\Theta_{11}(a) = \begin{bmatrix} \theta_{11(0)} & \theta_{12(0)} & 0 & R_1 & 0 \\ \star & \theta_{22(0)} & \theta_{23} & \theta_{24} & N_1 \\ \star & \star & \theta_{33} & \theta_{34} & 0 \\ \star & \star & \star & \theta_{44} & M_1 \\ \star & \star & \star & \star & -R_2 \end{bmatrix} \tag{2.261}$$

$$\Theta_{11}(b) = \begin{bmatrix} \theta_{11(0)} & \theta_{12(0)} & 0 & R_1 & 0 \\ \star & \theta_{22(0)} & \theta_{23} & \theta_{24} & M_2 \\ \star & \star & \theta_{33} & \theta_{34} & N_2 \\ \star & \star & \star & \theta_{44} & 0 \\ \star & \star & \star & \star & -R_2 \end{bmatrix} \tag{2.262}$$

$$\Theta_{12} = \begin{bmatrix} \delta A^T R_1 & A^T R_2 \\ \delta A_d^T R_1 & A_d^T R_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Theta_{22} = \begin{bmatrix} -R_1 & 0 \\ 0 & -R_2 \end{bmatrix}$$

The elements of the $\Theta_{11}(a)$ *and* $\Theta_{11}(b)$ *are as follows,*

$\theta_{11(0)} = PA + A^T P + \Sigma_{i=1}^3 Q_i - R_1$, $\theta_{12(0)} = PA_d$,

$\theta_{22(0)} = -(1-\mu)Q_2 + d_{lu}^{-1}(M_2 + M_2^T - N_1 - N_1^T)$, $\theta_{23} = d_{lu}^{-1}(-M_2 + N_2^T)$,

$\theta_{24} = d_{lu}^{-1}(-M_1^T + N_1)$, $\theta_{33} = d_{lu}^{-1}(-N_2^T - N_2) - Q_3$,

$\theta_{44} = d_{lu}^{-1}(M_1^T + M_1) - Q_1 - R_1$, $\delta = \frac{d_l + d_u}{2}$, $d_{lu} = (d_u - \delta)$

Proof Considering the similar type of LK functional as in [47],

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t) + V_6(t) \tag{2.263}$$

where the individual functionals are as follows,

$$V_1(t) = x^T P x(t), \quad V_2(t) = \int_{t-\delta}^{t} x^T(s) Q_1 x(s) ds, \quad V_3(t) = \int_{t-d(t)}^{t} x^T(s) Q_2 x(s) ds$$

$$V_4(t) = \int_{t-d_u}^{t} x^T(s) Q_3 x(s) ds, \quad V_5(t) = \delta \int_{-\delta}^{0} \int_{t+\theta}^{t} x^T(\theta) R_1 x(\theta) d\theta ds$$

$$V_6(t) = d_{lu}^{-1} \int_{-d_u}^{-\delta} \int_{t+\theta}^{t} x^T(\theta) R_2 x(\theta) d\theta ds$$

Note that, the $V_6(t)$ functional term selected here contains a factor d_{lu}^{-1} whereas in [47] it appears as d_{lu}, this modification is required in the functional in order to use the proposed bounding inequality described in Sect. 2.5.3.

$$\begin{aligned} \dot{V}(t) = {} & 2\dot{x}^T(t) P x(t) + x^T(t)(\sum_{i=1}^{3} Q_i) x(t) - (1-\mu) x^T(t-d(t)) Q_2 x(t-d(t)) \\ & - x^T(t-d_u) Q_3 x(t-d_u) - x^T(t-\delta) Q_3 x(t-\delta) + \delta^2 \dot{x}^T(t) R_1 \dot{x}(t) \\ & + \dot{x}^T(t) R_2 \dot{x}(t) - \delta \int_{t-\delta}^{t} \dot{x}^T(s) R_1 \dot{x}(s) ds \\ & - d_{lu}^{-1} \int_{t-d_u}^{t-\delta} \dot{x}^T(s) R_2 \dot{x}(s) ds \end{aligned} \tag{2.264}$$

Define augmented state space vector as,

$$\zeta(t) = \left[x^T(t)\ x^T(t-d(t))\ x^T(t-d_u)\ x^T(t-\delta) \right]^T$$

Now, one can rearrange the terms in (2.264) in view of augmented state space vector as,

$$\dot{V}(t) \leq \zeta^T(t)\Xi\zeta(t) - d_{lu}^{-1} \int_{t-d_u}^{t-\delta} \dot{x}^T(s)R_2\dot{x}(s)ds$$
$$-\delta \int_{t-\delta}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds$$

$$\dot{V}(t) \leq \zeta^T(t)\Xi\zeta(t) - \delta \int_{t-\delta}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds$$
$$-d_{lu}^{-1} \int_{t-d(t)}^{t-\delta} \dot{x}^T(s)R_2\dot{x}(s)ds - d_{lu}^{-1} \int_{t-d_u}^{t-d(t)} \dot{x}^T(s)R_2\dot{x}(s)ds \quad (2.265)$$

where, $\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & 0 & 0 \\ \star & \Xi_{22} & 0 & 0 \\ \star & \star & -Q_3 & 0 \\ 0 & 0 & 0 & -Q_1 \end{bmatrix}$

The elements of 'Ξ' matrix are as follows:

$\Xi_{11} = PA(t) + A^T(t)P + \sum_{i=1}^{3} Q_i + A^T(t)(\delta^2 R_1 + R_2)A(t),$

$\Xi_{12} = PA_d(t) + A^T(t)(\delta^2 R_1 + R_2)A_d(t),$

$\Xi_{22} = -(1-\mu)Q_2 + A_d^T(t)(\delta^2 R_1 + R_2)A_d(t)$

The integral terms in (2.265) are approximated using the proposed tighter bounding inequality condition as discussed in Theorem 2.20. One can obtain the following expression as,

$$-\delta \int_{t-\delta}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds \leq \begin{bmatrix} x(t) \\ x(t-\delta) \end{bmatrix}^T \begin{bmatrix} -R_1 & R_1 \\ \star & -R_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\delta) \end{bmatrix} \quad (2.266)$$

$$-\delta\int_{t-\delta}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds \le \zeta^T\begin{bmatrix}-R_1 & 0 & 0 & R_1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \star & 0 & 0 & -R_1\end{bmatrix}\zeta(t) \tag{2.267}$$

$$\begin{aligned}-d_{lu}^{-1}\int_{t-d(t)}^{t-\delta}\dot{x}^T(s)R_2\dot{x}(s)ds \le & \begin{bmatrix}x(t-\delta)\\ x(t-d(t))\end{bmatrix}^T\\ & \left\{d_{lu}^{-1}\begin{bmatrix}M_1+M_1^T & -M_1+N_1^T\\ \star & -N_1-N_1^T\end{bmatrix}\right.\\ & \left.\times\,\varrho\begin{bmatrix}M_1\\ N_1\end{bmatrix}R_2^{-1}\begin{bmatrix}M_1\\ N_1\end{bmatrix}^T\right\}\\ & \times\begin{bmatrix}x(t-\delta)\\ x(t-d(t))\end{bmatrix}\end{aligned} \tag{2.268}$$

and,

$$\begin{aligned}-d_{lu}^{-1}\int_{t-d_u}^{t-d(t)}\dot{x}^T(s)R_2\dot{x}(s)ds \le & \begin{bmatrix}x(t-d(t))\\ x(t-d_u)\end{bmatrix}^T\\ & \left\{d_{lu}^{-1}\begin{bmatrix}M_2+M_2^T & -M_2+N_2^T\\ \star & -N_2-N_2^T\end{bmatrix}\right.\\ & \left.\times\,(1-\varrho)\begin{bmatrix}M_2\\ N_2\end{bmatrix}R_2^{-1}\begin{bmatrix}M_2\\ N_2\end{bmatrix}^T\right\}\\ & \times\begin{bmatrix}x(t-d(t))\\ x(t-d_u)\end{bmatrix}\end{aligned} \tag{2.269}$$

where, $\varrho = \frac{d(t)-\delta}{d_{lu}}$ and $0 \le \varrho \le 1$. Substituting the value of integral (2.267), (2.268) and (2.269) in (2.265), and carrying out some algebraic manipulations yields resulting expression as,

$$\dot{V}(t) \le \zeta^T(t)\{\Gamma(t) + \varrho\Psi_1R_2^{-1}\Psi_1^T + (1-\varrho)\Psi_2R_2^{-1}\Psi_2^T\}\zeta(t) \tag{2.270}$$

where, $\Psi_1 = \begin{bmatrix}0 & N_1^T & 0 & M_1^T\end{bmatrix}^T$, and $\Psi_2 = \begin{bmatrix}0 & M_2^T & N_2^T & 0\end{bmatrix}^T$.

One can observe convex combination of matrices $\Psi_1R_2^{-1}\Psi_1^T$ and $\Psi_2R_2^{-1}\Psi_2^T$ in (2.270). For asymptotic stability of the system (2.15), $\dot{V}(t)$ must be negative-definite for which one must have,

$$\Gamma(t) + \varrho\Psi_1R_2^{-1}\Psi_1^T + (1-\varrho)\Psi_2R_2^{-1}\Psi_2^T < 0 \tag{2.271}$$

The above expression can be further rewritten as,

$$\varrho(\Gamma(t)+\Psi_1 R_2^{-1}\Psi_1^T)+(1-\varrho)(\Gamma(t)+\Psi_2 R_2^{-1}\Psi_2^T)<0 \tag{2.272}$$

In view of the condition $0 \le \varrho \le 1$, one may write the above inequality as,

$$\Gamma(t)+\Psi_1 R_2^{-1}\Psi_1^T<0 \tag{2.273}$$

$$\Gamma(t)+\Psi_2 R_2^{-1}\Psi_2^T<0 \tag{2.274}$$

where,

$$\Gamma(t)=\begin{bmatrix} \gamma_{11} & \gamma_{12} & 0 & R_1 \\ \star & \gamma_{22} & \gamma_{23} & \gamma_{24} \\ \star & \star & \gamma_{33} & 0 \\ \star & \star & \star & \gamma_{44} \end{bmatrix} \tag{2.275}$$

and,
$\gamma_{11}=PA(t)+A^T(t)P+\sum_{i=1}^{3}Q_i-R_1+A^T(t)(\delta^2R_1+R_2)A(t)$,
$\gamma_{12}=PA_d(t)+A^T(t)(\delta^2R_1+R_2)A_d(t)$,
$\gamma_{22}=-(1-\mu)Q_2+d_{lu}^{-1}(M_2+M_2^T-N_1-N_1^T)+A_d^T(t)(\delta^2R_1+R_2)A_d(t)$, $\gamma_{23}=d_{lu}^{-1}(-M_2+N_2^T)$,
$\gamma_{24}=d_{lu}^{-1}(-M_1^T+N_1)$, $\gamma_{33}=d_{lu}^{-1}(-N_2^T-N_2)-Q_3$, $\gamma_{44}=d_{lu}^{-1}(M_1^T+M_1)-Q_1-R_1$

Using Schur-Complement one can write (2.273) and (2.274) as

$$\Theta(t)(a)=\begin{bmatrix} \theta_{11} & \theta_{12} & 0 & R_1 & 0 \\ \star & \theta_{22} & \theta_{23} & \theta_{24} & N_1 \\ \star & \star & \theta_{33} & \theta_{34} & 0 \\ \star & \star & \star & \theta_{44} & M_1 \\ \star & \star & \star & \star & -R_2 \end{bmatrix} \tag{2.276}$$

$$\Theta(t)(b)=\begin{bmatrix} \theta_{11} & \theta_{12} & 0 & R_1 & 0 \\ \star & \theta_{22} & \theta_{23} & \theta_{24} & M_2 \\ \star & \star & \theta_{33} & \theta_{34} & N_2 \\ \star & \star & \star & \theta_{44} & 0 \\ \star & \star & \star & \star & -R_2 \end{bmatrix} \tag{2.277}$$

where, $\theta_{11}=\gamma_{11}$, $\theta_{12}=\gamma_{12}$, $\theta_{22}=\gamma_{22}$, $\theta_{23}=\gamma_{23}$, $\theta_{24}=\gamma_{24}$, $\theta_{33}=\gamma_{33}$, $\theta_{44}=\gamma_{44}$
Once again using Schur-complement on (2.276) and (2.277) one can get,

$$\Phi(t)_{11}(a)=\begin{bmatrix} \Theta(t)_{11}(a) & \Theta(t)_{12} \\ \star & \Theta_{22} \end{bmatrix},\ \Phi(t)_{11}(b)=\begin{bmatrix} \Theta(t)_{11}(b) & \Theta(t)_{12} \\ \star & \Theta_{22} \end{bmatrix} \tag{2.278}$$

where,

$$\Theta(t)_{12} = \begin{bmatrix} \delta A^T(t)R_1 & A^T(t)R_2 \\ \delta A_d(t)^T R_1 & A_d^T(t)R_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Theta_{22} = \begin{bmatrix} -R_1 & 0 \\ 0 & -R_2 \end{bmatrix}$$

$$\Theta(t)_{11}(a) = \begin{bmatrix} \theta_{11(0)} & \theta_{12(0)} & 0 & R_1 & 0 \\ \star & \theta_{22(0)} & \theta_{23} & \theta_{24} & N_1 \\ \star & \star & \theta_{33} & \theta_{34} & 0 \\ \star & \star & \star & \theta_{44} & M_1 \\ \star & \star & \star & \star & -R_2 \end{bmatrix}$$

$$\Theta(t)_{11}(b) = \begin{bmatrix} \theta_{11(0)} & \theta_{12(0)} & 0 & R_1 & 0 \\ \star & \theta_{22(0)} & \theta_{23} & \theta_{24} & M_2 \\ \star & \star & \theta_{33} & \theta_{34} & N_2 \\ \star & \star & \star & \theta_{44} & 0 \\ \star & \star & \star & \star & -R_2 \end{bmatrix}$$

where,

$\theta_{11(0)}=PA(t) + A^T(t)P + \Sigma_{i=1}^3 Q_i - R_1$, $\theta_{12(0)}=PA_d(t)$,
$\theta_{22(0)}=-(1-\mu)Q_2 + M_2 + M_2^T - N_1 - N_1^T$, $\theta_{23}=d_{lu}^{-1}(-M_2 + N_2^T)$,
$\theta_{24}=d_{lu}^{-1}(-M_1^T + N_1)$, $\theta_{33}=d_{lu}^{-1}(-N_2^T - N_2) - Q_3$,
$\theta_{44}=d_{lu}^{-1}(M_1^T + M_1) - Q_1 - R_1$

Next replace the matrices $A(t)$ and $A_d(t)$ with the uncertain matrices in (2.17) and the structure of matrices $\Delta A(t)$ and $\Delta A_d(t)$ are decomposed as $DF(t)E_a$ and $DF(t)E_d$ respectively. Using Lemma 2.6 for elimination of uncertain matrices and separating nominal and uncertain matrices the LMIs (2.258) and (2.259) are obtained. This completes the proof. □

The estimated delay upper bound using the stability condition in Theorem 2.27 for the system described in Numerical Example 2.6 is illustrated in Table 2.16.

Remark 2.30 One can observe from the results presented in Table 2.16 that, the proposed bounding inequalities in conjunction with the delay partitioning method gives less conservative estimate of the delay upper bound compared to the existing results [47, 49, 50], for different cases of delay lower bounds.

Table 2.16 d_u results of Example 2.6 for $0 \leq \mu < 1$ for specified d_l

Stability methods	d_l	d_u
For $\mu = 0.5$		
Theorem 2.27	0	0.5563
[47]		0.4760
[49]		0.4243
[50]		0.4783
Theorem 2.27	0.1	0.5935
[49]		0.4767
Theorem 2.27	0.2	0.6294
[49]		0.5429
Theorem 2.27	0.3	0.6642
[49]		0.6059
Theorem 2.27	0.4	0.6982
[49]		0.6656
Theorem 2.27	0.5	0.7315
[49]		0.7238
For $\mu = 0.9$		
Theorem 2.27	0	0.5563
[47]		0.4760
[50]		0.4783
Theorem 2.27	0.1	0.5935
	0.2	0.6294
	0.3	0.6642
	0.4	0.6982
	0.5	0.7315

2.9 Conclusion

This chapter first deals with the review of existing literature on development of stability analysis of linear time-delay systems using LK functional approach in an LMI framework. The new stability conditions have been presented by introducing new LK functional, improved bounding inequalities and free weighting matrices. Unlike other methods, some useful terms (integral of quadratic form of $\dot{x}(t)$) in the derivative of LK functional are not ignored and their presence is taken into account by using tighter bounding of the integral term, this in turn, results less conservative results. The stability conditions for three classes of time-delay systems have been proposed:

(i) Delay-dependent stability condition for TDS with single time-varying delay
(ii) Delay-dependent stability condition for TDS with two-additive time-varying delays

(iii) Delay-range-dependent stability condition for TDS with time-varying delay.

The effectiveness of the proposed stability criteria are successfully verified by numerical examples. Tables 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9 and 2.10 show that the proposed delay-dependent stability criteria provide less conservative results for delay upper bound estimate consistently for different delay derivatives compared to the existing methods.

The second part of this chapter deals with the robust stability analysis of an uncertain time-delay systems where the structure of the uncertainty is assumed to be of norm-bounded type. New and improved robust stability conditions for the following problems have been obtained by adopting the same procedure as discussed in the first part of this chapter,

(i) Delay-dependent robust stability condition for TDS with single time-varying delay
(ii) Delay-range-dependent robust stability condition for TDS with single time-varying delay
(iii) Delay-range-dependent robust stability condition for TDS with single time-varying delay using delay partitioning approach.

Tables 2.12, 2.13, 2.14, 2.15 and 2.16 show that the proposed delay-dependent robust stability methods give less conservative results for delay upper bound estimate compared to the existing methods.

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