Chapter Two: Projective Geometry

Section 1. Synthetic Approach.

We begin with projections between two coplanar lines.

Let ℓ and ℓ' be two lines intersecting at a point F, and V be a point not on either of them. A projection from V of ℓ to ℓ' may be defined as follows. For each point P on ℓ, draw the line VP. If this line intersects ℓ' at a point P', we say that P' is the image of P under the projection from V. In particular, $F' = F$.

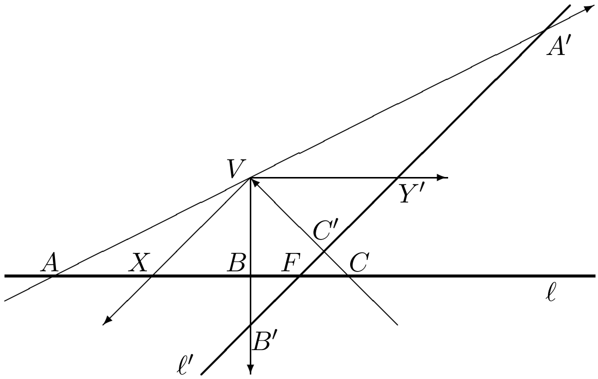

Figure 2.1

Note that the point X on ℓ where VX is parallel to ℓ' does not have an image. Also, the point Y' on ℓ' where VY' is parallel to ℓ is not the image of any point on ℓ. A simple and simple-minded remedy is to define Y' as the image of X, but this is not a satisfactory solution. We want projections to be **continuous** transformations. In other words, if a point P moves along ℓ towards another point Q, we want P' to move along ℓ' towards Q'.

Let B and C' be the respective feet of perpendicular from V to ℓ and ℓ'. As P moves from C towards B, P' does move from C' towards B'. However, as P moves on towards X, P' certainly does not move towards Y' but towards an imaginary point at infinity. Such a point does not exist in Euclidean Geometry. In Projective Geometry, we make the line ℓ' into a **projective line** $\overline{\ell'}$ by adding this point at infinity. We call it the **ideal point** of $\overline{\ell'}$, and it is taken to be the image X' of X. The other points of $\overline{\ell'}$ are called **ordinary points**.

Do we need another ideal point at the other end of $\overline{\ell'}$? As P moves from A towards X, P' moves away from A' towards this ideal point. However, it is also going to be the image of X, so that it should coincide with the X' already chosen. Thus we need only one ideal point per projective line.

A. Liu, *S.M.A.R.T. Circle Minicourses*, Springer Texts in Education, https://doi.org/10.1007/978-3-319-71743-2_2

It is now clear that the exceptional point Y' on $\overline{\ell'}$ must be the image of the ideal point Y of $\overline{\ell}$.

Note that in Figure 2.1, it appears that B is between A and C on $\overline{\ell}$ while B' is not between A' and C' on $\overline{\ell'}$. Thus betweenness does not seem to be preserved by projections between intersecting lines. This is hardly surprising since betweenness is not even a meaningful concept on a closed curve, and the addition of the ideal point turns a projective line into a closed curve.

Let A, B, C and D be four points on a closed curve. We say that A and B **separate** C and D if in going from A to B along the curve, we must pass over C in one direction and over D in the other. It is easy to see that in that case, C and D also separate A and B, and that separation is preserved by projection.

How would the conversion of parallel Euclidean lines ℓ and ℓ' into projective lines affect projections between them? It is easy to see that there is a one-to-one correspondence between the ordinary points of $\overline{\ell}$ and $\overline{\ell'}$. Let Z be the ideal point of $\overline{\ell}$. The line VZ is supposed to intersect $\overline{\ell'}$ at its ideal point Z'. In order to come up with Z'. we must specify how we are to join an ideal point to an ordinary point. To do this, we must systematically convert the Euclidean plane into the **projective plane**.

In the Euclidean plane, every two points determine a unique line, but not every two lines determine a unique point. The exceptional cases are when the two lines are parallel. If we postulate that parallel lines meet at the same ideal point, we have eliminated this anomaly. Although these lines now meet at an ideal point, we will continue to call them parallel lines.

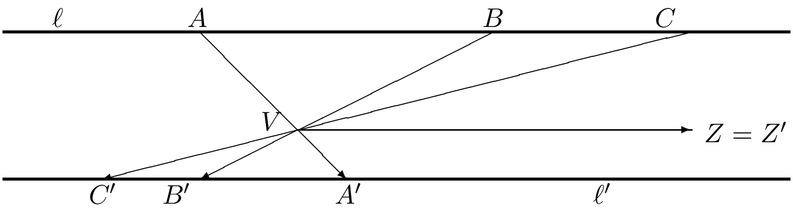

Figure 2.2

We may think of ideal points as the directions of the families of parallel lines. The line VZ is simply the line through V parallel to ℓ. Since it is also parallel to ℓ', they intersect at $Z' = Z$. This agrees with the coincidence of F and F' in Figure 2.1. There, $X' \neq Y$, so that intersecting lines have different ideal points. Otherwise, two such lines will determine more than just one point.

Is it still true that every two points determine a unique line? If both are ordinary points, we have no problem. If one is an ordinary point and the other is an ideal point, they still determine a unique line since we have an ordinary point on it, and we also know its direction. However, two ideal points do not determine any line as yet.

To fix this, we add a new projective line called the **ideal line**, consisting of all ideal points. We now check that every two projective lines still determine a unique point. If both are ordinary, parallel or otherwise, we have no problem. If one is an ordinary line and the other is the ideal line, they determine the ideal point on the ordinary line. Since we have only one ideal line, there are no other cases, and the construction of the projective plane is now completed.

In pure Projective Geometry, there is no distinction between ideal points and ordinary points, or between the ideal line and ordinary lines. Any of the lines may be taken as the ideal line, and the points on it as ideal points. However, our primary interest is to use projective methods to solve Euclidean problems. This is why we retain the notion of parallel lines in the Euclidean sense.

Note that the point V may be an ideal point. In that case, the lines from V are simply parallel lines whose direction is specified by the ideal point V. Projections from an ideal point are often called **parallel projections**, and those from an ordinary point **central projections**. However, this distinction is inconsequential. In many ways, the projective plane is actually simpler than the Euclidean plane.

We can extend Euclidean space into projective space by making each Euclidean plane into a projective plane, and adding an **ideal plane** which contains all ideal lines of the projective planes. In Solid Geometry, a plane becomes a subset of space. Thus it requires an implicit definition, and this is done by listing its basic properties.

1. Two points in a plane lies on a unique line which lies in that plane.
2. Two planes meeting at a point meets along a unique line passing through that point.
3. A line and a point which is not on that line lie on a unique plane.
4. A plane and a line which is not on that plane meet at a unique point.
5. Three points not all on a line lies on a unique plane.
6. Three planes not all meeting along a line meet at a unique point.
7. Two lines on the same plane meet at a unique point.
8. Two lines which meet at a point lie on a unique plane.

Lines and planes that meet at a point are said to be **concurrent**. Points that lie on a line and planes that meet along a line are said to be **collinear**. Points and lines that lie in a plane are said to be **coplanar**. Note that it is possible for two lines not to meet, at either an ordinary or an ideal point. An example is a straight bridge over a straight river. Two such lines are said to be a pair of **skew lines**.

We will retain the Euclidean concept of parallelism. Planes that are parallel meet along their common ideal line. A line and a plane which are parallel meet at the ideal point of the line. A plane is parallel to itself.

Parallel Line-Plane Theorem.
Let AB be a line in a plane Π and CD be a line not in Π. If AB and CD are parallel, then CD is parallel to Π.

Proof:
Suppose to the contrary that CD and Π meet at an ordinary point P. Since AB and CD are parallel, P is not on AB. Moreover, AB and CD lie on a plane. This plane must contain P which is on CD. However, the unique plane containing P and AB is Π, but this contradicts the assumption that CD is not in Π.

While we are primarily interested in Plane Geometry, the embedding of a plane in space gives us new ways of tackling problems. An important tool is the projection of a plane Π to another plane Π' from a point V not on either. There are two basic properties of such a projection. First, a straight line ℓ on Π becomes a straight line ℓ' in Π', ℓ' being the intersection of Π' and the plane determined by V and ℓ. Secondly, a point P in Π lies on ℓ if and only if its image P' in Π' lies on ℓ'.

Let ℓ be any line on a plane be Π. Choose a point V not on Π and let Ω be the unique plane determined by V and ℓ. Let Π' be any plane parallel to but different from Ω. For any point P on ℓ, its image P' is in Ω as well as Π'. Since these two planes are parallel, P' is an ideal point and the image ℓ' of ℓ is the ideal line of Π'. We have projected ℓ to infinity.

Desargues' Theorem.
Let ABC and $A'B'C'$ be two triangles. Suppose AA', BB' and CC' are concurrent at some point V, BC and $B'C'$ meet at L, CA and $C'A'$ meet at M, and AB and $A'B'$ meet at N. Then L M and N are collinear.

Proof:
Project MN to infinity. Then AB and $A'B'$ become parallel lines, as do AC and $A'C'$. Hence VAB and $VA'B'$ are similar triangles, so that $\frac{VB'}{VB} = \frac{VA'}{VA}$. Similarly, $\frac{VC'}{VC} = \frac{VA'}{VA}$. It follows that $\frac{VB'}{VB} = \frac{VC'}{VC}$, so that VBC and $VB'C'$ are also similar triangles. This means that BC and $B'C'$ are parallel, so that L is an ideal point. Hence L, M and N are collinear.

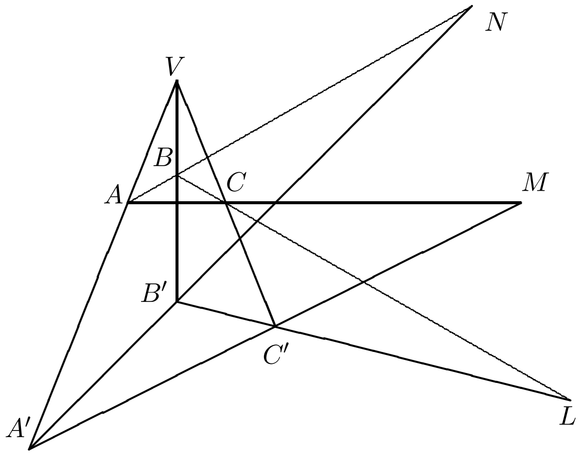

Figure 2.3

Converse of Desargues' Theorem.
Let ABC and $A'B'C'$ be two triangles. Suppose BC and $B'C'$ meet at L, CA and $C'A'$ meet at M, AB and $A'B'$ meet at N, where L, M and N are collinear. Then AA', BB' and CC' are concurrent.

Proof:
Project MN to infinity. Then the corresponding sides of ABC and $A'B'C'$ are parallel, so that the triangles are similar. Let AA' and BB' meet at V. Then VAB and $VA'B'$ are also similar triangles. Thus $\frac{B'C'}{BC} = \frac{A'B'}{AB} = \frac{VB'}{VB}$ while $\angle VB'C' = \angle VBC$. Hence BBC and $VB'C'$ are similar triangles, so that $\angle BVC = \angle B'VC'$. It follows that C, C' and V are collinear, so that AA', BB' and CC' are concurrent.

The Pappus-Pascal Theorem.
A, C and E are collinear points, as are B, D and F. If AB and DE meet at N, BC and EF meet at L, and CD and FA meet at M, then L, M and N are also collinear.

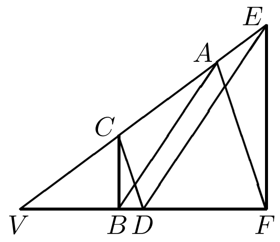

Figure 2.4

Proof:
Project MN to infinity. Then AB and DE are parallel lines, as are CD and FA. Let AE cut BF at V. Then VAB and VED are similar triangles, as are VAF and VCD. Hence $\frac{VA}{VB} = \frac{VE}{VD}$ and $\frac{VA}{VF} = \frac{VC}{VD}$. It follows that $\frac{VC}{VB} = \frac{VE}{VF}$, so that VCB and VEF are similar triangles. This means that BC and EF are parallel, so that L is an ideal point. Hence L, M and N are collinear.

The Pappus-Brianchon Theorem.
AB, CD and EF are concurrent lines, as are BC, DE and FA. Then AD, BE and CF are also concurrent.

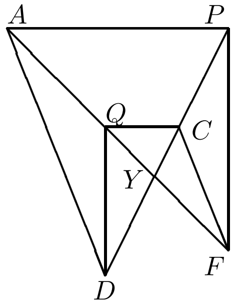

Figure 2.5

Proof:
Let AB, CD and EF be concurrent at P and let BC, DE and FA be concurrent at Q. Let CD cut FA at Y and project BE to infinity. Then AP and QC become parallel lines, as do FP and QD. Hence YAP and YQC are similar triangle, and so are YFP and YQD. It follows that $\frac{YQ}{YA} = \frac{YC}{YP}$ and $\frac{YF}{YQ} = \frac{YP}{YD}$, so that $\frac{YF}{YA} = \frac{YC}{YD}$. Hence YCF and YDA are similar triangles, so that AD and CF are parallel lines. It follows that they are concurrent with the ideal line BE.

An **inaccessible point** is one which is well-defined, usually as the intersection of two lines, but the point itself is not to be drawn.

Here is a construction problem involving an inaccessible point. The point N of intersection of two given lines ℓ and m is inaccessible. Using only a straight-edge, construct the line joining N to a given point M.

Take V not on either ℓ or m. Draw a line through V, cutting ℓ at A and m at D. Draw a second line through V, cutting ℓ at B and m at E. Draw a third line through V, cutting AM at C and DM at F. Let BC cut EF at L. By Desargues' Theorem, the line joining M and N passes through L.

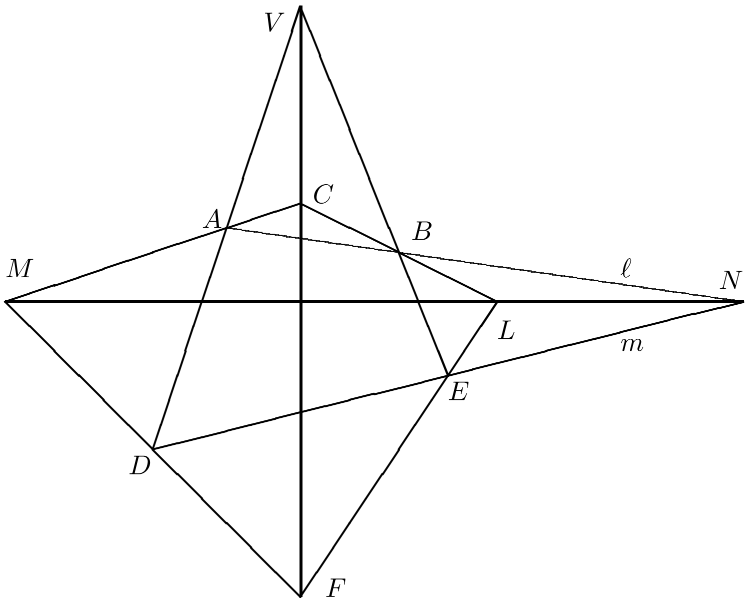

Figure 2.6

An **inaccessible line** is one which is well-defined, usually by two points on it, but the line itself is not to be drawn.

Here is a construction problem involving an inaccessible line. The line ℓ joining two given points A and B is inaccessible. Using only a straight-edge, construct the point of intersection of ℓ with a given line m.

Take V not on either ℓ or m. Let m cut VA at D and VB at E. Draw a third line through V. On this line, take a point C not on ℓ and a point F not on m. Let CA cut FD at M, CB cut FE at L, and LM cut m at N. By Desargues' Theorem, N also lies on ℓ.

Figure 2.6 serves for both constructions. In the former, the point N is to be omitted. In the latter, the line ℓ is to be omitted.

Constructions using a straight edge but without a compass are known as Poncelet-Steiner constructions.

Exercises

1. (a) Let Π and Π' be two planes and V be a point not in either of them. Let triangle ABC in Π be projected from V onto triangle $A'B'C'$ in Π'. Prove that the lines BC and $B'C'$ will intersect, as will CA and $C'A'$, as well as AB and $A'B'$, in three collinear points.
   (b) Deduce Desargues' Theorem from (a).
2. (a) Deduce Desargues' Theorem and its converse from each other.
   (b) Deduce the Pappus-Pascal Theorem and the Pappus-Brianchon Theorem from each other.
3. (a) The point L of intersection of two given lines ℓ and ℓ' is inaccessible, as is the point M of intersection of two given lines m and m'. Using only a straight-edge, construct the line joining L and M.
   (b) The line ℓ joining two given points A and B is inaccessible, as is the line ℓ' joining two given points A' and B'. Using only a straight-edge, construct the point of intersection of ℓ and ℓ'.

Section 2. Metric Approach.

In our synthetic treatment of projective geometry, we have completely avoided the measurement of distances. There are two reasons. First, distances to the ideal elements are necessarily infinite. Secondly, distance is not preserved by projection. However, our goal is to enhance with projective methods the contents of Euclidean Geometry, where distance is an essential concept. In this section, we will find ways to overcome the two difficulties mentioned above.

For a line, we choose arbitrarily either direction as being *positive*, so that the opposite direction is *negative*. Suppose AB is a segment on this line. we denote by $\overline{AB}$ the directed segment going from A to B. Just as the symbol AB also denotes distance, the symbol $\overline{AB}$ also denotes the signed distance. In particular, $\overline{AA} = 0$ and $\overline{AB} + \overline{BA} = 0$.

Let the Cartesian coordinates of A and B be 0 and 1 respectively, so that $\overline{AB} = 1$. For any point P on the line AB, we define the **Apollonian ratio** of P with respect to A and B to be $\dfrac{\overline{AP}}{\overline{PB}}$. Thus if the Cartesian coordinate of P is x, then its Apollonian ratio $\frac{x}{1-x}$. In particular, the Apollonian ratio of A is 0 and that of B is ∞. In Figure 2.7, the Cartesian coordinates are above the line while the Apollonian ratios are below the line.

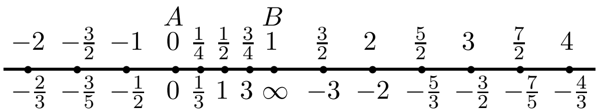

Figure 2.7

If P lies between A and B, both $\overline{AP}$ and $\overline{PB}$ are positive so that its Apollonian ratio is positive. As P moves from A towards B, the numerator increases while the denominator decreases. Hence its Apollonian ratio increases without bound.

Beyond B, $\overline{AP}$ is still positive but $\overline{PB}$ is now negative. Hence the Apollonian ratio of P is negative. As P moves towards B, the absolute value of its Apollonian ratio increases without bound. As P moves away from B, the absolute values of both the numerator and the denominator increase but the former is always 1 greater than the latter. Hence the absolute value of the Apollonian ratio of P decreases to 1 eventually, so that -1 is the natural choice as the Apollonian ratio of the ideal point of the line AB.

On the other side of A, $\overline{AP}$ is negative while $\overline{PB}$ is positive. Hence the Apollonian ratio of P is negative. As P moves towards A, the absolute value of its Apollonian ratio decreases to 0.

As P moves away from A, the absolute values of both the numerator and the denominator increase but the former is always 1 less than the latter. Hence the absolute value of the Apollonian ratio of P increases to 1 eventually. This agrees with the previous choice of -1 as the Apollonian ratio of the ideal point.

Suppose a point P starts from B, moves away from A, reappears at the other end of the line via the ideal point, continues past A and get back to B. During this movement, its Apollonian ratio is always increasing.

It should be pointed out that the choice of the Cartesian coordinates of A and B is immaterial. The Apollonian ratio of the ideal point with respect to any two points on the line will always be -1. Thus the Apollonian ratio is a useful concept in which the ideal point has a finite measure.

Let ABC be any triangle. Let D, E and F be points on the lines BC, CA and AB, respectively, not coinciding with any of A, B and C. The product of the Apollonian ratios $\frac{\overline{AF}}{\overline{FB}}$, $\frac{\overline{BD}}{\overline{DC}}$ and $\frac{\overline{CE}}{\overline{EA}}$ is called a **Cevian ratio**, and is denoted by (ABC, FDE).

We claim that Cevian ratios are preserved by projections. To prove this, we introduce the concept of signed angles. Angles are considered positive if measured in the counterclockwise direction, and negative if measured in the clockwise direction. Like signed distances, signed angles are marked with an overline.

Let V be a point not on the plane of ABC and let d be the distance from V to AB. Then $d\overline{AF} = \pm VA \cdot VF \sin \overline{AVF}$ since the absolute value of each side is twice the area of triangle VAF. Similarly, we also have $d\overline{FB} = \pm VF \cdot VB \sin \overline{FVB}$. Since the right sides of these two equations have the same sign, $\frac{\overline{AF}}{\overline{FB}} = \frac{VA \sin \overline{AVF}}{VB \sin \overline{FVB}}$. It follows that

$$(ABC, FDE) = \frac{\sin \overline{AVF} \sin \overline{BVD} \sin \overline{CVE}}{\sin \overline{FVB} \sin \overline{DVC} \sin \overline{EVA}}.$$

Since angles at V are not affected by projections from V, the claim is justified.

Menelaus' Theorem.
Let ABC be any triangle. Let D, E and F be points on the lines BC, CA and AB, respectively, not coinciding with any of A, B and C. Then D, E and F are collinear if and only if $(ABC, FDE) = -1$.

Proof:
Project EF to infinity. Then $\frac{\overline{AF}}{\overline{FB}} = -1 = \frac{\overline{CE}}{\overline{EA}}$. Suppose D, E and F are collinear. Then D is an ideal point, and $\frac{\overline{BD}}{\overline{DC}} = -1$. It follows that we have $(ABC, FDE) = -1$. Conversely, if $(ABC, FDE) = -1$, then $\frac{\overline{BD}}{\overline{DC}} = -1$. It follows that D must be the ideal point of the line BC, and D, E and F are indeed collinear.

Ceva's Theorem.
Let ABC be any triangle. Let D, E and F be points on the lines BC, CA and AB, respectively, not coinciding with any of A, B and C. Then AD, BE and CF are concurrent if and only if $(ABC, FDE) = 1$.

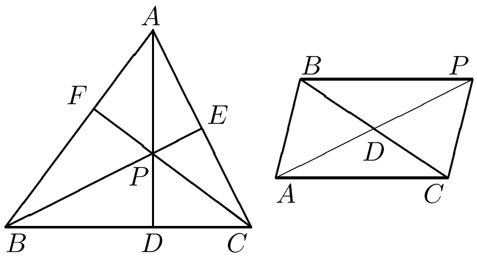

Figure 2.8

Proof:
Let BE and CF meet at P. Project EF to infinity. Then $\frac{\overline{AF}}{\overline{FB}} = -1 = \frac{\overline{CE}}{\overline{EA}}$. Moreover, $ABPC$ has become a parallelogram. Suppose AD also passes through P. Then D is the point of intersection of the diagonals AP and BC of the parallelogram. Hence $\frac{\overline{BD}}{\overline{DC}} = 1$ and $(ABC, FDE) = 1$. Conversely, if $(ABC, FDE) = 1$, then $\frac{\overline{BD}}{\overline{DC}} = 1$. It follows that D must be the midpoint of BC. Hence D lies on AP, so that AD, BE and CF are indeed concurrent.

The "only if" parts of these two theorems are often referred to as their respective converses. We now use these theorems to solve a problem.

AD is an altitude of triangle ABC. E and F are points on CA and AB respectively such that BE, CF and AD are concurrent. Prove that AD bisects $\angle FDE$.

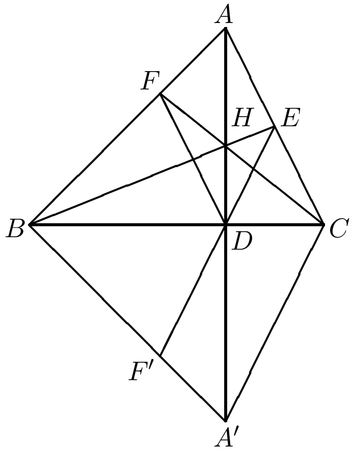

Figure 2.9

Reflect triangle ABC across BC and let A' and F' be the respective images of A and F. Since AD is an altitude, A, D and A' are collinear. Let H be the point of concurrency of AD, BE and CF. Applying Ceva's Theorem to triangle ABH, we have $(ABH, FED) = 1$. It follows that $(A'BH, F'ED) = -1$. Applying the converse of Menelaus' Theorem to triangle $A'BH$, E, D and F' are collinear. Hence

$$\angle ADE = \angle A'DF' = \angle ADF.$$

Note that we can derive Ceva's Theorem from Menelaus' Theorem as follows. Suppose AD, BE and CF are concurrent at P. Applying Menelaus' Theorem to ABD and ADC, we have $(ABD, FCP) = (ADC, PBE) = -1$. Multiplying these together, we have $(ABC, FDE) = 1$. Conversely, let AD cut BE at Q and CF at R. Applying Menelaus's Theorem to ABD and ADC, we have $(ABD, FCQ) = (ADC, RBE) = -1$. Multiplying these together with $1 = (ABC, FDE)$, we have $\dfrac{\overline{AQ}}{\overline{QD}} = \dfrac{\overline{AR}}{\overline{RD}}$. Hence $Q = R$, so that AD, BE and CF are indeed concurrent.

Let A, B, C and D be four distinct collinear points. The **cross ratio** (AB, CD) is defined as the product of the Apollonian ratios $\dfrac{\overline{AC}}{\overline{CB}}$ and $\dfrac{\overline{BD}}{\overline{DA}}$.

In the same way as for Cevian ratios, we can prove that cross ratios are also preserved by projections. Suppose a projection from V sends collinear points A, B, C and D into collinear points A', B', C' and D' respectively. Then $(AB, CD) = (A'B', C'D')$. We incorporate the point V into the equation by writing $(AB, CD)V(A'B', C'D')$. The converse of this result has the following important special case.

Cross Ratio Concurrency Theorem.
Let B, C and D be three points on a line. Let B', C' and D' be three points on another lines. If these lines intersects at A and $(AB, CD) = (AB', C'D')$, then BB', CC' and DD' are concurrent.

Proof:
Let BB' intersect CC' at V. Suppose VD intersects AB' at E'. Then we have $(AB, CD)V(AB', C'E')$, so that $(AB', C'D') = (AB', C'E')$. Hence $\frac{\overline{B'D'}}{\overline{D'A}} = \frac{\overline{B'E'}}{\overline{E'A}}$, so that $D' = E'$. It follows that VD passes through D', and BB', CC' and DD' are concurrent at V.

We now use this result to prove the theorems in Section 1.

Let MN cut VA, VB and VC at X, Y and Z respectively. Then we have $(VZ, CC')M(VX, AA')N(VY, BB')$. By the Cross Ratio Concurrency Theorem, $BC, B'C'$ and XY are concurrent. It follows that L, M and N are collinear. This establishes Desaegues' Theorem.

Let AA' cut BC, $B'C'$ and MN at X, X' and Y respectively. Then we have $(LB, XC)A(LN, YM)A'(LB', X'C')$. By the Cross Ratio Concurrency Theorem, AA', BB' and CC' are concurrent. This establishes the converse of Desargues' Theorem.

Let P be the point of concurrency of AB, CD and EF, and Q be that of BC, DE and FA. Let AB cut CF at X, BC cut EF at Y, CD cut FA at Z, and BE cut CF at R. Then $(RX, FC)B(EP, FY)Q(DP, ZC)$. By the Cross Ratio Concurrency Theorem, RD, XP and FZ are concurrent. It follows that AD, BE and CF are also concurrent. This establishes the Pappus-Pascal Theorem.

Let CD cut EF at X, DE cut FA at Y and AC cut BD at Z. Then $(XL, EF)C(DB, ZF)A(DN, EY)$. By the Cross Ratio Concurrency Theorem, XD, LN and FY are concurrent. It follows that L, M and N are collinear. This establishes the Pappus-Brianchon Theorem.

Cross Ratio Equality Theorem.
$(AB, CD) = (DC, BA) = (CD, AB) = (BA, DC)$ for any four collinear points A, B, C and D.

Proof:
Take a point V not on AD and join it to A, B and C. Draw a line through D, cutting VC at E, VB at F and VA at G. Let GC cut VB at M. Then $(AB, CD)G(VB, MF)C(ED, GF)V(CD, AB)$. Let FA cut VC at N. Then $(AB, CD)F(NV, CE)A(FG, DE)V(BA, DC)$. Let FC cut VA at L. Then $(AB, CD)F(AV, LG)C(DE, FG)V(DC, BA)$.

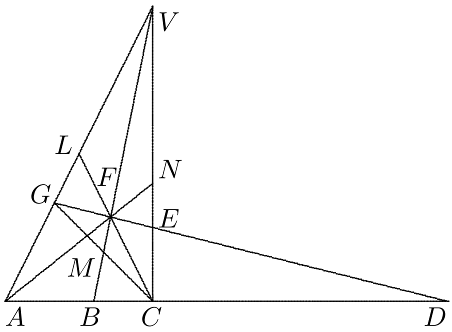

Figure 2.10

There are 24 permutations of the points A, B, C and D, so that they give rise to 24 cross ratios. In general, they form six groups of four and take on distinct values.

We tend not to make use of the actual value of cross ratios. An exception is the special case where $(AB, CD) = -1$. We say that C are D are **harmonic conjugates** of each other respect to A and B, and the four points A, C, B and D form a **harmonic range**. The harmonic conjugate of the midpoint of AB with respect to A and B is the ideal point of AB.

Harmonic Range Theorem.
If A, C, B and D form a harmonic range, then $(AB, CD) = (AB, DC)$.

Proof:
Given A, B and C, we first give a Poncelet-Steiner construction of D. Take a point V not on AB and a point E on VC. Let EA cut VB at F and EB cut VA at G. The point D on intersection of FG and AB is the desired conjugate. By Ceva's Theorem, $(VAB, GCF) = 1$. By Menelaus' Theorem, $(VBA, FDG) = -1$. It follows that $(AB, CD) = -1$. Let VC cut FG at H. It follows from the Cross Ratio Equality Theorem that we have $(AB, CD) = (BA, DC)E(GF, DH)V(AB, DC)$.

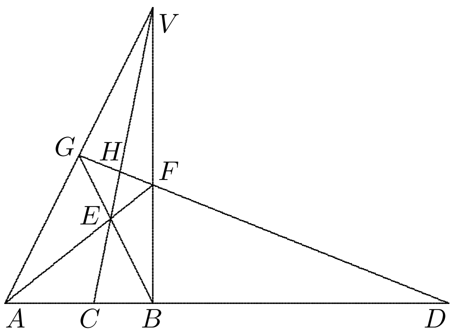

Figure 2.11

Exercises:

4. Deduce from Menelaus' Theorem

   (a) Desargues' Theorem;

   (b) the converse of Desargues' Theorem;

   (c) the Pappus-Pascal Theorem;

   (d) the Pappus-Brianchon Theorem.

5. (a) The bisector of the exterior angle at the vertex A of triangle ABC meets the extension of BC at K while the bisector of $\angle CAB$ meets BC at D. A line through K cuts CA at E and AB at F. Prove that AD, BE and CF are concurrent.

   (b) The bisector of the exterior angle at the vertex A of triangle ABC meets the extension of BC at K while the bisector of $\angle CAB$ meets BC at D. P is a point on AD. The extensions of BP and CP cut CA and AB at E and F, respectively. Prove that E, F and K are collinear.

   (c) D, E and F are points on the sides BC, CA and AB of triangle ABC respectively. The lines BC and EF meet at L, the lines CA and FD meet at M, and the lines AB and DE meet at N. If L, M and N are collinear, prove that AD, BE and CF are concurrent.

6. (a) Determine (DC, BA) in terms of (AB, CD).

   (b) Determine (AB, DC) in terms of (AB, CD).

   (c) P is a point inside triangle ABC. AP cuts BC at D, BP cuts CA at E, CP cuts AB at F, EF cuts BC at X, FD cuts CA at Y and DE cuts AB at Z. Prove that X, Y and Z are collinear.

Section 3. Analytic Approach.

We begin with some analytic geometry but in a rather unusual fashion. Consider a typical line $3x + 2y - 6 = 0$ which does not pass through the origin. It is a collection of points (x, y) where $y = 2 - \frac{2}{3}x$. We can plot this line by computing points on it.

x	-2	-1	0	1	2	3	4	5
y	$\frac{10}{3}$	$\frac{8}{3}$	2	$\frac{4}{3}$	$\frac{2}{3}$	0	$-\frac{2}{3}$	$-\frac{4}{3}$

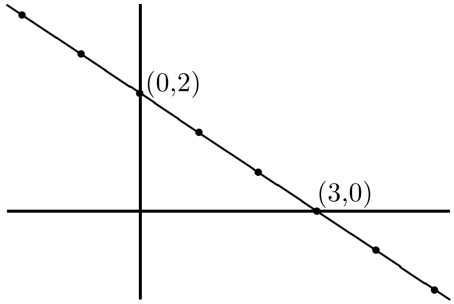

Figure 2.12

Since the constant term is non-zero, we divide the equation by it so that it now has the form $-\frac{1}{3}x - \frac{1}{2}y + 1 = 0$. The general form is $ux + vy + 1 = 0$. Here $u = -\frac{1}{3}$ and $v = -\frac{1}{2}$. Note that $-\frac{1}{u} = 3$ is the x-intercept of the line while $-\frac{1}{v} = 2$ is the y-intercept of the line.

We have seen in Section 1 the symmetry between points and lines in the projective plane. If we can give Cartesian coordinates to points, we should be able to give some kind of coordinates to lines too. From the symmetry in the equation $ux + vy + 1 = 0$ between x and y on the one hand versus u and v on the other hand, it is reasonable to define its **line coordinates** to be $[u, v]$. We use square brackets to remind ourselves that we are not talking about point coordinates.

In the equation $-\frac{1}{3}x - \frac{1}{2}y + 1 = 0$, we have treated x and y as variables and give specific numerical values to u and v. What we get is a line. What will we get if we treat u and v as variables and give specific numerical values to x and y? For instance, what is represented by $3u - 2v + 1 = 0$?

Just as $-\frac{1}{3}x - \frac{1}{2}y + 1 = 0$ represent all the points on the line $2x + 3y - 6 = 0$, we expect $3u - 2v + 1 = 0$ to represent all the lines passing through some point. Clearly, this point should be $(3, -2)$. Let us verify this graphically.

u	-1	1	$\frac{1}{3}$	$\frac{1}{5}$
v	-1	2	1	$\frac{4}{5}$

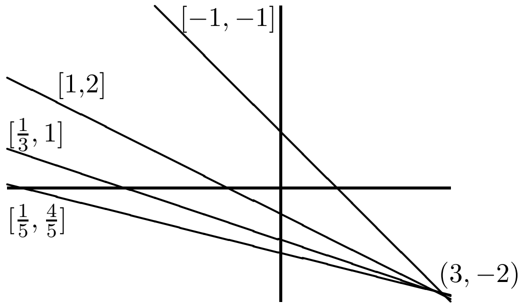

Figure 2.13

Consider the horizontal line passing through $(3,-2)$. The x-intercept approaches infinity, so that u approaches 0. Setting $u=0$ in $3u-2v+1=0$, we have $v=\frac{1}{2}$. Hence the desired line coordinates are $[0,\frac{1}{2}]$. Note that its y-intercept is indeed $-\frac{1}{v}=-2$. Similarly, the line coordinates of the vertical line passing through $(3,-2)$ is $[-\frac{1}{3},0]$.

In general, $[u,0]$ is the vertical line with x-intercept $-\frac{1}{u}$ while $[0,v]$ is the horizontal line with y-intercept $-\frac{1}{v}$. Note that [0,0] does not represent any line. So now every line not passing through the origin has line coordinates.

The equation in point coordinates of the unit circle is $x^2+y^2=1$, representing all points which lie on the circle. What should the equation be in line coordinates? What are the lines which should satisfy this equation? Lines can define a circle just as well as points. If we draw all lines tangent to the unit circle, they will form what is called an *envelope* of the circle.

Solving $x^2+y^2=1$ for x, we have $x=\pm\sqrt{1-y^2}$. Substituting this into $ux+vy+1=0$, we have $vy+1=\mp u\sqrt{1-y^2}$. Squaring both sides, we have $v^2y^2+2vy+1=u^2(1-y^2)$. This may be rewritten as a quadratic equation in y, namely, $(u^2+v^2)y^2+2vy+(1-u^2)=0$. If it has two real roots, they are the y coordinates of the points of intersection of the unit circle with this line. Since we want this line to be a tangent, the discriminant of the equation should be zero. Now $4v^2-4(u^2+v^2)(1-u^2)=0$ simplifies to $u^2(u^2+v^2-1)=0$. Apart from the singularities represented by the two horizontal tangents, $u\neq 0$, so that the equation of the unit circle in line coordinates is $u^2+v^2=1$, another triumph of the symmetry between points and lines.

However, as in Section 1, this symmetry is incomplete if we confine ourselves to the Euclidean plane. Lines passing through the origin do not have line coordinates. To achieve complete symmetry, we must once again extend to the projective plane, but then we need some way of coordinatizing ideal elements. This can be done by means of **homogeneous coordinates**.

In the Euclidean plane, a line is represented by an equation of the form $Ax + By + C = 0$, which equation is non-homogeneous if $C \neq 0$. This is because two of the terms are of degree 1 while the third term is of degree 0.

We now coordinatize a point (x, y) as $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{z}$ is any non-zero number such that $\frac{\overline{x}}{\overline{z}} = x$ and $\frac{\overline{y}}{\overline{z}} = y$. Just as both $\frac{3}{5}$ and $\frac{6}{10}$ represent the same fraction, both (3,5,1) and (6,10,2) represent the same point, namely (3,5). In general, $(\overline{\lambda x}, \overline{\lambda y}, \overline{\lambda z})$ represent the same point for any non-zero real number λ.

At first, it may seem that we are moving in the wrong direction, in that things become somewhat more complicated than before. If we insist that $\overline{z}$ be non-zero, then this is indeed much ado for nothing. However, by letting $\overline{z}$ tend to 0, both x and y tend to infinity provided $\overline{x}$ and $\overline{y}$ are non-zero. We have found a way to coordinatize ideal points, by setting $\overline{z} = 0$.

Note also that $Ax + By + C = 0$ becomes $A\overline{x} + B\overline{y} + C\overline{z} = 0$, which is always a homogeneous equation. This is the reason for the name homogeneous coordinates. As we shall see, there are significant advantages working with a homogeneous equation over working with a non-homogeneous one.

We first establish homogeneous point coordinates. We have no use for the triple (0,0,0). Every other triple represents some point in the projective plane. As we have seen, if $\overline{z} \neq 0$, $(\overline{x}, \overline{y}, \overline{z})$ represent an ordinary point (x, y) where $x = \frac{\overline{x}}{\overline{z}}$ and $y = \frac{\overline{y}}{\overline{z}}$.

The ideal point on a line with slope $\frac{y}{x}$ is represented by $(\overline{x}, \overline{y}, 0)$. The ideal point on the x-axis is represented by $(\overline{x}, 0, 0)$ and the ideal point on the y-axis is represented by $(0, \overline{y}, 0)$. The last two points are more conveniently represented by (1,0,0) and (0,1,0) respectively.

We now turn our attention to homogeneous line coordinates. In particular, we have to take care of lines passing through the origin, as well as the ideal line. It is perhaps not surprising that $[u, v]$ now becomes $[\overline{u}, \overline{v}, \overline{w}]$, where $\frac{\overline{u}}{\overline{w}} = u$ and $\frac{\overline{v}}{\overline{w}} = v$. Once again, we have no use for the triple [0,0,0], and $[\overline{\lambda u}, \overline{\lambda v}, \overline{\lambda w}]$ represent the same line for any non-zero real number λ.

If $\overline{w} \neq 0$, $[\overline{u}, \overline{v}, \overline{w}]$ is the ordinary line $[u, v]$ which does not pass through the origin. The homogeneous coordinates for the ideal points suggest that the homogeneous equation for the ideal line should be $\overline{wz} = 0$, so that its line coordinates may be taken as [0,0,1]. Thus we have found a use for the pair [0,0] which was left out before.

The homogeneous equation for a line passing through the origin has the form $\overline{ux} + \overline{yv} = 0$ so that its line coordinates are $[\overline{u}, \overline{v}, 0]$. In particular, the line with slope $\frac{3}{5}$ has line coordinates $[3, -5, 0]$. The line coordinates for the x-axis is [0,1,0] and those for the y-axis are [1,0,0].

We are now all set to do some analytic geometry in homogeneous coordinates. Note that the equation $\overline{ux}+\overline{vy}+\overline{wz}=0$ holds if and only if the point $(\overline{x},\overline{y},\overline{z})$ lies on the line $[\overline{u},\overline{v},\overline{w}]$. The expression $\overline{ux}+\overline{vy}+\overline{wz}$ is called the *dot product* of $(\overline{x},\overline{y},\overline{z})$ and $[\overline{u},\overline{v},\overline{w}]$, the sum of the products of corresponding terms. Naturally, it is denoted by a dot, and is a commutative operation.

Our first task is to find the homogeneous coordinates $(\overline{x},\overline{y},\overline{z})$ of the point of intersection of two lines whose homogeneous coordinates are $[\overline{u_1},\overline{v_1},\overline{w_1}]$ and $[\overline{u_2},\overline{v_2},\overline{w_2}]$. That these are lines means that neither triple is [0,0,0]. That they are distinct lines means that $\frac{\overline{u_1}}{\overline{u_2}}=\frac{\overline{v_1}}{\overline{v_2}}=\frac{\overline{w_1}}{\overline{w_2}}$ does not hold. By symmetry, we may assume that $\overline{u_1v_2}\neq\overline{v_1u_2}$.

The incidence relations yield a system of simultaneous equations,namely, $\overline{xu_1}+\overline{yv_1}+\overline{zw_1}=0$ and $\overline{xu_2}+\overline{yv_2}+\overline{zw_2}=0$. Solving for $\overline{x}$ and $\overline{y}$ in terms of $\overline{z}$, we have $\overline{x}=\frac{\overline{v_1w_2}-\overline{w_1v_2}}{\overline{u_1v_2}-\overline{v_1u_2}}\overline{z}$ and $\overline{y}=\frac{\overline{w_1u_2}-\overline{u_1w_2}}{\overline{u_1v_2}-\overline{v_1u_2}}\overline{z}$. It follows that we have $(\overline{x},\overline{y},\overline{z})=(\overline{v_1w_1}-\overline{w_1v_2},\overline{w_1u_2}-\overline{u_1w_2},\overline{u_1v_2}-\overline{v_1u_2})$.

This triple is called the *cross product* of $[\overline{u_1},\overline{v_1},\overline{w_1}]$ and $[\overline{u_2},\overline{v_2},\overline{w_2}]$. Naturally, it is denoted by a cross. It is anti-commutative in that if we reverse the order of the two factors, the terms in the product change signs but retain the same numerical values. Thus in homogeneous coordinates, it does not matter. Figure 2.14 shows a schematic computation of the cross product.

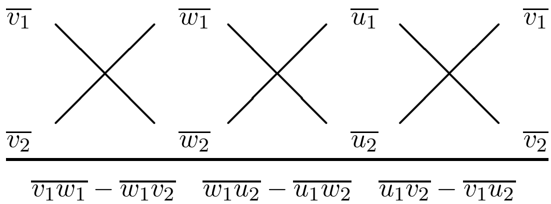

Figure 2.14

The companion task of finding the homogeneous coordinates $[\overline{u},\overline{v},\overline{w}]$ of the line passing through the two points whose homogeneous coordinates are $(\overline{x_1},\overline{y_1},\overline{z_1})$ and $(\overline{x_2},\overline{y_2},\overline{z_2}]$ parallels the above task, with the cross product of these two triples as the end result, on the assumption that $\overline{x_1y_2}\neq\overline{y_1x_2}$.

Let us use homogeneous coordinates to prove Desargues' Theorem. In Figure 2.3, choose V to be (1,1,1), A to be (1,0,0), B to be (0,1,0) and C to be (0,0,1). Then the line VA is $(1,1,1)\times(1,0,0)=[0,1,-1]$, which is $\overline{y}=\overline{z}$. Now A' is another point on this line. Hence its coordinates are $(a,1,1)$, with $a\neq 1$ to make is distinct from A. Similarly, B' is $(1,b,1)$ and C' is $(1,1,c)$, with $b\neq 1\neq c$.

There is one other condition, that these three points are not collinear. Now the line $A'B'$ is $(a,1,1)\times(1,b,1)=[1-b,1-a,ab-1]$. If C' lies on it, then $(1,1,1)\cdot[1-b,1-a,ab-1]=0$. It follows that we must have $(1-b)+(1-a)+c(ab-1)=abc+2-a-b-c\neq 0$.

The line AB is $(1,0,0) \times (0,1,0) = [0,0,1]$. The point N is therefore $[0,0,1] \times [1-b, 1-a, ab-1] = (a-1, 1-b, 0)$. Similarly, the point M is $(a-1, 0, 1-c)$ and the point L is $(0, b-1, 1-c)$. The line MN is $(a-1,0,1-c) \times (a-1,1-b,0) = [(b-1)(1-c), (a-1)(1-c), -(a-1)(b-1)]$. Since $(0, b-1, 1-c) \cdot [(b-1)(1-c), (a-1)(1-c), -(a-1)(b-1)] = 0$, L indeed lies on MN. Hence L, M and N are collinear, proving Desargues' Theorem.

We point out that the coordinates (1,1,1), (1,0,0), (0,1,0) and (0,0,1) can be assigned arbitrarily to any four points provided no three of them are collinear.

We now use homogeneous coordinates to prove the Pappus-Pascal Theorem. In Figure 2.15, choose E to be (1,1,1), C to be (1,0,0), F to be (0,1,0) and N to be (0,0,1). Then the line CE is $(1,1,1) \times (1,0,0) = [0,1,-1]$, and as before, A is $(a,1,1)$ for some $a \neq 1$. Similarly, L is $(1,\ell,1)$ and D is $(1,1,d)$ for some $\ell \neq 1 \neq d$.

The line AN is $(a,1,1) \times (0,0,1) = [1,-a,0]$. On the other hand, the line CL is $(1,0,0) \times (1,\ell,1) = [0,-1,\ell]$. It follows that the point B is $[1,-a,0] \times [0,-1,\ell] = (-a\ell, -\ell, -1)$. Since this point lies on DF, which is $(0,1,0) \times (1,1,d) = [-d,0,1]$, we must have $(-a\ell, -\ell, -1) \cdot [-d,0,1] = 0$. This simplifies to $\ell ad - 1 = 0$.

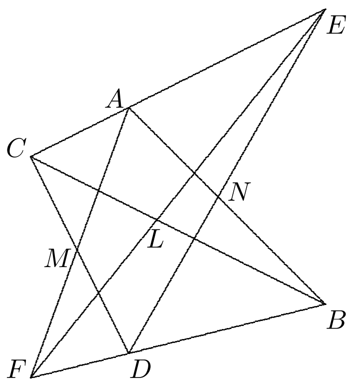

Figure 2.15

The line AF is $(a,1,1) \times (0,1,0) = [-1,0,a]$. On the other hand, the line CD is $(1,0,0) \times (1,1,d) = [0,-d,1]$. It follows that the point M is $[-1,0,a] \times [0,-d,1] = (ad,1,d)$. Now LN is $(1,\ell,1) \times (0,0,1) = [\ell,-1,0]$. Since $(ad,1,d) \cdot [\ell,-1,0] = \ell ad - 1 = 0$, M indeed lies on LN. Hence L, M and N are collinear, proving the Pappus-Pascal Theorem.

This theorem has a surprising algebraic consequence. In Figure 2.4, let $VC = VE = x$, $VA = VY = y$ and $VB = 1$. Now $\frac{VE}{VD} = \frac{VA}{VB}$ or $VE = yx$. On the other hand, $\frac{VE}{VF} = \frac{VC}{VB}$ or $VE = xy$. Thus the Pappus-Pascal Theorem implies that $xy = yx$, the commutative law of multiplication!

Exercises:

7. The equation $2y^2 = x$ in point coordinates represents a parabola, a curve which we will study in the next chapter. Find its equation in line coordinates.

8. Use homogeneous coordinates to prove the converse of Desargues' Theorem.

9. Use homogeneous coordinates to prove the Pappus-Brianchon Theorem.

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Solution to Exercises

1. (a) Note that B' and C' lie on the plane VBB', so that the coplanar lines BC and $B'C'$ will intersect at some point L. Similarly, CA cuts $C'A'$ at some point M, and AB cuts $A'B'$ at some point N. Now Π and Π' meet along a line, as shown in Figure 2.3. Since L lies in both Π and Π', it lies on this line. Similarly, so do M and N. Hence these three points are collinear.

   (b) Through V, draw a line not on the plane Π of ABC. Take on this line points D and D'. Now BCD and $B'C'D'$ are two non-coplanar triangles satisfying the hypothesis of Desargues' Theorem in Space. Hence BD meets $B'D'$ at some point Y, and CD meets $C'D'$ at some point Z. Moreover, L, Y and Z are collinear. Similarly, AD meets $A'D'$ at some point X, and M, Z and X are collinear, as are N, X and Y. Note that none of X, Y and Z is in Π. A plane Ω containing X, Y and Z will also contain L, M and N, and L, M and N all lie on the line of intersection of Π and Ω.

2. (a) We first deduce the converse from Desargues' Theorem. Consider triangles $BB'N$ and $CC'M$. Note that $B'N$ cuts $C'M$ at D while NB cuts MC at A. Since BC cuts $B'C'$ at L, the lines BB' and CC' are coplanar, and will meet at some point V. Since BC, $B'C'$ and MN are concurrent at L, it follows from Desargues' Theorem that V, A and A' are collinear, which is equivalent to the desired result.

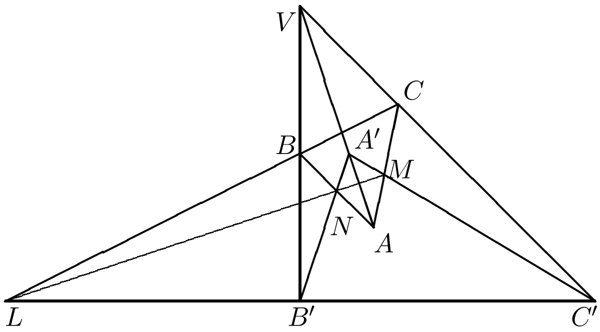

Figure 2.16

We now deduce Desargues' Theorem from its converse. Consider triangles $BB'N$ and $CC'M$. BB' cuts CC' at V, $B'N$ cuts $C'M$ at A', and NB cuts MC at A. Since V, A' and A are collinear, $BC, B'C'$ and MN are concurrent by the converse of Desargues' Theorem. Since BC and $B'C'$ meet at L, L, M and N are collinear.

(b) We first deduce the Pappus-Brianchon Theorem from the Pappus-Pascal Theorem. Let the two given points of concurrency be P and Q respectively, and let BE and CF meet at R. Now E, F and P are collinear, as are B, C and Q. A is the point of intersection of BP and FQ, and D is the point of intersection of PC and QE. It follows from the Pappus-Pascal Theorem that A, D and R are collinear. In other words, AD is concurrent with BE and CF.

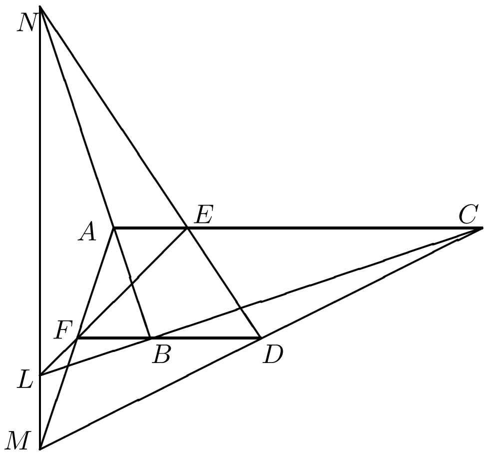

Figure 2.17

We now deduce the Pappus-Pascal Theorem from the Pappus-Brianchon Theorem. Note that AE, LB and DM are concurrent at C while EL, BD and MA are concurrent at F. By the Pappus-Brianchon Theorem, AB, ED and ML are also concurrent. Since AB and DE meet at N, L, M and N are collinear.

3. (a) Let C be the point of intersection of ℓ and m, and C' be that of ℓ' and m'. Take a third point V on CC'. Draw a line through V, cutting ℓ at A and ℓ' at A'. Draw a second line through V, cutting m at B and m' at B'. Let AB cut $A'B'$ at N. Apply Desargues' Theorem to triangles ABC and $A'B'C'$. Since AA', BB' and CC' are concurrent at V, L, M and N are collinear. Repeat this construction varying the position of V on CC'. The line joining the old and new N is the desired line LM.

   (b) Let AA' cut BB' at V and draw a third line m through V. Take C and C' on m. Let AC cut $A'C'$ at M and BC cut $B'C'$ at L. Apply the converse of Desargues' Theorem to triangles MAA' and LBB'. Since C, C' and V ar collinear, LM is concurrent with ℓ and ℓ'. Repeat this construction varying the positions of C and C' on m. The point of intersection of the old and the new LM is the desired point of intersection of ℓ and ℓ'.

4. (a) Applying Menelaus' Theorem to BCV, CAV and ABV, we have $(BCV, LC'B') = (CAV, MA'C') = (ABV, NB'A') = -1$. Multiplying these together, we have $(ABC, NLM) = -1$. By the converse of Menelaus' Theorem, L, M and N are collinear.

   (b) Let AA' and BB' meet at V. Applying Menelaus' Theorem to NBB', NBL and $NB'L$, we have $(NBB', AVA') = -1$ and $(NBL, ACM) = (NB'L, A'C'M) = -1$. Multiplying these together, we have $(LBB', CVC') = -1$. C, C' and V are collinear by the converse of Menelaus' Theorem, so that AA', BB' and CC' are concurrent.

   (c) Let BC cut DE at X, DE cut FA at Y, and FA cut BC at Z. By Menelaus' Theorem, $(XYZ, EFL) = (XYZ, DMC) = -1$ and $(XYZ, NAB) = (XZY, CAE) = (XZY, BFD) = -1$. Multiplying these together, we have $(XYZ, NML) = -1$. By the converse of Menelaus' Theorem, L, M and N are collinear.

   (d) Let BC cut EF at X, DE cut FA at Y, EF cut AB at Z, and AD cut CF at R. We have $(ADY, RCF) = (DQY, EFP) = -1$ as well as $(BCP, QYA) = (PCB, DQZ) = (QZA, EPF) = -1$ by Menelaus' Theorem. Multiplying these together, we then have $(ADZ, REB) = -1$. By the converse of Menelaus' Theorem, B, E and R are collinear, so that AD, BE and CF are concurrent at R.

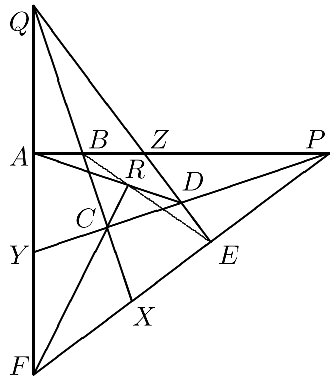

Figure 2.18

5. (a) In ABC, Menelaus' Theorem implies $(ABC, FKE) = -1$. Since BK and BD are angle bisectors, $-\frac{\overline{BK}}{\overline{KC}} = \frac{AB}{AC} = \frac{\overline{BD}}{\overline{DC}}$. Hence $(ABC, FDE) = 1$. By the converse of Ceva's Theorem, AD, BE and CF are concurrent.

   (b) Applying Ceva's Theorem to ABC, we have $(ABC, FDE) = 1$. Since BD and BK are angle bisectors, $\frac{\overline{BD}}{\overline{DC}} = \frac{AB}{AC} = -\frac{\overline{BK}}{\overline{KC}}$. Hence $(ABC, FKE) = -1$. By the converse of Menelaus' Theorem, K, E and F are collinear.

   (c) Menelaus' Theorem implies $(ABC, FLE) = (ABC, FDM) = -1$ and $(ABC, NDE) = -1$. Multiplying these together, we have $(ABC, NLM)(ABC, FDE)^2 = -1$. Since L, M and N are collinear, $(ABC, NLM) = -1$. Hence $(ABC, FDE) = \pm 1$. Since D, E and F all lie on the perimeter of ABC, they cannot be collinear. Hence $(ABC, FDE) \neq -1$. By the converse of Ceva' Theorem, AD, BE and CF are concurrent.

6. (a) We have $(DC, BA) = \frac{\overline{DB} \cdot \overline{AC}}{\overline{BC} \cdot \overline{DA}} = \frac{\overline{DB} \cdot \overline{AC}}{\overline{CB} \cdot \overline{AD}} = (AB, CD)$.

   (b) We have $(AB, DC) = \frac{\overline{AD} \cdot \overline{CB}}{\overline{DB} \cdot \overline{AC}} = \frac{1}{(AB,CD)}$.

   (c) We have $(BD, CX)P(EQ, FX)$ where Q is the point of intersection of EF and AD. Now $(EQ, FX)D(EA, YC)$. By (a) and (b), $(EA, YC) = \frac{1}{(EA,CY)} = \frac{1}{(YC,AE)} = (YC, EA) = (AE, CY)$. From $(BD, CX) = (AE, CY)$, AB, DE and XY are concurrent. Since AB cuts DE at Z, X, Y and Z are collinear.

7. We have $2uy^2 + vy + 1 = 0$, a quadratic equation with discriminant $v^{-}4(2u) = 0$. It follows that the desired equation in line coordinates is $v^2 - 8u = 0$.

8. In Figure 2.16, omit the line VA. Choose LN to be [1,1,1], BC to be [1,0,0], CA to be [0,1,0] and AB to be [0,0,1]. Then the point L is $[1,1,1] \times [1,0,0] = (0,1,-1)$. Now $B'C'$ is another line passing through N. Hence its coordinates are $[a,1,1]$ for some $a \neq 1$. Similarly, $C'A'$ is $[1,b,1]$ and $A'B'$ is $[1,1,c]$, with $b \neq 1 \neq c$. Note that C' is $[a,1,1] \times [1,b,1] = (1-b, 1-a, ab-1)$. On the other hand, C is $[1,0,0] \times [0,1,0] = (0,0,1)$. It follows that the line CC' is given by $(0,0,1) \times (1-b,1-a,ab-1) = [a-1,1-b,0]$. Similarly, the line BB' is $[a-1,0,1-c]$ and the line AA' is $[0,b-1,1-c]$. Let V be the point of intersection of BB' and CC'. Then its homogeneous coordinates are $[a-1,0,1-c] \times [a-1,1-b,0] = ((b-1)(1-c),(a-1)(1-c),-(a-1)(b-1))$. Since $[0,b-1,1-c] \cdot ((b-1)(1-c),(a-1)(1-c),-(a-1)(b-1)) = 0$, AA' indeed passes through V. Hence AA', BB' and CC' are concurrent, proving the converse of Desargues' Theorem.

9. In Figure 2.18, omit the line CF and the points X, Y and Z. Choose R to be (1,1,1), A to be (1,0,0), C to be (0,1,0) and E to be (0,0,1). Then AF is $(1,0,0) \times (1,1,1) = [0,-1,1]$. Since Q is another point on this line, take Q to be $(q,1,1)$ for some $q \neq 1$. Similarly, P may be taken as $(1,1,p)$ for some $p \neq 1$. We perform the following computations.

$$
\begin{array}{ccccccc}
QC & : & (0,0,1) & \times & (q,1,1) & = & [1,0,-q] \\
PC & : & (0,1,0) & \times & (1,1,p) & = & [p,0,-1] \\
PA & : & (1,0,0) & \times & (1,1,p) & = & [0,-p,1] \\
QE & : & (0,0,1) & \times & (q,1,1) & = & [-1,q,0] \\
B & : & [0,-p,1] & \times & [1,0,-q] & = & (pq,1,p) \\
D & : & [-1,q,0] & \times & [p,0,-1] & = & (-q,-1,-pq) \\
AD & : & (1,0,0) & \times & (-q,-1,-pq) & = & [0,pq,-1] \\
BE & : & (0,0,1) & \times & (pq,1,p) & = & [-1,pq,0] \\
R & : & [0,pq,-1] & \times & [-1,pq,0] & = & (pq,q,pq) \\
CF & : & (0,1,0) & \times & (1,1,1) & = & (1,0,-1)
\end{array}
$$

Since $(1,0,-1) \cdot [pq,0,pq] = 0$, R indeed lies of CF. Hence AD, BE and CF are concurrent, proving the Pappus-Brianchon Theorem.

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S.M.A.R.T. Circle Minicourses
Liu, A.C.-F.
2018, XIV, 244 p. 173 illus., Softcover
ISBN: 978-3-319-71742-5