A Model Problem: 1D Elastostatics

2

2.1 Introduction: A Model Problem

In most problems of mathematical physics the true solutions are nonsmooth; i.e., they are not continuously differentiable. *Thus, we cannot immediately apply a Galerkin approach.* For example in the equation of static mechanical equilibrium[1]

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}, \tag{2.1}$$

there is an implicit requirement that the stress, $\boldsymbol{\sigma}$, is differentiable in the classical sense. Virtually the same mathematical structure form holds for other partial differential equations of mathematical physics describing diffusion, heat conduction, etc. *In many applications, differentiability is too strong a requirement, and in many locations it does not hold (the solution "jumps").* Therefore, when solving such problems we have two options: (1) enforcement of solution jump conditions at all of these locations (often they are not even known a priori) or (2) weak formulations (weakening the regularity requirements). Weak forms, which are designed to accommodate irregular data and solutions, are usually preferred. *Numerical techniques employing weak forms, such as the finite element method, have been developed with the essential property that whenever a smooth classical solution exists, it is also a solution to the weak form problem.* Therefore, we lose nothing by reformulating a problem in a more general way, by weakening the a priori smoothness requirements of the solution.

In the following few chapters, we shall initially consider a one-dimensional structure which occupies an open bounded domain in $\Omega \in I\!R$, with boundary $\partial\Omega$. The boundary consists of Γ_u on which the displacements (u), or any other primal variable (temperature in heat conduction applications, concentration in diffusion

[1] Here $\boldsymbol{f}$ are the body forces.

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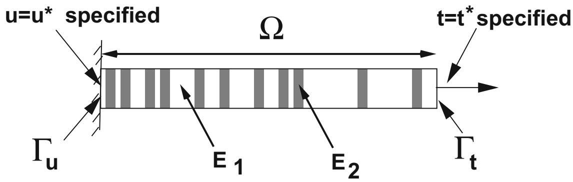

Fig. 2.1 A one-dimensional body

applications, etc. (see Appendix B)), are prescribed and a part Γ_t on which tractions ($t \stackrel{\text{def}}{=} \sigma n$, n being the outward normal) are prescribed ($t = t^*$, Fig. 2.1). We now focus on weak forms of a one-dimensional version of Eq. 2.1

$$\frac{d\sigma}{dx} + f = 0, \qquad (\sigma = E\frac{du}{dx}), \tag{2.2}$$

where $E = E(x)$ is a spatially varying coefficient (Fig. 2.1). Thereafter, we will discuss three-dimensional problems.

2.2 Weak Formulations in One Dimension

To derive a direct weak formulation for a body, we take Eq. 2.2 (denoted the strong form), form a product with an arbitrary smooth scalar-valued function ν, and integrate over the body

$$\int_\Omega (\frac{d\sigma}{dx} + f)\nu\, dx = \int_\Omega r\nu\, dx = 0, \tag{2.3}$$

where r is the residual. We call ν a "test" function. If we were to add a condition that we do this for all ($\stackrel{\text{def}}{=} \forall$) possible "test" functions then

$$\int_\Omega (\frac{d\sigma}{dx} + f)\nu\, dx = \int_\Omega r\nu\, dx = 0\, \forall\nu, \tag{2.4}$$

implies $r(x) = 0$. Therefore, if every possible test function were considered, then $r = \frac{d\sigma}{dx} + f = 0$ on any finite region in (Ω). Consequently, the weak and strong statements would be equivalent, provided the true solution is smooth enough to have a strong solution. Clearly, r can never be nonzero over any finite region in the body, because the test function will "find" them (Fig. 2.2). Using the product rule of differentiation on $\sigma\nu$ yields

$$\frac{d}{dx}(\sigma\nu) = (\frac{d\sigma}{dx})\nu + \sigma\frac{d\nu}{dx} \tag{2.5}$$

which leads to, $\forall\nu$

$$\int_\Omega (\frac{d}{dx}(\sigma\nu) - \sigma\frac{d\nu}{dx})\, dx + \int_\Omega f\nu\, dx = 0, \tag{2.6}$$

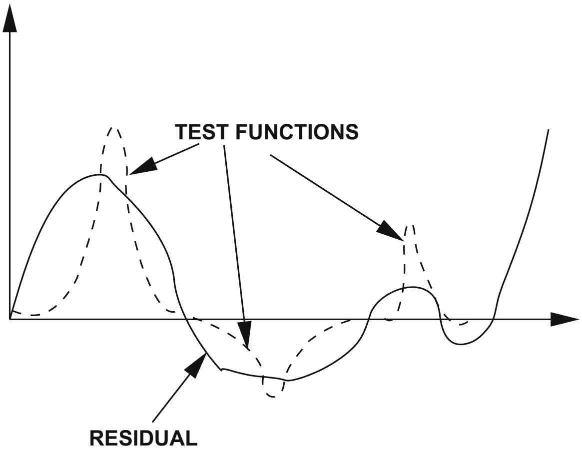

Fig. 2.2 Test functions actions on residuals

where we choose the ν from an admissible set, to be discussed momentarily. This leads to, $\forall\nu$

$$\int_\Omega \frac{d\nu}{dx}\sigma\, dx = \int_\Omega f\nu\, dx + \sigma\nu|_{\partial\Omega}, \tag{2.7}$$

On Γ_t, the stress σ on this boundary is known, $\sigma = t^*$ (Fig. 2.1), and is unknown on Γ_u, and thus, we decide to restrict our choices of ν's to those that attain $\nu|_{\Gamma_u} = 0$. We note the use of the symbol t^* stems from the identification of stresses on the boundary as "tractions." Also, choosing a priori for the solution from those functions such that $u|_{\Gamma_u} = u^*$, where u^* is the applied boundary displacement, on a displacement part of the boundary, Γ_u, we have

$$\boxed{\begin{aligned} &\text{Find } u, u|_{\Gamma_u} = u^*, \quad \text{such that } \forall\nu, \nu|_{\Gamma_u} = 0 \\ &\underbrace{\int_\Omega \frac{d\nu}{dx} E \frac{du}{dx}\, dx}_{\stackrel{\text{def}}{=}\mathcal{B}(u,\nu)} = \underbrace{\int_\Omega f\nu\, dx + t^*\nu|_{\Gamma_t}}_{\stackrel{\text{def}}{=}\mathcal{F}(\nu)}. \end{aligned}} \tag{2.8}$$

This is called a *weak* form because it does not require the differentiability of σ. In other words, the differentiability requirements have been *weakened*. It is clear that we are able to consider problems with quite irregular solutions. We observe that if we test the solution with all possible test functions of sufficient smoothness, then the weak solution is equivalent to the strong solution. *We emphasize that provided the true solution is smooth enough, the weak and strong forms are equivalent, which can be seen by the above constructive derivation.* To explain the point more clearly, we consider a simple example.

2.3 An Example

Let us define a one-dimensional continuous function $r \in C^0(\Omega)$, on a one-dimensional domain, $\Omega = (0, L)$. Our claim is that

$$\int_{\Omega} r\nu \, dx = 0, \tag{2.9}$$

$\forall \nu \in C^0(\Omega)$, implies $r = 0$. This can be easily proven by contradiction. Suppose $r \neq 0$ at some point $\zeta \in \Omega$. Since $r \in C^0(\Omega)$, there must exist a subdomain (subinterval), $\omega \in \Omega$, defined through δ, $\omega \stackrel{\text{def}}{=} \zeta \pm \delta$ such that r has the same sign as at point ζ. Since ν is arbitrary, we may choose ν to be zero outside of this interval, and with the same sign as r inside (Fig. 2.3). This would imply that

$$0 < \int_{\Omega} r\nu \, dx = \int_{\omega} r\nu \, dx = 0, \tag{2.10}$$

which is a contradiction. Now select

$$r = \frac{d\sigma}{dx} + f \in C^0(\Omega) \Rightarrow \frac{d}{dx}\left(E\frac{du}{dx}\right) + f \in C^0(\Omega) \Rightarrow u \in C^2(\Omega). \tag{2.11}$$

Therefore, for this model problem, the equivalence of weak and strong forms occurs if $u \in C^2(\Omega)$.

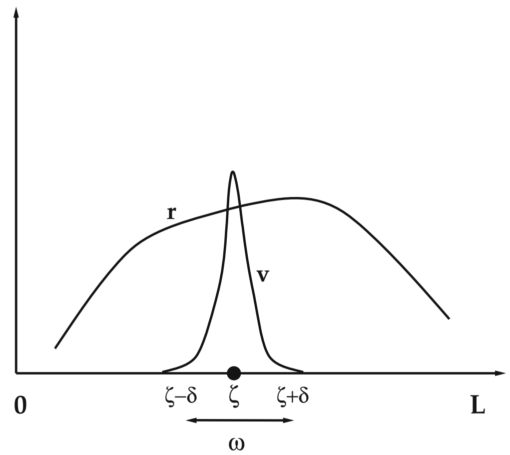

Fig. 2.3 A residual function and a test function

2.4 Some Restrictions

A key question is the selection of the sets of functions in the weak form. Somewhat naively, the answer is simple; the integrals must remain finite. Therefore, the following restrictions hold $(\forall \nu)$, $\int_{\Omega} \frac{d\nu}{dx}\sigma\, dx < \infty$, $\int_{\Omega} f\nu\, dx < \infty$, $\int_{\partial\Omega} t\nu\, dx < \infty$ and govern the selection of the approximation spaces. In order to make precise statements one must have a method of "book keeping." Such a system is to employ so-called Hilbertian Sobolev spaces. We recall that a norm has three main characteristics for any vectors u and ν such that $||u|| < \infty$ and $||\nu|| < \infty$ are (1) $||u|| > 0$, $||u|| = 0$ if and only if $u = 0$ ("positivity"), (2) $||u + \nu|| \leq ||u|| + ||\nu||$ (triangle inequality), and (3) $||\alpha u|| = |\alpha|||u||$, where α is a scalar constant ("scalability"). Certain types of norms, so-called Hilbert space norms, are frequently used in mathematical physics. Following standard notation, we denote $H^1(\Omega)$ as the usual space of scalar functions with generalized partial derivatives of order ≤ 1 in $L^2(\Omega)$; i.e., it is square integrable. In other words, $u \in H^1(\Omega)$ if

$$||u||^2_{H^1(\Omega)} \stackrel{\text{def}}{=} \int_{\Omega} \frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\, dx + \int_{\Omega} uu\, dx < \infty. \tag{2.12}$$

Using these definitions, a complete boundary value problem can be written as follows. The input data loading is assumed to be such that for body forces $f \in L^2(\Omega)$ and boundary traction $\sigma = t^* \in L^2(\Gamma_t)$, but less smooth data can be considered without complications. In summary we assume that our solutions obey these restrictions, leading to the following weak form

$$\boxed{\begin{aligned} &\text{Find } u \in H^1(\Omega), u|_{\Gamma_u} = u^*, \quad \text{such that } \forall \nu \in H^1(\Omega), \nu|_{\Gamma_u} = 0 \\ &\int_{\Omega} \frac{d\nu}{dx} E \frac{du}{dx}\, dx = \int_{\Omega} f\nu\, dx + t^*\nu|_{\Gamma_t}. \end{aligned}} \tag{2.13}$$

We note that if the data in (2.13) are smooth and if (2.13) possesses a solution u that is sufficiently regular, then u is the solution of the classical problem in strong form

$$\boxed{\begin{aligned} &\tfrac{d}{dx}(E\tfrac{du}{dx}) + f = 0, \qquad \forall x \in \Omega, \\ &u = u^*, \qquad \forall x \in \Gamma_u, \\ &\sigma = E\tfrac{du}{dx} = t^*, \qquad \forall x \in \Gamma_t. \end{aligned}} \tag{2.14}$$

2.5 Remarks on Nonlinear Problems

The treatment of nonlinear problems is outside the scope of this introductory monograph. However, a few comments are in order. The literature of solving nonlinear problems with the FEM is vast. This is a complex topic that is best illustrated with an extremely simple one-dimensional example with material nonlinearities. Starting with

$$\frac{d}{dx}\underbrace{\left(E(\underbrace{\frac{du}{dx}}_{\stackrel{\text{def}}{=}\epsilon})^p\right)}_{\stackrel{\text{def}}{=}\sigma}+f=0 \tag{2.15}$$

the weak form reads

$$\int_0^L \frac{d\nu}{dx}\sigma\,dx = \int_0^L f\nu\,dx + t^*v|_{\Gamma_t}. \tag{2.16}$$

Using a Taylor series expansion of $\sigma(\epsilon(u))$ about a trial solution $u^{(k)}$ yields (k will be used as an iteration counter)

$$\begin{aligned}\sigma(u^{(k+1)}) &= E(\epsilon(u^{(k+1)}))^p\\ &\approx E\left((\epsilon(u^{(k)}))^p + p(\epsilon(u^{(k)}))^{p-1}\times\left(\epsilon(u^{(k+1)})-\epsilon(u^{(k)})\right)+\mathcal{O}(||u^{(k+1)}-u^{(k)}||^2)\right)\end{aligned} \tag{2.17}$$

and substituting this into the weak form yields

$$\begin{aligned}\int_0^L \frac{d\nu}{dx}\underbrace{\left(Ep(\epsilon(u^{(k)}))^{p-1}\right)}_{E^{tan}}\epsilon(u^{(k+1)})\,dx &= \int_0^L f\nu\,dx + t^*\nu|_{\Gamma_t}\\ &\quad -\int_0^L \frac{d\nu}{dx}E\left((\epsilon(u^{(k)}))^p - p((\epsilon(u^{(k)}))^p)\right)dx.\end{aligned} \tag{2.18}$$

One then iterates $k = 1, 2, ...,$ until $||u^{(k+1)} - u^{(k)}|| \leq TOL$. Convergence of such a Newton-type formulation is of concern. We refer the reader to the seminal book of Oden [1], which developed and pioneered nonlinear formulations and convergence analysis. For example, consider a general abstract nonlinear equation of the form

$$\boldsymbol{\Pi}(\boldsymbol{u}) = 0, \tag{2.19}$$

and the expansion

$$\boldsymbol{\Pi}(\boldsymbol{u}^{(k+1)}) = \boldsymbol{\Pi}(\boldsymbol{u}^{(k)}) + \nabla_{\boldsymbol{u}} \boldsymbol{\Pi}(\boldsymbol{u}^{(k)}) \cdot (\boldsymbol{u}^{(k+1)} - \boldsymbol{u}^{(k)}) + \mathcal{O}(||\boldsymbol{u}^{(k+1)} - \boldsymbol{u}^{(k)}||^2) \approx 0. \tag{2.20}$$

The Newton update can be written in the following form

$$\boldsymbol{u}^{(k+1)} = \boldsymbol{u}^{(k)} - \left(\boldsymbol{\Pi}^{TAN}(\boldsymbol{u}^{(k)})\right)^{-1} \cdot \boldsymbol{\Pi}(\boldsymbol{u}^{(k)}), \tag{2.21}$$

where $\boldsymbol{\Pi}^{TAN}(\boldsymbol{u}) \stackrel{\text{def}}{=} \nabla_{\boldsymbol{u}} \boldsymbol{\Pi}(\boldsymbol{u})$ is the so-called tangent operator. One immediately sees a potential difficulty, due to the possibility of a zero, or near zero, tangent when employing a Newton's method to a system that may have a nonmonotonic response, for example those involving material laws with softening. Specialized techniques can be developed for such problems, and we refer the reader to the state of the art found in Wriggers [2].

References

1. Oden, J. T. (1972). *Finite elements of non-linear continua*. New York: McGraw-Hill.
2. Wriggers, P. (2008). *Nonlinear finite element analysis*. Berlin: Springer.

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