

Chapter 2

Fuzzy Graphs

A graph represents a particular relationship between elements of a set V . It gives an idea about the extent of the relationship between any two elements of V . We can solve this problem by using a weighted graph if proper weights are known. But in most of the situations, the weights may not be known, and the relationships are ‘fuzzy’ in a natural sense. Hence, a fuzzy relation can deal with the situation in a better way. As an example, if V represents certain locations and a network of roads is to be constructed between elements of V , then the costs of construction of the links are fuzzy. But the costs can be compared, to some extent using the terrain and local factors and can be modeled as fuzzy relations. Thus, fuzzy graph models are more helpful and realistic in natural situations.

Kaufman [91] gave the first definition of a fuzzy graph. But it was Rosenfeld [154] and Yeh and Bang [186] who laid the foundations for fuzzy graph theory. Rosenfeld introduced fuzzy analogs of several basic graph-theoretic concepts, including sub-graphs, paths, connectedness, cliques, bridges, cutvertices, forests, and trees. Yeh and Bang independently introduced many connectivity concepts including vertex and edge connectivity in fuzzy graphs and applied fuzzy graphs for the first time in clustering of data.

In this chapter, we discuss fundamentals of fuzzy graph theory. We provide formal definitions, basic concepts, and properties of fuzzy graphs. For simplicity, we consider only undirected fuzzy graphs, unless otherwise specified. Thus, the edges of the fuzzy graph are unordered pairs of vertices.

2.1 Definitions and Basic Properties

Let V be a nonempty set. Define the relation \sim on $V \times V$ by for all $(x, y), (u, v) \in V \times V$, $(x, y) \sim (u, v)$ if and only if $x = u$ and $y = v$ or $x = v$ and $y = u$. Then it is easily shown that \sim is an equivalence relation on $V \times V$. For all $x, y \in V$, let

$[(x, y)]$ denote the equivalence class of (x, y) with respect to \sim . Then $[(x, y)] = \{(x, y), (y, x)\}$. Let $\mathcal{E}_V = \{[(x, y)] \mid x, y \in V, x \neq y\}$. For simplicity, we often write \mathcal{E} for \mathcal{E}_V when V is understood. Let $E \subseteq \mathcal{E}$. A graph is a pair (V, E) . The elements of V are thought of as vertices of the graph and the elements of E as the edges. For $x, y \in V$, we let xy denote $[(x, y)]$. Then clearly $xy = yx$. We note that graph (V, E) has no loops or parallel edges.

Definition 2.1.1 A **fuzzy graph** $G = (V, \sigma, \mu)$ is a triple consisting of a nonempty set V together with a pair of functions $\sigma : V \rightarrow [0, 1]$ and $\mu : \mathcal{E} \rightarrow [0, 1]$ such that for all $x, y \in V$, $\mu(xy) \leq \sigma(x) \wedge \sigma(y)$.

The fuzzy set σ is called the **fuzzy vertex set** of G and μ the **fuzzy edge set** of G . Clearly μ is a fuzzy relation on σ . We consider V as a finite set, unless otherwise specified. For notational convenience, we use simply G or (σ, μ) to represent the fuzzy graph $G = (V, \sigma, \mu)$. Also, σ^* and μ^* , respectively, represent the supports of σ and μ , also denoted by $\text{Supp}(\sigma)$ and $\text{Supp}(\mu)$.

Example 2.1.2 Let $V = \{a, b, c\}$. Define the fuzzy set σ on V as $\sigma(a) = 0.5$, $\sigma(b) = 1$ and $\sigma(c) = 0.8$. Define a fuzzy set μ of \mathcal{E} such that $\mu(ab) = 0.5$, $\mu(bc) = 0.7$ and $\mu(ac) = 0.1$. Then $\mu(xy) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$. Thus, $G = (\sigma, \mu)$ is a fuzzy graph. If we redefine $\mu(ab) = 0.6$, then it is no longer a fuzzy graph.

It follows from the Definition 2.1.1 that any unweighted graph (V, E) is trivially a fuzzy graph with $\sigma(x) = 1$ for all $x \in V$ and $\mu(xy) = 0$ or 1 for all $x, y \in V$. Also, we write (V, μ) to denote a fuzzy graph with $\sigma(x) = 1$ for all $x \in V$.

Definition 2.1.3 Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then a fuzzy graph $H = (V, \tau, \nu)$ is called a **partial fuzzy subgraph** of G if $\tau \subseteq \sigma$ and $\nu \subseteq \mu$. Similarly, the fuzzy graph $H = (P, \tau, \nu)$ is called a **fuzzy subgraph** of G induced by P if $P \subseteq V$, $\tau(x) = \sigma(x)$ for all $x \in P$ and $\nu(xy) = \mu(xy)$ for all $x, y \in P$. We write $\langle P \rangle$ to denote the fuzzy subgraph induced by P .

Example 2.1.4 Let $G = (\tau, \nu)$, where $\tau^* = \{a, b, c\}$ and $\mu^* = \{ab, bc\}$ with $\tau(a) = 0.4$, $\tau(b) = 0.8$, $\tau(c) = 0.5$, $\nu(ab) = 0.3$ and $\nu(bc) = 0.2$. Then clearly G is a partial fuzzy subgraph of the fuzzy graph in Example 2.1.2. Also, if $P = \{a, b\}$ and $H = (\tau, \nu)$, where $\tau(a) = 0.5$, $\tau(b) = 1$ and $\nu(ab) = 0.5$, then H is the induced fuzzy subgraph of G in Example 2.1.2, induced by P .

Definition 2.1.5 Let $G = (\sigma, \mu)$ be a fuzzy graph. Then a partial fuzzy subgraph (τ, ν) of G is said to **span** G if $\sigma = \tau$. In this case, we call (τ, ν) a **spanning fuzzy subgraph** of (σ, μ) .

In fact a fuzzy subgraph $H = (\tau, \nu)$ of a fuzzy graph $G = (\sigma, \mu)$ induced by a subset P of V is a particular partial fuzzy subgraph of G . Take $\tau(x) = \sigma(x)$ for all $x \in P$ and 0 for all $x \notin P$. Similarly, take $\nu(xy) = \mu(xy)$ if xy is in a set of edges involving elements from P , and 0 otherwise.

Definition 2.1.6 Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Let $0 \leq t \leq 1$. Let $\sigma^t = \{x \in \sigma^* \mid \sigma(x) \geq t\}$ and $\mu^t = \{uv \in \mu^* \mid \mu(uv) \geq t\}$.

Clearly, $\mu^t \subseteq \{xy \mid \sigma(x) \geq t, \sigma(y) \geq t\}$ and hence $H = (\sigma^t, \mu^t)$ is a graph with vertex set σ^t and edge set μ^t . H is called the **threshold graph** of the fuzzy graph G , corresponding to t .

Proposition 2.1.7 Let $G = (\sigma, \mu)$ be a fuzzy graph and $0 \leq s < t \leq 1$. Then the threshold graph (σ^t, μ^t) is a subgraph of the threshold graph (σ^s, μ^s) . Also, if $H = (\nu, \tau)$ is a partial fuzzy subgraph of G and $t \in [0, 1]$, then (ν^t, τ^t) is a subgraph of (σ^t, μ^t) .

2.2 Connectivity in Fuzzy Graphs

We mainly discuss the concepts of fuzzy cutvertices and fuzzy bridges in this section. Most of the results are due to Sunitha and Vijayakumar [167, 168]. Also, Theorem 2.2.1, by Rosenfeld gives a very strong characterization for a fuzzy bridge.

A **path** P in a fuzzy graph $G = (\sigma, \mu)$ is a sequence of distinct vertices x_0, x_1, \dots, x_n (except possibly x_0 and x_n) such that $\mu(x_{i-1}x_i) > 0, i = 1, \dots, n$. Here n is called the **length** of the path. The consecutive pairs are called the **edges** of the path. The **diameter** of $x, y \in V$, written $\text{diam}(x, y)$, is the length of the longest path joining x to y . The **strength** of P is defined to be $\wedge_{i=1}^n \mu(x_{i-1}x_i)$. In words, the strength of a path is defined to be the weight of the weakest edge. We denote the strength of a path P by $d(P)$ or $s(P)$. The strength of connectedness between two vertices x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $\mu^\infty(x, y)$ or $\text{CONN}_G(x, y)$. A strongest path joining any two vertices x, y has strength $\mu^\infty(x, y)$. It can be shown that if (τ, ν) is a partial fuzzy subgraph of (σ, μ) , then $\nu^\infty \subseteq \mu^\infty$. We call P a **cycle** if $x_0 = x_n$ and $n \geq 3$. Two vertices that are joined by a path are called **connected**. It follows that this notion of connectedness is an equivalence relation. The equivalence classes of vertices under this equivalence relation are called **connected components** of the given fuzzy graph. They are just its maximal connected partial fuzzy subgraphs.

Let $G = (\sigma, \mu)$ be a fuzzy graph, let x, y be two distinct vertices and let G' be the partial fuzzy subgraph of G obtained by deleting the edge xy . That is, $G' = (\sigma, \mu')$, where $\mu'(xy) = 0$ and $\mu' = \mu$ for all other pairs. We call xy a **fuzzy bridge** in G if $\mu'^\infty(u, v) < \mu^\infty(u, v)$ for some u, v in σ^* . In words, the deletion of the edge xy reduces the strength of connectedness between some pair of vertices in G . Thus, xy is a fuzzy bridge if and only if there exists vertices u, v such that xy is an edge of every strongest path from u to v .

Theorem 2.2.1 ([154]) Let $G = (\sigma, \mu)$ be a fuzzy graph. Then the following statements are equivalent.

- (i) xy is a fuzzy bridge.
- (ii) $\mu^\infty(x, y) < \mu(xy)$.
- (iii) xy is not the weakest edge of any cycle.

Proof (ii) \Rightarrow (i) If xy is not a fuzzy bridge, then $\mu^\infty(x, y) = \mu^\infty(x, y) \geq \mu(xy)$.

(i) \Rightarrow (iii) If xy is the weakest edge of a cycle, then any path P involving edge xy can be converted into a path P' not involving xy but at least as strong as P , by using the rest of the cycle as a path from x to y . Thus, xy cannot be a fuzzy bridge.

(iii) \Rightarrow (ii) If $\mu^\infty(x, y) \geq \mu(xy)$, then there is a path from x to y not involving xy , that has strength $\geq \mu(xy)$, and this path together with xy forms a cycle of G in which xy is a weakest edge. ■

Let w be any vertex and let G' be the partial fuzzy subgraph of G obtained by deleting the vertex w . That is, $G' = (\sigma', \mu')$ is the partial fuzzy subgraph of G such that $\sigma'(w) = 0$, $\sigma = \sigma'$ for all other vertices, $\mu'(wz) = 0$ for all vertices z , and $\mu' = \mu$ for all other edges. We call w a **fuzzy cutvertex** in G if $\mu^\infty(u, v) < \mu^\infty(u, v)$ for some u, v in V such that $u \neq w \neq v$. In words, w is a fuzzy cutvertex if deleting the vertex w reduces the strength of connectedness between some other pair of vertices. Hence, w is a fuzzy cutvertex if and only if there exists u, v distinct from w such that w is on every strongest path from u to v . G' is called **nonseparable** or a **block** if it has no fuzzy cutvertices. Although in a fuzzy graph, a block may have fuzzy bridges, this cannot happen for crisp graphs. Sometimes we refer to a block in a fuzzy graph as a fuzzy block.

A **maximum spanning tree** of a connected fuzzy graph (σ, μ) is a fuzzy spanning subgraph $T = (\sigma, \nu)$ of G , which is a tree, such that $\mu^\infty(u, v)$ is the strength of the unique strongest $u - v$ path in T for all $u, v \in G$. We next characterize fuzzy cutvertices and fuzzy bridges of fuzzy graphs using maximum spanning trees.

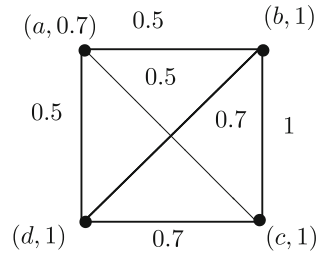
Theorem 2.2.2 ([168]) *A vertex w of a fuzzy graph $G = (\sigma, \mu)$ is a fuzzy cutvertex if and only if w is an internal vertex of every maximum spanning tree of G .*

Proof Let $G = (\sigma, \mu)$ be a fuzzy graph and w be a fuzzy cutvertex of G . Then there exists u, v distinct from w such that w is on every strongest $u - v$ path. Now, each maximum spanning tree of G contains a unique strongest $u - v$ path and hence w is an internal vertex of every maximum spanning tree of G .

Conversely, let w be an internal vertex of every maximum spanning tree. Let T be a maximum spanning tree and let uw and wv be edges in T . Note that the path u, w, v is a strongest $u - v$ path in T . If possible assume that w is not a fuzzy cutvertex. Then between every pair of vertices u, v , there exists at least one strongest $u - v$ path not containing w . Consider one such $u - v$ path P which clearly contains edges not in T . Now, without loss of generality, let $\mu^\infty(u, v) = \mu(uw)$ in T . Then edges in P have strength $\geq \mu(uw)$. Removal of uw and adding P in T will result in another maximum spanning tree of G , of which w is an end vertex, which contradicts our assumption. ■

From Theorem 2.2.2, it can be seen that the end vertices of a maximum spanning tree T of G cannot be fuzzy cutvertices of G . Thus, we have the following corollary.

Fig. 2.1 A noncomplete fuzzy graph satisfying $\mu^\infty = \mu$



Corollary 2.2.3 Every fuzzy graph G has at least two vertices which are not fuzzy cutvertices.

Corollary 2.2.4 An edge uv of a fuzzy graph $G = (\sigma, \mu)$ is a fuzzy bridge if and only if uv is in every maximum spanning tree of G .

Proof Let uv be a fuzzy bridge of G . Then the edge uv is the unique strongest $u - v$ path and hence is in every maximum spanning tree of G .

Conversely, let uv be in every maximum spanning tree of G and assume that uv is not a fuzzy bridge. Then uv is the weakest edge of some cycle in G and $\mu^\infty(u, v) \geq \mu(uv)$, which implies that there is at least one maximum spanning tree of G not containing uv . ■

In 1989, Bhutani [40] introduced the concept of a complete fuzzy graph as follows.

A **complete fuzzy graph (CFG)** is a fuzzy graph $G = (\sigma, \mu)$ such that $\mu(uv) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in \sigma^*$. If $G = (\sigma, \mu)$ is a complete fuzzy graph, then $\mu^\infty = \mu$ and G has no fuzzy cutvertices.

Example 2.2.5 Let $V = \{a, b, c, d\}$ and $X = \{ab, bc, cd, da, ac, bd\}$. Let $\sigma(a) = 0.7$ and $\sigma(b) = \sigma(c) = \sigma(d) = 1$. Let μ be the fuzzy subset of X defined by $\mu(ab) = \mu(da) = \mu(ac) = 0.5$, $\mu(bc) = 1$ and $\mu(cd) = \mu(bd) = 0.7$. Then $\mu^\infty = \mu$, and G has no fuzzy cutvertices, but G is not complete (Fig. 2.1).

Example 2.2.6 We show that a complete fuzzy graph may have a bridge. Let $V = \{a, b, c, d\}$ and $X = \{ab, bc, cd, da, ac, bd\}$. Let $\sigma(a) = 0.7$, $\sigma(b) = 0.8$, $\sigma(c) = 1$ and $\sigma(d) = 0.6$. Let μ be the fuzzy subset of X defined by $\mu(ab) = \mu(ac) = 0.7$, $\mu(bd) = \mu(cd) = \mu(ad) = 0.6$ and $\mu(bc) = 0.8$. Clearly, (σ, μ) is complete and bc is a fuzzy bridge (Fig. 2.2).

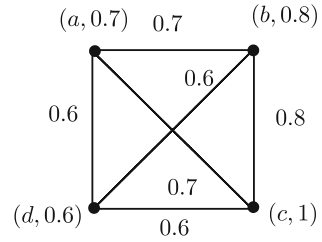
Theorem 2.2.7 If $G = (\sigma, \mu)$ is a complete fuzzy graph, then for any edge $uv \in \mu^*$, $\mu^\infty(u, v) = \mu(uv)$.

Proof By definition, $\mu^2(u, v) = \bigvee_{z \in \sigma^*} \{\mu(uz) \wedge \mu(zv)\} = \bigvee \{\sigma(u) \wedge \sigma(v) \wedge \sigma(z)\} = \sigma(u) \wedge \sigma(v) = \mu(uv)$.

Similarly, $\mu^3(u, v) = \mu(uv)$ and in the same way one can show that $\mu^k(u, v) = \mu(uv)$ for all positive integers k . Thus,

$$\mu^\infty(u, v) = \sup\{\mu^k(u, v) \mid \text{for all integers } k \geq 1\} = \mu(uv). \quad \blacksquare$$

Fig. 2.2 A complete fuzzy graph with a fuzzy bridge



Corollary 2.2.8 *A complete fuzzy graph has no fuzzy cutvertices.*

Theorem 2.2.7 says that every edge uv in a CFG is a strongest $u - v$ path. Also, a CFG can have at most one fuzzy bridge (Theorem 4, [168]), even though it has no fuzzy cutvertices. This bridge can be easily located as seen from the following theorem.

Theorem 2.2.9 *Let $G = (\sigma, \mu)$ be a CFG with $|\sigma^*| = n$. Then G has a fuzzy bridge if and only if there exists an increasing sequence $\{t_1, t_2, \dots, t_n\}$ such that $t_{n-2} < t_{n-1} \leq t_n$, where $t_i = \sigma(u_i)$ for $i = 1, 2, \dots, n$. Also, the edge $u_{n-1}u_n$ is the fuzzy bridge of G .*

Proof Assume that $G = (\sigma, \mu)$ is a complete fuzzy graph and that G has a fuzzy bridge uv . Now, $\mu(uv) = \sigma(u) \wedge \sigma(v)$. Without loss of generality let, $\sigma(u) \leq \sigma(v)$, so that $\mu(uv) = \sigma(u)$. Note that uv is not a weakest edge of any cycle in G . It is required to prove that $\sigma(u) > \sigma(w)$ for all $w \neq v$. On the contrary, assume that there is at least one vertex $w \neq v$ such that $\sigma(u) \leq \sigma(w)$. Now, consider the cycle $C : u, v, w, u$. Then $\mu(uv) = \mu(uw) = \sigma(u)$ and $\mu(vw) = \sigma(v)$ if $\sigma(u) = \sigma(v)$ or $\sigma(u) < \sigma(v) \leq \sigma(w)$ and $\mu(vw) = \sigma(w)$ if $\sigma(u) < \sigma(w) < \sigma(v)$. In either case, the edge uv becomes the weakest edge of a cycle and by Theorem 2.2.1, uv cannot be a fuzzy bridge, a contradiction.

Conversely, let $t_1 \leq t_2 \leq \dots \leq t_{n-2} \leq t_{n-1} \leq t_n$ and $t_i = \sigma(u_i)$ for all i .

Claim. Edge $u_{n-1}u_n$ is the fuzzy bridge of G .

We have, $\mu(u_{n-1}u_n) = \sigma(u_{n-1}) \wedge \sigma(u_n) = \sigma(u_{n-1})$ and by hypothesis, all other edges of G will have strength strictly less than that of $\sigma(u_{n-1})$. Thus, the edge $u_{n-1}u_n$ is not the weakest edge of any cycle in G and by Theorem 2.2.1, is a fuzzy bridge. ■

We next present more connectivity properties of fuzzy cutvertices and fuzzy bridges.

Theorem 2.2.10 *Let $G = (\sigma, \mu)$ be a fuzzy graph such that (σ^*, μ^*) is a cycle. Then a vertex of G is a fuzzy cutvertex if and only if it is a common vertex of two fuzzy bridges.*

Proof Let w be a fuzzy cutvertex of G . Then there exists u and v , other than w such that w is on every strongest $u-v$ path. Because $G^* = (\sigma^*, \mu^*)$ is a cycle, there exists

only one strongest u - v path containing w and all its edges are fuzzy bridges. Thus, w is a common vertex of two fuzzy bridges.

Conversely, let w be a common vertex of two fuzzy bridges uw and wv . Then both uw and wv are not weakest edges of G . Also, the path from u to v not containing edges uw and wv has strength less than $\mu(uw) \wedge \mu(wv)$. Hence, the strongest u - v path is the path u, w, v and $\mu^\infty(u, v) = \mu(uw) \wedge \mu(wv)$. Thus, w is a fuzzy cutvertex. ■

Theorem 2.2.11 *If w is a common vertex of at least two fuzzy bridges, then w is a fuzzy cutvertex.*

Proof Let u_1w and wu_2 be two fuzzy bridges. Then there exists u, v such that u_1w is on every strongest u - v path. If w is distinct from u and v , then it follows that w is a fuzzy cutvertex. Next suppose one of v, u is w so that u_1w is on every strongest u - w path or wu_2 is on every strongest w - v path. Suppose that w is not a fuzzy cutvertex. Then between every two vertices there exists at least one strongest path not containing w . In particular, there exists at least one strongest path P joining u_1 and u_2 , not containing w . This path together with u_1w and wu_2 forms a cycle.

We now consider two cases. First, suppose that u_1, w, u_2 is not a strongest path. Then clearly one of u_1w, wu_2 or both become weakest edges of a cycle, which contradicts that u_1w and wu_2 are fuzzy bridges.

Second, suppose that u_1, w, u_2 is also a strongest path joining u_1 to u_2 . Then $\mu^\infty(u_1, u_2) = \mu(u_1w) \wedge \mu(wu_2)$, the strength of P . Thus, edges of P are at least as strong as $\mu(u_1w)$ and $\mu(wu_2)$, which implies that u_1w, wu_2 are both weakest edges of a cycle, which again is a contradiction. ■

Example 2.2.12 This example shows that the condition in Theorem 2.2.11 is not necessary. Let $V = \{a, b, c, d\}$ and $X = \{ab, bc, cd, da, ac, db\}$. Let $\sigma(x) = 1$ for all $x \in V$ and let μ be the fuzzy subset of X defined by $\mu(ac) = \mu(bd) = 0.9$, $\mu(da) = \mu(cd) = 0.3$ and $\mu(ab) = \mu(bc) = 0.8$. Clearly, b is a fuzzy cutvertex. However, ac and db are the only fuzzy bridges.

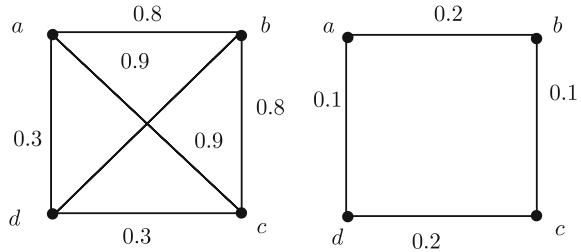
Example 2.2.13 Consider the fuzzy graph $G = (V, X)$, where $V = \{a, b, c, d\}$. Let $X = \{ab, bc, cd, ad\}$. Let $\sigma(s) = 1$ for all $s \in V$ and let μ be the fuzzy subset of X defined by $\mu(ab) = \mu(cd) = 0.2$, and $\mu(bc) = \mu(ad) = 0.1$. Note that ab and cd are fuzzy bridges and no vertex is a fuzzy cutvertex. This is a significant difference from the crisp graph theory.

The fuzzy graphs in Examples 2.2.12 and 2.2.13 are given in Fig. 2.3.

Theorem 2.2.14 *If uv is a fuzzy bridge, then $\mu^\infty(u, v) = \mu(uv)$.*

Proof Suppose that uv is a fuzzy bridge and that $\mu^\infty(u, v) > \mu(uv)$. Then there exists a strongest u - v path with strength greater than $\mu(uv)$ and all edges of this strongest path have strength greater than $\mu(uv)$. Now, this path together with the edge uv forms a cycle in which uv is the weakest edge, contradicting that uv is a fuzzy bridge. ■

Fig. 2.3 Fuzzy graphs given in Examples 2.2.12 and 2.2.13



2.3 Forests and Trees

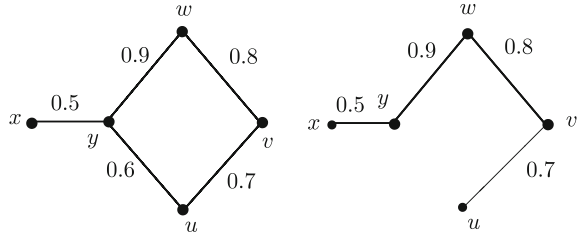
Rosenfeld first studied the concepts of fuzzy forests and fuzzy trees in [154]. The results in this section are by Rosenfeld [154], Mordeson and Nair [127] and Sunitha and Vijayakumar [167, 168]. The structure of fuzzy trees is significantly different from that of trees. In fact, a fuzzy tree can contain cycles in the classical sense. There are several characterizations for fuzzy trees in this book.

A crisp graph that has no cycles is called **acyclic** or a **forest**. A connected forest is a **tree**. A fuzzy graph is called a **forest** if the graph consisting of its nonzero edges is a forest, and a **tree** if this graph is also connected. If $G = (\sigma, \mu)$ is a fuzzy graph, we call G a **fuzzy forest** if it has a partial fuzzy spanning subgraph $F = (\sigma, \nu)$, which is a forest, where for all edges xy not in F , i.e., such that $\nu(xy) = 0$, we have $\mu(xy) < \nu^\infty(x, y)$. In words, if xy is in G , but is not in F , there is a path in F between x and y whose strength is greater than $\mu(xy)$. Clearly, a forest is a fuzzy forest.

Theorem 2.3.1 *A fuzzy graph G is a fuzzy forest if and only if in any cycle of G there is an edge xy such that $\mu(xy) < \mu'^\infty(x, y)$, where $G' = (\sigma, \mu')$ is the partial fuzzy subgraph obtained by deleting the edge xy from G .*

Proof Suppose xy is an edge, belonging to a cycle which has the property of the theorem and for which xy is the smallest. (If there are no cycles, G is a forest and we are done.) If we delete xy , the resulting partial fuzzy subgraph satisfies the path property of a fuzzy forest. If there are still cycles in this graph, we can repeat the process. Now, at each stage, no previously deleted edge is stronger than the edge being currently deleted. Thus, the path guaranteed by the property of the theorem involves only edges that have not yet been deleted. When no cycles remain, the resulting partial fuzzy subgraph is a forest F . Let xy not be an edge of F . Then xy is one of the edges that we deleted in the process of constructing F , and there is a path from x to y that is stronger than $\mu(xy)$ and that does not involve xy nor any of the edges deleted prior to it. If this path involves edges that were deleted later, it can be diverted around them using a path of still stronger edges; if any of these were deleted later, the path can be further diverted; and so on. This process eventually stabilizes with a path consisting entirely of edges of F . Thus, G is a fuzzy forest.

Fig. 2.4 A fuzzy tree G and its spanning tree F



Conversely, if G is a fuzzy forest and P is any cycle, then some edge xy of P is not in F . Thus, by definition of a fuzzy forest, we have $\mu(xy) < \nu^\infty(x, y) \leq \mu'(xy)$. ■

We see that if G is connected, then so is F as determined by the construction in the first part of the proof. The tree F thus constructed plays a very important role in the study of fuzzy trees. Also, Theorem 2.3.1 allows fuzzy forests to have cycles, as mentioned before.

Proposition 2.3.2 *If there is at most one strongest path between any two vertices of G , then G is a fuzzy forest.*

Proof Suppose G is not a fuzzy forest. Then by the previous theorem, there is a cycle P in G such that $\mu(xy) \geq \mu'(xy)$ for all edges xy of P . Thus, xy is a strongest path from x to y . If we choose xy to be a weakest edge of P , it follows that the rest of P also is a strongest path from x to y , a contradiction. ■

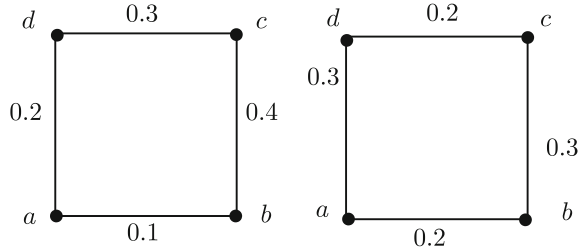
We note that the converse of the previous proposition does not hold.

Example 2.3.3 Consider the fuzzy graphs $G = (\sigma, \mu)$ and $F = (\tau, \nu)$ given in Fig. 2.4 with $V = \{x, y, u, v, w\}$. Define $\sigma, \tau : V \rightarrow [0, 1]$ by for all $z \in V$, $\sigma(z) = 1 = \tau(z)$. Define $\mu, \nu : V \times V \rightarrow [0, 1]$ as $\mu(xy) = 0.5$, $\mu(yw) = 0.9$, $\mu(wv) = 0.8$, $\mu(vu) = 0.7$, $\mu(yu) = 0.6$, $\nu(xy) = 0.5$, $\nu(yw) = 0.9$, $\nu(wv) = 0.8$, $\nu(uv) = 0.7$. It is clear that G is a fuzzy forest. F is the spanning tree of G . But note that both x, xy, y, yu, u, uv, v and x, xy, y, yw, w, wv, v are strongest $x - v$ paths in G .

Proposition 2.3.4 *If $G = (\sigma, \mu)$ is a fuzzy forest, then the edges of $F = (\tau, \nu)$ are precisely the bridges of G .*

Proof An edge xy not in F cannot be a bridge because $\mu(xy) < \nu^\infty(x, y) \leq \mu'(xy)$. Suppose that xy is an edge in F . If it were not a bridge, we would have a path P from x to y , not involving xy , of strength greater than or equal to $\mu(xy)$. This path must involve edges not in F because F is a forest and has no cycles. However, by definition, any such edge $u_i v_i$ can be replaced by a path P_i in F of strength greater than $\mu(u_i v_i)$. Now, P_i cannot involve xy because all its edges are strictly stronger than $\mu(u_i v_i) \geq \mu(xy)$. Thus, by replacing each $u_i v_i$ by P_i , we can construct a path in F from x to y that does not involve xy , giving a cycle in F , a contradiction. ■

Fig. 2.5 A fuzzy tree and a fuzzy cycle



Definition 2.3.5 Let $G = (\sigma, \mu)$ be a fuzzy graph. Then

- (i) G is called a **tree** if $(\text{Supp}(\sigma), \text{Supp}(\mu))$ is a tree.
- (ii) G is called a **fuzzy tree** if G has a fuzzy spanning subgraph $F = (\sigma, \nu)$, which is a tree, such that for all $uv \in \text{Supp}(\mu) \setminus \text{Supp}(\nu)$, $\mu(uv) < \nu^\infty(u, v)$. That is, there exists a path in (σ, ν) between u and v whose strength is greater than $\mu(uv)$.

Definition 2.3.6 Let $G = (\sigma, \mu)$ be a fuzzy graph. Then

- (i) G is called a **cycle** if $(\text{Supp}(\sigma), \text{Supp}(\mu))$ is a cycle.
- (ii) G is called a **fuzzy cycle** if $(\text{Supp}(\sigma), \text{Supp}(\mu))$ is a cycle and \nexists unique $xy \in \text{Supp}(\mu)$ such that $\mu(xy) = \wedge\{\mu(uv) \mid uv \in \text{Supp}(\mu)\}$.

Example 2.3.7 Let $V = \{a, b, c, d\}$ and $X = \{ab, ac, ad, bc, cd, db\}$. Let $\sigma(x) = 1$ for all $x \in V$ and let μ be the fuzzy subset of X defined by $\mu(ab) = 0.9$, $\mu(bc) = \mu(cd) = 0.7$, $\mu(bd) = 0.3$. Then (σ, μ) is neither a fuzzy cycle nor a fuzzy tree.

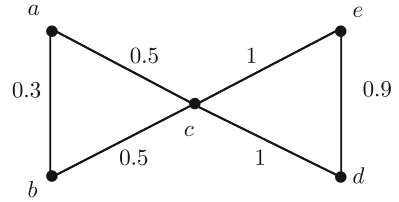
Example 2.3.8 Let $V = \{a, b, c, d\}$ and $X = \{ab, ac, ad, bc, bd, cd\}$. Let $\sigma(x) = 1$ for all $x \in V$ and μ, μ' be fuzzy subsets of X defined by $\mu(ab) = 0.1$, $\mu(bc) = 0.4$, $\mu(cd) = 0.3$, $\mu(ad) = 0.2$ and $\mu'(ab) = 0.2$, $\mu'(bc) = 0.3$, $\mu'(cd) = 0.2$, $\mu'(ad) = 0.3$. Then (σ, μ) is a fuzzy tree, but not a cycle and not a fuzzy cycle while (σ, μ') is a fuzzy cycle, but not a fuzzy tree (Fig. 2.5).

Theorem 2.3.9 ([127]) Let $G = (\sigma, \mu)$ be a cycle. Then G is a fuzzy cycle if and only if G is not a fuzzy tree.

Proof Suppose that G is a fuzzy cycle. Then there exists edges $x_1y_1, x_2y_2 \in \text{Supp}(\mu)$ such that $\mu(x_1y_1) = \mu(x_2y_2) = \wedge\{\mu(uv) \mid uv \in \text{Supp}(\mu)\}$. If (σ, ν) is any spanning tree of (σ, μ) , then $\text{Supp}(\mu) \setminus \text{Supp}(\nu) = \{uv\}$ for some $u, v \in V$ because (σ, μ) is a cycle. Hence, \nexists a path in (σ, ν) between u and v of greater strength than $\mu(uv)$. Thus, (σ, μ) is not a fuzzy tree.

Conversely, suppose that (σ, μ) is not a fuzzy tree. Because (σ, μ) is a cycle, we have for all $uv \in \text{Supp}(\mu)$, (σ, ν) is a fuzzy spanning subgraph of (σ, μ) , which is a tree, and $\nu^\infty(u, v) \leq \mu(uv)$, where $\nu(uv) = 0$ and $\nu(xy) = \mu(xy)$ for all $xy \in \text{Supp}(\mu) \setminus \{uv\}$. Hence, μ does not attain $\wedge\{\mu(xy) \mid xy \in \text{Supp}(\mu)\}$ uniquely. Thus, (σ, μ) is a fuzzy cycle. ■

Fig. 2.6 Fuzzy tree in Example 2.3.11



Theorem 2.3.10 Let $G = (\sigma, \mu)$ be a fuzzy graph. If there exists $t \in (0, 1]$ such that $(\text{Supp}(\sigma), \mu^t)$ is a tree, then G is a fuzzy tree. Conversely, if G is a cycle and G is a fuzzy tree, then there exists $t \in (0, 1]$ such that $(\text{Supp}(\sigma), \mu^t)$ is a tree.

Proof Suppose that there exists $t \in (0, 1]$ such that $(\text{Supp}(\sigma), \mu^t)$ is a tree. Let ν be the fuzzy subset of $V \times V$ such that $\nu = \mu$ on μ^t and $\nu(xy) = 0$ if $xy \in E \setminus \mu^t$. Then (σ, ν) is a spanning fuzzy subgraph of (σ, μ) such that (σ, ν) is a fuzzy tree because $(\text{Supp}(\sigma), \text{Supp}(\nu))$ is a tree. Suppose that $uv \in E$ and $uv \notin \mu^t$. Then \exists a path between u and v of strength $\geq t > \mu(uv)$. Thus, (σ, μ) is a fuzzy tree. For the converse, we note that because (σ, μ) is a cycle and a fuzzy tree, \exists unique $xy \in \text{Supp}(\mu)$ such that $\mu(xy) = \bigwedge \{\mu(uv) \mid uv \in \text{Supp}(\mu)\}$. Let t be such that $\mu(xy) < t \leq \bigwedge \{\mu(uv) \mid uv \in \text{Supp}(\mu) \setminus \{xy\}\}$. Then $(\text{Supp}(\sigma), \mu^t)$ is a tree. ■

Example 2.3.11 Let $V = \{a, b, c, d, e\}$ and $X = \{ab, bc, ac, cd, de, ec\}$. Let $\sigma(x) = 1$ for all $x \in V$ and let μ be the fuzzy subset of X defined by $\mu(ab) = 0.3$, $\mu(bc) = \mu(ac) = 0.5$, $\mu(ec) = \mu(cd) = 1$, $\mu(de) = 0.9$. Then $\nexists t \in (0, 1]$ such that $(\text{Supp}(\sigma), \mu^t)$ is a tree. However, (σ, μ) is a fuzzy tree (see Fig. 2.6).

Theorem 2.3.12 If $G = (\sigma, \mu)$ is a fuzzy tree and (σ^*, μ^*) is not a tree, then there exists at least one edge uv in $\text{Supp}(\mu)$ for which $\mu(uv) < \mu^\infty(u, v)$.

Proof If G is a fuzzy tree, then by definition there exists a fuzzy spanning subgraph $F = (\sigma, \nu)$, which is a tree and $\mu(uv) < \nu^\infty(u, v)$ for all edges uv not in F . Also, $\nu^\infty(u, v) \leq \mu^\infty(u, v)$. Thus, $\mu(uv) < \mu^\infty(u, v)$ for all uv not in F and by hypothesis there exists at least one edge uv not in F . ■

The rest of the results in this section are from [167, 168].

Theorem 2.3.13 Let $G = (\sigma, \mu)$ be a connected fuzzy graph with no fuzzy cycle. Then G is a fuzzy tree.

Proof If G^* has no cycles, then G^* is a tree and G is a fuzzy tree. So assume that G has cycles and by hypothesis no cycle is a fuzzy cycle. That is, every cycle in G will have exactly one weakest edge in it. Remove the weakest edge say e in a cycle C of G . If there are still cycles in the resulting fuzzy graph, repeat the process, which will eventually results in a fuzzy subgraph, which is a tree, and which is the required spanning subgraph F . ■

When we delete a bridge, different fuzzy graph structures behave differently. Consider the case of fuzzy trees in the following theorem.

Theorem 2.3.14 *If G is a fuzzy tree, then the removal of any fuzzy bridge reduces the strength of connectedness between its end vertices and also between some other pair of vertices.*

Proof Let $G = (\sigma, \mu)$ be a fuzzy tree and let uv be a fuzzy bridge of G . Then by Proposition 2.3.4 uv is an edge of the maximum spanning tree T of G and T contains unique strongest paths joining every pair of vertices. So removal of uv reduces the strength of connectedness between some other pair of vertices x, y , where x is adjacent to u and y is adjacent to v if uv is an internal edge of T , and $u = x$ or $v = y$ otherwise. ■

Note that when G^* is K_2 , its unique edge is a fuzzy bridge and its removal reduces the strength of connectedness between its end vertices alone.

Theorem 2.3.15 *If $G = (\sigma, \mu)$ is a fuzzy tree, then G is not complete.*

Proof If possible, let G be a complete fuzzy graph. Then $\mu^\infty(u, v) = \mu(uv)$ for all u, v . Now, G being a tree, $\mu(uv) < \nu^\infty(u, v)$ for all u, v not in F . Thus, $\mu^\infty(u, v) < \nu^\infty(u, v)$, which is impossible. ■

Theorem 2.3.16 *If G is a fuzzy tree, then the internal vertices of F are fuzzy cutvertices of G .*

Proof Let w be any vertex in G , which is not an end vertex of F . Then w is the common vertex of at least two edges in F , which are fuzzy bridges of G and by Theorem 2.2.11, w is a fuzzy cutvertex. Also, if w is an end vertex of F , then w is not a fuzzy cutvertex; else there would exist u, v distinct from w such that w is on every u - v path and one such path certainly lies in F . But because w is an end vertex of F , this is not possible. ■

Corollary 2.3.17 *A fuzzy cutvertex of a fuzzy tree is the common vertex of at least two fuzzy bridges.*

Theorem 2.3.18 ([167]) *Let $G = (\sigma, \mu)$ be a fuzzy graph. Then G is a fuzzy tree if and only if the following conditions are equivalent for all $u, v \in V$.*

- (i) uv is a fuzzy bridge.
- (ii) $\mu^\infty(u, v) = \mu(uv)$.

Proof Let $G = (\sigma, \mu)$ be a fuzzy tree and suppose that uv is a fuzzy bridge. Then $\mu^\infty(u, v) = \mu(uv)$ by Theorem 2.2.14. Now, let uv be an edge in G such that $\mu^\infty(u, v) = \mu(uv)$. If G^* is a tree, then clearly uv is a fuzzy bridge; otherwise, it follows from Theorem 2.3.12 that uv is in F and uv is a bridge.

Conversely, assume that (i) and (ii) are equivalent. Construct a maximum spanning tree $T = (\sigma, \nu)$ for G [38]. If uv is in T , by an algorithm in [38], $\mu^\infty(u, v) = \mu(uv)$ and hence uv is a fuzzy bridge. Now, these are the only fuzzy bridges for G ; for, if possible, let $u'v'$ be a fuzzy bridge of G , which is not in T . Consider a cycle C consisting of $u'v'$ and the unique $u'-v'$ path in T . Now, edges of this $u'-v'$ path are fuzzy bridges and so they are not weakest edges of C and thus $u'v'$ must be the weakest edge of C and cannot be a fuzzy bridge.

Moreover, for all edges $u'v'$ not in T , we have $\mu'(u'v') < \nu'(u'v')$; for if possible let $\mu(u'v') \geq \nu^\infty(u', v')$. But $\nu^\infty(u', v') < \mu^\infty(u', v')$, were strict inequality holds because $u'v'$ is not a fuzzy bridge. Hence, $\nu^\infty(u', v') < \mu^\infty(u', v')$, which gives a contradiction because $\nu^\infty(u', v')$ is the strength of the unique $u'-v'$ path in T and by the algorithm in [36], $\mu^\infty(u', v') = \nu^\infty(u', v')$. Thus, T is the required spanning subgraph F , which is a tree and hence G is a fuzzy tree. ■

From the previous theorem, it follows that the spanning fuzzy subgraph of a fuzzy tree is unique. Also, it follows that F is nothing but the maximum fuzzy spanning tree of G . Thus, we have the following theorem.

Theorem 2.3.19 *A fuzzy graph is a fuzzy tree if and only if it has a unique maximum fuzzy spanning tree.*

If G is a fuzzy graph such that G^* is not a tree and T is the maximum fuzzy spanning tree of G , then there is at least one edge in G which is not a fuzzy bridge. Also, edges not in T are not fuzzy bridges of G . So we have the following result.

Theorem 2.3.20 *If $G = (\sigma, \mu)$ is a fuzzy graph with $\text{Supp}(\sigma) = V$ and $|V| = p$, then G has at most $p - 1$ fuzzy bridges.*

If G is a fuzzy graph with T as its unique maximum fuzzy spanning tree, then end vertices of T are not fuzzy cutvertices G . Thus, every fuzzy graph will have at least two vertices which are not fuzzy cutvertices.

2.4 Fuzzy Cut Sets

This section is based on [127], a work by Mordeson and Nair in 1996. We begin by some topics in graph theory, which can be found in [83]. When G is a graph, one can associate G with two vector spaces over the field of scalars $\mathbb{Z}_2 = \{0, 1\}$, where addition and multiplication are modulo 2. Note that for $1 \in \mathbb{Z}_2$, $1 + 1 = 0$. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. A **0-chain** of G is a formal linear combination $\sum \epsilon_i v_i$ of vertices and a **1-chain** is a formal linear combination of edges $\sum \epsilon_i e_i$, where $\epsilon_i \in \mathbb{Z}_2$. The **boundary operator** ∂ is a linear function which maps 1-chains to 0-chains such that if $e = xy$, then $\partial(e) = x + y$. The **coboundary operator** δ is a linear function which maps 0-chains to 1-chains such that $\delta(v) = \sum \epsilon_i e_i$ whenever e_i is incident with v .

Example 2.4.1 Let $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_6\}$ and $E(G) = \{e_1, e_2, \dots, e_9\}$, where $e_1 = v_1v_2$, $e_2 = v_1v_3$, $e_3 = v_2v_3$, $e_4 = v_2v_4$, $e_5 = v_2v_5$, $e_6 = v_3v_5$, $e_7 = v_3v_6$, $e_8 = v_4v_5$ and $e_9 = v_5v_6$. The 1-chain $\gamma_1 = e_1 + e_2 + e_4 + e_9$ has boundary

$$\begin{aligned}\partial(\gamma_1) &= (v_1 + v_2) + (v_1 + v_3) + (v_2 + v_4) + (v_5 + v_6) \\ &= v_3 + v_4 + v_5 + v_6.\end{aligned}$$

The 0-chain $\gamma_0 = v_3 + v_4 + v_5 + v_6$ has coboundary

$$\begin{aligned}\delta(\gamma_0) &= (e_2 + e_3 + e_6 + e_7) + (e_4 + e_8) + (e_5 + e_6 + e_8 + e_9) + (e_7 + e_9) \\ &= e_2 + e_3 + e_4 + e_5.\end{aligned}$$

A 1-chain with boundary 0 is called a **cycle vector** of G which can be visualized as a set of edge disjoint cycles. The collection of all cycle vectors is called the **cycle space** of G and it is clearly a vector space over \mathbb{Z}_2 . A cut set of a connected graph is a collection of edges whose removal results in a disconnected graph. A **cocycle** is a minimal cutset. A **coboundary** of G is the coboundary of some 0-chain in G . The coboundary of a subset of V is the set of all edges joining a point in this subset to a point not in the subset. Hence, every coboundary is a cutset. Because any minimal cutset is a coboundary, a cocycle is just a minimal nonzero coboundary. The collection of all coboundaries of G is a vector space over \mathbb{Z}_2 and is called the **cocycle space** of G . A basis of this space which consists entirely of cocycles is called a **cocycle basis** for G .

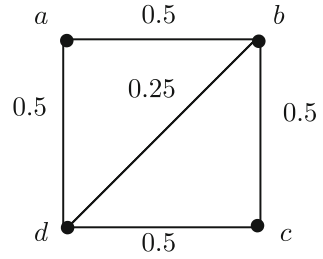
Let G be a connected graph. A **chord** of a spanning tree T of G is an edge of G which is not in T . The subgraph of G consisting of T and any chord of T has only one cycle. The set $C(T)$ of cycles obtained in this way is independent. Every cycle C depends on the set $C(T)$ because C is the symmetric difference of the cycles determined by the chords of T which lie in C . The **cycle rank** $m(G)$ is defined to be the number of cycles in a basis for the cycle space of G . Thus, we have the following result.

Theorem 2.4.2 *The cycle rank of a connected graph G is equal to the number of chords of any spanning tree in G .*

Similar results can be derived for the cocycle space. Assume that G is a connected graph. The **cotree** T' of a spanning tree T of G is the spanning subgraph of G containing exactly those edges which are not in T . A cotree of G is the cotree of some spanning tree T . The edges of G which are not in T' are called its **twigs**. The subgraph of G consisting of T' and any one of its twigs contains exactly one cocycle. The collection of cocycles obtained by adding twigs to T' , one at a time is a basis for the cocycle space of G . The cocycle rank $m'(G)$ is the number of cocycles in a basis for the cocycle space of G .

Theorem 2.4.3 *The cocycle rank of a connected graph G is the number of twigs in any spanning tree T of G .*

Fig. 2.7 Fuzzy graph having no fuzzy bridges



Definition 2.4.4 Let $G = (\sigma, \mu)$ be a fuzzy graph. Let $x \in V$ and let $t \in [0, 1]$. Define the fuzzy subset x_t of V by for all $y \in V$, $x_t(y) = 0$ if $y \neq x$ and $x_t(y) = t$ if $y = x$. Then x_t is called a **fuzzy singleton** in V . If $xy \in E$, then $xy_{\mu(xy)}$ denotes a fuzzy singleton in E .

Definition 2.4.5 Let $G = (\sigma, \mu)$ be a fuzzy graph and let S be a subset of $\text{Supp}(\mu)$. Then

- (i) $\{xy_{\mu(xy)} \mid xy \in S\}$ is called a **cut set** of (σ, μ) if S is a cut set of $(\text{Supp}(\sigma), \text{Supp}(\mu))$.
- (ii) $\{xy_{\mu(xy)} \mid xy \in S\}$ is called a **fuzzy cut set** of (σ, μ) if $\exists u, v \in \text{Supp}(\sigma)$ such that $\mu'^{\infty}(u, v) < \mu^{\infty}(u, v)$, where μ' is the fuzzy subset of E defined by $\mu' = \mu$ on $\text{Supp}(\mu)$ and $\mu'(xy) = 0$ for all $xy \in S$.

When S is a singleton set, a cut set is called a **bridge** and a fuzzy cut set is a **fuzzy bridge**.

Example 2.4.6 In this example, we show there is a fuzzy graph (σ, μ) that has no fuzzy bridges and μ is not a constant function. Let $V = \{a, b, c, d\}$ and $x = \{ab, bc, cd, da, bd\}$. Let $\sigma(x) = 1$ for all $x \in V$, $\mu(ab) = \mu(bc) = \mu(cd) = \mu(da) = 1$ and $\mu(bd) = 0.25$. Then μ is not a constant, but (σ, μ) does not have a fuzzy bridge because the strength of connectedness between any pair of vertices of (σ, μ) remains 0.5 even after the removal of an edge as seen from Fig. 2.7.

Theorem 2.4.7 Let $G = (\sigma, \mu)$ be a fuzzy graph. Let $V = \{v_1, \dots, v_n\}$ and $C = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$, $n \geq 3$.

- (i) Suppose that $C \subseteq \text{Supp}(\mu)$ and that for all $v_jv_k \in \text{Supp}(\mu) \setminus C$, $\mu(v_jv_k) < \bigvee \{\mu(v_i v_{i+1}) \mid i = 1, \dots, n\}$, where $v_{n+1} = v_1$. Then either μ is a constant function on C or G has a fuzzy bridge.
- (ii) If $\emptyset \neq \text{Supp}(\mu) \subset C$, then G has a fuzzy bridge.

Proof (i) Suppose μ is not constant on C . Let $v_h v_{h+1} \in C$ be such that $\mu(v_h v_{h+1}) = \bigvee \{\mu(v_i v_{i+1}) \mid i = 1, \dots, n\}$. Because μ is not constant on C , the strength of the path $C \setminus \{v_h v_{h+1}\}$ between v_h and v_{h+1} is strictly less than $\mu(v_h v_{h+1})$. The strength of any other path P between v_h and v_{h+1} is also strictly less than $\mu(v_h v_{h+1})$ because P must contain an edge from $\text{Supp}(\mu) \setminus C$. Thus, $v_h v_{h+1} \mu(v_h v_{h+1})$ is a fuzzy bridge.

(ii) The result here is immediate. ■

Theorem 2.4.8 *Let $G = (\sigma, \mu)$ be a fuzzy graph. Suppose that the dimension of the cycle space of $(\text{Supp}(\sigma), \text{Supp}(\mu))$ is 1. Then G does not have a fuzzy bridge if and only if G is a cycle and μ is a constant function.*

Proof Suppose it is not the case that (σ, μ) is a cycle and μ is a constant function. If (σ, μ) is not a cycle, then there exists $xy \in \text{Supp}(\mu)$ which is not part of a cycle. Then $xy_{\mu(xy)}$ is a bridge and hence a fuzzy bridge. Suppose that (σ, μ) is a cycle, but μ is not a constant function. Let $xy \in \text{Supp}(\mu)$ be such that $\mu(xy)$ is maximal. Then $xy_{\mu(xy)}$ is a fuzzy bridge.

Conversely, suppose that (σ, μ) is a cycle and μ is not a constant function. Then the deletion of an edge $v_i v_{i+1}$ yields a unique path between v_i and v_{i+1} of strength equal to $\mu(v_i v_{i+1})$. Thus, $v_i v_{i+1}_{\mu(v_i v_{i+1})}$ is not a fuzzy bridge. ■

Several other concepts like fuzzy chords, fuzzy cotrees, and fuzzy twigs can also be found in [127].

2.5 Bridges, Cutsets, and Blocks

In 1985, Delgado, Verdegay, and Vila [62], defined the notions of connectedness, fuzzy cycles, and fuzzy trees differently than Rosenfeld. They used the notion of level sets to define these terms. They pointed out some valid reasons for their definitions. For example, they noted that a fuzzy graph may have different degrees of connectedness and that two fuzzy graphs may share the property that neither is connected, but there is a t -cut of one which is connected while no t -cut of the other is connected.

Later in 2002, Mordeson and Yao [131] studied this further and obtained several new results on connectedness by levels. The work in this section is from [131]. Connectivity analysis by levels is important in any interconnection network. The structural properties of finite fuzzy graphs provide tools for the solutions of Operations Research problems. In this section, several connectedness properties of various types of fuzzy graph structures are discussed. Level graphs are used to define different variants.

Definition 2.5.1 Let $d(\mu) = \wedge\{\mu(xy) \mid xy \in \mu^*\}$ and $h(\mu) = \vee\{\mu(xy) \mid xy \in \mu^*\}$. Then $d(\mu)$ is called the **depth** of μ and $h(\mu)$ is called the **height** of μ .

Note that $d(\mu)$ and $h(\mu)$ are undefined in Definition 2.5.1 if $\mu^* = \emptyset$.

Definition 2.5.2 Let $xy \in \mu^*$. Then

- (i) xy is called a **bridge** if xy is a bridge of (σ^*, μ^*) .
- (ii) xy is called a **fuzzy bridge** if $\mu'^{\infty}(u, v) < \mu^{\infty}(u, v)$ for some $uv \in \mu^*$, where μ' is μ restricted to $E \setminus \{xy\}$.
- (iii) xy is called a **weak fuzzy bridge** if $\exists t \in (0, h(\mu)]$ such that xy is a bridge for G^t .

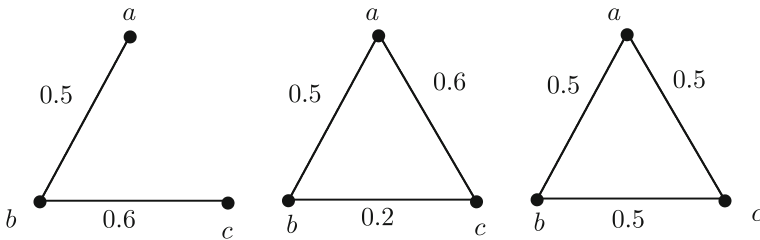


Fig. 2.8 Different types of bridges in a fuzzy graph

- (iv) xy is called a **partial fuzzy bridge** if xy is a bridge for G^t for all $t \in (d(\mu), h(\mu)] \cup \{h(\mu)\}$.
- (v) xy is called a **full fuzzy bridge** if xy is a bridge for G^t for all $t \in (0, h(\mu)]$.

We note that in (iv) of Definition 2.5.2, $(d(\mu), h(\mu)] = \emptyset$ if $d(\mu) = h(\mu)$.

Example 2.5.3 Let $V = \{a, b, c\}$. Define the fuzzy subsets σ of V and μ of $E = \{ab, bc\}$ as follows: $\sigma(a) = \sigma(b) = \sigma(c) = 1$ and $\mu(ab) = 0.5$, $\mu(bc) = 0.6$. Then $d(\mu) = 0.5$ and $h(\mu) = 0.6$. For $0 < t \leq 0.5$, $G^t = (V, \{ab, bc\})$ and for $0.5 < t \leq 0.6$, $G^t = (V, \{bc\})$. Hence, bc is a full fuzzy bridge and ab is a weak fuzzy bridge, but not a partial fuzzy bridge. Both ab and bc are bridges and fuzzy bridges.

Example 2.5.4 Let $V = \{a, b, c\}$. Define the fuzzy sets σ of V and μ of $E = \{ab, bc, ac\}$ as follows: $\sigma(a) = \sigma(b) = \sigma(c) = 1$, $\mu(ab) = 0.5$, $\mu(ac) = 0.6$, and $\mu(bc) = 0.2$. Then $d(\mu) = 0.2$ and $h(\mu) = 0.6$. For $0 < t \leq 0.2$, $G^t = (V, \{ab, bc, ac\})$, for $0.2 < t \leq 0.5$, $G^t = (V, \{ab, ac\})$, and for $0.5 < t \leq 0.6$, $G^t = (V, \{ac\})$. Then ac is a fuzzy bridge and a partial fuzzy bridge, but not a full fuzzy bridge and not a bridge. The edge bc is not any of the five types of bridges.

Example 2.5.5 Let $V = \{a, b, c\}$. Define the fuzzy subsets σ of V and μ of $E = \{ab, bc, ac\}$ as follows: $\sigma(a) = \sigma(b) = \sigma(c) = 1$ and $\mu(ab) = \mu(bc) = \mu(ac) = 0.5$. Then G has no bridges of any of the five types.

The fuzzy graphs of Examples 2.5.3–2.5.5 are given in Fig. 2.8.

Example 2.5.6 Let $V = \{a, b, c, d\}$. Define the fuzzy subsets σ of V and μ of $E = \{ab, bc, cd, ac\}$ as follows: $\sigma(a) = \sigma(b) = \sigma(c) = \sigma(d) = 1$ and $\mu(ab) = 0.3 = \mu(bc)$, $\mu(ac) = 0.9 = \mu(cd)$. Then $d(\mu) = 0.3$ and $h(\mu) = 0.8$. For $0 < t \leq 0.3$, $G^t = (V, \{ab, bc, cd, ac\})$ and for $0.3 < t \leq 0.8$, $G^t = (V, \{ac, cd\})$. Hence, cd is a full fuzzy bridge and ac is a partial fuzzy bridge, but not a full fuzzy bridge (Fig. 2.9).

Proposition 2.5.7 xy is a full fuzzy bridge if and only if xy is a bridge for G^* and $\mu(xy) = h(\mu)$.

Fig. 2.9 Fuzzy graph with a full fuzzy bridge

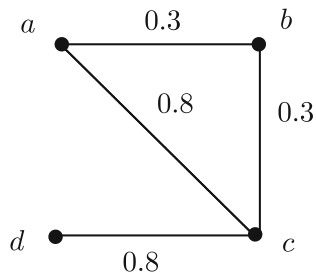
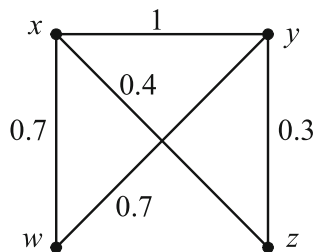


Fig. 2.10 Fuzzy graph in Example 2.5.9



Proof Suppose xy is a full fuzzy bridge. Then xy is a bridge for G^t for all $t \in (0, h(\mu)]$. Hence, $xy \in \mu^{h(\mu)}$ and so $\mu(xy) = h(\mu)$. Because xy is a bridge for G^t for all $t \in (0, h(\mu)]$, it follows that xy is a bridge for G^* because $\sigma^* = \sigma^{d(\mu)}$ and $\mu^* = \mu^{d(\mu)}$. Conversely, suppose that xy is a bridge for G^* and $\mu(xy) = h(\mu)$. Then $xy \in \mu^t$ for all $t \in (0, h(\mu)]$. Thus, because xy is also a bridge for G^* , xy is a bridge for G^t for all $t \in (0, h(\mu)]$, because each G^t is a subgraph of G^* . Hence, xy is a full fuzzy bridge. ■

Proposition 2.5.8 Suppose that xy is not contained in a cycle of G^* . Then the following conditions are equivalent.

- (i) $\mu(xy) = h(\mu)$.
- (ii) xy is a partial fuzzy bridge.
- (iii) xy is a full fuzzy bridge.

Proof Because xy is not contained in a cycle of G^* , xy is a bridge of G^* . Hence, by Proposition 2.5.7, (i) \Leftrightarrow (iii). Clearly, (iii) \Rightarrow (ii). Suppose that (ii) holds. Then xy is a bridge for G^t for all $t \in (d(\mu), h(\mu)]$ and so $xy \in \mu^{h(\mu)}$. Hence, $\mu(xy) = h(\mu)$, i.e., (i) holds. ■

Example 2.5.9 Consider the fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{x, y, z, w\}$ and $\mu^* = \{xz, xw, wy, yz\}$ (Fig. 2.10). Define the fuzzy subsets σ and μ as follows: $\sigma(x) = \sigma(y) = \sigma(z) = \sigma(w) = 1$, $\mu(xy) = 1$, $\mu(xz) = 0.4$, $\mu(xw) = 0.7$, $\mu(wy) = 0.7$, and $\mu(yz) = 0.3$. Then $d(\mu) = 0.3$ and $h(\mu) = 1$. For $0 < t \leq 0.3$, $G^t = (V, \{xy, xz, xw, wy, yz\})$. For $0.3 < t \leq 0.4$, $G^t = (V, \{xy, xz, xw, wy\})$, for $0.4 < t \leq 0.7$, $G^t = (V, \{xy, xw, yz\})$, and for $0.7 < t \leq 1$, $G^t = (V, \{xy\})$. Then xy is in a cycle of G^* , xy is not a partial fuzzy bridge, and $\mu(xy) = h(\mu)$. Also, xy is a weak fuzzy bridge and a fuzzy bridge, but not a bridge (Fig. 2.10).

Proposition 2.5.10 *If xy is a bridge, then xy is a weak fuzzy bridge and a fuzzy bridge.*

Proof xy is a bridge $\Leftrightarrow xy$ is a bridge for G^* $\Leftrightarrow xy$ is a bridge for $G^{d(\mu)}$, because $G^* = G^{d(\mu)} \Rightarrow xy$ is a weak fuzzy bridge. xy is a bridge implies that its removal disconnects G^* and so xy is a fuzzy bridge. ■

Theorem 2.5.11 *xy is a fuzzy bridge if and only if xy is a weak fuzzy bridge.*

Proof Suppose xy is a weak fuzzy bridge. Then $\exists t \in (0, h(\mu)]$ such that xy is a bridge for G^t . Hence, the removal of xy disconnects G^t . Thus, any path from x to y in G has an edge uv with $\mu(uv) < t$. Hence, the removal of xy results in $\mu'^\infty(x, y) < t \leq \mu^\infty(x, y)$. Thus, xy is a fuzzy bridge. Conversely, suppose xy is a fuzzy bridge. Then $\exists u, v$ such that removal of xy results $\mu'^\infty(u, v) < \mu^\infty(u, v)$. Hence, xy is on every strongest path connecting u and v and in fact, $\mu(xy)$ is greater than or equal to this value. Thus, there does not exist a path (other than xy) connecting x and y in $G^{\mu(xy)}$, else this other path without xy would be of strength $\geq \mu(xy)$ and would be part of a strongest path connecting u and v , contrary to the fact xy is on every such path. Hence, xy is a bridge of $G^{\mu(xy)}$ and $0 < \mu(xy) \leq h(\mu)$. Thus, $\mu(xy)$ is a desired t . ■

Definition 2.5.12 Let $x \in V$.

- (i) x is called a **cutvertex** if x is a cutvertex of G^* .
- (ii) x is called a **fuzzy cutvertex** if $\exists u, v \in V \setminus \{x\}$ such that $\mu'^\infty(u, v) < \mu^\infty(u, v)$, where μ' is μ restricted to $E \setminus \{xz, zx \mid z \in V\}$.
- (iii) x is called a **weak fuzzy cutvertex** if $\exists t \in (0, h(\mu)]$ such that x is a cutvertex for G^t .
- (iv) x is called a **partial fuzzy cutvertex** if x is a cutvertex for G^t for all $t \in (d(\mu), h(\mu)] \cup \{h(\mu)\}$.
- (v) x is called a **full fuzzy cutvertex** if x is a cutvertex for G^t for all $t \in (0, h(\mu)]$.

Example 2.5.13 Consider the fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{x, y, z\}$ and $\mu^* = \{xy, xz, yz\}$. Let the fuzzy subsets σ and μ be defined as $\sigma(x) = \sigma(y) = \sigma(z) = 1$, $\mu(xy) = 0.5$, $\mu(xz) = 0.4$, and $\mu(yz) = 0.3$. Then $d(\mu) = 0.3$ and $h(\mu) = 0.5$. For $0 < t \leq 0.3$, $G^t = (V, \{xy, xz, yz\})$, for $0.3 < t \leq 0.4$, $G^t = (V, \{xy, xz\})$, and for $0.4 < t \leq 0.5$, $G^t = (V, \{xy\})$. Thus, x is a fuzzy cutvertex and a weak fuzzy cutvertex, but neither a cutvertex nor a partial cutvertex.

Example 2.5.14 Consider the fuzzy graph $G = (V, \sigma, \mu)$ with $V = \{x, y, z\}$. Define the fuzzy subsets σ of V and μ of $E = \{xy, xz, yz\}$ as follows: $\sigma(x) = \sigma(y) = \sigma(z) = 1$ and $\mu(xy) = \mu(xz) = 0.7$ and $\mu(yz) = 0.1$. Then $d(\mu) = 0.1$ and $h(\mu) = 0.7$. For $0 < t \leq 0.1$, $G^t = (V, \{xy, xz, yz\})$ and for $0.1 < t \leq 0.7$, $G^t = (V, \{xy, xz\})$. Thus, x is a fuzzy cutvertex and a partial fuzzy cutvertex, but neither a cutvertex nor a full fuzzy cutvertex.

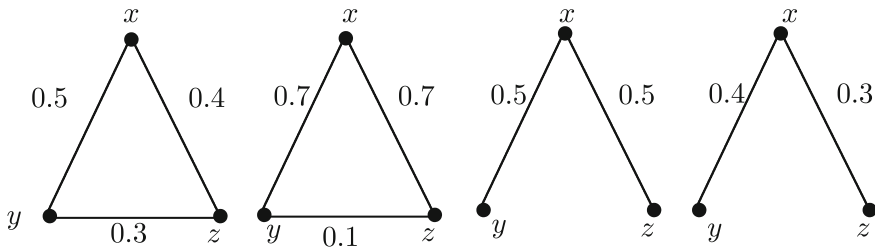


Fig. 2.11 Fuzzy Graphs in Examples 2.5.13–2.5.16

Example 2.5.15 Let $V = \{x, y, z\}$. Define the fuzzy subsets σ of V and μ of $E = \{xy, xz\}$ as follows: $\sigma(x) = \sigma(y) = \sigma(z) = 1$ and $\mu(xy) = \mu(xz) = 0.5$. Then $d(\mu) = h(\mu) = 0.5$. For $0 < t \leq 0.5$, $G^t = (V, \{xy, xz\})$. Thus, x is a full fuzzy cutvertex, a fuzzy cutvertex, and a cutvertex.

Example 2.5.16 Let $V = \{x, y, z\}$. Define the fuzzy subsets σ of V and μ of $E = \{xy, xz\}$ as follows: $\sigma(x) = \sigma(y) = \sigma(z) = 1$ and $\mu(xy) = 0.4$ and $\mu(xz) = 0.3$. Then $d(\mu) = 0.3$ and $h(\mu) = 0.4$. For $0 < t \leq 0.3$, $G^t = (V, \{xy, xz\})$ and for $0.3 < t \leq 0.4$, $G^t = (V, \{xy\})$. Thus, x is a cutvertex, a fuzzy cutvertex, and a weak fuzzy cutvertex, but not a partial fuzzy cutvertex.

The fuzzy graphs in Examples 2.5.13–2.5.16 are given in Fig. 2.11.

- Definition 2.5.17** (i) G is called a **block** if G^* is a block.
(ii) G is called a **fuzzy block** if it has no fuzzy cutvertices.
(iii) G is called a **weak fuzzy block** if $\exists t \in (0, h(\mu)]$ such that G^t is a block.
(iv) G is called a **partial fuzzy block** if G^t is a block for all $t \in (d(\mu), h(\mu)] \cup \{h(\mu)\}$.
(v) G is called a **full fuzzy block** if G^t is a block for all $t \in (0, h(\mu)]$.

Example 2.5.18 Consider the fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{x, y, z\}$ and $\mu^* = \{xy, yz, xz\}$. The fuzzy subsets σ and μ are defined as follows: $\sigma(x) = \sigma(y) = \sigma(z) = 1$ and $\mu(xy) = \mu(yz) = 0.6$, and $\mu(xz) = 0.7$. Then $d(\mu) = 0.6$ and $h(\mu) = 0.7$. For $0 < t \leq 0.6$, $G^t = (V, \{xy, yz, xz\})$ and for $0.6 < t \leq 0.7$, $G^t = (V, \{xz\})$. Thus, G is a block, a fuzzy block, and a weak fuzzy block. G is not a partial fuzzy block because G^t is not a block for $0.5 < t \leq 0.9$; it is not connected.

Example 2.5.19 Consider the fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{x, y, z\}$ and $\mu^* = \{xy, yz, xz\}$. The fuzzy subsets σ and μ are defined as follows: $\sigma(x) = \sigma(y) = \sigma(z) = 1$ and $\mu(xy) = \mu(xz) = 0.8$ and $\mu(yz) = 0.7$. Then $d(\mu) = 0.7$ and $h(\mu) = 0.8$. For $0 < t \leq 0.7$, $G^t = (V, \{xy, xz, yz\})$ and for $0.7 < t \leq 0.8$, $G^t = (V, \{xy, xz\})$. Thus, G is a block and a weak fuzzy block because G is a block for $0 < t \leq 0.7$. However, G is not a fuzzy block because x is a fuzzy cutvertex of G . Also, G is not a partial fuzzy block because x is a cutvertex for G^t for $0.7 < t \leq 0.8$.

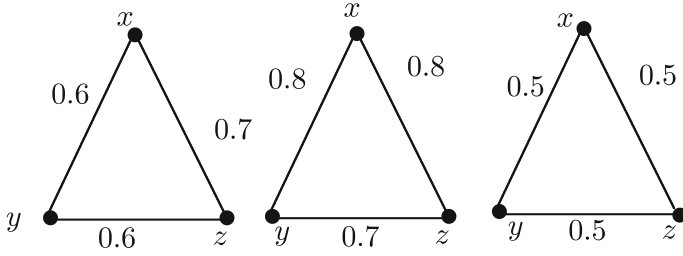
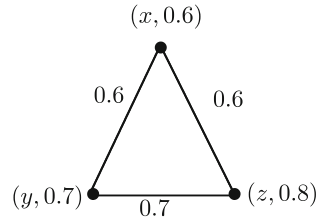


Fig. 2.12 Examples given in Examples 2.5.18–2.5.20

Fig. 2.13 A fuzzy block which is not firm



Example 2.5.20 Consider the fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{x, y, z\}$ and $\mu^* = \{xy, yz, xz\}$. The fuzzy subsets σ and μ are defined as follows: $\sigma(x) = \sigma(y) = \sigma(z) = 1$ and $\mu(xy) = \mu(xz) = \mu(yz) = 0.5$. Then $d(\mu) = h(\mu) = 0.5$. For $0 < t \leq 0.5$, $G^t = (V, \{xy, xz, yz\})$. Thus, G is a block, a fuzzy block and a full fuzzy block.

The fuzzy graphs of Examples 2.5.18–2.5.20 are given in Fig. 2.12.

Definition 2.5.21 G is said to be **firm** if $\bigwedge \{\sigma(x) \mid x \in V\} \geq \bigvee \{\mu(xy) \mid xy \in \mu^*\}$.

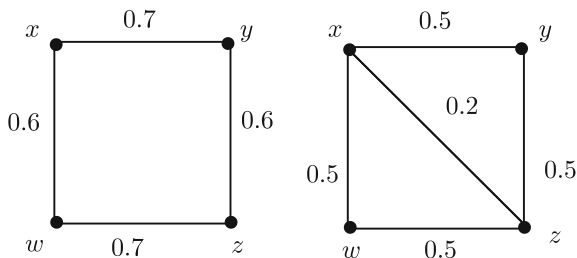
To this point all examples of fuzzy graphs except Fig. 2.2 in Example 2.2.6 have been firm.

Example 2.5.22 Consider the fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{x, y, z\}$ and $\mu^* = \{xy, yz, xz\}$. The fuzzy subsets σ and μ are defined as follows: $\sigma(x) = 0.6$, $\sigma(y) = 0.7$, $\sigma(z) = 0.8$, and $\mu(xy) = \mu(xz) = 0.6$, and $\mu(yz) = 0.7$. Then $d(\mu) = 0.6$ and $h(\mu) = 0.7$. For $0 < t \leq 0.6$, $G^t = (V, \{xy, xz, yz\})$ and for $0.6 < t \leq 0.7$, $G^t = (V, \{yz\})$. Thus, G is a block, a fuzzy block, and a full fuzzy block. We note that G is not firm (Fig. 2.13).

2.6 Cycles and Trees

In this section, we discuss the connectedness properties of cycles and trees in fuzzy graphs by levels. This is a continuation of results from [131].

Fig. 2.14 A fuzzy cycle and a partial fuzzy cycle



Definition 2.6.1 (i) G is called a **cycle** if G^* is a cycle.

(ii) G is called a **fuzzy cycle** if G^* is a cycle and there does not exist unique $xy \in \mu^*$ such that $\mu(xy) = \bigwedge \{\mu(uv) \mid uv \in \mu^*\}$.

(iii) G is called a **weak fuzzy cycle** if $\exists t \in (0, h(\mu)]$ such that G^t is a cycle.

(iv) G is called a **partial fuzzy cycle** if G^t is a cycle for all $t \in (d(\mu), h(\mu)] \cup \{h(\mu)\}$.

(v) G is called a **full fuzzy cycle** if G^t is a cycle for all $t \in (0, h(\mu)]$.

Example 2.6.2 Consider $V = \{x, y, z, w\}$. Let σ be the fuzzy subset of V and μ the fuzzy subset of $E = \{xy, wz, xw, yz\}$ defined as follows: $\sigma(x) = \sigma(y) = \sigma(z) = \sigma(w) = 1$, $\mu(xy) = \mu(wz) = 0.7$ and $\mu(xw) = \mu(yz) = 0.6$. Then $d(\mu) = 0.6$ and $h(\mu) = 0.7$. For $0 < t \leq 0.6$, $G^t = (V, \{xy, xw, yz, wz\})$ and for $0.6 < t \leq 0.7$, $G^t = (V, \{xy, wz\})$. Thus, G is a fuzzy cycle and a weak fuzzy cycle, but G is not a partial fuzzy cycle.

Example 2.6.3 Let $V = \{x, y, z, w\}$. Let σ be the fuzzy subset of V and μ the fuzzy subset of $E = \{xy, yz, zw, wx\}$ defined as follows: $\sigma(x) = \sigma(y) = \sigma(z) = \sigma(w) = 1$, $\mu(xy) = \mu(yz) = \mu(zw) = \mu(wx) = 0.5$, and $\mu(x, z) = 0.2$. Then G is not a cycle. Now, $d(\mu) = 0.2$ and $h(\mu) = 0.5$. For $0 < t \leq 0.2$, $G^t = (V, \{xy, yz, zw, wx, xz\})$ which is not a cycle and for $0.2 < t \leq 0.5$, $G^t = (V, \{xy, yz, zw, wx\})$ which is a cycle. Thus, G is a partial fuzzy cycle, but not a full fuzzy cycle.

Fuzzy graphs in Examples 2.6.2 and 2.6.3 are given in Fig. 2.14.

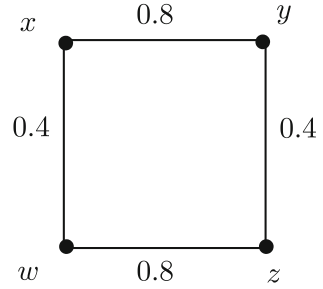
Proposition 2.6.4 Suppose G is a cycle. Then G is a partial fuzzy cycle if and only if G is a full fuzzy cycle.

Proof Suppose G is a partial fuzzy cycle. Let $t \in (0, d(\mu)]$. Then $G^t = G^*$ and G^* is given to be a cycle. Hence, G is a full fuzzy cycle. ■

Proposition 2.6.5 G is a full fuzzy cycle if and only if G is a cycle and μ is constant on μ^* .

Proof Suppose G is a full fuzzy cycle. Then $G^* = G^{d(\mu)}$ is a cycle. Suppose $\exists t_1$ and $t_2 \in \text{Im}(\mu)$ with $0 < t_1 < t_2$. Then $\exists xy \in \mu^*$ such that $\mu(xy) = t_1$. Hence, $xy \notin \mu^{t_2}$. Thus, G^{t_2} is not a cycle, a contradiction. Hence, μ is constant on μ^* . The converse is immediate. ■

Fig. 2.15 A partial fuzzy forest



Corollary 2.6.6 *If G is a full fuzzy cycle, then G is a fuzzy cycle.*

Proposition 2.6.7 *G is a partial fuzzy cycle if and only if $G^{h(\mu)}$ is a cycle and $|\text{Im}(\mu) \setminus \{0\}| \leq 2$.*

Proof Suppose G is a partial fuzzy cycle. Then clearly $G^{h(\mu)}$ is a cycle and in fact G^t is a cycle for all $t \in (d(\mu), h(\mu)] \cup \{h(\mu)\}$. Suppose $|\text{Im}(\mu) \setminus \{0\}| > 2$. Then $\exists t$ such that $0 < d(\mu) < t < h(\mu)$. Hence, $\exists xy \in \mu^*$ such that $\mu(xy) = t$. Thus, $xy \notin \mu^{h(\mu)}$ and so $G^{h(\mu)}$ is not a cycle, a contradiction. Conversely, suppose $G^{h(\mu)}$ is a cycle and $|\text{Im}(\mu) \setminus \{0\}| \leq 2$. If $|\text{Im}(\mu) \setminus \{0\}| = 1$, then G is a full fuzzy cycle by Proposition 2.6.5. Suppose $|\text{Im}(\mu) \setminus \{0\}| = 2$. Then $\text{Im}(\mu) \setminus \{0\} = \{d(\mu), h(\mu)\}$. Because $G^t = G^{h(\mu)}$ for $d(\mu) < t \leq h(\mu)$, it follows that G is a partial fuzzy tree. ■

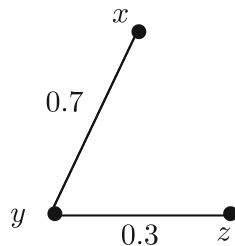
Definition 2.6.8 (i) G is called a **forest (tree)** if G^* is a forest (tree).

- (ii) G is called a **fuzzy forest (tree)** if G has a fuzzy spanning subgraph (σ, ν) which is a forest (tree) such that for all $uv \in \mu^* \setminus \nu^*$, $\mu(uv) < \nu^\infty(uv)$.
- (iii) G is called a **weak fuzzy forest (tree)** if $\exists t \in (0, h(\mu)]$ such that G^t is a forest (tree).
- (iv) G is called a **partial fuzzy forest (tree)** if G^t is a forest (tree) for all $t \in (d(\mu), h(\mu)] \cup \{h(\mu)\}$.
- (v) G is called a **full fuzzy forest (tree)** if G^t is a forest (tree) for all $t \in (0, h(\mu)]$.

The definition of a weak fuzzy forest in Definition 2.6.8(iii) is equivalent to the definition of a fuzzy graph being acyclic by t -cuts in Chap. 5. We will show that the definition of a full fuzzy forest here and the one in Chap. 5 are equivalent, but that this is not the case for the notion of a full fuzzy tree.

Example 2.6.9 Consider the fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{x, y, z, w\}$ and $\mu^* = \{xw, yz, xy, wz\}$. The fuzzy subsets σ and μ are defined as follows: $\sigma(x) = \sigma(y) = \sigma(z) = \sigma(w) = 1$ and $\mu(xw) = \mu(yz) = 0.4$, $\mu(xy) = \mu(wz) = 0.8$. Then $d(\mu) = 0.4$ and $h(\mu) = 0.8$. For $0 < t \leq 0.4$, $G^t = (V, \{xw, yz, xy, wz\})$ and for $0.4 < t \leq 0.8$, $G^t = (V, \{xy, wz\})$. Hence, G is a partial fuzzy forest, but is neither a fuzzy forest nor a full fuzzy forest (Fig. 2.15).

Proposition 2.6.10 *G is a full fuzzy forest if and only if G is a forest.*

Fig. 2.16 A full fuzzy forest

Proof Suppose G is a full fuzzy forest. Then $G^* = G^{d(\mu)}$ is a forest. Conversely, suppose G is a forest. Then G^* is a forest and hence so must be G^t for all $t \in (0, h(\mu)]$ because each such G^t is a subgraph of G^* . ■

Example 2.6.11 Consider the fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{x, y, z\}$ and $\mu^* = \{xy, yz\}$. The fuzzy subsets σ and μ are defined as follows: $\sigma(x) = \sigma(y) = \sigma(z) = 1$, $\mu(xy) = 0.7$ and $\mu(yz) = 0.3$. Then $d(\mu) = 0.3$ and $h(\mu) = 0.7$. For $0 < t \leq 0.3$, $G^t = (V, \{xy, yz\})$ and for $0.3 < t \leq 0.7$, $G^t = (V, \{xy\})$. Hence, G is a forest (and a full fuzzy forest) without being a constant on μ^* (Fig. 2.16). Note that $G^{h(\mu)}$ has more connected components than G^* .

Proposition 2.6.12 G is a weak fuzzy forest if and only if G does not contain a cycle whose edges are of strength $h(\mu)$.

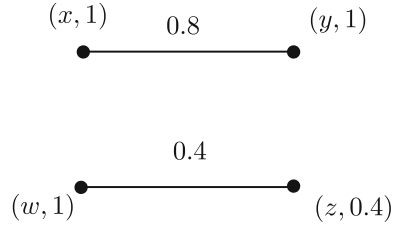
Proof Suppose G contains a cycle whose edges are of strength $h(\mu)$. Then G^t , $t \in (0, h(\mu)]$, contains this cycle and so is not a forest. Thus, G is not a weak fuzzy forest. Conversely, suppose G does not contain a cycle all of whose edges are of strength $h(\mu)$. Then $G^{h(\mu)}$ does not contain a cycle and so is a forest. ■

Corollary 2.6.13 If G is a fuzzy forest, then G is a weak fuzzy forest.

Proof G cannot have a cycle all of whose edges are of strength $h(\mu)$, else it could not have a fuzzy spanning forest with the property that for all $uv \in \mu^* \setminus \nu^*$, $\mu(uv) < \nu^\infty(u, v)$. ■

Theorem 2.6.14 G is a forest and μ is a constant on μ^* if and only if G is a full fuzzy forest, G^* and $G^{h(\mu)}$ have the same number of connected components, and G is firm.

Proof Suppose that G is a forest and μ is constant on μ^* . Then for all $t \in (0, h(\mu)]$, $G^t = G^*$ and so G is a full fuzzy forest and G^* and $G^{h(\mu)}$ have the same number of connected components. Clearly, G is firm because μ is a constant on μ^* . Conversely, suppose G is a full fuzzy forest, G^* and $G^{h(\mu)}$ have the same number of connected components, and G is firm. Suppose $\exists t_1, t_2 \in \text{Im}(\mu)$ such that $0 < t_1 < t_2$. Then $\exists xy \in \mu^*$ such that $\mu(xy) = t_1$. Now, $xy \in \mu^{t_1}$, $xy \notin \mu^{t_2}$. Hence, G^{t_2} has more connected components than G^{t_1} because G is firm, i.e., no vertices were lost. Thus, $G^{h(\mu)}$ has more connected components than G^* , a contradiction. ■

Fig. 2.17 A full fuzzy forest

Example 2.6.15 Consider $V = \{x, y, z, w\}$. Define the fuzzy subsets σ of V and μ of $E = \{xy, zw\}$ as follows: $\sigma(x) = \sigma(y) = \sigma(z) = 1$, $\sigma(w) = 0.4$ and $\mu(xy) = 0.8$, $\mu(zw) = 0.4$. Then $d(\mu) = 0.4$ and $h(\mu) = 0.8$. For $0 < t \leq 0.4$, $G^t = (V, \{xy, zw\})$ and for $0.4 < t \leq 0.8$, $G^t = (\{x, y, w\}, \{xy\})$. Thus, G^* and $G^{h(\mu)}$ are forests with the same number of connected components. G is a full fuzzy forest (Fig. 2.17), μ is not constant on μ^* , and G is not firm.

Corollary 2.6.16 G is a tree and μ is constant on μ^* if and only if G is a full fuzzy tree and G is firm.

Proof Suppose G is a full fuzzy tree and G is firm. Because G^t is a tree for all $t \in (0, h(\mu)]$, G^* is a tree and so $G^{h(\mu)}$ and G^* have the same number of connected components. The desired result now follows from Theorem 2.6.14. ■

Example 2.6.17 Consider the fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{x, y, z\}$ and $\mu^* = \{xy, yz\}$. The fuzzy subsets σ and μ are defined as follows: $\sigma(x) = \sigma(y) = 1$, $\sigma(z) = 0.6$ and $\mu(xy) = 0.8$, $\mu(yz) = 0.6$. Then $d(\mu) = 0.6$ and $h(\mu) = 0.8$. For $0 < t \leq 0.6$, $G^t = (V, \{(x, y), (y, z)\})$ and for $0.6 < t \leq 0.8$, $G^t = (\{x, y\}, \{xy\})$. Thus, G is a tree, G is a full fuzzy tree, and G^* and $G^{h(\mu)}$ have the same number of connected components. However, G is not firm and μ is not constant on μ^* .

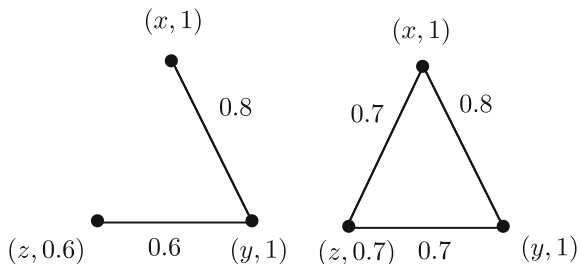
Example 2.6.18 Let $V = \{x, y, z\}$. Define the fuzzy subsets σ of V and μ of $E = \{xy, xz, yz\}$ as follows: $\sigma(x) = \sigma(y) = 1$, $\sigma(z) = 0.7$ and $\mu(xy) = 0.8$, $\mu(xz) = \mu(yz) = 0.7$. Then $d(\mu) = 0.7$ and $h(\mu) = 0.8$. For $0 < t \leq 0.7$, $G^t = (V, \{xy, xz, yz\})$ and for $0.7 < t \leq 0.8$, $G^t = (\{x, y\}, \{xy\})$. Thus, G is a partial fuzzy tree, but not a full fuzzy tree. G is not a fuzzy tree. Hence, it is not the case that if G is a weak fuzzy tree, then G is a fuzzy tree. G is not firm.

The fuzzy graphs in Examples 2.6.17 and 2.6.18 are given in Fig. 2.18.

Definition 2.6.19 For all $t \in (0, 1]$ define $\sigma^{(t)} : \sigma^t \rightarrow [0, 1]$ and $\mu^{(t)} : \mu^t \rightarrow [0, 1]$ by $\sigma^{(t)}(x) = \sigma(x)$ for all $x \in \sigma^t$; $\sigma^{(t)}(x) = 0$ otherwise, and $\mu^{(t)}(xy) = \mu(xy)$ for all $xy \in \mu^t$ and $\mu^{(t)}(xy) = 0$ otherwise. Let $G^{(t)} = (\sigma^{(t)}, \mu^{(t)})$ for all $t \in (0, 1]$.

Proposition 2.6.20 Suppose that G is firm. If G is a weak fuzzy tree, then G is a fuzzy tree.

Fig. 2.18 Fuzzy graphs in Examples 2.6.17 and 2.6.18



Proof There exist $t \in (0, h(\mu)]$ such that G^t is a tree. Because G is firm, $G^{(t)}$ is a fuzzy spanning subgraph of G which is a tree. If uv is in $\mu^* \setminus \mu^t$, then $\mu(uv) < t$ and so it follows that G is a fuzzy tree. ■

Example 2.6.21 Consider the fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{x, y, z, w\}$ and $\mu^* = \{xy, yz, xz, zw, xw\}$. The fuzzy subsets σ and μ are defined as follows: $\sigma(x) = \sigma(y) = \sigma(z) = \sigma(w) = 1$ and $\mu(xy) = \mu(yz) = 0.8$, $\mu(xz) = \mu(zw) = 0.5$, $\mu(xw) = 0.2$. Then $d(\mu) = 0.2$ and $h(\mu) = 0.8$. For $0 < t \leq 0.2$, $G^t = (V, \{xy, yz, zw, wx, xz\})$, for $0.2 < t \leq 0.5$, $G^t = (V, \{xy, yz, zw, xz\})$, and for $0.5 < t \leq 0.8$, $G^t = (V, \{xy, yz\})$. We see that G is not a weak fuzzy tree. However, it is a fuzzy tree because (σ, ν) is a fuzzy spanning subgraph of G , which is a tree, where $\nu(xy) = \nu(yz) = 0.8$ and $\nu(zw) = 0.5$.

Example 2.6.22 Consider $V = \{x, y, z, w\}$. Let σ be the fuzzy subset of V and μ the fuzzy subset of $E = \{xy, yz, zw\}$ defined as follows: $\sigma(x) = \sigma(y) = \sigma(z) = \sigma(w) = 1$ and $\mu(xy) = \mu(yz) = 0.8$, $\mu(zw) = 0.6$. Then G is a tree, a fuzzy tree, a weak fuzzy tree, but not a partial fuzzy tree (if we were to define, $\mu(w) = 0.6$, then G would be a full fuzzy tree, but not firm).

Example 2.6.23 Let $V = \{x, y, z\}$. Let σ be the fuzzy subset of V and μ the fuzzy subset of $E = \{xy, yz, xz\}$ defined as follows: $\sigma(x) = \sigma(y) = \sigma(z) = 1$ and $\mu(xy) = \mu(xz) = 0.8$, $\mu(yz) = 0.2$. Then G is a fuzzy tree, but not a tree. G is a partial fuzzy tree, but not a full fuzzy tree.

The fuzzy graphs in Examples 2.6.21–2.6.23 are given in Fig. 2.19.

- Definition 2.6.24** (i) G is called **connected** if G^* is connected.
(ii) G is called **fuzzy connected** if G is a fuzzy block.
(iii) G is called **weakly connected** if $\exists t \in (0, h(\mu)]$ such that G^t is connected.
(iv) G is called **partially connected** if G^t is connected for all $t \in (d(\mu), h(\mu)] \cup \{h(\mu)\}$.
(v) G is called **fully connected** if G^t is connected for all $t \in (0, h(\mu)]$.

Example 2.6.25 Consider the fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{x, y, z, w\}$ and $\mu^* = \{xy, zw\}$. The fuzzy subsets σ and μ are defined as follows: $\sigma(x) = \sigma(y) = 1$, $\sigma(z) = \sigma(w) = 0.6$ and $\mu(xy) = 0.8$ and $\mu(zw) = 0.6$. Then G is not connected. G is partially connected, but not fully connected. We see that G is not firm (Fig. 2.20).

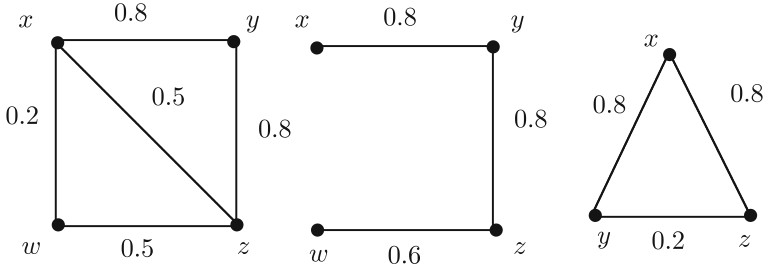


Fig. 2.19 Fuzzy graphs in Examples 2.6.21–2.6.23

Fig. 2.20 Fuzzy graph in Example 2.6.25

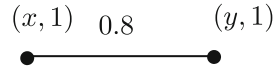
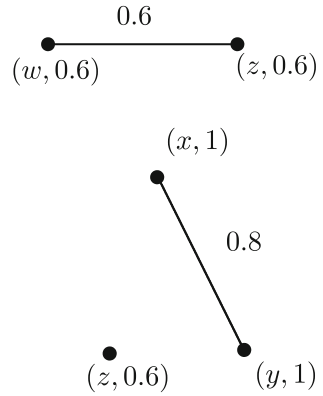


Fig. 2.21 Fuzzy graph in Example 2.6.26



Example 2.6.26 Consider $V = \{x, y, z\}$. Let σ be the fuzzy subset of V and μ be the fuzzy subset of $E = \{xy\}$ defined as follows: $\sigma(x) = \sigma(y) = 1, \sigma(z) = 0.6$ and $\mu(xy) = 0.8$. Then G is not connected. G is partially connected, but not fully connected. Note that G is not firm (Fig. 2.21).

Proposition 2.6.27 *If G is connected, then G is weakly connected. Conversely, if G is firm and weakly connected, then G is connected.*

Proof G connected implies G^* is connected. Now, $G^* = G^{d(\mu)}$ and so G is weakly connected. Conversely, if G^t is connected for some $t \in (0, h(\mu)]$, then G^* is connected because G is firm. ■

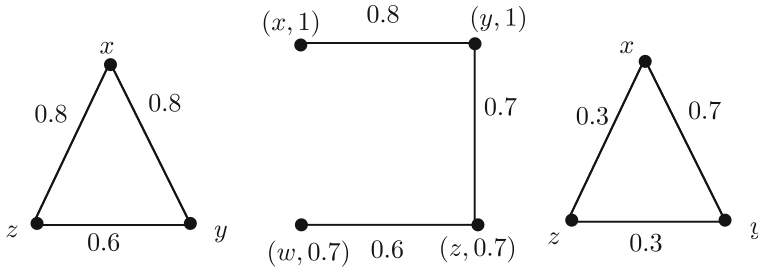


Fig. 2.22 Fuzzy graphs in Examples 2.6.28–2.6.30

Example 2.6.28 Consider the fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{x, y, z\}$ and $\mu^* = \{xy, yz, xz\}$. The fuzzy subsets σ and μ are defined as follows: $\sigma(x) = \sigma(y) = \sigma(z) = 1$ and $\mu(xy) = \mu(xz) = 0.8$, $\mu(yz) = 0.6$. Then G is fully connected, but μ is not constant on μ^* .

Example 2.6.29 Consider the fuzzy graph $G = (\sigma, \mu)$ with $\sigma^* = \{x, y, z, w\}$ and $\mu^* = \{xy, yz, zw\}$. The fuzzy subsets σ and μ are defined as follows: $\sigma(x) = \sigma(y) = 1$, $\sigma(z) = \sigma(w) = 0.7$ and $\mu(xy) = 0.8$, $\mu(yz) = 0.7$, $\mu(zw) = 0.6$. Then $d(\mu) = 0.6$ and $h(\mu) = 0.8$. For $0 < t \leq 0.6$, $G^t = (V, \{xy, yz, zw\})$, for $0.6 < t \leq 0.7$, $G^t = (V, \{xy, yz\})$, and for $0.7 < t \leq 0.8$, $G^t = (\{x, y\}, \{xy\})$. Thus, G is weakly connected, but not partially connected. G is connected but G is not firm.

Example 2.6.30 Consider $V = \{x, y, z\}$. Let σ be the fuzzy subset of V and μ be the fuzzy subset of $E = \{xy, yz, xz\}$ be defined as: $\sigma(x) = \sigma(y) = \sigma(z) = 1$ and $\mu(xy) = 0.7$, $\mu(yz) = \mu(xz) = 0.3$. Then G is weakly fuzzy connected because G^t is connected for $0 < t \leq 0.3$. G is a weak fuzzy forest because G^t is a forest for $0.3 < t \leq 0.7$. However G is not a weak fuzzy tree because G^t is not a tree for any t such that $0 < t \leq 0.7$.

The fuzzy graphs in Examples 2.6.28–2.6.30 are given in Fig. 2.22.

- Proposition 2.6.31** (i) If G is a weak fuzzy tree, then G is weakly connected and G is a weak fuzzy forest. Conversely, if $\exists t_1, t_2 \in (0, h(\mu)]$ with $t_1 < t_2$ such that G^{t_1} is a forest and G^{t_2} is connected, then G is a weak fuzzy tree.
- (ii) G is a tree if and only if G is a forest and G is connected.
- (iii) G is a partial fuzzy tree if and only if G is a partial fuzzy forest and G is partially connected.
- (iv) G is a full fuzzy tree if and only if G is a full fuzzy forest and G is fully connected.

Proof (i) If G^t is a tree for some $t \in (0, h(\mu)]$, then G^t is connected and is a forest. For the converse, we note that G^{t_2} must also be a forest. Because also G^{t_2} is connected, G^{t_2} is a tree.

(ii), (iii), (iv): Immediate. ■

Proposition 2.6.32 G is firm if and only if $G^{(t)}$ is firm for all $t \in (0, h(\mu)]$.

Proof Suppose G is firm. Let $t \in (0, h(\mu)]$. Let $xy \in \mu^t$. Then $t \leq \mu(xy) \leq \wedge \{\sigma(x) \mid x \in \sigma^*\} \leq \wedge \{\sigma(x) \mid x \in \sigma^t\}$. Hence, $\vee \{\mu(xy) \mid x, y \in \mu^t\} \leq \wedge \{\sigma(x) \mid x \in \sigma^t\}$. Thus, if we note that $\mu^{(t)*} = \mu^t$ and $\sigma^{(t)*} = \sigma^t$, we see that $G^{(t)}$ is firm. Conversely, suppose $G^{(t)}$ is firm for all $t \in (0, h(\mu)]$. Let $\wedge \{\sigma(x) \mid x \in \sigma^*\} = t_0$. Then $t_0 > 0$. Now, $\vee \{\mu(xy) \mid xy \in \mu^{t_0}\} \leq t_0$ because $G^{(t_0)}$ is firm and $\sigma^* = \sigma^{t_0} = \sigma^{(t_0)*}$. Let $xy \in \mu^* \setminus \mu^{t_0}$. Then $\mu(xy) < t_0$. Thus, $\vee \{\mu(xy) \mid xy \in \mu^*\} \leq t_0 = \wedge \{\sigma(x) \mid x \in \sigma^*\}$. Hence, G is firm. ■

2.7 Blocks in Fuzzy Graphs

The definition of a nonseparable fuzzy graph first appeared in Rosenfeld's classic paper in 1975 [154]. But a formal study of blocks in fuzzy graphs was made by Sunitha and Vijayakumar in 2005 [168]. Later Mathew and Sunitha [110] studied blocks further in 2010 and characterized a class of blocks in fuzzy graphs. In graph theory, a graph without cutvertices is called a **block** (or **nonseparable**). This concept is generalized to fuzzy graph theory. A fuzzy graph is said to be a block if it has no fuzzy cutvertices. It is clear that a block in fuzzy graphs has no cutvertices. Thus, a fuzzy block is trivially a block in the classical sense, but the converse is not true. In contrast to the conventional concept of a block in graphs, the study of blocks in fuzzy graphs is challenging due to the complexity of its cutvertices. Note that the cutvertices of a fuzzy graph are those vertices which reduce the strength of connectedness between some pair of vertices rather than the total disconnection of the fuzzy graph on its removal from the fuzzy graph.

Rosenfeld [154] observed that a block may have a fuzzy bridge. Sunitha and Vijayakumar [168] identified that a fuzzy graph can have more than one fuzzy bridge as seen from the example below.

Example 2.7.1 Let $V = \{u, v, w, x, y\}$. Let σ be the fuzzy subset of V and μ be the fuzzy subset of $V \times V$ defined as follows. $\sigma(u) = \sigma(v) = \sigma(w) = \sigma(x) = \sigma(y) = 1$ and $\mu(uv) = \mu(xy) = 0.9$, $\mu(vy) = \mu(uy) = \mu(ux) = 0.5$ and $\mu(wx) = \mu(wy) = 0.3$. It can be verified easily that $G = (\sigma, \mu)$ is a block. But note that both uv and xy are fuzzy bridges (Fig. 2.23).

Fig. 2.23 A block with two fuzzy bridges

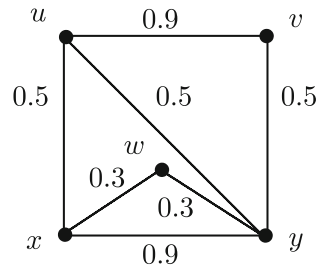
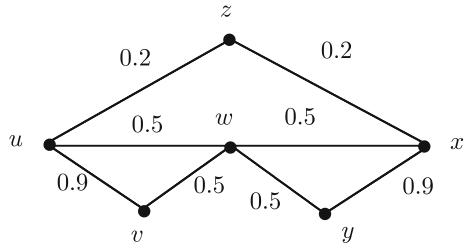


Fig. 2.24 A non fuzzy block having two fuzzy bridges



It is obvious from the definition that no two fuzzy bridges in a block can have a common vertex. A complete fuzzy graph is clearly a block. As pointed out in Theorem 2.3.14, the removal of a fuzzy bridge from a fuzzy tree reduces the strength of connectedness between some pair of vertices other than its end vertices. But, the situation is different in blocks as seen from the following theorem.

Theorem 2.7.2 *If $G = (\sigma, \mu)$ is a block with at least one fuzzy bridge, then removal of any fuzzy bridge reduces the strength of connectedness between its end vertices alone.*

Proof Let $G = (\sigma, \mu)$ be a block and uv be a fuzzy bridge of G . Assume on the contrary that removal of uv reduces the strength of connectedness between some other pair of vertices u_l and v_l .

Case 1: Both u_l and v_l are distinct from u and v .

Without loss of generality let $u_l \neq u$ and $v_l \neq v$. By assumption, every strongest $u_l - v_l$ path contains the edge uv . Thus, clearly removal of either u or v reduces the strength of connectedness between u_l and v_l , which shows that u and v are fuzzy cutvertices of G , contradicting that G is a block.

Case 2: One of u or v is u_l or v_l .

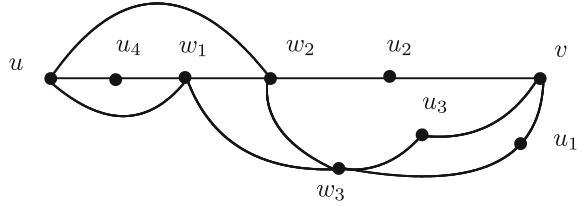
Let $v_l = v$ and $u_l \neq u$. Then as before removal of v reduces the strength of connectedness between u_l and v_l showing that v is a fuzzy cutvertex of G and similarly if $u_l = u$ and $v_l \neq v$, then u becomes a fuzzy cutvertex, both contradict the hypothesis that G is a block. Thus, the only possibility is that $u_l = u$ and $v_l = v$ and hence the theorem. ■

The condition in Theorem 2.7.2 is not sufficient as seen from Example 2.7.3.

Example 2.7.3 Let $V = \{u, v, w, x, y, z\}$. Let σ be a fuzzy subset of V and μ be a fuzzy subset of \mathcal{E} defined as $\sigma(s) = 1$ for all $s \in V$ and $\mu(uv) = \mu(xy) = 0.9$, $\mu(vw) = \mu(yw) = \mu(uw) = \mu(xw) = 0.5$, $\mu(xz) = \mu(uz) = 0.2$. In G , uv and xy are fuzzy bridges, but their removal does not reduce the strength of connectedness between any pair of vertices other than their endvertices. Clearly, G is not a block as w is a fuzzy cutvertex of G (See Fig. 2.24).

In [154], it is proposed that if every pair of vertices in a fuzzy graph G are joined by strongest paths, then G is a block and the converse is not true. Also, if an edge uv of a fuzzy graph is a bridge, then it is the unique strongest $u - v$ path. Sunitha and

Fig. 2.25 Case I of Theorem 2.7.4



Vijayakumar [168] proved that the converse of Rosenfeld's observation is true only for blocks having no fuzzy bridges. Hence, we have the following result.

Theorem 2.7.4 ([168]) *The following statements are equivalent for a fuzzy graph $G = (\sigma, \mu)$.*

- (i) G is a block.
- (ii) Any two vertices u and v such that uv is not a fuzzy bridge are joined by two internally disjoint strongest paths.
- (iii) For every three distinct vertices of G , there is a strongest path joining any two of them not containing the third.

Proof (i) \Rightarrow (ii) Let $G = (\sigma, \mu)$ be a block. Let u and v be any two vertices such that $\mu(uv) \geq 0$ and uv is not a fuzzy bridge. If there exists a unique strongest $u - v$ path of length greater than or equal to 2, then the vertices on this path other than u and v are fuzzy cutvertices of G . Hence, there exist more than one strongest $u - v$ paths. If these strongest $u - v$ paths are internally disjoint, then we are done. Note that all strongest $u - v$ paths do not have a common vertex, if so, that vertex becomes a fuzzy cutvertex. So consider the following cases.

Case 1:

Let $P_1 : u - w_2 - w_3 - u_1 - v$, $P_2 : u - u_4 - w_1 - w_2 - u_2 - v$ and $P_3 : u - w_1 - w_3 - u_3 - v$ be strongest $u - v$ paths. Let w_2 be the last common vertex of P_1 and P_2 (Fig. 2.25). Then $u - w_2$ subpath in P_1 together with $w_2 - u_2 - v$ subpath in P_2 is a path (say) P disjoint from P_3 .

Claim: P is a strongest $u - v$ path.

Let e_1, e_2 and e_3 be weakest edges in P_1, P_2 and P_3 , respectively, and let $\mu(e_1) = \mu(e_2) = \mu(e_3) = \mu^\infty(u, v)$. Then e_1 should be in $u - w_2$ subpath of P_1 or e_2 should be in $w_2 - u_2 - v$ subpath of P_2 ; for if not, then strength of $P > \mu^\infty(u, v)$, contradiction. Hence, P is a strongest $u - v$ path.

Case 2:

Let $P_1 : u - u_1 - w_1 - w_2 - v$, $P_2 : u - w_1 - w_3 - u_2 - v$ and $P_3 : u - w_2 - w_3 - v$ be strongest $u - v$ paths. Let w_2 be the first common vertex of P_1 and P_3 . Then $u - w_2$ subpath in P_3 together with $w_2 - v$ subpath in P_1 is a path disjoint from P_2 (Fig. 2.26). As in Case 1, it can be proved that P is a strongest $u - v$ path.

Fig. 2.26 Case II of
Theorem 2.7.4

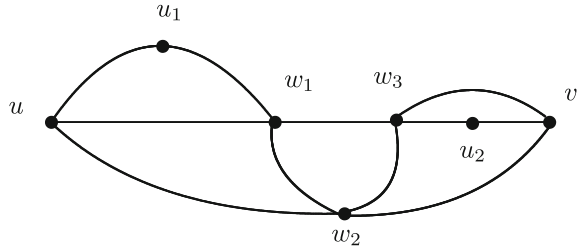
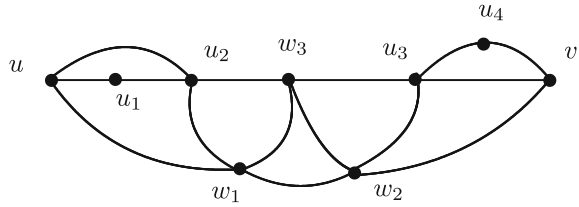


Fig. 2.27 Case III of
Theorem 2.7.4



Case 3:

Let $P_1 : u - u_2 - w_1 - w_2 - u_3 - u_4 - v$, $P_2 : u - u_1 - u_2 - w_3 - u_3 - v$ and $P_3 : u - w_1 - w_3 - w_2 - v$ be strongest $u - v$ paths. Let w_1 and w_2 be the first and last common vertices of P_1 and P_3 , respectively (Fig. 2.27). Then $u - w_1$ subpath in P_3 and $w_1 - w_2$ subpath in P_1 together with $w_2 - v$ subpath in P_3 will give a strongest $u - v$ path disjoint from P_2 .

(ii) \Rightarrow (iii)

Let $u \neq v \neq w$ be any three vertices of G . Choose any two (say) u and v . If edge uv is a fuzzy bridge, then it is the strongest $u - v$ path and (iii) holds. So assume uv is not a fuzzy bridge. Now, by (ii), there exist two internally disjoint strongest $u - v$ paths and hence w cannot be in both.

(iii) \Rightarrow (i) If possible let w be a fuzzy cutvertex of G . Then by definition there exist u, v different from w such that w is on every strongest $u - v$ path. But this contradicts (iii). ■

A fuzzy analogue of the characterization of blocks in graphs given in [83] with all six conditions is not possible. But in any fuzzy graph there exists a strongest path between every pair of vertices. We will discuss a characterization using strongest strong paths in this section.

An edge xy is said to be a strong if its membership value is at least as great as the connectedness of its end vertices when the edge is deleted. That is, if $\mu(xy) \geq \mu'^\infty(x, y)$ or $\mu(xy) \geq CONN_{G-xy}(x, y)$. A detailed discussion of strong edges will be made in Chap. 3. From the following example, it can be seen that a block can contain edges which are even not strong.

Example 2.7.5 Let $G = (\sigma, \mu)$ with $\sigma^* = \{u, v, w, x\}$, $\sigma(s) = 1$ for all $s \in \sigma^*$ and $\mu(uv) = \mu(xu) = \mu(vw) = \mu(wx) = 0.9$, $\mu(vx) = 0.2$. Then G is a block. Here vx is the unique weakest edge of the cycle $uvxvu$ which is not strong (Fig. 2.28).

Fig. 2.28 A block with a non strong edge

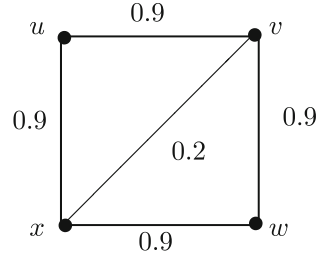
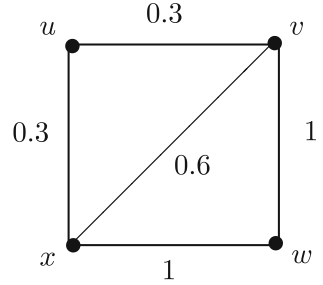


Fig. 2.29 Fuzzy graph in Example 2.7.6



A path in a fuzzy graph G is strong if all its edges are strong. Recall that an $x - y$ path P in a fuzzy graph G is said to be a strongest $x - y$ path if $d(P) = CONN_G(x, y)$. In a block, a strongest path need not be strong and a strong path need not be strongest. But there exists a strongest strong path between any two vertices of G .

Example 2.7.6 Let $G = (\sigma, \mu)$ with $\sigma^* = \{u, v, w, x\}$, $\sigma(s) = 1$ for all $s \in \sigma^*$ and $\mu(uv) = 0.3 = \mu(xu)$, $\mu(vw) = \mu(wx) = 1$, $\mu(vx) = 0.6$. Edge xv is not strong because $0.6 = \mu(xv) < CONN_{G-xv}(x, v) = 1$. Thus, $P : uxv$ is not a strong path even though it is a strongest $u - v$ path. $d(P) = 0.3 = CONN_G(u, v)$. Also, the $x - v$ path $Q : xuv$ is not a strongest $x - v$ path even if it is a strong $x - v$ path (Fig. 2.29).

Definition 2.7.7 A cycle in a fuzzy graph G is called a **strong cycle** if all its edges are strong.

Example 2.7.8 Consider the fuzzy graph in Example 2.7.6. The cycle $uvwxu$ is a strong cycle whereas $vwxv$ is not.

Next we have a characterization of blocks having no fuzzy bridges.

Theorem 2.7.9 ([110]) Let $G = (\sigma, \mu)$ be a fuzzy graph with at least three vertices and having no fuzzy bridges. Then the following statements are equivalent.

- (i) G is a block.
- (ii) For any two vertices x, y of G , there exists a cycle containing the vertices x and y which is formed by two strongest strong $x - y$ paths.

- (iii) For each vertex u and each strong edge vw of G , there exists a cycle containing the vertex u and the edge vw which is formed by two strongest strong $u - v$ paths or $u - w$ paths.
- (iv) For each pair of strong edges xy and uv of G , there exists a cycle containing the edges xy and uv which is formed by two strongest strong $x - u$ or $y - u$ paths.
- (v) For every three distinct vertices of G there exists a strongest strong path joining any two of them not containing the third.

Proof (i) \Rightarrow (ii) Suppose G is a block. Consider a maximum spanning tree T of G . Clearly, every edge in a maximum spanning tree is strong. Also, every $x - y$ path in T is a strongest $x - y$ path in G . Thus, between any two vertices of G there exists a strongest strong path. Let P be a strongest strong $x - y$ path in G . Assume that P is a unique $x - y$ path in G . Then P should belong to all maximum spanning trees. Also, note that the length of P is at least two because G has no fuzzy bridges. Thus, all internal vertices of P are internal vertices of every maximum spanning tree and by Theorem 2.2.2, they are all fuzzy cutvertices contradicting the fact that G is a block. Thus, it follows that the strongest strong $x - y$ path P does not belong to all maximum spanning trees. Hence, there exists a maximum spanning tree say T_1 not containing P . Let P_1 be a strongest strong $x - y$ path in T_1 . This strongest strong path P_1 together with P gives a cycle in G containing the vertices x and y as required. Note that P and P_1 should be internally disjoint because otherwise the common vertices of P and P_1 become fuzzy cutvertices of G .

(ii) \Rightarrow (iii)

Let u be a vertex and vw , a strong edge of G . Let C_1 be the cycle containing u and v satisfying the conditions in (ii) and C_2 be the cycle containing u and w satisfying the conditions in (ii). If w is a neighbor of v in C_1 or v is a neighbor of w in C_2 then we are done.

So suppose that vw is neither in C_1 nor in C_2 . Let P_1 and P_2 be the strongest strong $u - v$ paths in C_1 and Q be a strongest strong $u - w$ path in C_2 . Let z be the vertex in Q before w and nearest to it at which Q meets P_1 or P_2 (note that z can be the vertex u itself). Without loss of generality suppose that P_2 is the $u - v$ path which meets Q at z . Let P be the union of $w - z$ sub path of Q and $z - u$ sub path of P_2 . Then let $C = P_1 \cup vw \cup P$.

Claim: C is the required cycle.

Let xy be an edge in P_1 such that $s(P_1) = s(P_2) = \mu(xy)$. Then three cases arise.

Case 1: $\mu(vw) > \mu(xy)$.

Sub Claim 1: $s(P) = \mu(xy)$.

We have, $s(P)$ cannot exceed $\mu(xy)$, for otherwise $P \cup vw$ will become a strong $u - v$ path having strength more than the strongest $u - v$ path P_1 , a contradiction. Therefore, $s(P) \leq \mu(xy)$. Because $s(P_2) = \mu(xy)$, the strength of the $u - z$ sub path of P_2 is greater than or equal to $\mu(xy)$. Hence, if $s(P) < \mu(xy)$, then the strength of $z - w$ sub path of $Q < \mu(xy)$ and thus $s(Q) < \mu(xy)$. Thus, we have $P_1 \cup vw$ is a strong $u - w$ path which is stronger than the strongest $u - w$ path Q , a contradiction. Hence, the only possibility is that $s(P) = \mu(xy)$.

Now, we have two strongest strong paths between u and v namely P_1 and $P \cup vw$ whose union gives the required cycle containing the vertex u and the edge vw .

Case 2: $\mu(vw) = \mu(xy)$.

If $s(P) < \mu(xy)$, then as in Case-1, $s(Q) < \mu(xy)$ and hence $P_1 \cup vw$ becomes a strong $u - w$ path which is stronger than the strongest $u - w$ path Q , a contradiction. Thus, $s(P) \geq \mu(xy)$ and hence we have two strongest strong $u - v$ paths namely P_1 and $P \cup vw$ whose union gives the required cycle.

Case 3: $\mu(vw) < \mu(xy)$.

Sub Claim 2: $s(P) = \mu(vw)$.

If $s(P) > \mu(vw)$, then all edges in P and P_1 have strength more than $\mu(vw)$ and thus vw becomes the unique weakest edge of the cycle $P_1 \cup vw \cup P$, contradicting our assumption that vw is a strong edge. If $s(P) < \mu(vw)$, then the strength of $z - w$ sub path of $Q < \mu(vw)$ because the strength of the $u - z$ sub path of $P_2 \geq \mu(xy) > \mu(vw)$. Therefore, $s(Q) < \mu(vw) < \mu(xy) = s(P_1)$ and hence $P_1 \cup vw$ is a strong path having strength more than that of Q , which is a contradiction to the fact that Q is a strongest strong $u - w$ path. Thus, $s(P) = \mu(vw)$.

Sub Claim 3: P is a strongest $u - w$ path.

To prove Sub Claim 3, it is sufficient to prove that $s(Q) = \mu(vw)$ because Q is a strongest $u - w$ path. Clearly, $s(Q) \geq \mu(vw)$. If $s(Q) < \mu(vw)$, then P will become a strong $u - w$ path which is stronger than the strongest $u - w$ path Q , a contradiction. If $s(Q) > \mu(vw)$ because $\mu(vw) < \mu(xy)$, all edges in $P_1 \cup Q$ have strength more than $\mu(vw)$ and hence vw becomes the unique weakest edge of the cycle $P_1 \cup vw \cup Q$ contradicting that vw is a strong edge. Thus, $s(Q) = \mu(vw)$ and the sub Claim 3 is proved.

Now, we have two strongest strong $u - w$ paths namely P and $P_1 \cup vw$ whose union gives the required cycle containing the vertex u and edge vw . Thus, in all the three cases the claim is proved.

(iii) \Rightarrow (iv)

Let xy and uv be any two given strong edges of G . Let C_1 be a cycle containing the vertex x and the strong edge uv and let C_2 be a cycle containing the vertex u and the edge xy . If y is a neighbor of x in C_1 or v is a neighbor of u in C_2 , we are done. Suppose not. That is, xy is not in C_1 and uv is not in C_2 . Without loss of generality suppose that C_1 is the union of two strongest strong $x - u$ paths P_1 and P_2 with P_2 containing the strong edge uv and C_2 is the union of two strongest strong $u - y$ paths. Let Q be the strongest strong $u - y$ path in C_2 not containing the edge xy . Let z be the vertex nearest to u at which Q meets P_1 or P_2 . Without loss of generality let Q meets P_2 at z . Now, let P be the union of $y - z$ sub path of Q and $z - u$ sub path (containing the strong edge vu) of P_2 .

Claim: $P_1 \cup P \cup xy$ is the required cycle.

Let $x'y'$ be an edge in P_1 such that $s(P_1) = \mu(x'y')$. Then three cases arise.

Case 1: $\mu(xy) > \mu(x'y')$.

Sub Claim 1: $s(Q) = \mu(x'y')$.

If $s(Q) < \mu(x'y')$, then $P_1 \cup xy$ is a strong $u - y$ path having strength more than the strength of Q , a contradiction to the assumption that Q is a strongest strong $u - y$ path. If $s(Q) > \mu(x'y')$, then the $u - x$ path $Q \cup yx$ has strength greater than

$\mu(x'y')$ because, $\mu(xy) > \mu(x'y')$, which contradicts the fact that P_1 is a strongest strong $u - x$ path. Therefore, only possibility is that $s(Q) = \mu(x'y')$ and hence Sub Claim 1 is proved.

Thus, we have strength of the $y - z$ sub path of $Q \geq \mu(x'y')$. Also, the strength of $z - u$ sub path of $P_2 \geq \mu(x'y')$, because $s(P_2) = \mu(x'y')$. Now, if both these sub paths are of strength greater than $\mu(x'y')$, then $s(P)$ is greater than $\mu(x'y')$, which contradicts the fact that Q is a strongest strong $u - y$ path. Thus, at least one of this sub paths should have strength equal to $\mu(x'y')$ and $s(P)$ is equal to $\mu(x'y')$. Hence, we have two strongest strong $x - u$ paths namely P_1 and $P \cup xy$ whose union gives the required cycle containing the edges xy and uv .

Case 2: $\mu(xy) = \mu(x'y')$.

Because the strength of the $y - u$ path passing through x and P_1 is $\mu(xy)$, we have the strength of Q is at least $\mu(xy)$. Also, because $s(P_2)$ being $\mu(xy)$, the $z - u$ sub path of P_2 has strength at least $\mu(xy)$. Thus, $s(P)$ is at least $\mu(xy)$ and hence we have two strongest strong $x - u$ paths namely P_1 and $P \cup xy$ whose union gives the required cycle containing the edges xy and uv .

Case 3: $\mu(xy) < \mu(x'y')$.

Sub Claim 2: $s(Q) = \mu(xy)$.

Clearly, $s(Q) \leq \mu(xy)$, for otherwise all edges in $P_1 \cup Q$ have strength more than $\mu(xy)$ and hence xy becomes the unique weakest edge of the cycle $P_1 \cup Q \cup xy$, contradicting our assumption that xy is a strong edge. Now, if $s(Q) < \mu(xy)$ we have a strong path from x to u namely $xy \cup P_1$ which is stronger than the strongest $y - u$ path Q , a contradiction. Thus, $s(Q) = \mu(xy)$.

Sub Claim 3: The strength of $y - z$ sub path of Q is precisely $\mu(xy)$.

Because $s(Q) = \mu(xy)$, the strength of $y - z$ sub path of Q is at least $\mu(xy)$, but if the strength of the $y - z$ sub path of Q is greater than $\mu(xy)$, then the $x - z$ sub path of P_2 has strength $\geq \mu(x'y') > \mu(xy)$ and so we have all the edges in the two sub paths have strength greater than $\mu(xy)$ and xy becomes the unique weakest edge of the cycle formed by the edge xy , the $x - z$ sub path of P_2 and the $y - z$ sub path of Q , which contradicts the assumption that xy is a strong edge.

Also, because P_2 is a strongest strong $x - u$ path, $s(P_2) = \mu(x'y')$. Therefore, the strength of $z - u$ sub path of $P_2 \geq \mu(x'y') > \mu(xy)$. Thus, the strength of the $y - u$ path P is $\mu(xy)$ and hence it is a strongest strong $y - u$ path in G . Also, note that the strength of the $y - u$ path $P_1 \cup xy$ is $\mu(xy)$ because $\mu(x'y') > \mu(xy)$. Hence, it follows that $P \cup P_1 \cup xy$ is a cycle containing the edges xy and uv formed by the union of two strongest strong $y - u$ paths as required.

$(iv) \Rightarrow (v)$

Let x, u and w be any three distinct vertices of G . Let P be a strongest strong $x - u$ path in G with strength α (say). Let y be a strong neighbor of x and v a strong neighbor of u in P . Then xy and uv are strong edges of G . By (iv), there exists a cycle C containing the edges xy and uv formed by two strongest strong $x - u$ paths or $y - u$ paths.

Case 1: C is the union of two strongest strong $x - u$ paths.

In this case, because there exist two internally disjoint strongest strong $x - u$ paths, at least one of them will not contain w .

Case 2: C is the union of two strongest strong $y - u$ paths.

Let P_1 be the $y - u$ strongest strong path containing the edge uv and let $C - P_1$ be the other path containing edge xy . Because P is a strongest strong $x - u$ path, containing y with strength α , we have $\mu(xy) \geq \alpha$. Also, we have $s(P_1) \geq \alpha$ for otherwise the $y - u$ sub path of P will become a strong path stronger than the strongest. But $s(P_1)$ cannot exceed α because $C - P_1$ is also a strongest strong $y - u$ path which passes through the edge xy with $\mu(xy) = \alpha$. Thus, only possibility is that $s(P_1) = \alpha$. In this case, $s(P_1 \cup xy)$ is also α and hence $P_1 \cup xy$ becomes a strongest strong $x - u$ path. Also, the strength of $C - \{P_1 \cup xy\}$ cannot exceed α for otherwise it is a contradiction to the fact that P is a strongest strong $x - u$ path. That is,

$$s(C - \{P_1 \cup xy\}) \leq \alpha \quad (2.1)$$

Also, because $C - P_1$ is a strongest strong $y - u$ path, it follows that

$$s(C - \{P_1 \cup xy\}) \geq \alpha. \quad (2.2)$$

From (2.1) and (2.2), $s(C - \{P_1 \cup xy\}) = \alpha$. Thus, we have two internally disjoint strongest strong $x - u$ paths namely $P_1 \cup xy$ and $C - \{P_1 \cup xy\}$ with at least one of them not containing the vertex w .

$(v) \Rightarrow (i)$

Assume (v) . Let w be a vertex in G . By (v) , between any two vertices u and v other than w there exists a strongest strong $u - v$ path not containing w . Thus, w is not in all strongest paths between any pair of vertices and hence is not a fuzzy cutvertex. Hence, G is a block. ■

The remaining statements of the characterization of blocks in graphs given in Harary [83] cannot be extended to fuzzy graphs as seen from the following example.

Example 2.7.10 Let $G = (\sigma, \mu)$ with $\sigma^* = \{u, v, w, x, y, z\}$ and $\mu(uv) = \mu(vw) = \mu(wz) = \mu(zu) = 0.8$, $\mu(ux) = \mu(xw) = \mu(uy) = \mu(yw) = 0.1$. G is a block. But there is no strongest strong $v - w$ path containing the vertex x and no strongest strong $v - w$ path containing the strong edge ux (Fig. 2.30).

Fig. 2.30 A fuzzy block different from block

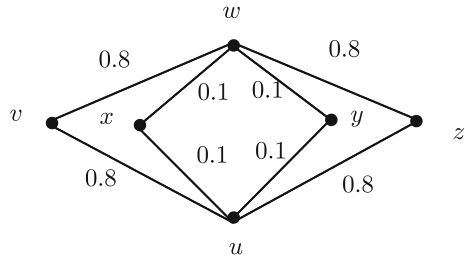
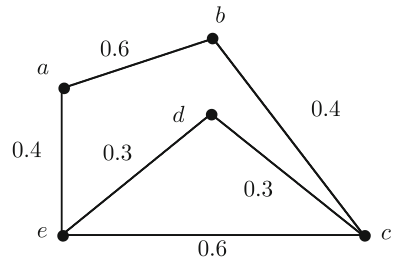


Fig. 2.31 Strongest strong cycles in a fuzzy graph



2.8 Strongest Strong Cycles and θ -Fuzzy Graphs

In the previous section, we observed that blocks in fuzzy graphs cannot be fully characterized even by cycles formed by two strongest strong paths. When the underlying structure of a fuzzy graph is a cycle, we can see that it is a block only when it is strong and is the union of two different strongest paths. A cycle is called a **locamin cycle** if every vertex of the cycle lies on a weakest edge.

Definition 2.8.1 The **strength of a cycle** C in a fuzzy graph G is defined as the weight of a weakest edge in C .

Definition 2.8.2 A cycle C in a fuzzy graph G is said to be a **strongest strong cycle (SSC)** if C is the union of two strongest strong $u - v$ paths for every pair of vertices u and v in C except when uv is a fuzzy bridge of G in C .

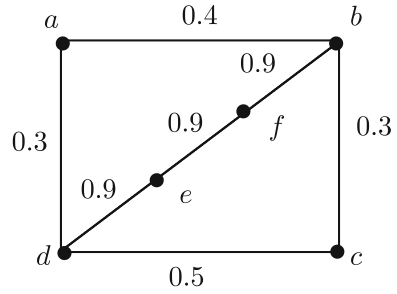
Note that in Definition 2.8.2, it is possible that edge uv can be a fuzzy bridge of G . But the condition that C is the union of two strongest strong $u - v$ paths can be relaxed for those vertices which are the end vertices of fuzzy bridges of G which are in C . Also, $CONN_G(x, y) = CONN_C(x, y)$ for all vertices x, y in C . The concept of SSC is illustrated below.

Example 2.8.3 Let $G = (\sigma, \mu)$ with $\sigma^* = \{a, b, c, d, e\}$, $\sigma(x) = 1$ for all $x \in \sigma^*$, $\mu(ab) = \mu(ce) = 0.6$, $\mu(ae) = \mu(bc) = 0.4$ and $\mu(cd) = \mu(de) = 0.3$ (Fig. 2.31). Here ab and ce are the fuzzy bridges of G . $C_1 = a, b, c, e, a$ and $C_2 = e, c, d, e$ are strongest strong cycles while $C_3 = a, b, c, d, e, a$ is not, because C_3 is not a union of two strongest strong $c - e$ paths. Here, $CONN_G(c, e) = 0.6$. But none of the $c - e$ paths in C_3 is strongest. Also, note that ce is a fuzzy bridge of G which is not in C_3 .

If the underlying graph is a cycle, then the concepts of strongest strong cycle and locamin cycle coincide and is equal to a block as seen from the next theorem.

Theorem 2.8.4 Let $G = (\sigma, \mu)$ a fuzzy graph such that G^* is a cycle. Then the following are equivalent.

Fig. 2.32 A fuzzy graph with a locamin cycle



- (i) G is a block.
- (ii) G is an SSC.
- (iii) G is a locamin cycle.

Proof (i) \Rightarrow (ii) Suppose that G is a block, where G^* is a cycle. Then by Theorem 2.7.4, any two vertices u and v such that uv is not a fuzzy bridge are joined by two internally disjoint strongest paths. Clearly every edge in G is strong, otherwise G will have exactly one non strong edge, whose removal from G results in a tree, with all internal vertices fuzzy cutvertices, contradicting the assumption that G is a fuzzy block. Thus, G is the union of two strongest strong $u - v$ paths for every pair of vertices u and v in G except when uv is a fuzzy bridge of G . Thus, G is an SSC.

(ii) \Rightarrow (iii) Suppose that G is an SSC. If possible suppose that G is not locamin. Then there exists some vertex w such that w is not on a weakest edge of G . Let uw and wv be the two edges incident on w , which are not weakest edges. This implies that the path uwv is the unique strongest $u - v$ path in G , contradiction to the assumption that G is an SSC.

(iii) \Rightarrow (i) Because we consider only simple fuzzy graphs, G will have at least three edges and the proof follows from Theorem 2.7.4. ■

Generally in a fuzzy graph, a locamin cycle need not be an SSC and an SSC need not be a locamin cycle. (See Examples 2.8.5 and 2.8.6).

Example 2.8.5 Let $G = (\sigma, \mu)$ with $\sigma^* = \{a, b, c, d, e, f\}$, $\sigma(x) = 1$ for all $x \in \sigma^*$ and $\mu(ab) = 0.4$, $\mu(bc) = \mu(da) = 0.3$, $\mu(cd) = 0.5$, $\mu(de) = \mu(e f) = \mu(fb) = 0.9$ (Fig. 2.32). Here, $C_1 = a, b, c, d, a$ is a locamin cycle but it is not an SSC because there do not exist two strongest $b - d$ paths in C_1 . Note that $CONN_G(b, d) = 1$ (strength of the path b, f, e, d).

Example 2.8.6 Let $G = (\sigma, \mu)$ with $\sigma^* = \{a, b, c, d, e, f\}$, $\sigma(x) = 1$ for all $x \in \sigma^*$ and $\mu(ab) = \mu(bc) = \mu(ca) = \mu(cd) = \mu(de) = \mu(e f) = \mu(fb) = 0.8$, $\mu(da) = 0.5$ (See Fig. 2.33). Here G contains no locamin cycles but there are several strongest strong cycles. Note that in G , any cycle not containing the weakest edge da is a strongest strong cycle.

Fig. 2.33 Fuzzy graph with SSC but no locamin cycles

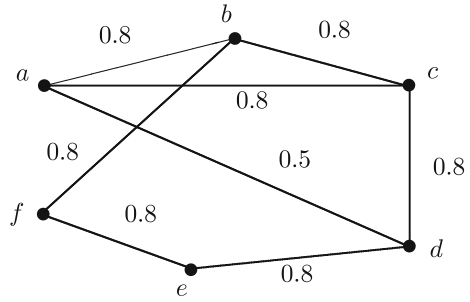
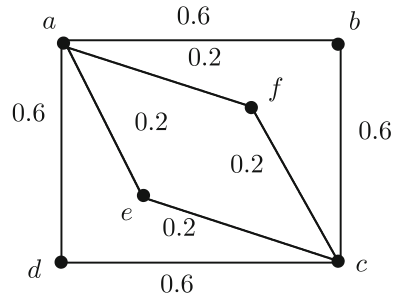


Fig. 2.34 Counter example for the converse of Theorem 2.8.7



Next we discuss a sufficient condition for a fuzzy graph to be a block.

Theorem 2.8.7 *If any two vertices of a fuzzy graph G lie on a common SSC, then G is a block.*

Proof Let $G = (\sigma, \mu)$ be a fuzzy graph satisfying the condition of the theorem. Clearly G is connected. Let w be a vertex in G . For any two vertices x and y such that $x \neq w \neq y$, there exists an SSC containing x and y . That is, there exist two internally disjoint strongest $x - y$ paths in G . At most one of these paths can contain the vertex w and hence w cannot be a fuzzy cutvertex of G . Because w is arbitrary, it follows that G is a block. ■

The converse of the above result is not true in general as seen from the next example, but is true for a sub family of fuzzy graphs which will be discussed soon.

Example 2.8.8 Let $G = (\sigma, \mu)$ with $\sigma^* = \{a, b, c, d, e, f\}$, $\sigma(x) = 1$ for all $x \in \sigma^*$ and $\mu(ab) = \mu(bc) = \mu(cd) = \mu(da) = 0.6$, $\mu(ae) = \mu(ec) = \mu(cf) = \mu(fa) = 0.2$ (Fig. 2.34). Here the vertices b and e do not belong to a common SSC, but G is a block.

A fuzzy graph is said to be **edge disjoint** when no two cycles share a common edge. When the fuzzy graph is edge disjoint, the blocks can be easily characterized as in the following theorem.

Theorem 2.8.9 *Let $G = (\sigma, \mu)$ be an edge disjoint fuzzy graph with at least three vertices. Then the following are equivalent.*

- (i) G is a block.
- (ii) G is an SSC.
- (iii) G is a locamin cycle.

Proof (i) \Rightarrow (ii) Suppose G is a block. In view of Theorem 2.8.4, it is sufficient to prove that G^* is a cycle. Suppose G^* is not a cycle. Because G is a block, each vertex must be on a cycle. Thus, G is a union of more than one cycle with a unique vertex in common. Let w be this common vertex of intersection. Then clearly w is a cutvertex, a contradiction to (i).

(ii) \Rightarrow (iii) Let G be an edge disjoint fuzzy graph such that G is an SSC. To prove G is locamin. Because G is an SSC, G^* is a cycle and hence by Theorem 2.8.4, G is a locamin cycle.

(iii) \Rightarrow (i) Let G be a locamin cycle. Then G^* is a cycle and hence by Theorem 2.8.4, G is a block. ■

The relevance of the above theorem is that any connected edge disjoint fuzzy graph with more than one cycle will always have a fuzzy cutvertex. The following are a set of necessary conditions for a fuzzy graph to be a block.

Theorem 2.8.10 *If $G = (\sigma, \mu)$ is a block, then the following conditions hold and are equivalent.*

- (i) Every two vertices of G lie on a common strong cycle.
- (ii) Each vertex and a strong edge of G lie on a common strong cycle.
- (iii) Any two strong edges of G lie on a common strong cycle.
- (iv) For any two given vertices and a strong edge in G , there exists a strong path joining the vertices containing the edge.
- (v) For every three distinct vertices of G , there exist strong paths joining any two of them containing the third.
- (vi) For every three vertices of G , there exist strong paths joining any two of them which does not contain the third.

Proof (i) Suppose that G is a block. Let u and v be any two vertices in G such that there exists a unique strong path between u and v . Now, two cases arise. (1) uv is a strong edge. (2) uv is either weakest edge of a cycle or there exists a $u - v$ path of length more than one in G .

Case 1: uv is a strong edge.

Because uv is not on any strong cycle, uv is an edge in every maximum spanning tree of G and hence it is a fuzzy bridge. If u is an end vertex in all maximum spanning trees, then clearly u is a fuzzy end vertex of G and hence v is a fuzzy cutvertex of G or vice versa, contradicting our assumption that G is a block.

Now, suppose that u is an end vertex in some maximal spanning tree T_1 and v is an end vertex in some maximal spanning tree T_2 . Let u' be a strong neighbor of u in T_2 . Because u is an end vertex and v is an internal vertex in T_1 , there exists a strong path P in T_1 from u to u' passing through v . The path P together with the strong edge uu' forms a strong cycle in G , a contradiction.

Case 2: Either uv is the weakest edge of a cycle or there exists a strong $u - v$ path of length more than one in G .

If uv is a weakest edge of a cycle, then there exists a strong path between u and v . Because there is a unique strong uv path P in G , P belongs to all maximum spanning trees. Thus, all internal vertices in P are internal vertices in all the maximum spanning trees and hence all of them are fuzzy cutvertices in G , contradiction to the assumption that G is a block. Thus, condition (i) holds. Next we show that each of the given conditions are equivalent.

(i) \Rightarrow (ii) Suppose that every two vertices of G lie on a common strong cycle. To prove that a given vertex and a strong edge lie on a common strong cycle. Let u be a vertex and vw be an edge in G . Let C be a strong cycle containing u and v . If w is a neighbor of v in C , then there is nothing to prove. Now, suppose that w is not a neighbor of v in C . Let C_1 be a strong cycle containing u and w . Let P_1 and P_2 be the strong $u - v$ paths in C and P'_1 and P'_2 the strong $u - w$ paths in C_1 . Let x_1 be the vertex at which P'_1 leaves P_1 . Then clearly $u \dots (P_1) \dots x_1 \dots (P'_1) \dots wv \dots (P_2)u$ is a strong cycle containing u and vw . If $x = u$ then $u \dots (P'_1) \dots wv \dots (P_2)u$ is the required cycle. If $x_1 = v$, then let x_2 be the vertex at which P'_2 leaves P_2 . Then $u \dots (P_1) \dots vw \dots (P'_2) \dots x_2 \dots (P_2)u$ is the required strong cycle. If $x_2 = u$, then $u \dots (P'_2) \dots wv \dots (P_1) \dots u$ is the required strong cycle. Because P_1 and P_2 are internally disjoint, both x_1 and x_2 cannot be the same as v .

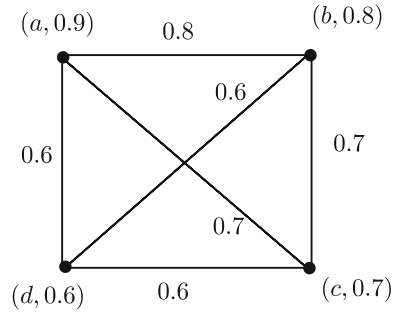
(ii) \Rightarrow (iii) Suppose that each vertex and strong edge lie on a common strong cycle. To prove any two strong edges lie on a common strong cycle. Let uv and xy be two strong edges of G . Let P_1 and P_2 be two internally disjoint strong paths between v and x and Q_1 and Q_2 be two internally disjoint strong paths between u and y . If P_1 , P_2 , Q_1 and Q_2 are internally disjoint, then $uv \dots (P_1) \dots xy \dots (Q_2) \dots u$ is a strong cycle containing uv and xy . If Q_1 and Q_2 intersect P_1 or P_2 , then a strong cycle containing uv and xy can be extracted from the parts of the four cycles P_1 , P_2 , Q_1 and Q_2 .

(iii) \Rightarrow (iv) Let x and y be two vertices and let uv be a strong edge in G . Let x' be a strong neighbor of x and y' be a strong neighbor of y . Now, there exists a strong cycle C_1 containing xx' and uv and a strong cycle C_2 containing yy' and uv . Now, $xx' \dots (C_1) \dots uv \dots (C_2) \dots y'y$ is a strong $x - y$ path containing the edge uv .

(iv) \Rightarrow (v) Let G be a block and u, v, w be three distinct vertices of G . Let v' be a strong neighbor of v . Then u and w are distinct vertices and vv' is a strong edge of G . By (iv), there exists a strong path from u to w containing the edge uv' (Even if $v' = u$ or w). Thus, we have a strong path between the two given vertices containing the third.

(v) \Rightarrow (vi) Let u, v, w be three distinct vertices of G . Let P be a strong path between u and w containing v . Then clearly the $u - v$ strong sub path say P' does not contain w .

(vi) \Rightarrow (i) Let u and v be two given vertices. Let w be a third vertex in G . Let P_1 be the strong path joining u and v not containing w . Let P_2 be the strong path joining u and w not containing v and let P_3 be the strong path joining v and w not containing u . Then $P_1 \cup P_2 \cup P_3$ will contain a strong cycle containing u and v . ■

Fig. 2.35 Cycle connectivity

So far, we have been trying to characterize fuzzy blocks as in classical graph theory. But, in a graph, the strength of every cycle is 1 whereas in fuzzy graphs, cycles of different strengths can pass through pairs of vertices. So a similar generalization is impossible. But we can find a subfamily where all the characterizations are valid. In this section, two new connectivity concepts in fuzzy graphs, namely θ -evaluation and cycle connectivity and a new subfamily of fuzzy graphs called θ -fuzzy graphs are discussed. Blocks in θ -fuzzy graphs are fully characterized.

Consider the following definition of θ -evaluation of a pair of vertices in a fuzzy graph.

Definition 2.8.11 Let $G = (\sigma, \mu)$ be a fuzzy graph. Then for any two vertices u and v of G , there associated a set say $\theta(u, v)$ called the **θ -evaluation** of u and v and is defined as $\theta(u, v) = \{\alpha \mid \alpha \in (0, 1]\}$, where α is the strength of a strong cycle passing through both u and v .

Note that if there are no strong cycles passing both u and v , then $\theta(u, v) = \emptyset$.

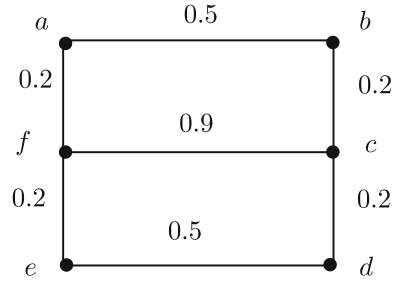
Using θ -evaluation, a new measure of connectivity in graphs, namely cycle connectivity can be introduced.

Definition 2.8.12 Let $G = (\sigma, \mu)$ be a fuzzy graph. Then $\bigvee \{\alpha \mid \alpha \in \theta(u, v), u, v \in \sigma^*\}$ is defined as the **cycle connectivity** between u and v in G and denoted by $C_{u,v}^G$. If $\theta(u, v) = \emptyset$ for some pair of vertices u and v , define the cycle connectivity between u and v to be 0.

θ -evaluation and cycle connectivity are illustrated in the following example.

Example 2.8.13 Let $G = (\sigma, \mu)$ with $\sigma^* = \{a, b, c, d\}$ with $\sigma(a) = 0.9$, $\sigma(b) = 0.8$, $\sigma(c) = 0.7$, $\sigma(d) = 0.6$ and $\mu(ab) = 0.8$, $\mu(bc) = \mu(ac) = 0.7$, $\mu(bd) = \mu(cd) = \mu(da) = 0.6$ (Fig. 2.35). Here G is a complete fuzzy graph. $\theta\{a, c\} = \{0.6, 0.7\}$ and hence $C_{a,c}^G = 0.7$

Cycle connectivity is a measure of connectedness in a fuzzy graph and it is always less than or equal to the strength of connectedness between any two vertices u and v . In a crisp graph the cycle connectivity between two vertices u and v is 1 if u and v belong to a common cycle and 0 otherwise.

Fig. 2.36 A θ -fuzzy graph

A new subclass of fuzzy graphs called θ -fuzzy graphs, with the property that every cycle passing through a particular pair of vertices have the same strength, is discussed below.

Definition 2.8.14 Let $G = (\sigma, \mu)$ be a fuzzy graph. G is said to be a **θ -fuzzy graph** if θ - evaluation of each pair of vertices in G is either empty or a singleton set. In other words G is called a θ -fuzzy graph if for each pair of vertices u and v , either there is no strong cycle passing through u and v or all strong cycles passing through u and v have the same strength.

Consider the following example.

Example 2.8.15 Let $G = (\sigma, \mu)$ with $\sigma^* = \{a, b, c, d, e, f\}$, $\sigma(x) = 1$ for all $x \in \sigma^*$ and $\mu(ab) = \mu(de) = 0.5$, $\mu(bc) = \mu(cd) = \mu(ef) = \mu(fa) = 0.2$, $\mu(cf) = 0.9$ (Fig. 2.36). G is clearly a θ -fuzzy graph because all strong cycles in this graph have strength 0.1 and hence $\theta(u, v) = \{0.1\}$ for any two vertices u and v .

Proposition 2.8.16 *Edge disjoint fuzzy graphs are θ -fuzzy graphs.*

Proof Let G be an edge disjoint fuzzy graph. First we show that two distinct vertices cannot be in two different cycles of G . Because C_1 and C_2 share no edge in common, C_1 and C_2 intersect at u and v (say). Let P_1 and P_2 be the two $u - v$ paths in C_1 and Q_1 and Q_2 be the two $u - v$ paths in C_2 . Then clearly $P_1 \cup Q_1$, $P_1 \cup Q_2$, $P_2 \cup Q_1$, $P_2 \cup Q_2$ are all cycles and some of them will definitely share edges with C_1 or C_2 which is not possible. Thus, between any pair of vertices u and v of G there exists at most one strong cycle and hence it follows that θ - evaluation of any two vertices in G is either empty set or a singleton. Thus, G is a θ -fuzzy graph. ■

In a CFG, the cycle connectivity between vertices can be easily evaluated using the following result.

Theorem 2.8.17 *Let $G = (\sigma, \mu)$ be a complete fuzzy graph. Then for any two vertices u and v in G , $C_{u,v}^G = \vee \{ \wedge \{ \sigma(u), \sigma(v), \sigma(w) \} \mid w \in \sigma^* \}$.*

Proof Let $G = (\sigma, \mu)$ be a CFG. By Theorem 2.2.7, all edges of G are strong and hence all cycles in G are strong cycles. Now, let $u, v \in \sigma^*$ and consider a strong cycle $C : u = u_1 u_2 u_3 \dots u_n = vu$.

Then $s(C) = \wedge\{\mu(u_1 u_2), \mu(u_2 u_3), \dots, \mu(u_{n-1} u_n), \mu(uv)\}$.

Because G is complete, $\mu(xy) = \sigma(x) \wedge \sigma(y)$ for all x and y and hence $s(C) = \wedge\{\sigma(u_1), \sigma(u_2), \dots, \sigma(u_n)\} = \sigma(u_l)$ for some $l \in \{1, 2, \dots, n\}$. Thus, the strength of $C = \wedge\{\sigma(u), \sigma(v), \sigma(w)\}$ for some vertex $w \in \sigma^*$, and hence for any two vertices u and v in G , $C_{u,v}^G = \vee\{\wedge\{\sigma(u), \sigma(v), \sigma(w)\} \mid w \in \sigma^*\}$. ■

In other words, the cycle connectivity between u and v in a complete fuzzy graph is the maximum of strengths of all triangles passing through u and v .

The next lemma is the key for the characterization of blocks in θ -fuzzy graphs. It gives the relationship between strong paths and strongest paths in θ -fuzzy graphs which are blocks.

Lemma 2.8.18 *Let G be a θ -fuzzy graph which is a block. Then any strong $u - v$ path such that uv is not a fuzzy bridge is a strongest $u - v$ path and hence any strong cycle in G is a strongest strong cycle.*

Proof Let $G = (\sigma, \mu)$ be a θ -fuzzy graph which is a block. Let $u, v \in \sigma^*$ be such that uv is not a fuzzy bridge. Let P be a strong $u - v$ path in G . If P is not a strongest $u - v$ path, then because G is a block, there exist two internally disjoint strongest strong $u - v$ paths say P_1 and P_2 . Then $P_1 \cup P$ is a strong cycle with strength less than that of the cycle $P_1 \cup P_2$. Both these cycles pass through u and v and hence $\theta(u, v)$ is not a singleton or empty which is a contradiction to the fact that G is a θ -fuzzy graph. Thus, P must be a strongest strong $u - v$ path.

To prove the second assertion of the lemma, let C be a strong cycle in G . Let u, v be two vertices in C such that uv is not a fuzzy bridge. Then by first part both these $u - v$ paths in C are strongest $u - v$ paths. Thus, G is a strongest strong cycle. ■

Thus, in a θ -fuzzy graph which is a block, the concepts of strong path and strongest path coincide and as a result, the concepts of strong cycle and SSC also coincide. Thus, all the six necessary and sufficient conditions for blocks in graphs can be generalized to blocks in θ -fuzzy graphs.

Next we have the awaited characterization for blocks in θ -fuzzy graphs.

Theorem 2.8.19 *Let $G = (\sigma, \mu)$ be a θ -fuzzy graph. Then the following statements are equivalent.*

- (i) G is a block.
- (ii) Every pair of vertices of G lie on a common strongest strong cycle.
- (iii) Each vertex and a strong edge of G lie on a common strongest strong cycle.
- (iv) Any two strong edges of G lie on a common strongest strong cycle.
- (v) For any two given vertices u and v such that uv is not a fuzzy bridge and a strong edge xy in G , there exists a strongest strong $u - v$ path containing the edge xy .

- (vi) For every three distinct vertices $u_i, i = 1, 2, 3$ of G such that $u_i u_j, i \neq j$ is not a fuzzy bridge, there exist strongest strong paths joining any two of them containing the third.
- (vii) For every three distinct vertices $u_i, i = 1, 2, 3$ of G such that $u_i u_j, i \neq j$ is not a fuzzy bridge, there exist strongest strong paths joining any two of them not containing the third.

Proof Theorem 2.8.10 and Lemma 2.8.18 together give all the required implications except $(vii) \Rightarrow (i)$

To prove $(vii) \Rightarrow (i)$, note that for any vertex w of G and for every pair of vertices x, y , other than w , there exists a strongest $x - y$ path not containing the vertex w . That is, w is not on every strongest $x - y$ path for any x and y and hence w is not a fuzzy cutvertex. Because w is arbitrary, it follows that G is a block. ■

2.9 Fuzzy Line Graphs

The contents of this section are from [125]. The study of fuzzy line graphs was carried out by Mordeson in 1993. It was one of the first theoretical topics studied in fuzzy graphs after Rosenfeld's introductory paper.

The line graph, $L(G)$, of a (crisp) graph G is the intersection graph of the set of edges of G . Hence, the vertices of $L(G)$ are the edges of G with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are. We present the notion of a fuzzy line graph. Let $G = (V, X)$ and $G' = (V', X')$ be graphs.

Definition 2.9.1 Let (σ, μ) and (σ', μ') be partial fuzzy subgraphs of G and G' , respectively. Let f be a one-to-one function of V onto V' . Then

- (i) f is called a **(weak) vertex-isomorphism** of (σ, μ) onto (σ', μ') if and only if for all $v \in V$, $(\sigma(v) \leq \sigma'(f(v))$ and $\text{Supp}(\sigma') = f(\text{Supp}(\sigma))$, $\sigma(v) = \sigma'(f(v))$;
- (ii) f is called a **(weak) line-isomorphism** of (σ, μ) onto (σ', μ') if and only if for all $u, v \in V$, $\mu(uv) \leq \mu'(f(u)f(v))$ and $\text{Supp}(\mu') = \{(f(u)f(v)) \mid u, v \in \text{Supp}(\mu)\}$, $\mu(uv) = \mu'(f(u)f(v))$.

If f is a (weak) vertex-isomorphism and a (weak) line-isomorphism of (σ, μ) onto (σ', μ') , then f is called a **(weak) isomorphism** of (σ, μ) onto (σ', μ') . If (σ, μ) is isomorphic to (σ', μ') , then we write $(\sigma, \mu) \simeq (\sigma', \mu')$.

Let $G = (V, X)$ be a graph, where $V = \{v_1, \dots, v_n\}$. Let $S_i = \{v_i, x_{i1}, \dots, x_{iq_i}\}$, where $x_{ij} \in X$ and x_{ij} has v_i as a vertex, $j = 1, \dots, q_i$; $i = 1, \dots, n$. Let $S = \{S_1, \dots, S_n\}$. Let $T = \{S_i S_j \mid S_i, S_j \in S, S_i \cap S_j \neq \emptyset, i \neq j\}$. Then $\mathcal{I}(S) = (S, T)$ is an intersection graph and $G \simeq \mathcal{I}(S)$. Any partial fuzzy subgraph (ι, γ) of $\mathcal{I}(S)$ with $\text{Supp}(\gamma) = T$ is called a **fuzzy intersection graph**.

Let (σ, μ) be a partial fuzzy subgraph of G . Let $\mathcal{I}(S)$ be the intersection graph described above. Define the fuzzy subsets ι, γ of S and T , respectively as follows:

For all $S_i \in S$, $\iota(S_i) = \sigma(v_i)$;

For all $S_i S_j \in T$, $\gamma(S_i S_j) = \mu(v_i v_j)$.

Proposition 2.9.2 *Let (σ, μ) be a partial fuzzy subgraph of G . Then*

- (i) (ι, γ) is a partial fuzzy subgraph of $\mathcal{I}(S)$;
- (ii) $(\sigma, \mu) \simeq (\iota, \gamma)$.

Proof (i) $\gamma(S_i S_j) = \mu(v_i v_j) \leq \sigma(v_i) \wedge \sigma(v_j) = \iota(S_i) \wedge \iota(S_j)$.

(ii) Define $f : V \rightarrow S$ by $f(v_i) = S_i, i = 1, \dots, n$. Clearly f is a one-to-one function of V onto S . Now, $v_i v_j \in X$ if and only if $S_i S_j \in T$ and so $T = \{f(v_i) f(v_j) \mid v_i v_j \in X\}$. Also, $\iota(f(v_i)) = \iota(S_i) = \sigma(v_i)$ and $\gamma(f(v_i) f(v_j)) = \gamma(S_i S_j) = \mu(v_i v_j)$. Thus, f is an isomorphism of (σ, μ) onto (ι, γ) . ■

Let $\mathcal{I}(S)$ be the intersection graph of (V, X) . Let (i, γ) be the fuzzy intersection graph of $\mathcal{I}(S)$ as defined above. We call (ι, γ) the *fuzzy intersection graph* of (σ, μ) . The previous proposition shows that any fuzzy graph is isomorphic to a fuzzy intersection graph.

The line graph of $G, L(G)$, is by definition the intersection graph $\mathcal{I}(X)$. That is, $L(G) = (Z, W)$, where $Z = \{\{x\} \cup \{u_x, v_x\} \mid x \in X, u_x, v_x \in V, x = u_x v_x\}$ and $W = \{S_x S_y \mid S_x \cap S_y \neq \emptyset, x, y \in X, x \neq y\}$ and where $S_x = \{x\} \cup \{u_x, v_x\}, x \in X$. Let (σ, μ) be a partial fuzzy subgraph of G . Define the fuzzy subsets λ, ω of Z, W , respectively, as follows:

For all $S_x \in Z, \lambda(S_x) = \mu(x)$;

For all $S_x S_y \in W, \omega(S_x S_y) = \mu(x) \wedge \mu(y)$.

Proposition 2.9.3 (λ, ω) is a fuzzy subgraph of $L(G)$ and is called the *fuzzy line graph* corresponding to (σ, μ) .

Proof $\omega(S_x S_y) = \mu(x) \wedge \mu(y) = \lambda(S_x) \wedge \lambda(S_y)$. ■

Every cutpoint of $L(G)$ is a bridge of G which is not a pendent edge, and conversely [83]. It is shown in [125] that the relationship between cutpoints in $L(G)$ and bridges in G does not carry over to the fuzzy case.

Let (σ, μ) and (σ', μ') be partial fuzzy subgraphs of G and G' , respectively. If f is a weak isomorphism of (σ, μ) onto (σ', μ') , then it can be shown that f is an isomorphism of $(\text{Supp}(\sigma), \text{Supp}(\mu))$ onto $(\text{Supp}(\sigma'), \text{Supp}(\mu'))$. If (λ, ω) is the fuzzy line graph of (σ, μ) , then it can also be shown that $(\text{Supp}(\lambda), \text{Supp}(\omega))$ is the fuzzy line graph of $(\text{Supp}(\sigma), \text{Supp}(\mu))$.

Example 2.9.4 Let $G = (V, X)$, where $V = \{v_1, v_2, v_3, v_4\}$ and $X = \{x_1 = v_1 v_2, x_2 = v_1 v_3, x_3 = v_2 v_3, x_4 = v_3 v_4\}$. Let $\sigma(v_i) = 1, i = 1, 2, 3, 4, \mu(x_1) = \mu(x_3) = 1$ and $\mu(x_2) = \mu(x_4) = 1/2$. Then $\lambda(S_{x_1}) = \lambda(S_{x_3}) = 1, \lambda(S_{x_2}) = \lambda(S_{x_4}) = 1/2$ and $\omega(S_{x_1} S_{x_2}) = 1, \omega(S_{x_2} S_{x_3}) = \omega(S_{x_3} S_{x_4}) = \omega(S_{x_2} S_{x_4}) = 1/2$. If we delete x_1 from G , then the strength of connectedness between v_1 and v_2 before the deletion of x_1 is $1 = \mu(x_1)$. Thus, x_1 is a bridge of (σ, μ) (not an endline of G). However, the strength of connectedness between any pair of vertices $S_{x_2}, S_{x_3}, S_{x_4}$ is $1/2$ before and after the deletion of S_{x_1} . Thus, S_{x_1} is not a cutvertex of (σ, μ) .

Proposition 2.9.5 *Let (σ, μ) and (σ', μ') be partial fuzzy subgraphs of G and G' , respectively. If f is a weak isomorphism of (σ, μ) onto (σ', μ') , then f is an isomorphism of $(\text{Supp}(\sigma), \text{Supp}(\mu))$ onto $(\text{Supp}(\sigma'), \text{Supp}(\mu'))$.*

Proof $v \in \text{Supp}(\sigma) \Leftrightarrow f(v) \in \text{Supp}(\sigma')$ and $uv \in \text{Supp}(\mu) \Leftrightarrow f(u)f(v) \in \text{Supp}(\mu')$. ■

Proposition 2.9.6 *If (λ, ω) is the fuzzy line graph of the fuzzy graph (σ, μ) , then $(\text{Supp}(\lambda), \text{Supp}(\omega))$ is the line graph of $(\text{Supp}(\sigma), \text{Supp}(\mu))$.*

Proof (σ, μ) is a partial fuzzy subgraph of G and (λ, ω) is a partial fuzzy subgraph of $L(G)$. Now, $\lambda(S_x) = \mu(x)$ for all $x \in X$ and so $S_x \in \text{Supp}(\lambda) \Leftrightarrow x \in \text{Supp}(\mu)$. Also, $\omega(S_x S_y) = \mu(x) \wedge \mu(y)$ for all $S_x S_y \in W$ and so $\text{Supp}(\omega) = \{S_x S_y \mid S_x \cap S_y \neq \emptyset, x, y \in \text{Supp}(\mu), x \neq y\}$. ■

Theorem 2.9.7 ([125]) *Let (λ, ω) be the fuzzy line graph corresponding to (σ, μ) . Suppose that $(\text{Supp}(\sigma), \text{Supp}(\mu))$ is connected. Then the following properties hold.*

- (i) *There exists a weak isomorphism of (σ, μ) onto (λ, ω) if and only if $(\text{Supp}(\sigma), \text{Supp}(\mu))$ is a cycle and σ and μ are constant functions on $\text{Supp}(\sigma)$ and $\text{Supp}(\mu)$, respectively, taking on the same value;*
- (ii) *If f is a weak isomorphism of (σ, μ) onto (λ, ω) , then f is an isomorphism.*

Proof Suppose that f is a weak isomorphism of (σ, μ) onto (λ, ω) . By Proposition 2.9.5, we can see that f is an isomorphism of $(\text{Supp}(\sigma), \text{Supp}(\mu))$ onto $(\text{Supp}(\lambda), \text{Supp}(\omega))$. By Proposition 2.9.6, $(\text{Supp}(\sigma), \text{Supp}(\mu))$ is a cycle. Let $\text{Supp}(\sigma) = \{v_1, v_2, \dots, v_n\}$ and $\text{Supp}(\mu) = \{v_1 v_2, v_2 v_3, \dots, v_n v_1\}$, where $v_1 v_2 \dots v_n v_1$ is a cycle. Let $\sigma(v_i) = s_i$ and $\mu(v_i v_{i+1}) = r_i$, $i = 1, 2, \dots, n$, where $v_{n+1} = v_1$. Then $s_{n+1} = s_1$ and

$$r_i \leq s_i \wedge s_{i+1}, \quad i = 1, 2, \dots, n. \quad (2.3)$$

Now, we have $\text{Supp}(\lambda) = \{S_{(v_i, v_{i+1})} \mid i = 1, 2, \dots, n\}$ and $\text{Supp}(\omega) = \{(S_{(v_i, v_{i+1})}, S_{(v_{i+1}, v_{i+2})}) \mid i = 1, 2, \dots, (n-1)\}$. Also, for $r_{n+1} = r_1$, $\lambda(S_{(v_i, v_{i+1})}) = \mu(v_i, v_{i+1}) = r_i$ and $\omega(S_{(v_i, v_{i+1})}, S_{(v_{i+1}, v_{i+2})}) = \mu(v_i, v_{i+1}) \wedge \mu(v_{i+1}, v_{i+2}) = r_i \wedge r_{i+1}$, $i = 1, 2, \dots, n$, where $v_{n+2} = v_2$. Because f is an isomorphism of $(\text{Supp}(\sigma), \text{Supp}(\mu))$ onto $(\text{Supp}(\lambda), \text{Supp}(\omega))$, f maps $\text{Supp}(\sigma)$ onto $\text{Supp}(\lambda) = \{S_{(v_1, v_2)}, \dots, S_{(v_n, v_1)}\}$. Also, f preserves adjacency. Hence, f induces a permutation π of $\{1, 2, \dots, n\}$ such that $f(v_i) = S_{(v_{\pi(i)}, v_{\pi(i)+1})}$ and

$$(v_i, v_{i+1}) \rightarrow (f(v_i), f(v_{i+1})) = (S_{(v_{\pi(i)}, v_{\pi(i)+1})}, S_{(v_{\pi(i+1)}, v_{\pi(i+1)+1})})$$

for $i = 1, 2, \dots, (n-1)$. Now,

$$s_i = \sigma(v_i) \leq \lambda(f(v_i)) = \lambda(S_{(v_{\pi(i)}, v_{\pi(i)+1})}) = r_{\pi(i)}$$

and

$$\begin{aligned} r_i &= \mu(v_i, v_{i+1}) \leq \omega(f(v_i), f(v_{i+1})) \\ &= \omega(S_{(v_{\pi(i)}, v_{\pi(i)+1})}, S_{(v_{\pi(i+1)}, v_{\pi(i+1)+1})}) = \mu(v_{\pi(i)}, \end{aligned}$$

$$v_{\pi(i)+1}) \wedge \mu(v_{\pi(i)+1}, v_{\pi(i+1)+1}) = r_{\pi(i)} \wedge r_{\pi(i+1)}, i = 1, 2, \dots, n.$$

That is,

$$s_i \leq r_{\pi(i)} \text{ and } r_i \leq r_{\pi(i)} \wedge r_{\pi(i+1)}, i = 1, 2, \dots, n. \quad (2.4)$$

By the second part of (2.4), we have that $r_i \leq r_{\pi(i)}$, $i = 1, 2, \dots, n$, and so $r_{\pi(i)} \leq r_{\pi(\pi(i))}$, $i = 1, 2, \dots, n$. Continuing we have that $r_i \leq r_{\pi(i)} \leq \dots \leq r_{\pi^j(i)} \leq r_i$ and so $r_i = r_{\pi(i)}$, $i = 1, 2, \dots, n$, where π^{j+1} is the identity map. By (2.4) again, we have $r_i \leq r_{\pi(i+1)} = r_{i+1}$, $i = 1, 2, \dots, n$, where $r_{n+1} = r_1$. Hence, by (2.3) and (2.4), $r_1 = r_2 = \dots = r_n = s_1 = s_2 = \dots = s_n$. Thus, we have not only proved the conclusion about σ and μ being constant functions, but we have also shown that (ii) holds.

Conversely, suppose that $(\text{Supp}(\sigma), \text{Supp}(\mu))$ is a cycle and for all $v \in \text{Supp}(\sigma)$ and $x \in \text{Supp}(\mu)$, $\sigma(v) = \mu(x)$. By Proposition 2.9.6, $(\text{Supp}(\lambda), \text{Supp}(\omega))$ is the line graph of $(\text{Supp}(\sigma), \text{Supp}(\mu))$. Because $(\text{Supp}(\sigma), \text{Supp}(\mu))$ is a cycle, $(\text{Supp}(\sigma), \text{Supp}(\mu)) \cong (\text{Supp}(\lambda), \text{Supp}(\omega))$ by Theorem 8.2 of [83]. This isomorphism induces an isomorphism of (σ, μ) onto (λ, ω) , because $\sigma(v) = \mu(x)$ for all $v \in V$ and $x \in X$ and so $\sigma = \mu = \lambda = \omega$ on their respective domains. ■

Theorem 2.9.8 *Let (σ, μ) and (σ', μ') be partial fuzzy subgraphs of G and G' , respectively, such that $(\text{Supp}(\sigma), \text{Supp}(\mu))$ and $(\text{Supp}(\sigma'), \text{Supp}(\mu'))$ are connected. Let (λ, ω) and (λ', ω') be the line graphs corresponding to (σ, μ) and (σ', μ') , respectively. Suppose that it is not the case that one of $(\text{Supp}(\sigma), \text{Supp}(\mu))$ and $(\text{Supp}(\sigma'), \text{Supp}(\mu'))$ is K_3 and the other is $K_{1,3}$. If $(\lambda, \omega) \simeq (\lambda', \omega')$, then (σ, μ) and (σ', μ') are line isomorphic.*

Proof Because $(\lambda, \omega) \simeq (\lambda', \omega')$, $(\text{Supp}(\lambda), \text{Supp}(\omega)) \simeq (\text{Supp}(\lambda'), \text{Supp}(\omega'))$ by Proposition 2.9.5. Because $(\text{Supp}(\lambda), \text{Supp}(\omega))$ and $(\text{Supp}(\lambda'), \text{Supp}(\omega'))$ are line graphs of $(\text{Supp}(\sigma), \text{Supp}(\mu))$ and $(\text{Supp}(\sigma'), \text{Supp}(\mu'))$, respectively, by Proposition 2.9.6, we have that $(\text{Supp}(\sigma), \text{Supp}(\mu)) \simeq (\text{Supp}(\sigma'), \text{Supp}(\mu'))$ by Theorem 8.3 of [83]. Let g denote the isomorphism of (λ, ω) onto (λ', ω') and f the isomorphism of $(\text{Supp}(\sigma), \text{Supp}(\mu))$ onto $(\text{Supp}(\sigma'), \text{Supp}(\mu'))$. Then $\lambda(S_{uv}) = \lambda'(g(S_{uv})) = \lambda'(g(S_{f(u)f(v)}))$, where the latter equality holds by the proof of Theorem 8.3 in [83] and so $\mu(uv) = \mu'(f(u)f(v))$. Hence, (σ, μ) and (σ', μ') are line isomorphic. ■

Proposition 2.9.9 *Let (τ, ν) be a partial fuzzy subgraph of $L(G)$. Then (τ, ν) is a fuzzy line graph of some partial fuzzy subgraph of G if and only if for all $S_x S_y \in W$, $\nu(S_x S_y) = \tau(x) \wedge \tau(y)$.*

Proof Suppose that $\nu(S_x S_y) = \tau(S_x) \wedge \tau(S_y)$ for all $S_x S_y \in W$. For all $x \in X$, define $\sigma(x) = \tau(S_x)$. Then $\nu(S_x S_y) = \tau(S_x) \wedge \tau(S_y) = \mu(x) \wedge \mu(y)$. Any σ that yields the property $\mu(uv) \leq \sigma(x) \wedge \sigma(y)$ will suffice, i.e., $\sigma(v) = 1$ for all $v \in V$. The converse is immediate. ■

Theorem 2.9.10 (σ, μ) is a fuzzy line graph if and only if $(\text{Supp}(\sigma), \text{Supp}(\mu))$ is a line graph and for all $uv \in \text{Supp}(\mu)$, $\mu(uv) = \sigma(u) \wedge \sigma(v)$.

Proof Suppose that (σ, μ) is a fuzzy line graph. Then the conclusion holds by Propositions 2.9.6 and 2.9.9. Conversely, suppose that $(\text{Supp}(\sigma), \text{Supp}(\mu))$ is a line graph and for all $uv \in \text{Supp}(\mu)$, $\mu(uv) = \sigma(u) \wedge \sigma(v)$. Then the conclusion holds from Proposition 2.9.9. ■

2.10 Fuzzy Interval Graphs

The results in this section are due to the important work of Craine in [61]. In [61], it was shown that a fuzzy graph without loops is the intersection graph of some family of fuzzy sets. It was shown that the characterization of interval graphs by Gilmore and Hoffman naturally extends to fuzzy interval graphs while that of Fulkerson and Gross does not. We present these results here.

As stated in [61], Roberts [153] cites applications of interval graphs in archaeology, developmental psychology, ecological modeling, mathematical sociology and organization theory. These disciplines all have components that are ambiguously defined, require subjective evaluation, or are satisfied to differing degrees. These are extremely active areas of application of fuzzy methods. It is therefore valuable to explore the extent that intersection graph results can be extended using fuzzy set theory.

The intersection graph of a family (perhaps with repeated members) of sets is a graph with a vertex representing each member of the family and an edge connecting two vertices if and only if the two sets have nonempty intersection. Generally loops are suppressed. If the family is composed of intervals or is the edge set of a hypergraph, then the intersection graph is called an *interval graph* or a *line graph*, respectively.

McAllister [119] used a different approach in defining a fuzzy intersection graph. However, his approach did not yield the usual definition of an intersection graph when applied to families of crisp sets. A different approach is taken in [61]. Each definition and theorem is a natural generalization of the crisp theory.

The t -norm minimum is used to define the fuzzy intersection graph of a family of fuzzy sets. We present a proof of a fuzzy analog of Marczewski's theorem, [61]. The proof shows that every fuzzy graph without loops is the intersection graph of some family of fuzzy sets. We also show that the natural generalization of the Fulkerson and Gross characterization of interval graphs fails, [61]. We then present a natural generalization of the Gilmore and Hoffman characterization.

Results characterizing a fuzzy property in terms of cut level set properties are significant, in that such theorems demonstrate the extent to which the crisp theory can be generalized. To accomplish this here, we provide the sequence of crisp cut level graphs given in [26]. Define the fundamental sequence of a fuzzy graph $G = (\sigma, \mu)$ to be the ordered set

$$\text{fg}(G) = \{\sigma(x) > 0 \mid x \in X\} \cup \{\mu(xy) > 0 \mid (x, y) \in X \times X\},$$

where the decreasing order inherited from the real interval $[0, 1]$ is used.

The first element listed in $\text{fg}(G)$ is the maximal vertex strength while the last element listed is the minimal nonzero edge strength.

2.10.1 Fuzzy Intersection Graphs

We now define a fuzzy intersection graph and prove some basic results. Let X be a set. Define the function $h : \mathcal{FP}(X) \rightarrow [0, 1]$ by for all $\alpha \in \mathcal{FP}(X)$, $h(\alpha) = \vee\{\alpha(x) \mid x \in X\}$. Then $h(\alpha)$ is called the **height** of α .

Definition 2.10.1 Let $\mathcal{F} = \{\alpha_1, \dots, \alpha_n\}$ be a finite family of fuzzy sets defined on a set X and consider \mathcal{F} a crisp vertex set. The **fuzzy intersection graph** of \mathcal{F} is the fuzzy graph $\text{Int}(\mathcal{F}) = (\sigma, \mu)$, where $\sigma : \mathcal{F} \rightarrow [0, 1]$ is defined by $\sigma(\alpha_i) = h(\alpha_i)$ and $\mu : \mathcal{E}_{\mathcal{F}} \rightarrow [0, 1]$ is defined by

$$\mu(\alpha_i \alpha_j) = \begin{cases} h(\alpha_i \wedge \alpha_j) & \text{if } i \neq j \\ 0 & \text{if } i = j, \end{cases}$$

where we recall that $\mathcal{E}_{\mathcal{F}} = \{[(\alpha_i, \alpha_j)] \mid (\alpha_i, \alpha_j) \in \mathcal{F} \times \mathcal{F}\}$.

The purpose of requiring $\mu(\alpha_i, \alpha_j) = 0$ for $i = j$, is to preclude loops. We note that an edge $\alpha_i \alpha_j$ has zero strength if and only if $\alpha_i \wedge \alpha_j$ is the zero function or $i = j$.

Let \mathcal{F} be a family of sets and $c \in [0, 1]$. Define $\mathcal{F}^c = \{\alpha^c \mid \alpha \in \mathcal{F}\}$, where α^c is the c -level set of α . Let $G = (\sigma, \mu)$ be a fuzzy graph. Let G^c denote (σ_c, μ_c) .

If $\mathcal{F} = \{\alpha_1, \dots, \alpha_n\}$ is a family of fuzzy sets and $c \in [0, 1]$, then $\text{Int}(\mathcal{F}^c) = (\text{Int}(\mathcal{F}))^c$. The graph $\text{Int}(\mathcal{F}^c)$ has a vertex representing $\alpha_i \in \mathcal{F}$ if and only if $h(\alpha_i) > c$. The pair $\{(\alpha_i)^c, (\alpha_j)^c\}$ is an edge of $\text{Int}(\mathcal{F}^c)$ if and only if $i \neq j$ and $h(\alpha_i \wedge \alpha_j) \geq c$. These conditions also characterize the graph $(\text{Int}(\mathcal{F}))^c$. In particular, if \mathcal{F} is a family of crisp subsets of X , then the fuzzy intersection graph and crisp intersection graph definitions coincide.

Theorem 2.10.2 [61] (Fuzzy analog of Marczewski's theorem [105]) *If $G = (\sigma, \mu)$ is a fuzzy graph without loops, then for some family of fuzzy sets \mathcal{F} , $G = \text{Int}(\mathcal{F}^c)$.*

Proof Let $G = (\sigma, \mu)$ be a fuzzy graph on V . For each $x \in V$ define the anti-reflexive, symmetric fuzzy subset $\alpha_x : \mathcal{E}_V \rightarrow [0, 1]$ by for all $y, z \in V$,

$$\alpha_x(yz) = \begin{cases} \sigma(x) & \text{if } y = x \text{ and } z = x \\ \mu(xz) & \text{if } y = x \text{ and } z \neq x \\ \mu(yx) & \text{if } y \neq x \text{ and } z = x \\ 0 & \text{if } y \neq x \text{ and } z \neq x. \end{cases}$$

We show that G is the fuzzy intersection graph of $F = \{\alpha_x \mid x \in V\}$. By definition, $\alpha_x(x, x) = \sigma(x) \geq \mu(xy)$ and so $h(\alpha_x) = \sigma(x)$ as required. For $x \neq y$ a nonzero value of $(\alpha_x \cap \alpha_y)(zw) = \alpha_x(zw) \wedge \alpha_y(zw)$ occurs only if $x = z$ and $y = w$ (or $y = z$ and $x = w$). Thus, $h(\alpha_x \cap \alpha_y) = (\alpha_x \cap \alpha_y)(xy) = \mu(xy)$ and the desired result holds. ■

2.10.2 Fuzzy Interval Graphs

The families of sets most often considered in connection with intersection graphs are families of intervals of a linearly ordered set. This class of interval graphs is central to many applications. In this section, we define fuzzy interval graphs and examine some of their basic properties.

In both the crisp and fuzzy cases, distinct families of sets can have the same intersection graph. In particular, the intersection properties of a finite family of real intervals (fuzzy numbers) can be characterized by a family of intervals (fuzzy intervals) defined on a finite set. Therefore, as is common in interval graph theory [120], we restrict our attention to intervals (fuzzy intervals) with finite support.

We generalize two characterizations of (crisp) interval graphs. Theorem 2.10.8 gives the Fulkerson and Gross characterization [74] and Theorem 2.10.16 provides the Gilmore and Hoffman characterization [78]. Both theorems make use of relationships between the finite number of points which define the intervals and the cliques of the corresponding interval graph.

Recall a clique is a maximal (with respect to set inclusion) complete subgraph. We adopt the convention of naming a clique by its vertex set. Clearly, if a vertex z is not a member of a clique K , then there exists an $x \in K$ such that xz is not an edge of G . We generalize this concept in Definition 2.10.7.

Definition 2.10.3 Let X be a linearly ordered set. A **fuzzy interval** \mathcal{I} on X is a normal, convex fuzzy subset of X . That is, there exists an $x \in X$ with $\mathcal{I}(x) = 1$ and the ordering $w \leq y \leq z$ implies that $\mathcal{I}(y) \geq \mathcal{I}(w) \wedge \mathcal{I}(z)$. A **fuzzy number** is a real fuzzy interval. A **fuzzy interval graph** is the fuzzy intersection graph of a finite family of fuzzy intervals.

By normality of the fuzzy intervals, the vertex set of a fuzzy interval graph is crisp.

Theorem 2.10.4 ([61]) *Let $G = \text{Int}(\mathcal{F})$ be a fuzzy interval graph. Then for all $c \in (0, 1]$, the level graph G^c is an interval graph.*

Proof Let $G = \text{Int}(\mathcal{F})$ for a family of fuzzy intervals $\mathcal{F} = \{\alpha_1, \dots, \alpha_n\}$. For all $c \in (0, 1]$, convexity implies that each $(\alpha_i)^c \in F^c$ is a crisp interval. Now, $G = (\text{Int}(\mathcal{F}))^c = \text{Int}(\mathcal{F}^c)$ and G^c is an interval graph. ■

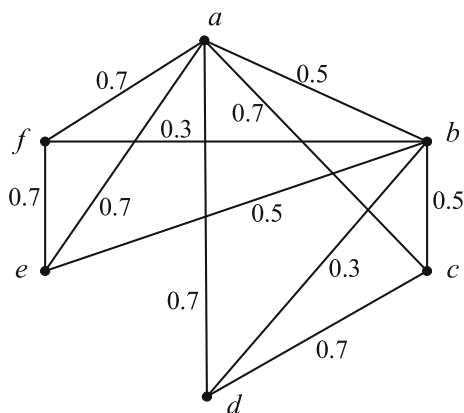


Fig. 2.37 A fuzzy graph which is not a fuzzy interval graph

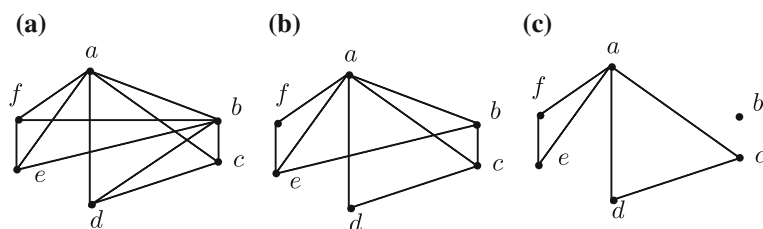


Fig. 2.38 $a G^{0.3}$, $b G^{0.5}$, $c G^{0.7}$

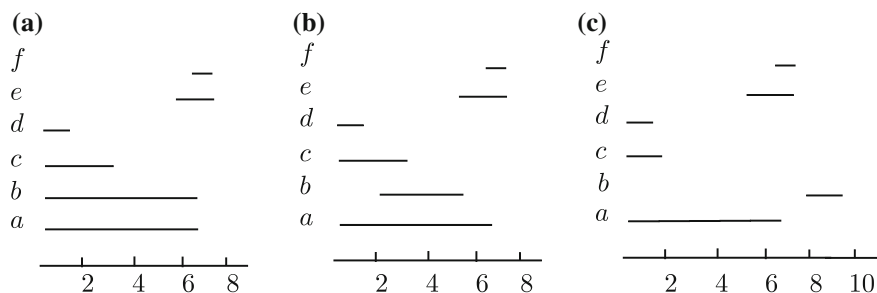


Fig. 2.39 Interval representation of the fuzzy graph in Fig. 2.37

Example 2.10.5 The converse of the above result is not true. Consider the fuzzy graph given in Fig. 2.37.

The level graphs of G are given in Fig. 2.38 and its interval representation in Fig. 2.39. G is not a fuzzy interval graph.

Consider $G^{0.7}$ in Fig. 2.38c. It has an interval representation. Let $S_a = \{a\} \cup \{af, ae, ad, ac\}$, $S_b = \{b\}$, $S_c = \{c\} \cup \{ac, cd\}$, $S_d = \{d\} \cup \{cd, ad\}$, $S_e = \{e\} \cup \{ef, ae\}$ and $S_f = \{f\} \cup \{af, ef\}$. Let $\{S_a, S_b, S_c, S_d, S_e, S_f\}$ be the vertex set.

We can see that the intervals in Fig. 2.39c gives an approximate representation. Similarly, Fig. 2.39b is an approximate representation of $G^{0.5}$ and Fig. 2.39a is that of $G^{0.3}$. Suppose that $G = \text{Int}(\mathcal{F})$, where the fuzzy interval $v \in \text{Int}(\mathcal{F})$ corresponds to vertex v of G . Because $h(c \cap e) = 0$, we can assume that $\text{Supp}(c)$ lies strictly to the left of $\text{Supp}(e)$. By Interval Graph Theorem, there exists x_1 such that $x_1 \in a^{0.7} \cap c^{0.7} \cap d^{0.7}$ because $\{a, c, d\}$ defines a clique of G^1 . Therefore, $a(x_1) \wedge c(x_1) \wedge d(x_1) \geq 0.7$. Similarly, there exists an x_5 such that $a(x_5) \wedge e(x_5) \wedge f(x_5) \geq 0.7$. Now, $h(b \cap d) = 0.3$ and $h(b \cap f) = 0.3$ implies $b(x_1) \leq 0.3$ and $b(x_5) \leq 0.3$, respectively.

Continuing $h(b \cap c) = 0.5$ and $h(b \cap e) = 0.5$ imply there exist x_2 and x_4 with $b(x_2) \geq 0.5$ and $b(x_4) \geq 0.5$. By the normality of b there exists x_3 such that $b(x_3) = 1$. By the convexity of the fuzzy intervals and the assumption that $\text{Supp}(c)$ lies strictly to the left of $\text{Supp}(e)$, the ordering of these points must be $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$, with $x_2 < x_4$.

Because $a(x_1) \geq 0.7$, $a(x_5) \geq 0.7$ and a is convex, it follows that $a(x_3) \geq 0.7$. Hence, $h(a \cap b) \geq 0.7$. This contradicts $h(a \cap b) = 0.6$. Hence, G is not a fuzzy interval graph.

2.10.3 The Fulkerson and Gross Characterization

The Fulkerson and Gross characterization makes use of a correspondence between the set of points on which the family of intervals is defined and the set of cliques of the corresponding interval graph. We provide natural generalizations of the (crisp) definitions and then show that for fuzzy graphs this relationship holds only in one direction.

The proof of the Fulkerson and Gross theorem rests on the following ideas for a crisp graph G^* . Suppose G^* is an interval graph. Any set of intervals defining a clique will have a common point. If one such point is associated with each clique, the linear ordering of these points induces a linear ordering on the cliques of G^* . Using this ordering the resulting vertex clique incidence matrix has convex rows.

Suppose there exists a linear ordering of the cliques of G^* for which the vertex clique incidence matrix has convex rows. Then each convex row naturally defines the characteristic function of a subinterval of the linearly ordered set of cliques. The graph G^* is the intersection graph of this family of intervals.

Theorem 2.10.6 [74] (Fulkerson and Gross) *A graph G is an interval graph if and only if there exists a linear ordering of the cliques of G for which the vertex clique incidence matrix has convex rows.*

Definition 2.10.7 Let $G = (\sigma, \mu)$ be a fuzzy graph. We say that a fuzzy subgraph \mathcal{K} defines a **fuzzy clique** of G if for each $t \in (0, 1]$, \mathcal{K}^t induces a clique of G^t . We associate with G a **vertex clique incidence matrix**, where the rows are indexed by the domain of σ , the column are indexed by the family of all cliques of G , and the x, \mathcal{K} entry is $\mathcal{K}(x)$.

Suppose that G is a fuzzy graph with $\text{fs}(G) = \{r_1, \dots, r_n\}$ and let \mathcal{K} be a fuzzy clique of G . The level sets of \mathcal{K} define a sequence $\mathcal{K}^{r_1} \subseteq \dots \subseteq \mathcal{K}^{r_n}$, where each \mathcal{K}^{r_i} is a clique of G^{r_i} . Conversely, any sequence $K_1 \subseteq \dots \subseteq K_n$, where each K_i is a clique of G^{r_i} defines a fuzzy clique \mathcal{K} , where $\mathcal{K}(x) = \vee \{r_i \mid x \in K_i\}$. Therefore, K is a clique of the t -level graph G^t if and only if $K = \mathcal{K}^t$ for some fuzzy clique \mathcal{K} .

Theorem 2.10.8 [61] (Fuzzy analog of Fulkerson and Gross) *Let $G = (V, \mu)$ be a fuzzy graph. Then the row of any vertex clique incidence matrix of G defines a family of fuzzy subsets \mathcal{F} for which $G = \text{Int}(\mathcal{F})$. Further, if there exists an ordering of the fuzzy cliques of G such that each row of the vertex clique incidence matrix is convex, then G is a fuzzy interval graph.*

Proof Let $I = \{\mathcal{K}_1, \dots, \mathcal{K}_p\}$ be an ordered family of the fuzzy cliques of G and let M be the vertex clique incidence matrix where the columns are given in this ordering. For each $x \in V$, define the fuzzy subset $\mathcal{I}_x : I \rightarrow [0, 1]$ by $\mathcal{I}_x(\mathcal{K}_i) = \mathcal{K}_i(x)$ and let $\mathcal{F} = \{\mathcal{I}_x \mid x \in V\}$. Because each vertex x has strength 1, x is contained in the 1-level cut of some fuzzy clique \mathcal{K}_i in I . Therefore, $\mathcal{I}_x(\mathcal{K}_i) = \mathcal{K}_i(x) = 1$ and \mathcal{I}_x is normal.

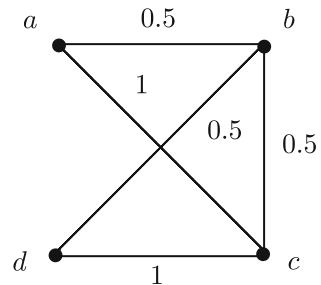
We must now show for $x \neq y \in V$ that $h(\mathcal{I}_x \cap \mathcal{I}_y) = \mu(xy)$. Also, assuming that each row is convex implies that each \mathcal{I}_x is a fuzzy interval and that G is a fuzzy interval graph. By definition, if $x \neq y$, then

$$\begin{aligned} h(\mathcal{I}_x \cap \mathcal{I}_y) &= \vee \{(\mathcal{I}_x \cap \mathcal{I}_y)(\mathcal{K}_i) \mid \mathcal{K}_i \in I\} h(\mathcal{I}_x \cap \mathcal{I}_y) \\ &= \vee \{(\mathcal{I}_x \cap \mathcal{I}_y)(\mathcal{K}_i) \mid \mathcal{K}_i \in I\} \\ &= \vee \{\mathcal{K}_i(x) \wedge \mathcal{K}_i(y) \mid \mathcal{K}_i \in I\} = \vee \{t \in [0, 1] \mid \mathcal{K}_i \in I \\ &\quad \text{and } xy \text{ is an edge of } (\mathcal{K}_i)^t\}. \end{aligned}$$

The edge strength $\mu(xy) = t$ is the maximal value where xy is an edge of G^t so is contained in a clique of G^t . Thus, $h(\mathcal{I}_x \cap \mathcal{I}_y) = \mu(xy)$ as required. ■

Example 2.10.9 The fuzzy graph G given in Fig. 2.40 shows that the converse of Theorem 2.10.8 does not hold.

Fig. 2.40 A fuzzy interval graph



Let the set \mathcal{F} of fuzzy intervals be defined by the rows of the matrix F given by

$$F = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} I_a \\ I_b \\ I_c \\ I_d \end{matrix} & \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0.5 & 1 \\ 1 & 1 & 1 & 0.5 \\ 0 & 0 & 1 & 0.5 \end{bmatrix} \end{matrix}$$

Then $G = \text{Int}(\mathcal{F})$. A vertex clique incidence matrix M is given below.

$$M = \begin{matrix} & \begin{matrix} \mathcal{K}_1 & \mathcal{K}_2 & \mathcal{K}_3 & \mathcal{K}_4 \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 1 \\ 1 & 0.5 & 1 & 0.5 \\ 0 & 0 & 1 & 0.5 \end{bmatrix} \end{matrix}$$

We can verify by exhaustion that no ordering of the fuzzy cliques produces a vertex clique incidence matrix M with convex rows.

2.10.4 The Gilmore and Hoffman Characterization

We begin with several graph theory definitions and state the Gilmore and Hoffman characterization. We then give corresponding fuzzy definitions, and conclude with the result that the Gilmore and Hoffman characterization generalizes exactly for fuzzy interval graphs.

Let $G = (X, E)$ be a connected graph. Recall that a **chord** of a spanning tree T is an edge of G which is not in T , and recall that a cycle of length n in $G = (X, E)$ is a sequence x_0, \dots, x_n of distinct vertices, where $x_0 x_n \in E$ and $1 \leq i \leq n$ implies $x_{i-1} x_i \in E$. A graph is **chordal** (**triangulated**) if each cycle with $n > 4$ has a chord. Formally, if there exist integers $j \neq 0$ or $k \neq n$ with $0 \leq j < k - 1 \leq n$ and $x_j x_k \in E$.

An **orientation** of a graph $G = (X, E)$ is a directed graph $G_A = (X, A)$ that has G as its underlying graph. We use the notation xy for an edge of G , and (x, y) for a directed edge of the corresponding orientation. That is, $xy \in E$ implies that $(x, y) \in A$ or $(y, x) \in A$ but not both. A graph G is **transitively orientable** if there exists an orientation of G for which $(u, v) \in A$ and $(v, w) \in A$ implies $(u, w) \in A$.

The proof of the following theorem can be found in [153].

Theorem 2.10.10 [78] (Gilmore and Hoffman) *A graph $G = (V, X)$ is an interval graph if and only if it satisfies the following two conditions.*

- (i) *Each subgraph of G induced by four vertices is chordal,*
- (ii) *G^c is transitively orientable.*

We now show that fuzzy interval graphs are chordal and have transitively orientable compliments.

Definition 2.10.11 A **cycle of length n** in a fuzzy graph is a sequence of distinct vertices x_0, x_1, \dots, x_n such that $\mu(x_0x_n) > 0$ and if $1 \leq i \leq n$, then $\mu(x_{i-1}x_i) > 0$. A fuzzy graph $G = (\sigma, \mu)$ is **chordal** if for each cycle with $n \geq 4$, there exist integers $j \neq 0$ or $k \neq n$ such that $0 \leq j < k - 1 \leq n$ and $\mu(x_jx_k) \geq \wedge\{\mu(x_{i-1}x_i) \mid i = 1, 2, \dots, n\} \wedge \mu(x_0x_n)$.

It is easily shown that a fuzzy graph $G = (\sigma, \mu)$ is chordal if and only if for each $t \in (0, 1]$ the t -level graph of G is chordal.

Theorem 2.10.12 ([61]) *If G is a fuzzy interval graph, then G is chordal.*

Proof By Theorem 2.10.4, each cut level graph G^t is an interval graph. As in the proof of Theorem 2.10.10, any interval graph is chordal. The result then follows from Definition 2.10.11. ■

To avoid confusion when dealing with cut level graphs, we base an orientation of a fuzzy graph on an orientation of its underlying graph.

Definition 2.10.13 Let $G = (\sigma, \mu)$ be a fuzzy graph with $\text{fs}(G) = \{r_1, \dots, r_n\}$ and let A be an orientation of G^n . Then the **orientation** of G by A is the fuzzy digraph $G_A = (\sigma, \mu_A)$, where

$$\mu_A((x, y)) = \begin{cases} \mu(xy) & \text{if } (x, y) \in A, \\ 0 & \text{if } (x, y) \notin A. \end{cases}$$

The fuzzy graph G is called **transitively orientable** if there exists an orientation which is transitive, i.e., $\mu_A((x, y)) \wedge \mu_A((y, z)) \leq \mu_A((x, z))$ for all $x, y, z \in V$.

The c level graph of G_A has edge set $\{(x, y) \mid \mu_A((x, y)) \geq c\}$. Therefore, an orientation of a fuzzy graph induces consistent orientations on each member of the fundamental sequence of cut level graphs. Conversely, it is possible to have a sequence of transitively oriented subgraphs $G_1 \subseteq G_2 \subseteq G_3$, where the transitive orientation of G_2 does not induce a transitive orientation of G_1 , and the transitive orientation of G_2 cannot be extended to a transitive orientation of G_3 .

Lemma 2.10.14 *Suppose that $G = \text{Int}(F)$ is a fuzzy interval graph. Then there exists an orientation A that induces a transitive orientation of G^c .*

Proof Suppose (α, β) is a nontrivial edge of G^c . Then $h(\alpha \cap \beta) < r_1 = 1$ and α^{r_1} and β^{r_1} are disjoint. We let $(\alpha, \beta) \in A$ if and only if α^{r_1} lies strictly to the left of β^{r_1} . Clearly A is a well-defined and transitive orientation of G^c . ■

Example 2.10.15 The fuzzy graph in Example 2.10.5 (Fig. 2.37) is not a fuzzy interval graph because any orientation of (d, e) shows that there is no transitive orientation of G^c . Figure 2.41 shows the cut level graphs of G^c with $(d, e) \in A$. Note that $(G^1)^c = (G^c)^{1-r_2}$, $(G^2)^c = (G^c)^{1-r_3}$ and $(G^3)^c = (G^c)^1$ where $r_1 = 1$ here.

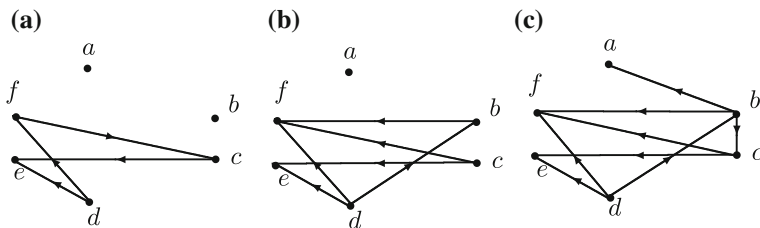


Fig. 2.41 a $(G^{r^3})^c$. b $(G^{r^2})^c$. c $(G^{r^1})^c$

Theorem 2.10.16 [61] (Fuzzy analog of Gilmore and Hoffman characterization) *A fuzzy graph $G = (\sigma, \mu)$ is a fuzzy interval graph if and only if the following conditions hold:*

- (i) *For all $x \in \text{Supp}(\sigma) = V$, $\sigma(x) = 1$ (σ is a crisp set);*
- (ii) *Each fuzzy subgraph of G induced by four vertices is chordal;*
- (iii) *G^c is transitively orientable.*

If G is a fuzzy interval graph, the three conditions follow from Definitions 2.10.1, 2.10.3, Theorem 2.10.12 and Lemma 2.10.14, respectively.

The following discussion is from [61]. For the remainder of the section, we assume that each fuzzy subgraph of $G = (V, \mu)$ induced by four vertices is chordal and that A is a transitive orientation of G^c . Because the proof that G is a fuzzy interval graph is quite involved we first outline the proof. Details are given in Definition 2.10.17 through Lemma 2.10.23; the algorithm is applied in Example 2.10.24. For notational convenience we let \mathcal{K}_{ij} denote the r_j cut level set of the fuzzy set \mathcal{K}_i .

Definition 2.10.17 Define the relation $<$ on the family of all fuzzy cliques of G as follows. Suppose $\mathcal{K} < \mathcal{L}$ if and only if $K^t <_t L^t$, where t is the smallest element of $fs(G)$ such that $K^t \neq L^t$.

The lexicographic ordering $<$ in the previous definition is clearly well defined, complete, and transitive. Therefore, $<$ defines a linear ordering on the family of all fuzzy cliques of G .

Definition 2.10.18 Let G satisfy the conditions of Theorem 2.10.16 and let $<$ be the relation of Definition 2.10.17. Let $t \in fs(G)$ and let $\mathcal{K} \neq \mathcal{L}$ be fuzzy cliques of G . We say \mathcal{K} and \mathcal{L} are **consistently ordered** by $<$ at level t provided $\mathcal{K}^t <_t \mathcal{L}^t$ if and only if $\mathcal{K} < \mathcal{L}$. We say the linear ordering $<$ is **cut level consistent** if for each pair of distinct fuzzy cliques of G and for each $t \in fs(G)$ the pair is consistently ordered by $<$ at level t .

By Theorem 2.10.8, the rows of any vertex clique incidence matrix of G define a family of fuzzy subsets that has G as its fuzzy intersection graph. If $<$ of Definition 2.10.17 is cut level consistent, then the rows of the vertex clique incidence matrix will be convex and the result follows from Theorem 2.10.8.

If $<$ is not level consistent, then some row is not convex. We modify this matrix in a “bottom up” construction using the notion of cut level consistent to determine which columns are modified or deleted from the vertex clique incidence matrix. We complete the proof by showing that in the modified matrix each row is normal and convex and that G is the fuzzy intersection graph of the family of fuzzy intervals defined by the rows.

By the discussion following Definitions 2.10.11 and 2.10.13 each level graph G^t is chordal and has a transitively orientable complement. Thus, each G^t is an interval graph and there exists a linear ordering $<_t$ on the family of all cliques of G^t . We now establish definitions that will be used extensively in the discussion to follow.

If the linear ordering $<$ is cut level consistent then each row of the vertex clique incidence matrix is convex (and G is a fuzzy interval graph by Theorem 2.10.8). We prove this statement by contrapositive. Assume there exists a row that is not convex. Suppose that there exists a vertex $x \in V$ and a sequence of fuzzy cliques $\mathcal{K} < \mathcal{L} < \mathcal{M}$ such that $\mathcal{L}(x) < \mathcal{K}(x) \wedge \mathcal{M}(x) = t$. Then $x \in \mathcal{K}^t$, $x \notin \mathcal{L}^t$ and $x \in \mathcal{M}^t$. As in Theorem 2.10.10 there exists $y \in \mathcal{L}^t$ such that $(x, y) \notin \mathcal{E}^t$. If $(x, y) \in A$, then $\mathcal{M}^t <_t \mathcal{L}^t$ with $\mathcal{L} < \mathcal{M}$. If $(y, x) \in A$ then $\mathcal{L}^t <_t \mathcal{K}^t$ with $\mathcal{K} < \mathcal{L}$. In either case the ordering $<$ is not cut level consistent.

By Example 2.10.9, there exist fuzzy interval graphs where no ordering of the fuzzy cliques is cut level consistent. We formalize a process that modifies or deletes “inconsistent” fuzzy cliques (matrix columns). The proof of the following lemma shows the “local” structure of noncut level consistent orderings. The lemma is also used to show the construction in Definition 2.10.20 is well defined.

Lemma 2.10.19 *Suppose that \mathcal{K} and \mathcal{L} are fuzzy cliques of G and that $s > t$. If $\mathcal{K}^s <_s \mathcal{L}^s$ and $\mathcal{L}^t <_t \mathcal{K}^t$, then there exists a clique M of G^t such that either*

- (i) $\mathcal{K}^s \subseteq M$ and $M <_t \mathcal{K}^t$ or
- (ii) $\mathcal{L}^s \subseteq M$ and $\mathcal{L}^t <_t M$.

Proof We check all possible edge configurations. Recall the edge set of the graph G^s is denoted by \mathcal{E}^s . Each case shares the general conditions shown in Fig. 2.42. By definition of $<_t$, there exist $x \in \mathcal{K}^t$ and $y \in \mathcal{L}^t$, with $xy \notin \mathcal{E}^t$ and $(x, y) \in A$. Similarly, there exist $x' \in \mathcal{K}^s$ and $y' \in \mathcal{L}^s$ with $x'y' \notin \mathcal{E}^s$ and $(y', x') \in A$. Then

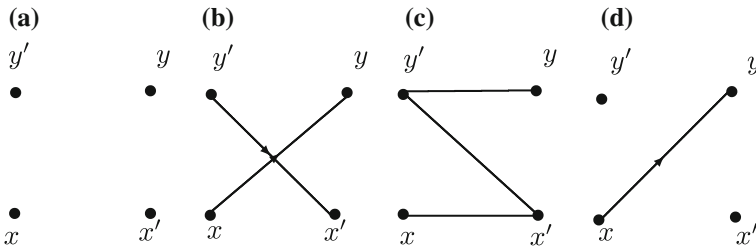


Fig. 2.42 Basic conditions for inconsistent cut level orderings

$s > t$ implies $xy \notin \mathcal{E}^s$, $xx' \in \mathcal{E}^s$ (or $x = x'$) and $yy' \in \mathcal{E}^t$ (or $y = y'$). Because $<_t$ is well defined, $x'y' \in \mathcal{E}^t$ and either $xx' \notin \mathcal{E}^s$ or $yy' \notin \mathcal{E}^s$.

If $y'x \notin \mathcal{E}^t$, then $y'x \notin \mathcal{E}^s$ and transitivity requires $(y', x) \in A$ and $xx' \notin \mathcal{E}^s$ (so $x \notin \mathcal{K}^s$). We claim for each $x'' \in \mathcal{K}^s \subseteq \mathcal{K}^t$ that $y'x'' \in \mathcal{E}^t$. For $(y', x'') \notin \mathcal{E}^t$ with $\mathcal{L}^t <_t \mathcal{K}^t$ implies $(y', x'') \in A$. However, $\mathcal{E}^s \subseteq \mathcal{E}^t$ and $\mathcal{K}^s <_s \mathcal{L}^s$ imply $(x'', y') \in A$; a contradiction. Therefore, $\{y'\} \cup \mathcal{K}^s$ is a complete subgraph of G^t and is contained in a clique M of G^t . Because $x \notin M$ and $(y', x) \in A$, we have $M <_t \mathcal{K}^t$. Thus, property (i) holds.

Similarly, if $yx' \notin \mathcal{E}^t$, we have that $(y, x') \in A$ and $yx' \notin \mathcal{E}^s$. By transitivity, $\{x'\} \cup \mathcal{L}^s$ is a complete subgraph of G^t and hence is contained in a clique M of G^t . Then $(y, x') \in A$ implies $\mathcal{L}^t <_t M$ and property (ii) holds.

If $y'x \in \mathcal{E}^t$ and $yx' \in \mathcal{E}^t$, then we show that $\mathcal{K}^s \cup \mathcal{L}^s$ is a complete subgraph of G^t . We need only to show for each $x'' \in \mathcal{K}^s$ and $y'' \in \mathcal{L}^s$ that $y''x'' \in \mathcal{E}^t$. Again $x''y'' \notin \mathcal{E}^t$ and $\mathcal{L}^t <_t \mathcal{K}^t$ implies $(y'', x'') \in A$ and $y''x'' \notin \mathcal{E}^s$. However, $\mathcal{K}^t <_s \mathcal{L}^s$ implies $(x'', y'') \in A$; a contradiction.

Therefore, $\mathcal{K}^s \cup \mathcal{L}^s$ induces a complete subgraph of G^t that is contained in some clique M of G^t . If $M <_t \mathcal{L}^t <_t \mathcal{K}^t$, property (i) holds. If $\mathcal{L}^t <_t M$ property (ii) holds. ■

We now construct a directed graph F and in turn a linearly ordered family of fuzzy subsets that define columns of an incidence matrix. These fuzzy subsets will either be fuzzy cliques of G or modifications of fuzzy cliques. The graph theory analogy of a forest with trees allows a good visualization of “vertically growing” cut level sets which define the required fuzzy sets.

We use the fuzzy clique ordering $<$ to recursively construct a forest F whose vertex set is the set of all cut level cliques of G and whose edges connect cut levels of fuzzy sets. We recursively build the forest by “vertically” adding cut level cliques as vertices of F and defining a set of edges between cut levels. In the recursion let i range from 1 to $n - 1$.

Definition 2.10.20 ([61]) Let $G = (V, \mu)$ with $fs(G) = \{r_1, r_2, \dots, r_n\}$ be a chordal fuzzy graph and let G^c be transitively oriented by A .

Level r_n : Linearly order the set of all cliques of G^{r_n} by the relation $<_{r_n}$ of Definition 2.10.18. Each of these cliques of G^{r_n} (vertices of F) represent the root of a tree in the forest.

Level r_{n-i} : Let $s = r_{n-i}$ and $t = r_{n-i+1}$. Linearly order the set of all cliques of G^s by the relation $<_s$. Let X^s be any set of edges that satisfy:

1. Each clique \mathcal{K}^s of G^s is a vertex of exactly one edge of X^s .
2. If $(\mathcal{K}^t, \mathcal{K}^s) \in X^s$ then \mathcal{K}^t is a clique of G^t , \mathcal{K}^s is a clique of G^s , and $\mathcal{K}^s \subseteq \mathcal{K}^t$.
Thus, an edge joins two level sets of (some) fuzzy clique.
3. For each pair of edges $(\mathcal{K}^t, \mathcal{K}^s) \in X^s$ and $(\mathcal{L}^t, \mathcal{L}^s) \in X^s$ we have $\mathcal{K}^s <_s \mathcal{L}^s$ if and only if $\mathcal{K}^t <_t \mathcal{L}^t$ or $\mathcal{K}^t = \mathcal{L}^t$.

Thus, when viewed as cut levels of a family of fuzzy cliques, the s level ordering is level consistent with the next “lower” level.

We continue with the discussion in [61]. We use Lemma 2.10.19 to demonstrate the existence of at least one such forest, and show in the last paragraph of this section that there may be a number of edge sets that satisfy these conditions. Let \mathcal{K}^s be the minimal (with respect to $<_s$) clique of G^s . Clearly there exists a minimal (with respect to $<_t$) clique \mathcal{K}^t of G^t where $\mathcal{K}^s \subseteq \mathcal{K}^t$. Let $< \mathcal{K}^t, \mathcal{K}^s > \in X^s$.

Next let \mathcal{L}^s be the successor of \mathcal{K}^s (with respect to $<_s$) and let \mathcal{L}^t be minimal (with respect to $<_t$) such that $\mathcal{L}^s \subseteq \mathcal{L}^t$ and $\mathcal{K}^t <_t \mathcal{L}^t$ or $\mathcal{K}^t = \mathcal{L}^t$. If \mathcal{L}^t does not exist, let L be maximal (with respect to $<_t$) with $\mathcal{L}^s \subseteq L$. Now, $\mathcal{K}^s <_s \mathcal{L}^s$ and $L <_t \mathcal{K}^t$ are the conditions of Lemma 2.10.19. However, property (1) contradicts the minimality of \mathcal{K}^t and property (2) contradicts the maximality of L . Therefore, \mathcal{L}^t exists and $(\mathcal{L}^t, \mathcal{L}^s) \in X^s$ is well defined.

We continue recursively to construct one edge for each clique of G^s . It may be that for some clique M_t of G^t , there is no edge from M_t . We call such a clique a dead branch of F .

Combining the edge sets $F^{r_{n-i}}$ for $i \in \{1, \dots, r-1\}$ defines a forest with edge set $\bigcup_{i=1}^{n-1} F^{r_{n-i}}$. As in Definition 2.10.18, we lexicographically order the set of paths from a root to a dead branch or a r_1 level clique. For notational convenience, we denote the t level vertex of path P_j by P_{jt} . To ensure convex rows in our (still undefined) incidence matrix, we add nonempty vertices “above” dead branches if “adjacent” cliques have nonempty intersection.

Suppose the path P_j ends with a dead branch at the t level. For each $s \in fs(G)$ with $s > t$, we continue the path P_j through the new vertex P_{js} , where $x \in P_{js}$ if and only if there exist $i < j < k$ with $x \in P_{is} \cap P_{ks}$. We call this final forest F . Each path in F has length n , and it is possible for a vertex P_{js} to be the empty set.

We complete the construction by letting paths in F define a linearly ordered family of fuzzy sets, say I . The fuzzy sets define columns of the *vertex forest matrix* of $(G, <)$; G is the interval graph of the family of rows.

Definition 2.10.21 Let G satisfy the conditions of Theorem 2.10.16. F be a forest for G as defined in Definition 2.10.20, and P_j be a path in F of length n . Associated with P_j define the fuzzy set $\mu_j \in I$ on the vertex set of G by $\mu_j(x) = \vee \{s \in fs(G) \mid x \text{ is an element of the } s \text{ level vertex of } P_j\}$.

We construct the *vertex forest matrix* of $(G, <)$ indexing rows by the vertex set of G , columns by the (ordered) fuzzy sets of I and defining the x, μ_i entry as $\mu_i(x)$. By construction each μ_j is either a fuzzy clique of G , or has a cut level set that is the intersection of two cut level cliques.

Let \mathcal{F} denote the family of fuzzy sets defined by the rows of the vertex forest matrix. We now complete the proof of Theorem 2.10.16 by showing that each member of \mathcal{F} is normal and convex (a fuzzy interval) and that $G = \text{Int } \mathcal{F}$.

Lemma 2.10.22 We assume the conditions and notation above. For each vertex x of G , define \mathcal{J}_x is a fuzzy interval.

Proof Let x be a vertex of G . Then x is a vertex in some clique of G^{r_1} , say K . By Definitions 2.10.20 and 2.10.21, K is the $r_1 = 1$ level cut of some fuzzy set μ in I . Therefore, $\mathcal{J}_x(\mu) = \mu(x) = 1$ and \mathcal{J}_x is normal.

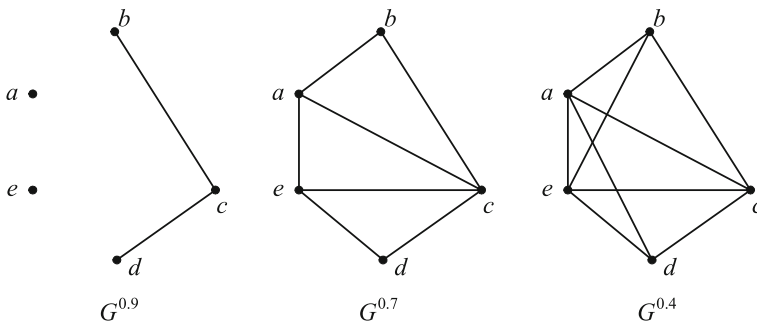


Fig. 2.43 Level graphs $G^{0.9}$, $G^{0.7}$, $G^{0.4}$

Each \mathcal{J}_x is convex if $i < j < k$ implies $\mathcal{J}_x(\mu_i) \wedge \mathcal{J}_x(\mu_k) \leq \mathcal{J}_x(\mu_j)$, or equivalently, if $\mu_i(x) \wedge \mu_k(x) \leq \mu_j(x)$. However, Definition 2.10.20 clearly provides these conditions. If μ_i, μ_j and μ_k are all fuzzy cliques, the result follows immediately from the discussion after Definition 2.10.18. Otherwise, the result follows by definition of the fuzzy sets μ_i, μ_j and μ_k . ■

Now, we conclude proof of Theorem 2.10.16, by next lemma.

Lemma 2.10.23 *Given the definitions and conditions of Theorem 2.10.16 through Lemma 2.10.22, $\mathcal{G} = \text{Int}(\mathcal{F})$.*

Proof There is a correspondence between the crisp vertex set V and the family of fuzzy intervals \mathcal{F} . Let x, y be distinct elements of V . We must show that $\mu(xy) = h(\mathcal{J}_x \cap \mathcal{J}_y)$. By definition, $h(\mathcal{J}_x \cap \mathcal{J}_y) = \vee\{\mathcal{J}_x(\mu_j) \wedge \mathcal{J}_y(\mu_j) \mid \mu_j \in I\} = \vee\{\mu_j(x) \wedge \mu_j(y) \mid \mu_j \in I\} = \vee\{s \in fs(G) \mid \{x, y\} \subseteq \mu_j^s\}$.

Because $\mu(xy) = t$ is the maximal value where xy is an edge of G^t , $\mu(xy)$ is the maximal value where xy is in a clique of G^t . By definition each clique of G^t is the t level set of some fuzzy set $\mu_j \in I$. Hence, $\mu(xy) = h(\mathcal{J}_x \cap \mathcal{J}_y)$ as required. ■

We provide an illustration for Theorem 2.10.16 in Example 2.10.24

Example 2.10.24 Consider the fuzzy graph G defined by the incidence matrix G below, where $fs(G) = \{s, t, u\} = \{0.9, 0.7, 0.4\}$. Figure 2.43 shows the cut level graphs of G and Fig. 2.44 shows transitive orientation A of G^c .

$$G = \begin{matrix} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 0.7 & 0.7 & 0.4 & 0.7 \\ 0.7 & 0 & 0.9 & 0 & 0.4 \\ 0.7 & 0.9 & 0 & 0.9 & 0.7 \\ 0.4 & 0 & 0.9 & 0.9 & 0.7 \\ 0.7 & 0.4 & 0.7 & 0.7 & 0 \end{bmatrix} \end{matrix}$$

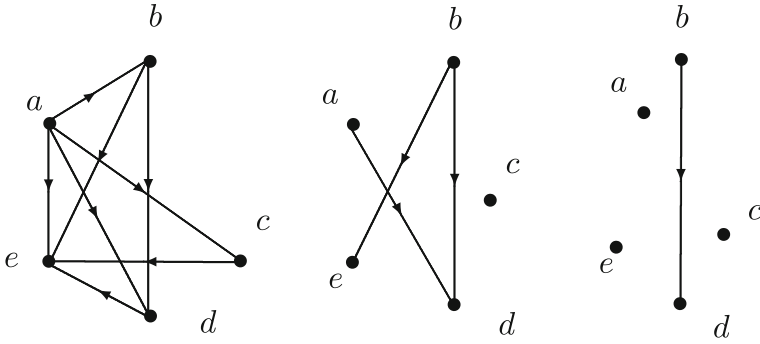


Fig. 2.44 Transitive orientations of the complements

Using Definition 2.10.18, we linearly order the cut level cliques by: $s = 0.9$, $\{a\} <_s \{b, c\} <_s \{c, d\} <_s \{e\}$, $t = 0.7$, $\{a, b, c\} <_t \{a, c, e\} <_t \{c, d, e\}$, $u = 0.4$, $\{a, b, c, e\} <_u \{a, c, d, e\}$.

There are eight fuzzy cliques of G ; subscripts indicate the order induced by Definition 2.10.18. The vertex clique incidence matrix M for G is given below. The only convex row is indexed by d . Thus, the fuzzy clique ordering is not cut level consistent.

$$M = \begin{array}{c} \begin{matrix} & \mathcal{K}_1 & \mathcal{K}_2 & \mathcal{K}_3 & \mathcal{K}_4 & \mathcal{K}_5 & \mathcal{K}_6 & \mathcal{K}_7 & \mathcal{K}_8 \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} \begin{bmatrix} 0.9 & 0.7 & 0.9 & 0.7 & 0.9 & 0.7 & 0.4 & 0.4 \\ 0.7 & 0.9 & 0.4 & 0.4 & 0 & 0 & 0 & 0 \\ 0.7 & 0.9 & 0.7 & 0.7 & 0.7 & 0.7 & 0.9 & 0.7 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.9 & 0.7 \\ 0.4 & 0.4 & 0.7 & 0.9 & 0.7 & 0.9 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

Following Definition 2.10.20, gives the forest F of Fig. 2.45 with the incidence matrix V , given below.

$$V = \begin{array}{c} \begin{matrix} & P_1 & P_2 & P_3 & P_4 & P_5 \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} \begin{bmatrix} 0.9 & 0.7 & 0.7 & 0.4 & 0.4 \\ 0.7 & 0.9 & 0.4 & 0 & 0 \\ 0.7 & 0.9 & 0.9 & 0.9 & 0.7 \\ 0 & 0 & 0 & 0.9 & 0.7 \\ 0.4 & 0.4 & 0.7 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

The paths P_1 , P_2 , P_3 , P_4 and P_5 correspond, respectively, to the fuzzy cliques \mathcal{K}_1 , \mathcal{K}_2 , \mathcal{K}_4 , \mathcal{K}_7 and \mathcal{K}_8 . The clique $\{a, c, e\}$ is a dead branch; so $P_{3s} = P_{2s} \cap P_{4s} = \{c\}$. The path P_3 is a modification of \mathcal{K}_4 , the fuzzy cliques \mathcal{K}_3 , \mathcal{K}_5 , and \mathcal{K}_6 are deleted.

The interval representation of a fuzzy graph G is not in general unique. The construction heavily depends on the orientation of G^c . Different orientations can give different vertex interval matrices. Slight modifications in Definition 2.10.20 can

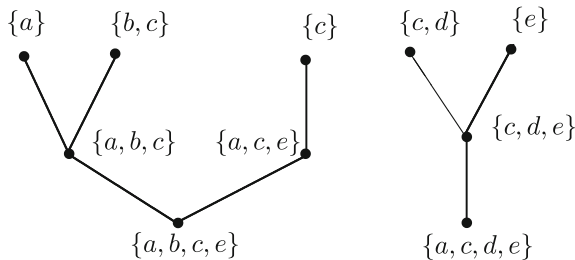


Fig. 2.45 A fuzzy interval representation for Example 2.10.24

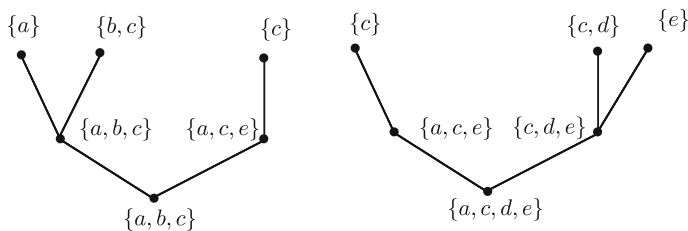


Fig. 2.46 Alternate fuzzy interval representation for Example 2.10.24

produce different vertex interval matrices. A left right construction was followed in the example given, while a right to left also will work nicely. Also, in the example, it is specified that each cut level clique will be the terminal vertex of only one edge. One can relax this condition as long as cut level consistency is maintained. Figure 2.46 gives an alternate interval representation for the fuzzy graph given in Example 2.10.24.

2.11 Operations on Fuzzy Graphs

Fuzzy graph operations were first studied in [130] by Mordeson and Peng in 1994. Later Sunitha and Vijayakumar [169] investigated the properties of compliments of fuzzy graphs with respect to these operations in 2002. By a **partial fuzzy subgraph** of a graph $G = (V, X)$, we mean a partial fuzzy subgraph of (χ_V, χ_X) , where χ_V and χ_X denote the characteristic functions of V and X , respectively. Let (σ_i, μ_i) be a partial fuzzy subgraph of the graph $G_i = (V_i, X_i)$, $i = 1, 2$. The operations of Cartesian product, composition, union, and join on (σ_1, μ_1) and (σ_2, μ_2) are given in [130]. If the graph G is formed from G_1 and G_2 by one of the these operations, necessary and sufficient conditions are given in [130] for an arbitrary partial fuzzy subgraph of G to also be formed by the same operation from partial fuzzy subgraphs of G_1 and G_2 . Recall that the Cartesian product $G = G_1 \times G_2$ of graphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ is given by $V = V_1 \times V_2$ and $X = \{(u, u_2)(u, v_2) \mid u \in$

$V_1, u_2 v_2 \in X_2\} \cup \{(u_1, w)(v_1 w) \mid w \in V_2, u_1 v_1 \in X_1\}$. Let σ_i be a fuzzy subset of V_i and μ_i be a fuzzy subset of $X_i, i = 1, 2$. Define the fuzzy subsets $\sigma_1 \times \sigma_2$ of V and $\mu_1 \mu_2$ of X as follows:

For all $(u_1, u_2) \in V, (\sigma_1 \times \sigma_2)(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2),$

For all $u \in V_1, \text{ for all } u_2 v_2 \in X_2, \mu_1 \mu_2((u, u_2)(u, v_2)) = \sigma_1(u) \wedge \mu_2(u_2 v_2),$

For all $w \in V_2, \text{ for all } u_1 v_1 \in X_1, \mu_1 \mu_2((u_1, w)(v_1, w)) = \sigma_2(w) \wedge \mu_1(u_1 v_1).$

Proposition 2.11.1 *Let G be the Cartesian product of graphs G_1 and G_2 . Let (σ_i, μ_i) is a partial fuzzy subgraph of $G_i, i = 1, 2$. Then $(\sigma_1 \times \sigma_2, \mu_1 \mu_2)$ is a partial subgraph of G .*

Proof We have

$$\begin{aligned} \mu_1 \mu_2((u, u_2)(u, u_2)) &= \sigma_1(u) \wedge \sigma_2(u_2 v_2) \leq \sigma_1(u) \wedge (\sigma_2(u_2) \wedge \sigma_2(v_2)) \\ &= (\sigma_1(u) \wedge \sigma_2(u_2)) \wedge (\sigma_1(u) \wedge \sigma_2(u_2)) \\ &= (\sigma_1 \times \sigma_2)(u, u_2) \wedge (\sigma_1 \times \sigma_2)(u, v_2). \end{aligned}$$

Similarly, $\mu_1 \mu_2((u_1, w)(v_1, w)) \leq (\sigma_1 \times \sigma_2)(u_1, w) \wedge (\sigma_1 \times \sigma_2)(v_1, w).$ ■

Theorem 2.11.2 *Suppose that G is a Cartesian product of two graphs G_1 and G_2 . Let (σ, μ) be a partial fuzzy subgraph of G . Then (σ, μ) is a Cartesian product of a partial fuzzy subgraph of G_1 and a partial fuzzy subgraph of G_2 if and only if the following three equations have solutions for x_i, y_j, z_{jk} , and w_{ih} , where $V_1 = \{v_{11}, v_{12}, \dots, v_{1n}\}$ and $V_2 = \{v_{21}, v_{22}, \dots, v_{2m}\}$:*

- (i) $x_i \wedge y_j = \sigma(v_{1i}, v_{2j}), i = 1, \dots, n; j = 1, \dots, m;$
- (ii) $x_i \wedge z_{jk} = \mu((v_{1i}, v_{2j})(v_{1i}, v_{2k})), i = 1, \dots, n; j, k \text{ such that } v_{2j} v_{2k} \in X_2;$
- (iii) $y_j \wedge w_{ik} = \mu((v_{1i}, v_{2j})(v_{1h}, v_{2j})), j = 1, \dots, m; i, h \text{ such that } v_{1i} v_{1h} \in X_1.$

Proof Suppose that a solution exists. Consider an arbitrary, but fixed j, k in equations (ii) and i, h in equations (iii). Let

$$\begin{aligned} \widehat{z}_{jk} &= \vee \{\mu((v_{1i}, v_{2j})(v_{1i}, v_{2k})) \mid i = 1, \dots, n\}, \\ \widehat{w}_{ik} &= \vee \{\mu((v_{1i}, v_{2j})(v_{1h}, v_{2j})) \mid j = 1, \dots, m\}. \end{aligned}$$

Set $J = \{(j, k) \mid j, k \text{ are such that } v_{2j} v_{2k} \in X_2\}$ and $I = \{(i, h) \mid i, h \text{ are such that } v_{1i} v_{1h} \in X_1\}$. Now, if $\{x_1, \dots, x_n\} \cup \{z_{jk} \mid (j, k) \in J\} \cup \{w_{ih} \mid (i, h) \in I\}$ is any solution to (i), (ii), and (iii), then $\{x_1, \dots, x_n\} \cup \{\widehat{z}_{jk} \mid (j, k) \in J\} \cup \{\widehat{w}_{ih} \mid (i, h) \in I\}$ is also a solution and in fact \widehat{z}_{jk} is a smallest possible z_{jk} and \widehat{w}_{ih} is a smallest w_{ih} . Fix such a solution and define the fuzzy subsets $\sigma_1, \sigma_2, \mu_1$, and μ_2 of V_1, V_2, X_1 , and X_2 , respectively, as follows:

$$\begin{aligned} \sigma_1(v_{1i}) &= x_i \text{ for } i = 1, \dots, n; \\ \sigma_2(v_{2j}) &= y_j \text{ for } j = 1, \dots, m; \\ \mu_2(v_{2j} v_{2k}) &= \widehat{z}_{jk} \text{ for } j, k \text{ such that } v_{2j} v_{2k} \in X_2; \\ \mu_1(v_{1i} v_{1h}) &= \widehat{w}_{ih} \text{ for } i, h \text{ such that } v_{1i} v_{1h} \in X_1. \end{aligned}$$

For any fixed j, k ,

$$\begin{aligned}\mu((v_{1i}, v_{2j})(v_{1i}, v_{2k})) &\leq \sigma(v_{1i}, v_{2j}) \wedge \sigma(v_{1i}, v_{2k}) \\ &= (\sigma_1(v_{1i}) \wedge \sigma_2(v_{2j}) \wedge (\sigma_1(v_{1i}) \wedge \sigma_2(v_{2k}))) \\ &\leq \sigma_2(v_{2j}) \wedge \sigma_2(v_{2k}), i = 1, \dots, n.\end{aligned}$$

Thus, $\hat{z}_{jk} = \vee \{\mu((v_{1i}, v_{2j})(v_{1i}, v_{2k})) \mid i = 1, \dots, n\} \leq \sigma_2(v_{2j}) \wedge \sigma_2(v_{2k})$. Hence, $\mu_2(v_{2j}v_{2k}) \leq \sigma_2(v_{2j}) \wedge \sigma_2(v_{2k})$. Thus, (σ_2, μ_2) is a partial fuzzy subgraph of G_2 . Similarly, (σ_1, μ_1) is a partial fuzzy subgraph of G_1 . Clearly, $\sigma = \sigma_1 \times \sigma_2$ and $\mu = \mu_1 \mu_2$.

Conversely, suppose that (σ, μ) is the Cartesian product of partial fuzzy subgraphs G_1 and G_2 . Then solutions to equations (i), (ii), and (iii) exist by definition of Cartesian product. ■

Remark 2.11.3 Consider an arbitrary fixed solution to the equations (i), (ii) and (iii) in the proof of Theorem 2.11.2 (if one exists). Then

- (i) Let $(j, k) \in J$ and let $I' = \{i_{jk} \in I \mid \hat{z}_{jk} = \mu((v_{1i_{jk}}, v_{2j})(v_{1i_{jk}}, v_{2k}))\}$ in Theorem 2.11.2. If $x_{i_{jk}} > \hat{z}_{jk}$ for some $i_{jk} \in I'$, then z_{jk} is unique for these particular x_1, x_2, \dots, x_n and equals \hat{z}_{jk} ; if $x_{i_{jk}} = \hat{z}_{jk}$ for all $i_{jk} \in I'$. Then $\hat{z}_{jk} \leq z_{jk} \leq 1$ for these particular x_1, x_2, \dots, x_n .
- (ii) Let $(i, h) \in I$ and let $J' = \{j_{ih} \in J \mid \hat{w}_{ih} = \mu((v_{1i}, v_{2j_{ih}})(v_{1h}, v_{2j_{ih}}))\}$ in Theorem 2.11.2. If $y_{j_{ih}} > \hat{w}_{ih}$ for some $j_{ih} \in J'$, then w_{ih} is unique for these particular y_1, y_2, \dots, y_m and equals with \hat{w}_{ih} ; if $y_{j_{ih}} = \hat{w}_{ih}$ for all $j_{ih} \in J'$, then $\hat{w}_{ih} \leq w_{ih} \leq 1$ for these particular y_1, y_2, \dots, y_m .

Example 2.11.4 Consider $V_1 = \{v_{11}, v_{12}\}$, $V_2 = \{v_{21}, v_{22}\}$, $X_1 = \{v_{11}v_{12}\}$ and $X_2 = \{v_{21}v_{22}\}$. If $\sigma((v_{11}, v_{21})) = 0.25$, $\sigma((v_{11}, v_{22})) = 0.5$, $\sigma((v_{12}, v_{21})) = 0.1$ and $\sigma((v_{12}, v_{22})) = 0.6$, then (σ, μ) is not a Cartesian product of partial fuzzy subgraphs of G_1 and G_2 for any choice of μ , because equation (i) in Theorem 2.11.2 is inconsistent; $x_1 \wedge y_1 = \sigma((v_{11}, v_{21})) = 0.25$, $x_1 \wedge y_2 = \sigma((v_{11}, v_{22})) = 0.5$, $x_2 \wedge y_1 = \sigma((v_{12}, v_{21})) = 0.1$, $x_2 \wedge y_2 = \sigma((v_{12}, v_{22})) = 0.6$, is impossible.

Note that examples satisfying Theorem 2.11.2(i) can be easily constructed, but either Theorem 2.11.2(ii) or Theorem 2.11.2(iii) may be inconsistent.

We now consider the composition of two fuzzy graphs. Let $G_1[G_2]$ denote the composition of graph $G_1 = (V_1, X_1)$ with graph $G_2 = (V_2, X_2)$. Then $G_1[G_2] = (V_1 \times V_2, X^0)$, where

$$\begin{aligned}X^0 &= \{(u, u_2)(u, v_2) \mid u \in V_1, u_2v_2 \in X_2\} \\ &\quad \cup \{(u_1, w)(v_1, w) \mid w \in V_2, u_1v_1 \in X_1\} \\ &\quad \cup \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in X_1, u_2 \neq v_2\}.\end{aligned}$$

Let σ_i be a fuzzy subset of V_i and μ_i a fuzzy subset of X_i , $i = 1, 2$. Define the fuzzy subsets $\sigma_1 \circ \sigma_2$ and $\mu_1 \circ \mu_2$ of $V_1 \times V_2$ and X^0 , respectively, as follows:

$$(\sigma_1 \circ \sigma_2)(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2) \text{ for all } (u_1, u_2) \in V_1 \times V_2,$$

$$(\mu_1 \circ \mu_2)((u, u_2)(u, v_2)) = \sigma_1(u) \wedge \mu_2(u_2 v_2) \text{ for all } u \in V_1, \text{ for all } u_2 v_2 \in X_2,$$

$$(\mu_1 \circ \mu_2)((u_1, w)(v_1, w)) = \sigma_2(w) \wedge \mu_1(u_1 v_1) \text{ for all } w \in V_2, \text{ for all } u_1 v_1 \in X_1,$$

$$(\mu_1 \circ \mu_2)((u_1, v_2)(v_1, v_2)) = \sigma_2(u_2) \wedge \sigma_2(v_2) \wedge \mu_1(u_1 v_1) \\ \text{for all } (u_1, v_2)(v_1, v_2) \in X^0 \setminus X,$$

where

$$X = \{(u, u_2)(u, v_2) \mid u \in V_1, u_2 v_2 \in X_2\} \cup \\ \{(u_1, w)(v_1, w) \mid w \in V_2, u_1 v_1 \in X_1\}.$$

We see that $\sigma_1 \circ \sigma_2 = \sigma_1 \times \sigma_2$ and $\mu_1 \circ \mu_2 = \mu_1 \mu_2$ on X .

Proposition 2.11.5 *Let G be the composition $G_1[G_2]$ of graph G_1 with graph G_2 . Let (σ_i, μ_i) be a partial subgraph of G_i , $i = 1, 2$. Then $(\sigma_1 \circ \sigma_2, \mu_1 \circ \mu_2)$ is a partial fuzzy subgraph of $G_1[G_2]$.*

Proof We have already seen in the proof of Proposition 2.11.1 that

$$(\mu_1 \circ \mu_2)((u_1, u_2)(v_1, v_2)) \leq (\sigma_1 \circ \sigma_2)((u_1, u_2)) \wedge (\sigma_1 \circ \sigma_2)((v_1, v_2))$$

for all $(u_1, u_2)(v_1, v_2) \in X$. Suppose that $(u_1, u_2)(v_1, v_2) \in X^0 \setminus X$ and so $u_1 v_1 \in X_1$ and $u_2 \neq v_2$. Then

$$\begin{aligned} (\mu_1 \circ \mu_2)((u_1, u_2)(v_1, v_2)) &= \sigma_2(u_2) \wedge \sigma_2(v_2) \wedge \mu_1(u_1 v_1) \\ &\leq \sigma_2(u_2) \wedge \sigma_2(v_2) \wedge \sigma_1(u_1) \wedge \sigma_1(v_1) \\ &= \sigma_1(u_1) \wedge \sigma_2(u_2) \wedge \sigma_1(u_1) \wedge \sigma_2(v_2) \\ &= (\sigma_1 \circ \sigma_2)((u_1, u_2)) \wedge (\sigma_1 \circ \sigma_2)((v_1, v_2)). \end{aligned}$$

■

The fuzzy graph $(\sigma_1 \circ \sigma_2, \mu_1 \circ \mu_2)$ of the previous proposition is called the **composition** of (σ_1, μ_1) with (σ_2, μ_2) .

Theorem 2.11.6 *Let G be the composition $G_1[G_2]$ of graph G_1 with graph G_2 . Let (σ, μ) be a partial subgraph of G . Consider the following equations:*

- (i) $x_i \wedge y_j = \sigma(v_{1i}, v_{2j}), i = 1, \dots, n; j = 1, \dots, m;$
- (ii) $x_i \wedge z_{jk} = \mu(v_{1i}, v_{2j})(v_{1h}, v_{2k}), i = 1, \dots, n; j, k \text{ such that } v_{2j} v_{2k} \in X_2;$
- (iii) $y_j \wedge w_{ih} = \mu((v_{1i} v_{2j})(v_{1h}, v_{2j})), j = 1, \dots, m; i, h \text{ such that } v_{1i} v_{1h} \in X;$
- (iv) $y_j \wedge y_k \wedge w_{ih} = \mu((v_{1i}, v_{2j})(v_{1h}, v_{2k})), \text{ where } (v_{1i}, v_{2j})(v_{1h}, v_{2k}) \in X^0 \setminus X \text{ for } X \text{ defined as above.}$

Suppose that a solution to equations (i)–(iv) exists. If

$$\widehat{w}_{ih} \geq \mu(((v_{1i}, v_{2j})(v_{1h}, v_{2k}))) \text{ for all } (i, h) \in I$$

such that $(v_{1i}, v_{2j})(v_{1h}, v_{2j}) \in X^0 \setminus X$, then (σ, μ) is a composition of partial fuzzy subgraphs of G_1 and G_2 .

Proof The necessary part of the theorem is clear. Suppose that a solution to equations (i)–(iv) exists. Then there exists a solution to equations (i)–(iv) as determined in the proof of Theorem 2.11.2 for equations (i)–(iii) because every $w_{ih} \geq \widehat{w}_{ih}$ and by the hypothesis concerning the \widehat{w}_{ih} . Thus, if μ_i , $i = 1, 2$, are defined as in the proof of Theorem 2.11.2, we have that (σ_i, μ_i) is a partial fuzzy subgraph of G_i , $i = 1, 2$, and $\sigma = \sigma_1 \circ \sigma_2$ and $\mu = \mu_1 \circ \mu_2$. ■

Example 2.11.7 Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be graphs and let $\sigma_1, \sigma_2, \mu_1, \mu_2$ be fuzzy subsets of V_1, V_2, X_1, X_2 , respectively. Then $(\sigma_1 \times \sigma_2, \mu_1 \mu_2)$ is a partial fuzzy subgraph of $G_1 \times G_2$, but (σ_i, μ_i) is not a partial fuzzy subgraph of G_i , $i = 1, 2$: Let $V_1 = \{u_1, v_1\}$, $V_2 = \{u_2, v_2\}$, $X_1 = \{u_1 v_1\}$, and $X_2 = \{u_2 v_2\}$. Define the fuzzy subsets $\sigma_1, \sigma_2, \mu_1$, and μ_2 as follows: $\sigma_1(u_1) = \sigma_1(v_1) = \sigma_2(u_2) = \sigma_2(v_2) = 1/2$ and $\mu_1(u_1 v_1) = \mu_2(u_2 v_2) = 3/4$. Then (σ_i, μ_i) is not a partial fuzzy subgraph of G_i , $i = 1, 2$. Now, $x \in V_1$ and $y \in V_2$, $\mu_1 \mu_2((x, u_2)(x, v_2)) = \sigma_1(x) \wedge \mu_2(u_2 v_2) = 1/2 = \sigma_1(x) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2) = (\sigma_1 \times \sigma_2)(x, u_2)) \wedge (\sigma_1 \times \sigma_2)(x, v_2))$ and similarly, $\mu_1 \mu_2((u_1, y)(v_1, y)) = (\sigma_1 \times \sigma_2)(u_1, y)) \wedge (\sigma_1 \times \sigma_2)(v_1, y))$. Thus, $(\sigma_1 \times \sigma_2, \mu_1 \mu_2)$ is a partial fuzzy subgraph of $G_1 \times G_2$. Note that for $x_1 y_1 \in X_1$ and $x_2 y_2 \in X_2$, $(\mu_1 \circ \mu_2)((x_1, x_2)(y_1, y_2)) = \sigma_2(x_2) \wedge \sigma_2(y_2) \wedge \mu_1(x_1 y_1) = 1/2 = (\sigma_1 \times \sigma_2)((x_1, x_2)) \wedge (\sigma_1 \times \sigma_2)((y_1, y_2))$. Thus, $(\sigma_1 \circ \sigma_2, \mu_1 \circ \mu_2)$ is a partial fuzzy subgraph of $G_1[G_2]$.

In the previous example, $(\sigma_1 \times \sigma_2, \mu_1 \mu_2)$ satisfies the conditions in Theorem 2.11.2. Hence, $(\sigma_1 \times \sigma_2, \mu_1 \mu_2)$ is the Cartesian product of partial fuzzy subgraphs (τ_i, ν_i) of G_i , $i = 1, 2$. In fact, τ_i and ν_i , $i = 1, 2$, are constant functions with value $1/2$.

Consider the union $G = G_1 \cup G_2$ of two graphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$. Let μ_i be a fuzzy subset of v_i and ρ_i a fuzzy subset of X_i , $i = 1, 2$. Define the fuzzy subsets $\sigma_1 \cup \sigma_2$ of $V_1 \cup V_2$ and $\mu_1 \cup \mu_2$ of $X_1 \cup X_2$ as follows:

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u) & \text{if } u \in V_1 \setminus V_2, \\ \sigma_2(u) & \text{if } u \in V_2 \setminus V_1, \\ \sigma_1(u) \vee \sigma_2(u) & \text{if } u \in V_1 \cap V_2, \end{cases}$$

$$(\mu_1 \cup \mu_2)(uv) = \begin{cases} \mu_1(uv) & \text{if } uv \in X_1 \setminus X_2, \\ \mu_2(uv) & \text{if } uv \in X_2 \setminus X_1, \\ \mu_1(uv) \vee \mu_2(uv) & \text{if } uv \in X_1 \cap X_2. \end{cases}$$

Proposition 2.11.8 Let G be the union of the graphs G_1 and G_2 . Let (σ_i, μ_i) be a partial fuzzy subgraph of G_i , $i = 1, 2$. Then $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is a partial fuzzy subgraph of G .

Proof Suppose that $uv \in X_1 \setminus X_2$. We have three different cases to consider.

- (i) Suppose $u, v \in V_1 \setminus V_2$. Then $(\mu_1 \circ \mu_2)(uv) = \mu_1(uv) \leq \sigma_1(u) \wedge \sigma_1(v) = (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v)$.
- (ii) Suppose $u \in V_1 \setminus V_2$ and $v \in V_1 \cap V_2$. Then $(\mu_1 \cup \mu_2)(uv) \leq (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1(v) \vee \sigma_2(v)) = (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v)$.
- (iii) Suppose $u, v \in V_1 \cap V_2$. Then

$$\begin{aligned} (\mu_1 \cup \mu_2)(uv) &\leq (\sigma_1(u) \vee \sigma_2(u)) \wedge (\sigma_1(v) \vee \sigma_2(v)) \\ &= (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v). \end{aligned}$$

Similarly, if $uv \in X_2 \setminus X_1$. Then $(\mu_1 \cup \mu_2)(uv) \leq (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v)$. Suppose that $uv \in X_1 \cap X_2$. Then

$$\begin{aligned} (\mu_1 \cup \mu_2)(uv) &= \mu_1(uv) \vee \mu_2(uv) \\ &\leq (\sigma_1(u) \wedge \sigma_1(v)) \vee \sigma_2(u) \wedge \sigma_2(v) \\ &\leq (\sigma_1(u) \vee \sigma_2(u)) \wedge \sigma_1(v) \vee \sigma_2(v) \\ &= (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v). \end{aligned}$$

■

The fuzzy subgraph $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ of Proposition 2.11.8 is called the **union** of (σ_1, μ_1) and (σ_1, μ_2) .

Theorem 2.11.9 *If G is a union of two fuzzy subgraphs G_1 and G_2 , then every partial fuzzy subgraph (σ, μ) is a union of a partial fuzzy subgraph of G_1 and a partial fuzzy subgraph of G_2 .*

Proof Define the fuzzy subsets $\sigma_1, \sigma_2, \mu_1$, and μ_2 of V_1, V_2, X_1 and X_2 , respectively, as follows:

$$\sigma_i(u) = \sigma(u) \text{ if } u \in V_i \text{ and } \mu_i(uv) = \mu(uv) \text{ if } uv \in X_i, i = 1, 2.$$

Then $\mu_i(uv) = \mu(uv) \leq \sigma(u) \wedge \sigma(v) = \sigma_i(u) \wedge \sigma_i(v)$ if $uv \in X_i, i = 1, 2$. Thus, (σ_i, μ_i) is a partial fuzzy subgraph of $G_i, i = 1, 2$. Clearly, $\sigma = \sigma_1 \cup \sigma_2$ and $\mu = \mu_1 \cup \mu_2$. ■

Consider the join $G = G_1 + G_2 = (V_1 \cup V_2, X_1 \cup X_2 \cup X')$ of graphs $G_1 = (V_1, X_2)$ and $G_2 = (V_2, X_2)$, where X' is the set of all edges joining the vertices of V_1 and V_2 and where we assume $V_1 \cap V_2 = \emptyset$. Let σ_i be a fuzzy subset of V_i a fuzzy subset of $X_i, i = 1, 2$. Define the fuzzy subsets $\sigma_1 + \sigma_2$ of $V_1 \cup V_2$ and $\mu_1 + \mu_2$ of $X_1 \cup X_2 \cup X'$ as follows:

$$(\sigma_1 + \sigma_2)(u) = (\sigma_1 \cup \sigma_2)(u) \text{ for all } u \in V_1 \cup V_2,$$

$$(\mu_1 + \mu_2)(uv) = \begin{cases} (\mu_1 \cup \mu_2)(uv) & \text{if } uv \in X_1 \cup X_2, \\ \sigma_1(u) \wedge \sigma_2(v) & \text{if } uv \in X', u \in V_1, v \in V_2. \end{cases}$$

Proposition 2.11.10 *Let G be the join of two graphs G_1 and G_2 . Let (σ_1, μ_i) be a partial fuzzy subgraph of G_i , $i = 1, 2$. Then $(\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ is a partial fuzzy subgraph of G .*

Proof Suppose that $uv \in X_1 \cup X_2$. Then the desired result follows from Proposition 2.11.8. Suppose that $uv \in X'$. Then

$$\begin{aligned} (\mu_1 + \mu_2)(uv) &= \sigma_1(u) \wedge \sigma_2(v) \\ &= (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v) \\ &= (\sigma_1 + \sigma_2)(u) \wedge (\sigma_1 + \sigma_2)(v). \end{aligned}$$

■

The fuzzy subgraph $(\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ of Proposition 2.11.10 is called the **join** of (σ_1, μ_1) and (σ_2, μ_2) .

Definition 2.11.11 Let (σ, μ) be a partial fuzzy subgraph of a graph $G = (V, X)$. Then (σ, μ) is called a **strong partial fuzzy subgraph** of G if $\mu(uv) = \sigma(u) \wedge \sigma(v)$ for all $uv \in X$.

Theorem 2.11.12 *If G is the join of two subgraphs G_1 and G_2 , then every strong partial fuzzy subgraph (σ, μ) of G is a join of a strong partial fuzzy subgraph of G_1 and a strong partial fuzzy subgraph of G_2 .*

Proof Define the fuzzy subsets $\sigma_1, \sigma_2, \mu_1$, and μ_2 of V_1, V_2, X_1 and X_2 , respectively, as follows: $\sigma_i(u) = \sigma(u)$ if $u \in V_i$ and $\mu_i(uv) = \mu(uv)$ if $uv \in X_i, i = 1, 2$. Then (σ_i, μ_i) is a fuzzy partial subgraph of $G_i, i = 1, 2$, and $\sigma = \sigma_1 + \sigma_2$ as in the proof of Theorem 2.11.9. If $uv \in X_1 \cup X_2$, then $\mu(uv) = (\mu_1 + \mu_2)(uv)$ as in the proof of Theorem 2.11.9. Suppose that $uv \in X'$, where $u \in V_1$ and $v \in V_2$. Then $(\mu_1 + \mu_2)(uv) = \sigma_1(u) \wedge \sigma_2(v) = \sigma(u) \wedge \sigma(v) = \mu(uv)$, where the latter equality holds because (σ, μ) is strong. ■

Example 2.11.13 Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be graphs and let $\sigma_1, \sigma_2, \mu_1, \mu_2$ be fuzzy subsets of V_1, V_2, X_1, X_2 , respectively. Then $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is a partial fuzzy subgraph of $G_1 \cup G_2$, but (σ_i, μ_i) is not a partial fuzzy subgraph of $G_i, i = 1, 2$: Let $V_1 = V_2 = \{u, v\}$ and $X_1 = X_2 = \{uv\}$. Define the fuzzy subsets $\sigma_1, \sigma_2, \mu_1, \mu_2$ be fuzzy subsets of V_1, V_2, X_1, X_2 , respectively, as follows: $\sigma_1(u) = 1 = \sigma_2(v), \sigma_1(v) = 1/4 = \sigma_2(u), \mu_1(uv) = 1/2 = \mu_2(uv)$. Then (σ_i, μ_i) is not a partial fuzzy subgraph of $G_i, i = 1, 2$. Now, $(\mu_1 \cup \mu_2)(uv) = \mu_1(uv) \vee \mu_2(uv) = 1/2 < 1 = (\sigma_1(u) \vee \sigma_2(u)) \wedge (\sigma_1(v) \vee \sigma_2(v)) = (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v)$. Thus, $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is a partial fuzzy subgraph of $G_1 \cup G_2$.

The above example can be extended to the case where $V_1 \not\subseteq V_2, V_2 \not\subseteq V_1$ and $X_1 \not\subseteq X_2, X_2 \not\subseteq X_1$ as follows: Let $V_1 = \{u, v, w\}, V_2 = \{u, v, z\}$ and $X_1 = \{uv, uw\}, X_2 = \{uv, vz\}$ and $\sigma_1(w) = \sigma_2(z) = 1 = \mu_1(uw) = \mu_2(uz)$.

Theorem 2.11.14 Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_1)$ be graphs. Suppose that $V_1 \cap V_2 = \emptyset$. Let $\sigma_1, \sigma_2, \mu_1$, and μ_2 be fuzzy subsets of V_1, V_2, X_1 and X_2 , respectively. Then $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is a partial fuzzy subgraph of $G_1 \cup G_2$ if and only if (σ_1, μ_1) and (σ_2, μ_2) are partial fuzzy subgraphs of G_1 and G_2 , respectively.

Proof Suppose that $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is a partial fuzzy subgraph of $G_1 \cup G_2$. Let $uv \in X_1$. Then $uv \notin X_2$ and $u, v \in V_1 \setminus V_2$. Hence, $\mu_1(uv) = (\mu_1 \cup \mu_2)(uv) \leq (\sigma_1 \cup \sigma_2)(u) \wedge (\sigma_1 \cup \sigma_2)(v) = \sigma_1(u) \wedge \sigma_1(v)$. Thus, (σ_1, μ_1) is partial fuzzy subgraph of G_1 . Similarly, (σ_2, μ_2) is partial fuzzy subgraph of G_2 . The converse is Proposition 2.11.8. ■

The following result follows from the proof of Theorem 2.11.12 and Proposition 2.11.10.

Theorem 2.11.15 Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_1)$ be graphs. Suppose that $V_1 \cap V_2 = \emptyset$. Let $\sigma_1, \sigma_2, \mu_1$, and μ_2 be fuzzy subsets of V_1, V_2, X_1 and X_2 , respectively. Then $(\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ is a partial fuzzy subgraph of $G_1 + G_2$ if and only if (σ_1, μ_1) and (σ_2, μ_2) are partial fuzzy subgraphs of G_1 and G_2 , respectively.

Definition 2.11.16 Let (σ, μ) be a partial fuzzy subgraph of (V, T) , where $T \subseteq V$. Define the fuzzy subsets σ' of V and μ' of T as follows: $\sigma' = \sigma$ and for all $uv \in T$, $\mu'(uv) = 0$ if $\mu(uv) > 0$ and $\mu'(uv) = \sigma(u) \wedge \sigma(v)$ if $\mu(uv) = 0$.

Clearly, $G' = (\sigma', \mu')$ is a fuzzy graph.

Definition 2.11.17 Let (σ, μ) be a partial fuzzy subgraph of $G = (V, X)$, Then (σ, μ) is said to be **complete** if $X = T$ and for all $uv \in X$, $\mu(uv) = \sigma(u) \wedge \sigma(v)$.

We use the notation $C_m(\sigma, \mu)$ for a complete fuzzy graph, where $|V| = m$.

Definition 2.11.18 (σ, μ) is called a **fuzzy bigraph** if and only if there exist partial fuzzy subgraphs (σ_i, μ_i) , $i = 1, 2$, of (σ, μ) such that (σ, μ) is the join $(\sigma_1, \mu_1) + (\sigma_2, \mu_2)$, where $V_1 \cap V_2 = \emptyset$ and $X_1 \cap X_2 = \emptyset$. A fuzzy bigraph is said to be **complete** if $\mu(uv) > 0$ for all $uv \in X'$.

We use the notation $C_{m,n}(\sigma, \mu)$ for a complete fuzzy bigraph, where $|V| = m$ and $|V_2| = n$.

Proposition 2.11.19 $C_{m,n}(\sigma, \mu) = C_m(\sigma, \mu)' + C_n(\sigma, \mu)'$.

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