Chapter 2
Theorems of Helmholtz and Kelvin

2.1 Introduction

Helmholtz' (1858) vortex theorems paved the way for the legendary Ludwig Prandtl (1921) to invent the lifting line theory, a crowning advancement in theoretical aerodynamics. Sommerfeld (1950) remarked that *the main contents of Helmholtz's theory* are *conservation laws: It is impossible to produce or destroy vortices, or, in more general terms, the vortex strength is constant in time*. Sommerfeld noted that Helmholtz' theorems are correct under the conditions *the fluid is inviscid and incompressible; the external forces possess a single-valued potential within the space filled by the fluid*. Sommerfeld also pointed out that in Helmholtz' theorems, *apart from the conservation of the vortex strength in time*, *there is also a spatial conservation: the vortex strength is constant along each vortex line or vortex tube, which must either be closed or end at the boundary of the fluid*. In the current literature, Helmholtz' theorem on the spatial conservation of the vorticity strength is called Helmholtz' first vortex theorem, and the theorem on the time conservation is called his second vortex theorem.

Lighthill (1986) discussed Kelvin's (1869) theorem on the persistence of circulation and pointed out that *Kelvin's theorem is exact for the Euler model*, which contains the key assumptions that the fluid is *inviscid and incompressible*. Lighthill noted: *One especially valuable deduction from Kelvin's theorem is concerned with "the movement of vortex lines". This is Helmholtz's theorem (also exact on the Euler model), which states that vortex lines move with the fluid.*

In vorticity dynamics, the strategy of partitioning the overall flow problem into its kinematic and kinetic aspects offers advantages in aerodynamic analyses (Wu 1981; Tuncer et al. 1990). In the present study, this strategy is used in a revisit of the vortex theorems of Helmholtz and Kelvin and in generalizing these theorems to flows of viscous fluids. It is learned from this revisit that Helmholtz' first vortex theorem on spatial conservation is a theorem of vorticity kinematics and, as such, is valid not only in the fluid region but also in the solid region. This observation leads

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to the portrayal of the vorticity distribution as a system of closed vorticity tubes. With this portrayal, classical aerodynamic theories, including the lifting line theory, are interpretable on the basis of the vorticity-moment theorem (Wu 1981, Chap. 6). This portrayal also brings forth opportunities for establishing new approaches for viscous and unsteady aerodynamic analysis (Wu et al. 2002, Chap. 7).

Helmholtz' second vortex theorem, or its equivalence Kelvin's theorem, is a vorticity-dynamic theorem based on both kinetics and kinematics. The generalized second vortex theorem states that the vorticity strength in the viscous fluid is not conserved in time; it diffuses at a predictable rate.

2.2 Kinematic-Kinetic Partition of Flow Problems

As discussed in Chap. 1, the present study deals with the incompressible flow of a viscous fluid as described by the following set of two differential equations:

$$\nabla \cdot \mathbf{v} = 0 \tag{2.1}$$

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \nabla q \tag{2.2}$$

where $\mathbf{v}$ is the flow velocity, p, ρ, and ν are respectively the pressure, the density, and the kinematic viscosity of the fluid, and ∇q represents a conservative body force.

Equation (2.1), the continuity equation, is a mathematical statement of the law of conservation of mass for the incompressible flow. Equation (2.2), Navier–Stokes' momentum equation, expresses Newton's second law of motion for the viscous fluid. For the idealized inviscid fluid, the last term in (2.2) is zero and one has Euler's momentum equation for the incompressible flow

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{\rho}\nabla p + \nabla q \tag{2.3}$$

The overall problem of the incompressible flow of a viscous fluid is partitioned into two aspects by ascribing (2.1) to kinematics and (2.2), or (2.3), to kinetics. Kinematics and kinetics, the two branches of dynamics, are typically defined as follows: *Kinematics* is *the study of motions of themselves, apart from their causes*. *Kinetics is the study of changes of motions produced by forces*. In ascribing (2.1) to kinematics, the view is taken that the law of mass conservation imposes a condition on the incompressible motion of the fluid. Namely, the incompressible velocity field must be divergence-free. This condition is imposed on the flow at each instant of time, and is not regarded as a cause of change of the flow with time. This view is generally accepted in classical studies of the incompressible flow.

In an elegant treatise, Truesdell (1954) stated: *In general the flow of a fluid, whether perfect or viscous, can be defined by kinematical conditions.* In Truesdell's view, *circulation preservation* in time is one of the kinematical conditions on the flow. He then presented extensive results based on this view. In the present study, circulation preservation is viewed not as a kinematic condition, but as a kinetic consequence. Based on this perspective, it is shown that, in the case of the viscous flow, the kinetics as described by (2.2) causes the circulation to diffuse. However, in regions of the flow where viscous effects are insignificant, (2.3) brings about circulation preservation with time.

Scholars in fluid dynamics with different research focuses often interpret the terms kinematics and kinetics differently. Differences in interpretations are, by themselves, not important to the present study. The recognition that the overall flow problem contains two aspects, each with its own physical and mathematical characteristics is, however, crucially important. Viewing (2.1) as a constraint and (2.2) as a cause–effect relationship makes it feasible to treat these two interlaced aspects individually before uniting the individual results. The advantages of this strategy are not obvious if (2.1) and (2.2) are both viewed as kinematic constraints. These advantages become pronounced in studies of the flow of the viscous fluid using concepts of vorticity dynamics.

The familiar definition of the vorticity field, as stated in Chap. 1, is

$$\nabla \times \mathbf{v} = \boldsymbol{\omega} \tag{2.4}$$

Equations (2.1), (2.2), and (2.4) constitute a set of vorticity-dynamic equations describing the flow of the viscous fluid. Of these equations, (2.1) and (2.4) are kinematic since they deal with motions without reference to forces that cause motions. The kinetic Eq. (2.2) can be restated in terms of $\boldsymbol{\omega}$ as

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla(h - q) - \omega \times \mathbf{v} - \nu \nabla \times \omega \tag{2.5}$$

where $h = p/\rho + (\mathbf{v} \cdot \mathbf{v})/2$ is the total head.

Taking the curl of each term in (2.5) yields the vorticity-transport equation:

$$\frac{\partial \omega}{\partial t} = -\nabla \times (\omega \times \mathbf{v}) - \nu \nabla \times (\nabla \times \omega) \tag{2.6}$$

For the viscous fluid, the flow kinetics is represented by either (2.5) or (2.6). Although (2.6) does not contain a force term explicitly, it is considered a kinetic equation because it is a forthwith consequence of (2.2). This equation describes the change of the vorticity field with time. The terms on the right-hand side of (2.6) represent physical processes causing the change. For the inviscid fluid, the flow kinetics is described by (2.7) or (2.8)

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla h - \omega \times \mathbf{v} \tag{2.7}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = -\nabla \times (\omega \times \mathbf{v}) \tag{2.8}$$

Several useful mathematical identities are derived in §2.3. These identities are not new. They are included here to facilitate the discussions of the theorems of Helmholtz and Kelvin. The reader wishing to verify the derivations may wish to refer to vector identities listed in §3.2.

2.3 Kinematic Preliminaries

In the following discussions, sufficient smoothness of lines (curves) and surfaces in space and sufficient differentiability of field variables are assumed so that the divergence theorem, Stokes' theorem, and other mathematical relations are meaningful.

A curve in space whose tangent at each point on it is in the direction of a vector field **f** is called a vector line of **f**. A surface comprising all the vector lines of **f** passing through a circuit (a closed curve) in space is called a vector tube of **f**. Vector lines and vector tubes of the velocity field **v** are called *streamlines* and *stream tubes*. Vectors lines and vector tubes of the vorticity field $\boldsymbol{\omega}$ are called vorticity lines and vorticity tubes.

A vector field **f** is said to be solenoidal if its divergence, $\nabla \cdot \mathbf{f}$, is zero and lamellar if its curl, $\nabla \times \mathbf{f}$, is zero. A solenoidal velocity field **v** is said to be incompressible and a lamellar velocity field irrotational. In the following discussions, the terms irrotational and lamellar are used interchangeably: **f** is said to be irrotational or lamellar if $\nabla \times \mathbf{f} = 0$.

With Helmholtz' decomposition, a general vector field **f** is expressible in the form

$$\mathbf{f} = \nabla g + \nabla \times \mathbf{h} \tag{2.9}$$

where g is a scalar potential function and **h** is a vector potential function. By virtual of the vector identities $\nabla \times (\nabla g) = 0$ and $\nabla \cdot (\nabla \times \mathbf{h}) = 0$, ∇g is lamellar and $\nabla \times \mathbf{h}$ is solenoidal. Thus (2.9) decomposes the general vector field **f** into a solenoidal field, denoted $\mathbf{f}^*$ in this study, and a lamellar field denoted $\mathbf{f}'$. A lamellar field $\mathbf{f}'$ is expressible as ∇g alone and a solenoidal field $\mathbf{f}^*$ as $\nabla \times \mathbf{h}$ alone.

Consider two circuits C_1 and C_2 that lie on the same vector tube of **f**, each circuit encircles the tube once. Let S_1 be a cap of C_1 and S_2 a cap of C_2. Denote the section of the vector tube between C_1 and c by S_e. The three surfaces S_1, S_2, and S_e together form a closed surface S_o bounding a simply connected region R_o. The divergence theorem then gives

$$\iint_{S_1} \mathbf{f}\cdot\mathbf{n}\,dS + \iint_{S_2} \mathbf{f}\cdot\mathbf{n}\,dS + \iint_{S_e} \mathbf{f}\cdot\mathbf{n}\,dS = \iint_{R_0} \nabla\cdot\mathbf{f}\,dR \tag{2.10}$$

where $\mathbf{n}$ is the unit normal vector on the surfaces S_o directed outward from R_o.

The field $\mathbf{f}\cdot\mathbf{n}$ is a measure of the strength of the normal component of $\mathbf{f}$ on S and is called the *flux of* $\mathbf{f}$ on S. Since $\mathbf{f}$ is tangent to the vector tube, $\mathbf{f}\cdot\mathbf{n} = 0$ on S_e. For the solenoidal field $\mathbf{f}^*$, the right-hand side of (2.10) is zero. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be two unit normal vectors on S_1 and S_2 respectively, with their positive directions given by the axial direction of the vector tube. With this orientation, if $\mathbf{n}_1 = \mathbf{n}$ on S_1, then $\mathbf{n}_2 = -\mathbf{n}$ on S_2. On the other hand, if $\mathbf{n}_1 = -\mathbf{n}$, then $\mathbf{n}_2 = \mathbf{n}$ on S_2. In either case, one obtains from (2.10)

$$\iint_{S_1} \mathbf{f}^*\cdot\mathbf{n}_1\,dS + \iint_{S_2} \mathbf{f}^*\cdot\mathbf{n}_2\,dS \tag{2.11}$$

The two integrals in (2.11) are called respectively the total flux of the field $\mathbf{f}^*$ across S_1 and S_2 or the strength of $\mathbf{f}^*$ of the vector tube at the S_1 and S_2. This equation states that the strength of the vector tube of $\mathbf{f}^*$ is constant along the path of the tube. Equation (2.11) expresses a spatial, or kinematic, conservation relationship. If the solenoidal vector field $\mathbf{f}^*$ is time-dependent, then the spatial conservation is valid at each instant of time. Representing $\mathbf{f}^*$ by $\nabla\times\mathbf{h}$, one obtains, using Stokes' theorem

$$\oint_C \mathbf{h}\cdot\boldsymbol{\tau}\,ds = \iint_S \mathbf{f}^*\cdot\mathbf{n}\,dS \tag{2.12}$$

In (2.12), $\boldsymbol{\tau}$ is a unit tangent vector on the circuit C. The positive senses of $\mathbf{n}$ and $\boldsymbol{\tau}$ are related by the right-handed screw convention. Namely, as a right-handed screw turns in the positive $\boldsymbol{\tau}$ direction, it advances in the positive $\mathbf{n}$ direction. The left-hand side of (2.12) is called the circulation of $\mathbf{h}$ around C. Equation (2.12) states the total flux of a solenoidal field $\mathbf{f}^*$ across S is equal to the circulation of the vector potential $\mathbf{h}$ of $\mathbf{f}^*$ around C, of which S is a cap. The three terms *total flux across a surface*, *strength of vector tube*, and *circulation* are equivalent. These terms are used interchangeably in the literature, each describing the surface integral, or its equivalent line integral, in (2.12).

Let C be a time-dependent circuit. Denote the positions of C at the time levels $t_1 = t$ and $t_2 = t + \delta t$ respectively by C_1 and C_2. Consider the time derivative of the integration of $\mathbf{f}\cdot\boldsymbol{\tau}$ over C, where $\mathbf{f}$ is time-dependent. One writes

$$\frac{d}{dt}\oint_C \mathbf{f}(\mathbf{r},t)\cdot\boldsymbol{\tau}\,ds \equiv \lim_{\delta t\to 0}\left\{\left[\oint_{C_2}\mathbf{f}(\mathbf{r},t+\delta t)\cdot\boldsymbol{\tau}\,ds-\oint_{C_1}\mathbf{f}(\mathbf{r},t)\cdot\boldsymbol{\tau}\,ds\right]/\delta t\right\}$$
$$=\lim_{\delta t\to 0}\oint_{C_2}\left[\frac{\mathbf{f}(\mathbf{r},t+\delta t)-\mathbf{f}(\mathbf{r},t)}{\delta t}\right]\cdot\boldsymbol{\tau}\,ds+\lim_{\delta t\to 0}\left\{\left[\oint_{C_2}\mathbf{f}(\mathbf{r},t)\cdot\boldsymbol{\tau}\,ds-\oint_{C_1}\mathbf{f}(\mathbf{r},t)\cdot\boldsymbol{\tau}\,ds\right]/\delta t\right\} \tag{2.13}$$

Let S_1 be a cap of C_1 and S_2 a cap of C_2. Using Stokes' theorem, the last two integrals in (2.13) can be restated as integrations of $(\nabla\times\mathbf{f})\cdot\mathbf{n}_i$, i = 1 or 2, over the surfaces S_i. Let S_e be a surface joining C_1 and C_2 at the time level t. The three surfaces S_1, S_2, and S_e together form a closed surface S_o bounding the region R_o. If $\boldsymbol{\tau}$ is in a direction such that $\mathbf{n}_1=\mathbf{n}$, the outward unit normal vector on S_1, then, with the right-handed screw convention, $\mathbf{n}_2=-\mathbf{n}$ on S_2. Thus

$$\oint_{C_2}\mathbf{f}(\mathbf{r},t)\cdot\boldsymbol{\tau}\,ds-\oint_{C_1}\mathbf{f}(\mathbf{r},t)\cdot\boldsymbol{\tau}\,ds=-\oint\!\!\oint_{S_0}(\nabla\times\mathbf{f})\cdot\mathbf{n}\,dS+\iint_{S_e}(\nabla\times\mathbf{f})\cdot\mathbf{n}\,dS \tag{2.14}$$

The first integral on the right-hand side of (2.14) is an integral over the closed surface S_o. This integral can be restated, using the divergence theorem, as an integration of $\nabla\cdot(\nabla\times\mathbf{f})$ over R_o and is therefore zero. On S_e, as $\delta t\to 0$, one has $\mathbf{n}dS\to(\mathbf{v}_c\delta t)\times(\boldsymbol{\tau}ds)$, where $\mathbf{v}_c$ is the velocity of C_1. One therefore has

$$\iint_{S_e}(\nabla\times\mathbf{f})\cdot\mathbf{n}\,dS=\delta t\oint_{C_1}(\nabla\times\mathbf{f})\cdot(\mathbf{v}_c\times\boldsymbol{\tau})dS=\delta t\oint_{C_1}[(\nabla\times\mathbf{f})\times\mathbf{v}_c]\cdot\boldsymbol{\tau}\,dS \tag{2.15}$$

Placing (2.15) into (2.14), and the result into (2.13), one obtains, upon noting that, as $\delta t\to 0$, the integrand in the first term on right-hand side of (2.13) gives $(\partial\mathbf{f}/\partial t)\cdot\boldsymbol{\tau}$,

$$\frac{d}{dt}\oint_{C(t)}\mathbf{f}(\mathbf{r},t)\cdot\boldsymbol{\tau}\,ds=\oint_C\left(\frac{\partial\mathbf{f}}{\partial t}\right)\cdot\boldsymbol{\tau}\,ds+\oint_C[(\nabla\times\mathbf{f})\times\mathbf{v}_c]\cdot\boldsymbol{\tau}\,ds \tag{2.16}$$

where the superfluous subscripts for the circuit C are dropped. The symbol C(t) is used, rather than C, in the left-hand side of (2.16) to emphasize the time dependency of the path of integration. The two integrals on the right-hand side of (2.16) are not differentiated with respect to time. Therefore the time dependency of the integration path C is not an issue of concern. The line integral of $\mathbf{f}\cdot\boldsymbol{\tau}$ over a moving (and/or deforming) loop C(t) changes with time because of two contributing factors:

the local change of **f** on C and the motion of the loop C. The right-hand side integrals in (2.16) express separately the contributions.

For the lamellar vector field $\mathbf{f}' = \nabla g$, the last term in (2.16) vanishes since $\nabla \times (\nabla g) = 0$. The integral on the left-hand side of (2.16) vanishes since it can be restated as an integral of $(\nabla \times \mathbf{f}') \cdot \mathbf{n}$ over a cap S of C. The first term on the right-hand side of (2.16) is similarly zero because $\partial \mathbf{f}'/\partial t = \nabla(\partial g/\partial t)$. Therefore (2.16) is trivial for the lamellar field. Only the solenoidal part of the general vector field **f**, as represented by (2.9), plays a role in (2.16).

Consider the time derivative of the total flux of $\mathbf{f}(\mathbf{r},t)$ across S(t)

$$\begin{aligned}\frac{d}{dt}\iint\limits_{S} \mathbf{f}(\mathbf{r},t)\cdot\mathbf{n}_1\, dS &\equiv \lim_{\delta t\to 0}\left\{\left[\iint\limits_{S_2} \mathbf{f}(\mathbf{r},t+\boldsymbol{\delta}\mathbf{t})\cdot\mathbf{n}_2\, dS - \iint\limits_{S_1} \mathbf{f}(\mathbf{r},t)\cdot\mathbf{n}_1\, dS\right]/\delta t\right\}\\ &= \lim_{\delta t\to 0}\iint\limits_{S_2}\left[\frac{\mathbf{f}(\mathbf{r},t+\delta t)-\mathbf{f}(\mathbf{r},t)}{\delta t}\right]\cdot\mathbf{n}_2\, dS\\ &+ \lim_{\delta t\to 0}\left\{\left[\iint\limits_{S_2} \mathbf{f}(\mathbf{r},t)\cdot\mathbf{n}_2\, dS - \iint\limits_{S_1} \mathbf{f}(\mathbf{r},t)\cdot\mathbf{n}_1\, dS\right]/\delta t\right\}\end{aligned} \tag{2.17}$$

With the surface S_e defined earlier, one obtains, using the divergence theorem

$$\iint\limits_{S_2} \mathbf{f}\cdot\mathbf{n}_2\, dS - \iint\limits_{S_1} \mathbf{f}\cdot\mathbf{n}_1\, dS = -\iiint\limits_{R_0} \nabla.\mathbf{f}\, dR + \iint\limits_{S_e} \mathbf{f}\cdot\mathbf{n}_e\, dS \tag{2.18}$$

As $\delta t \to 0$, $dR \to -(\mathbf{v}_{s1}\delta t)\cdot\mathbf{n}_1 dS$ and $\mathbf{n}_e dS \to (\mathbf{v}_c \times \boldsymbol{\tau}) ds\delta t$, where $\mathbf{v}_{s1}$ is the velocity of the cap S_1 and $\mathbf{v}_c$ is the velocity of the circuit C_1. One thus obtains

$$\iint\limits_{S_2} \mathbf{f}\cdot\mathbf{n}_2\, dS - \iint\limits_{S_1} \mathbf{f}\cdot\mathbf{n}_1\, dS = \delta t\iint\limits_{S_1} (\nabla\cdot\mathbf{f})(\mathbf{v}_s.\mathbf{n})dS + \delta t\oint\limits_{C_1} \mathbf{f}.\left(\mathbf{v}_C\times\boldsymbol{\tau}\right)dS \tag{2.19}$$

The integrand in the last integral of (2.19) can be restated as $(\mathbf{f} \times \mathbf{v}_c)\cdot\boldsymbol{\tau}$. Thus Stokes' theorem yields

$$\oint\limits_{C_1} \mathbf{f}\cdot(\mathbf{v}_c\times\boldsymbol{\tau})ds = \iint\limits_{S_1} \nabla\times(\mathbf{f}\times\mathbf{v}_{s1})\cdot\mathbf{n}\, dS \tag{2.20}$$

Placing (2.20) into (2.19) and the results into (2.17), one obtains, upon noting the integrand in the first integral in the right-hand side of (2.17) is $\partial\mathbf{f}/\partial t$ and dropping the subscripts for S:

$$\frac{d}{dt}\iint_{S(t)} \mathbf{f}(\mathbf{r},t)\cdot\mathbf{n}\,dS = \iint_{S}\frac{\partial\mathbf{f}}{\partial t}\cdot\mathbf{n}\,dS + \iint_{S}[(\nabla\cdot\mathbf{f})\mathbf{v}_s]\cdot\mathbf{n}\,dS + \iint_{S}[\nabla\times(\mathbf{f}\times\mathbf{v}_s)]\cdot\mathbf{n}\,dS \tag{2.21}$$

The total flux of **f** over S(t) changes with time because the field **f** is time-dependent and also because the surface S is time-dependent. With (2.21), the time rate of change of the flux of **f** over S(t) is given by the sum of the three integrals on the right-hand side. The first integral represents the local rate of change of **f** on S. The second and third integrals together represent the motion and deformation of S.

For the solenoidal field $\mathbf{f}^* = \nabla\times\mathbf{h}$, (2.21) reduces to

$$\frac{d}{dt}\iint_{S(t)} \mathbf{f}^*\cdot\mathbf{n}dS = \iint_{S}\frac{\partial\mathbf{f}^*}{\partial t}\cdot\mathbf{n}dS + \iint_{S}\nabla\times(\mathbf{f}^*\times\mathbf{v_s})\cdot\mathbf{n}dS \tag{2.22}$$

Consider the time derivative of the integral of $\mathbf{f}(\mathbf{r},t)$ over S(t), which is a cap of C(t)

$$\begin{aligned}\frac{d}{dt}\iint_{S}\mathbf{f}(\mathbf{r},t)dS &\equiv \lim_{\delta t\to 0}\left\{\left[\iint_{S_2}\mathbf{f}(\mathbf{r},t+\delta t)dS - \iint_{S_1}\mathbf{f}(\mathbf{r},t)dS\right]/\delta t\right\}\\ &= \lim_{\delta t\to 0}\iint_{S_2}\left[\frac{\mathbf{f}(\mathbf{r},t+\delta t)-\mathbf{f}(\mathbf{r},t)}{\delta t}\right]dS + \lim_{\delta t\to 0}\left\{\left[\iint_{S_2}\mathbf{f}(\mathbf{r},t)dS - \iint_{S_1}\mathbf{f}(\mathbf{r},t)dS\right]/\delta t\right\}\end{aligned} \tag{2.23}$$

Through analyses similar to those leading to (2.19) and (2.21), one obtains

$$\iint_{S_2}\mathbf{f}(\mathbf{r},t)dS - \iint_{S_1}\mathbf{f}(\mathbf{r},t)dS = \delta t\oint_{C_1}\mathbf{f}(\mathbf{v}_c\cdot\mathbf{n})ds \tag{2.24}$$

$$\frac{d}{dt}\iint_{S(t)}\mathbf{f}(\mathbf{r},t)dS = \iint_{S}\frac{\partial\mathbf{f}}{\partial t}dS + \oint_{C_1}\mathbf{f}(\mathbf{v}_c\cdot n)ds \tag{2.25a}$$

$$= \iint_{S}\left[\frac{\partial\mathbf{f}}{\partial t} + (\nabla\cdot\mathbf{v}_s)\mathbf{f} + (\mathbf{v}_s\cdot\nabla)\mathbf{f}\right]dS \tag{2.25b}$$

A similar identity for the time derivative of the integral of $\mathbf{f}(\mathbf{r},t)$ over the three-dimensional region R(t) bounded by S(t) is

$$\frac{d}{dt}\iiint_{R(t)}\mathbf{f}(\mathbf{r},t)dR = \iiint_{R}\frac{\partial\mathbf{f}}{\partial t}dR + \oiint_{S}\mathbf{f}(\mathbf{v}_s\cdot\mathbf{n})dS \tag{2.26a}$$

$$= \iiint_R \left[\frac{\partial \mathbf{f}}{\partial t} + (\nabla \cdot \mathbf{v}_v)\mathbf{f} + (\mathbf{v}_v \cdot \nabla)\mathbf{f} \right] dR \tag{2.26b}$$

where $\mathbf{v}_v$ is the velocity of points in R and $\mathbf{v}_s$ is the velocity of points on the boundary S.

Equations (2.25a) and (2.26a) remain valid with the vector field $\mathbf{f}(\mathbf{r}, t)$ replaced by the scalar field $f(\mathbf{r}, t)$. The mathematical identities presented in this §2.3 are valid independently of the material medium occupying the space R. Whether a solid body, a viscous fluid, an inviscid fluid, or a combination of these media is present in R does not alter these identities. They are kinematic identities in which C needs not be a material circuit and S needs not be a material surface. $\mathbf{v}_c$, $\mathbf{v}_s$, and $\mathbf{v}_v$ represent the motions of the circuit, the surface, and points in R and need not be the flow velocity $\mathbf{v}$.

2.4 Helmholtz' First Vortex Theorem

In (2.11), let $\mathbf{f}^*$ be the incompressible flow velocity $\mathbf{v}$. One obtains

$$\iint_{S_1} \mathbf{v} \cdot \mathbf{n}_1 dS = \iint_{S_2} \mathbf{v} \cdot \mathbf{n}_2 \, dS \tag{2.27}$$

Equation (2.27) states a well-known fact: The flow rate (total velocity flux) through a stream tube in an incompressible flow is constant.

Let $\mathbf{f}^*$ be $\mathbf{v}$ and $\mathbf{h}$ be the vector potential $\boldsymbol{\psi}$ of $\mathbf{v}$ in (2.12). One has

$$\iint_S \mathbf{v} \cdot \mathbf{n} \, dS = \oint_C \boldsymbol{\psi} \cdot \boldsymbol{\tau} \, ds \tag{2.28}$$

Equation (2.28) states that the total rate of an incompressible flow across the area S is identical to the circulation of the vector potential $\boldsymbol{\psi}$ around the circuit C, of which S is a cap. Equation (2.27) therefore states that the circulation of the vector potential $\boldsymbol{\psi}$ around a stream tube in the incompressible flow is constant along the path of the stream tube.

Let $\mathbf{f}^*$ be $\boldsymbol{\omega}$ in (2.11). One obtains

$$\iint_{S_1} \boldsymbol{\omega} \cdot \mathbf{n}_1 dS = \iint_{S_2} \boldsymbol{\omega} \cdot \mathbf{n}_2 \, dS \tag{2.29}$$

Let $\mathbf{f}^*$ be $\boldsymbol{\omega}$ and $\mathbf{h}$ be $\mathbf{v}$. Equation (2.12) yields

$$\iint_S \boldsymbol{\omega} \cdot \mathbf{n}\, dS = \oint_C \mathbf{v} \cdot \tau\, ds \tag{2.30}$$

Equation (2.30) states that the total vorticity flux across S is identical to the circulation (of velocity) around C, of which S is a cap. Therefore (2.29) states: *The circulation of a vorticity tube is constant along each vorticity tube*. This statement paraphrases the first part of Sommerfeld's (1950) statement about spatial conservation of vorticity: *The vortex strength* (circulation) *is constant along each vortex line or vortex tube, which must either be closed or end at the boundary of the fluid*.

In classical fluid dynamics, the term vortex is typically used in place of the word vorticity, often in the context of the inviscid fluid idealization. In the present study, the idea of a vortex always means an approximation of a part of a vorticity field in a real flow, not a singular element in an inviscid fluid. The terms *vorticity tube*, *vorticity line*, and *vorticity flux* are therefore preferred in the present study. Sommerfeld used the terms vortex line and vortex tube interchangeably. In the present study, except in direct quotation of classical literature, a *vortex filament* (not a line) gives the approximate position of a thin vorticity tube in space. The filament has an infinitesimal (and nonzero) cross-sectional area and is not a line. It represents (approximates) the vorticity tube and has the circulation of the tube.

If a vortex filament of finite strength Γ ends in the fluid, then the strength of the vorticity tube represented by the filament changes abruptly at the ending point from Γ to zero. The strength along the vorticity tube is then not a constant. The second part of Sommerfeld's statement, that a vortex tube must either be closed or end at the boundary of the fluid, is therefore a consequence, a corollary, of the first part of his statement.

Consider a solid body immersed and moving in an infinite fluid. Denote the solid region by R_s, the fluid region by R_f, the fluid–solid interface by S, and the infinite unlimited space occupied jointly by the fluid and solids by R_∞. The velocity $\mathbf{v}_r$ of a rigid body rotating at the angular velocity $\boldsymbol{\Omega}$ is defined by $\mathbf{v}_r = \mathbf{v}_a + \boldsymbol{\Omega} \times (\mathbf{r} - \mathbf{r}_a)$, where $\mathbf{v}_a$ is the translation velocity of the point $\mathbf{r} = \mathbf{r}_a$. It is simple to show that $\nabla \cdot \mathbf{v}_r = 0$ and $\nabla \times \mathbf{v}_r = 2\boldsymbol{\Omega}$. In other words, (2.1) is satisfied in R_s and, as defined by (2.4), $\boldsymbol{\omega} = 2\boldsymbol{\Omega}$ in R_s. Equations (2.1) and (2.4) therefore describe the kinematics of both the fluid and the solid; they are valid equations in the infinite unlimited region R_∞. It is not difficult to generalize the above discussions to flows containing multiple rigid bodies.

With the *no-slip condition*, the normal vorticity component is necessarily zero at the surface of a non-rotating rigid solid body. Lighthill (1963), on the basis of this observation, concluded: *In flows which do not contain rotating bodies, all vorticity appears in closed loops*. Lighthill's conclusion can also be obtained by observing that with non-rotating solids, the solid region R_s contains no vorticity. With the vorticity field present only in R_f, S can only be a part of the surfaces of vorticity tubes. In consequence, vorticity tubes cannot continue into R_s and must be closed in R_f.

With the no-slip condition, the tangential components of the velocity vector are continuous on S. Hence the normal component of the vorticity vector is continuous on S whether or not the solids are rotating. Approaching S from the side of the solid or the side of the fluid gives the same value of the normal vorticity on S. The kinetics of the motion in the solid region differs from that in the fluid. The gradients of the velocity components, hence also the tangential components of vorticity, are therefore discontinuous on S. The discontinuity occurs whether or not the solid is rotating. In either case, because the normal component of vorticity is continuous on S, vorticity flux is continuous on S and Lighthill's observation can be generalized to the statement: *All vorticity appears in closed loop in the infinite unlimited region* R_∞.

Vorticity lines need not be smooth for the idea of vorticity tubes and vorticity loops to be meaningful. Vorticity tubes may bend at finite angles; they only need to continue in space. A vorticity line, being a curve in space, has a zero cross-sectional area and is not associated with a total flux or a circulation. Along the path of a vorticity line, the value of the vorticity can change. The line may pass through points of zero vorticity, giving the impression that it ends in space. This occurrence does not mean that a vorticity tube ends in space.

Vorticity tubes may overlap, or merge. The merged tube appears as a single vorticity tube with a circulation that is equal to the sum of the circulations of the component tubes. For the flow containing rotating solids, some of the vorticity tubes pass through S. On S, a vorticity tube in R_s merges with a vorticity tube in R_f. In most aerodynamic applications, the circulation of the vorticity tube in R_s (the cross-sectional area of the tube times $2\mathbf{\Omega}$) is weak compared to the circulation of the tube in R_f. The merging of the two tubes near S gives the misleading appearance that the vorticity tube in R_s does not continue into R_f and vice versa. This appearance obscures the truth that, because the vorticity flux is continuous on S, individual vorticity tubes are continued on S; they do not end on S either from the fluid side or from the solid side. Helmholtz' first vortex theorem therefore generalizes to the statement: *The vorticity strength* (circulation) *is constant along each vorticity tube, which must be closed in space*. In other words, a vorticity field in R_∞ is portrayable in general as a set of closed vorticity tubes (vorticity loops), each with a constant circulation along its path. This generalized Helmholtz' first vortex theorem is a kinematic theorem stating the spatial conservation of circulation.

Equations (2.1) and (2.4) contain first order spatial derivatives of the velocity field $\mathbf{v}$. For these equations to be meaningful, $\mathbf{v}$ must be not only continuous but piecewise smooth (once differentiable). The generalized Helmholtz' first vortex theorem and many vorticity-kinematic issues involve only integrations of $\boldsymbol{\omega}$. Therefore the vorticity field $\boldsymbol{\omega}$ needs only be piecewise continuous. Smoothness requirements for the vorticity in kinetics are more stringent since the vorticity transport equation (2.6), contains derivatives of $\boldsymbol{\omega}$. The kinematic-kinetic partition of the overall flow problem is in this context important in flow analyses. Without this partition, it is difficult to resolve many well-known conceptual difficulties and paradoxes in classical fluid dynamics.

2.5 Vorticity Loops

As a consequence of the generalized Helmholtz' first vortex theorem, every vorticity field in the infinite unlimited space R_∞ occupied jointly by the fluid and the solid is portrayable by a set of vorticity loops (closed tubes). The cross-sectional area S of each loop varies along the path of the loop, but the total flux of vorticity (circulation) of the loop is constant along the path. Each vorticity loop can be divided into n thinner vorticity loops simply by partitioning S into smaller areas S_i, i = 1, 2, … , n, with vorticity strengths Γ_i. These thinner loops are contiguous and together they occupy the space of the original undivided vorticity loop. The sum of the vorticity strengths of the thinner loops is the total strength of the original vorticity loop. A very thin vorticity loop is accurately approximated by a closed vortex filament (vortex loop), as discussed in §2.4. Thus any vorticity field can be approximated by a large number of vortex loops in space.

Let the closed path of the vortex loop representing a thin vorticity tube be C. If C is divided into two open paths C_1 and C_2 and the two dividing points are connected by the path C_3, then there are two closed paths: a closed path $C_1{}^*$ formed by combing C_3 and C_1 and a second closed path $C_2{}^*$ formed by combining C_3 and C_2. Consider two vortex loops: $\Gamma_1{}^*$ on $C_1{}^*$ and $\Gamma_2{}^*$ on $C_2{}^*$. Let $\Gamma_1{}^* = \Gamma_2{}^* = \Gamma$. On the open path C_3, the directions of the loops $C_1{}^*$ and $C_2{}^*$ are opposite. Thus the combined circulation of the loops $\Gamma_1{}^*$ and $\Gamma_2{}^*$ is zero on C_3. The two smaller loops $\Gamma_1{}^*$ and $\Gamma_2{}^*$ together are therefore equivalent, kinematically, to the original loop Γ. Thus any vortex loop Γ is divisible into two smaller vortex loops with the same vortex strength. Successive divisions give a set of small loops, each with the circulation Γ. In aggregate, the small vortex loops are equivalent to the large loop Γ. Since the paths of division can be arbitrarily chosen, the set of small loops can be configured with great flexibility.

Under general circumstances, the vorticity field spreads over a relatively large flow region. Therefore a large number of vortex loops, each representing a thin vorticity loop, are needed to produce a reasonably accurate approximation of the vorticity field. In the limit as the cross-sectional area of each vorticity loop approaches zero, there is an infinite number of vortex loops and the approximation becomes precise.

Dividing thin vorticity loops into sets of small vorticity loops produces a large number of thin and small vorticity loops well suited for flow computations. Dealing with vorticity loops rather than the vorticity field ensures the satisfaction of the principle of total vorticity conservation (see Chap. 5), which is necessary for solution stability in three-dimensional flow computations. The use of a small number of vorticity loops is convenient, but significant computational inaccuracy is expected. For streamlined flows, however, the vorticity-loop concept is powerful in interpreting the lifting-line theory of aerodynamics, as discussed in Chap. 6.

This suggests the possibility of simplified solution procedures for computing certain types of flows, including unsteady flows. Additional discussions of vorticity-loop portrayals are given by Wu et al. (2002) in a study of the lift, the profile drag, and the induced drag on hovering rotors.

2.6 Kelvin's Theorem

In (2.16), let C(t) be a material circuit C_m moving with the fluid and $\mathbf{f} = \boldsymbol{\psi}$, the vector potential of an incompressible velocity field $\mathbf{v}$. The integrand of the last integral in (2.16) vanishes since $(\nabla \times \boldsymbol{\Psi}) \times \mathbf{v} = \mathbf{v} \times \mathbf{v} = 0$. Thus one has

$$\frac{d}{dt}\oint_{C_m} \boldsymbol{\Psi}(\mathbf{r}, t) \cdot \boldsymbol{\tau}\, ds = \oint_{C_m} \left(\frac{\partial \boldsymbol{\Psi}}{\partial t}\right) \cdot \boldsymbol{\tau}\, ds \tag{2.31}$$

Equation (2.31) states that the rate of change of the material circulation of the vector potential $\boldsymbol{\Psi}$ is caused by only the local change of $\boldsymbol{\Psi}$. The motion of the material circuit does not contribute to the change.

Let $\mathbf{f} = \mathbf{v}$, the velocity field of an incompressible flow. Equation (2.16) gives

$$\frac{d}{dt}\oint_{C_m} \mathbf{v} \cdot \boldsymbol{\tau}\, ds = \oint_{C_m} \left(\frac{\partial \mathbf{v}}{\partial t}\right) \cdot \boldsymbol{\tau}\, ds + \oint_{C_m} (\boldsymbol{\omega} \times \mathbf{v}) \cdot \boldsymbol{\tau}\, ds \tag{2.32}$$

where the left-hand side integral gives the *material circulation* Γ_m of the velocity field.

The first integral on the right-hand side of (2.32) represents the contribution of the local change of $\mathbf{v}$ to the change of the material circulation Γ_m. The last integral is the circulation of Lamb's vector $\boldsymbol{\omega} \times \mathbf{v}$ around the material circuit C_m and represents the contribution to the rate of change of Γ_m by the motion of the material circuit C_m.

Equation (2.32) is a kinematic equation obtained using only (2.1) and (2.4). Placing the kinetic equation (2.5) into (2.32) yields

$$\frac{d\Gamma_m}{dt} = -\oint_{C_m} [\nabla(h - q)] \cdot \tau\, ds - \nu \oint_{C_m} (\nabla \times \boldsymbol{\omega}) \cdot \tau\, ds \tag{2.33}$$

The first integral in (2.33) can be restated using Stokes' theorem as an integral over S_m, a cap of C_m, of $\nabla \times [\nabla(h - q)]$, which is identically zero. One thus has

$$\frac{d\Gamma_m}{dt} = -\nu \oint_{C_m} (\nabla \times \boldsymbol{\omega}) \cdot \tau\, ds \tag{2.34}$$

Equation (2.34) is a vorticity-dynamic relation based on the kinematic equations (2.1) and (2.4) and the kinetic equation (2.5). Placing (2.7) in (2.32) or letting $\nu = 0$ in (2.34) yields

$$\frac{d\Gamma_m}{dt} = 0 \tag{2.35}$$

Equation (2.35) is a statement of Kelvin's theorem on the persistence of circulation: *The circulation around a material circuit moving with an inviscid fluid remains constant.*

For the viscous fluid, (2.34) states: *The circulation around a material loop moving with a viscous fluid changes with time as the result of viscous diffusion.* It is known that the viscosities of air and water, the primary flow media of interest in aerodynamics, are very small. More precisely, the dimensionless Reynolds number is very large. Therefore, except in flow regions close to solid surfaces, where the gradient of the vorticity field can be very steep, the right-hand side of (2.34) is small and the circulation around the material circuit changes very slowly. In many regions far from solid surfaces, the vorticity gradients are so small that circulation changes are negligible slow. Such flow regions are inviscid flow regions even though the fluid is not inviscid.

2.7 Helmholtz' Second Vortex Theorem

Let $\mathbf{f}^* = \boldsymbol{\omega}$ and $\mathbf{v}_s = \mathbf{v}$. One obtains using (2.22) and (2.6) the following equations:

$$\frac{d}{dt}\iint\limits_{S_m} \boldsymbol{\omega}\cdot\mathbf{n}\,dS = \iint\limits_{S_m} \frac{\partial\boldsymbol{\omega}}{\partial t}\cdot\mathbf{n}\,dS + \iint\limits_{S_m} \nabla\times(\boldsymbol{\omega}\times\mathbf{v})\cdot\mathbf{n}\,dS \tag{2.36}$$

$$\frac{d}{dt}\iint\limits_{S_m} \boldsymbol{\omega}\cdot\mathbf{n}\,dS = -\nu\iint\limits_{S_m} [\nabla\times(\nabla\times\boldsymbol{\omega})]\cdot\mathbf{n}\,dS \tag{2.37a}$$

$$= -\nu\oint\limits_{C_m} (\nabla\times\omega)\cdot\tau\,dS \tag{2.37b}$$

Letting $\nu = 0$ in (2.37), or placing (2.7) into (2.36), one obtains

$$\frac{d}{dt}\iint\limits_{S_m} \boldsymbol{\omega}\cdot\mathbf{n}\,dS = 0 \tag{2.38}$$

Sommerfeld's statement (1950) of Helmholtz' second vortex theorem is: *The vortex strength* (total flux of vorticity across a material surface) *is constant in time*. In other words, the circulation around a material vorticity tube is independent of time in an inviscid fluid. Thus Helmholtz' second vortex theorem and Kelvin's theorem on the persistence of circulation are equivalent. In fact, by virtue of (2.12), (2.38) is equivalent to (2.35) and (2.37) is equivalent to (2.34). The proof provided for (2.37) in this Section is therefore redundant.

The surface S_m is a cap of the material loop C_m. There exist an infinite number of caps for each material loop in space. Thus (2.34) is more general than (2.37a). In other words, using Stokes' theorem, (2.34) can be restated as (2.37) with S_m replaced by S. S can be an arbitrary cross-sectional surface of a material vorticity tube, a cap of a material circuit, but not necessarily a material cap moving with the fluid.

Equation (2.38) states that the total vorticity flux across the cross-sectional area of a material vorticity tube in an inviscid flow is independent of time. Letting the cross-sectional area of a material tube approach zero, one arrives at Lighthill's (1986) interpretation of Helmholtz' second theorem: *Vortex lines move with the* (inviscid) *fluid.* Saffman (1992) reviewed the works of Lamb (1932) and others that culminated in Lighthill's statement. Alternative proofs of this statement are presented in the works of Lamb, Lighthill, Lugt (1996), Saffman, Whitham (1963), and others.

As discussed in §2.6, the circulation of a material vorticity tube in a viscous fluid changes with time because of viscous diffusion, a process that spreads the vorticity in the flow. Consider, for simplicity, a Cartesian coordinate system (x,y,z) and a planar flow with the velocity field $\mathbf{v}(x,y)$ and $\boldsymbol{\omega} = \omega\,\mathbf{k}$, where $\mathbf{k}$ is the unit vector in the z-direction. For this planar flow, the vorticity flux $\boldsymbol{\omega} \cdot \mathbf{n}$ is identical to ω since $\mathbf{k} = \mathbf{n}$. Thus (2.37) yields

$$\frac{d}{dt}\iint_{S_m} \omega \cdot dS = \nu \iint_{S_m} \left(\nabla^2\omega\right) dS \tag{2.39}$$

Using (2.25), (2.39) can be restated in the form

$$\iint_{S} \left(\frac{D\omega}{Dt} - \nu\nabla^2\omega\right) dS = 0 \tag{2.40}$$

where $D\omega/Dt = \partial\omega/\partial t + (\mathbf{v} \cdot \nabla)\omega$ is the material (substantial) derivative of ω, a time derivative following the motion of the fluid. Equation (2.39) is given by Lamb (1932) and a more general equation for a non-planer surface S_m by Wu and Wu (1998).

Equation (2.40) is obtainable directly by integrating the two-dimensional version of the vorticity transport equation (2.6), which is expressible in the form $D\omega/Dt = \nu\nabla^2\omega$. In this form, the vorticity is a scalar field and the vorticity transport equation is analogous to the familiar diffusion equation. Consider, for example, the heat conduction (diffusion) equation $\partial T/\partial t = \kappa\nabla^2 T$, where T is the temperature field and κ is the heat conductivity (diffusivity). In heat conduction, heat energy, measured in terms of temperature, is transported in a medium at rest. This process is irreversible. It equalizes the heat energy, spreading it from higher temperature regions to lower temperature regions. Observed in a reference frame moving with the fluid, the viscous diffusion of vorticity is analogous to heat conduction in a stationary medium. Without viscous diffusion, the total vorticity over any material surface in the plane of the flow is independent of time. This is analogous to the temperature associated with each material element of a stationary non-conducting solid remaining unchanged relative to time. In the flow of the viscous fluid, the total vorticity in a material surface changes with time as a result of viscous diffusion.

In three-dimensional flows, the vorticity is a vector field. Equation (2.37) is valid for an arbitrarily chosen material surface, including an elemental material surface δS_m. Therefore the total vorticity flux over δS_m is spread by viscous diffusion when viewed in a *material reference frame* moving with the fluid. It is worth underscoring that the total flux $\boldsymbol{\omega} \cdot \mathbf{n}\ \delta S_m$, not the vector $\boldsymbol{\omega}$, is conserved in the inviscid flow. This total flux is diffused in the viscous flow. In three-dimensions, the surface element δS_m translates and rotates. The total vorticity flux translates and rotates with δS_m.

2.8 Concluding Remarks

The theorems of Helmholtz and Kelvin are most conveniently interpreted using the idea of the vorticity flux $\boldsymbol{\omega} \cdot \mathbf{n}$. Helmholtz' first vortex theorem is recognized as a theorem of vorticity kinematics stating that *the total vorticity flux in a vorticity tube is constant along the path of the tube*. This theorem is valid in the infinite unlimited region occupied jointly by the solid and the fluid. In consequence, *in external aerodynamics, all vorticity fields are portrayable as sets of vorticity loops, each with a constant circulation along the path of the loop*.

Helmholtz' second vortex theorem, or its equivalence Kelvin's theorem, is a theorem of vorticity dynamics. A generalized statement of this theorem is: *The total vorticity flux in each material vorticity tube changes with time only as a result of vorticity diffusion across the boundary surface of the tube*. It is important to note that, because this generalized theorem is a vorticity-dynamic theorem, it is applicable in the interior of the fluid region, and not in the solid region. In an inviscid flow, the total vorticity flux in each vorticity tube remains unchanged with time. In many flow regions far from solid surfaces, the vorticity gradients are so small that circulation changes are negligible slow. Such flow regions are effectively inviscid even though the fluid is viscous. In such flow regions, vorticity lines are effectively material lines. It is convenient to think of vorticity loops portraying the vorticity field as moving with the fluid while retaining their strengths (circulation) in such regions.

In flow regions where viscous effects are important, vorticity lines are not material lines. They do not move with the fluid. Since the circulation of each material circuit changes with time, the strength of each vorticity loop is not conserved in time. It is not sufficient to merely keep track of the movements of the vorticity loops; the strength and the shape of each loop must be undated continually. At the present stage of development, potential applications of the generalized theorems of Helmholtz and Kelvin in aerodynamics are evolving. Based on the present understanding, new interpretations of classical aerodynamic theories and of the connection between two- and three-dimensional aerodynamics are possible. Discussions of these topics are presented by Wu (1981) and in Chap. 6.

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