Chapter 2
Basic Equations and Normal Modes

Abstract This chapter presents a systematic linearization of the equations of motion about a state of rest, followed by additional approximations, each carefully analyzed for estimated error, culminating in a full four-dimensional separation of variables. The general solution to these approximate equations is a superposition of normal modes. Each mode is the product of four one-dimensional factors. The latitudinal structure of the modes is a Hermite function or the sum of a pair of Hermite functions. The vertical structure functions are the eigenmodes of the vertical structure equation. The normal mode is the product of a factor of latitude multiplied by a function of depth z only and lastly multiplied by $\exp(ik(x - ct))$.

> A traveler who refuses to pass over a bridge until he has personally tested the soundness of every part of it is not likely to go far; something must be risked, even in mathematics.
>
> Sir Horace Lamb

2.1 Model

The fundamental assumption for studying fluid dynamics at low latitude is the "equatorial beta-plane". In this approximation, the factors of sine and cosine of latitude are approximated by power series about zero latitude, i.e., the cosine is replaced by one and the sine is replaced by latitude itself. The rapid increase of the Coriolis force as one goes away from the equator confines a broad class of waves and jets to very low latitudes where these geometric approximations are quite good. Typically, the atmospheric equatorial waves have little amplitude beyond 20° — 1/3 of a radian — north and south of the equator while the ocean motions are much narrower still. Since $\cos\theta = 1 - (1/2)\theta^2 + \ldots$ and $\sin(\theta) = \theta - \theta^3/6$, we see that the relative error in retaining only the lowest order terms is no worse than 5% for either function as long as $\theta <= 1/3$.

By geophysical standards, this is a very small error: more important still, it is a purely numerical error that does not change the qualitative behavior of the equations of motion. The great advantage of these trigonometric approximations is that they permit one to replace *spherical* coordinates latitude and longitude by *Cartesian*

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coordinates x and y, which are simply latitude and longitude multiplied by the radius of the earth.

McCreary (1980) is a very detailed treatment of the assumptions and approximations made in equatorial fluid mechanics [1]. The equatorial beta-plane is so basic that he does not even mention it, assuming correctly that his audience will already be familiar with it! Following his practice of rating assumptions with labels in "all caps", we will supply what he omitted for this one approximation.

Assumption 2.1 Equatorial beta-plane

EXCELLENT for all equatorial motions that are truly equatorial, i.e., confined to low latitudes, provided that the zonal length scale is not too small.[1]

The equations of motion on the equatorial beta-plane then take the form

$$u_t + uu_x + vu_y + wu_z - fv + (1/\rho)p_x = (\nu u_z)_z \tag{2.1}$$

$$v_t + uv_x + vv_y + wv_z + fu + (1/\rho)p_y = (\nu v_z)_z \tag{2.2}$$

$$(1/\rho)p_z = -g \tag{2.3}$$

$$\rho_t + (u\rho)_x + (v\rho)_y + (w\rho)_z = 0 \tag{2.4}$$

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = (\daleth \rho_z)_z \tag{2.5}$$

where subscripts denote differentiation with respect to the subscripted variable and ν is viscosity, $\daleth$ is the thermal diffusivity and p is pressure. All symbols are defined in the "Symbols and Notation" section of the frontmatter.

On the equatorial beta-plane, the Coriolis parameter $f = 2\Omega y$ where Ω is the angular frequency of the earth's rotation. Parenthetically, note that (2.1)–(2.2) are also the proper equations for the "mid-latitude beta-plane" in which the Taylor expansions of the sine and cosine of latitude are about a middle latitude, usually 45°, rather than the equator. The separation-of-variables procedure is quite unaffected by the approximation we use for f.

Even in these equations, simplified further below, there are implicit a number of additional assumptions including the following:

Assumption 2.2 Neglect of vertical Coriolis force.

These terms are normally extremely small and this assumption is always made even in numerical models. McCreary rates this as GOOD; I would say EXCELLENT.

This is part of the "traditional approximation" in meteorology [2, 3] and is in fact necessary for energetic consistency with other simplifications of the "traditional

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[1] In Sect. 3.12, it is shown that the latitudinal scale is controlled by $E + s^2$ where E is the nondimensional parameter known as"Lamb's parameter" (defined below as 2.34) and s is the integer zonal wavenumber. The classical equatorial beta-plane must be generalized by keeping some additional terms in order to capture the zonal wavenumber effect on the north-south width of the waves. Fortunately, because $\varepsilon \sim O(10^5)$ or larger for ocean baroclinic modes, it is only for wavenumbers $s \sim O(300)$, or equivalently for east-west wavelengths of a couple of hundred kilometers or less, that an "extended equatorial beta-plane" is needed.

approximation". For dissenting views, see [4] and the review [5] and also [6–14, 14].

Assumption 2.3 Horizontal viscosity is neglected.

This would be a bad assumption in a pipe flow, but the horizontal scale of large-scale motions in both the atmosphere and ocean is very large (25–100 times) the vertical scale, so the errors in ignoring horizontal viscosity are only $O(10^{-4})$ of the vertical viscosity effects even when the horizontal viscosity coefficient is as large as the vertical viscosity coefficient. McCreary describes this as SENSIBLE; I would say EXCELLENT. (It should be noted, however, that many numerical models keep horizontal viscosity for computational stability.)

Assumption 2.4 Hydrostatic approximation.

This is poor for very small scales (wavelengths of a couple of kilometers or less), but it is a terrific approximation for the large-scale waves of interest here in both the atmosphere and ocean. Another way of putting it is that the vertical acceleration dw/dt is important only for motions whose time scales are on the order of $2\pi/$ the Brunt–Vaisala frequency. McCreary rates hydrostaticity as GOOD.[2]

Assumption 2.5 Incompressibility.

Equation (2.4), which expresses continuity of mass, is almost never used in that form. By subtracting (2.5), which is the heat equation, from it we obtain

$$\rho(u_x + v_y + w_z) = -(\urcorner \rho_z)_z \tag{2.4*}$$

We now simplify this by ignoring the right-hand side of (2.4*):

$$u_x + v_y + w_z = 0 \tag{2.4**}$$

Veronis [15] has shown that this is VERY GOOD for a typical ocean wind-driven circulation.

An important note: Veronis shows that the right-hand side of the heat equation (2.5) generally cannot be neglected. However, some authors do replace (2.5) by an equation which the temperature rather than the density ρ is the actual unknown. Semtner and Holland [16] use the approximate equation of state

$$\rho = \rho_0[1 - \alpha(T - T_0)] \tag{2.6}$$
$$\alpha = 0.0002/degree \tag{2.7}$$

where ρ_0 and T_0 and the thermal expansion coefficient α are constants. Equation (2.6) ignores salinity, which can be very important in general oceanography, but salinity

[2]The hydrostatic approximation must fail sufficiently close to breaking for a nonlinear Kelvin wave as described in Chap. 16.

differences are largest in northern waters where calving icebergs can overlay the salty seas with relatively fresh water. Salinity differences seem to be of secondary importance for the equatorial ocean, so they have been largely ignored up to now, and we shall do the same. Equations (2.6) and (2.7) make it possible to replace (2.5) with the alternative (but equivalent) heat equation

$$T_t + uT_x + vT_y + wT_z = (\urcorner \rho_z)_z \tag{2.5*}$$

The system of Eqs. (2.1)–(2.3), (2.4**) and either (2.5) or (2.5*) are the basic set of nonlinear equations that has been the basis for multi-level numerical simulations of the ocean. Atmospheric modellers use an equation of the same form except that (i) z is replaced as a vertical coordinate by pressure or log-pressure (ii) vertical viscosity is usually ignored and (iii) the eddy conductivity term in (2.5*) is normally replaced by an explicit heating function.

To obtain *analytical* models of ocean flow, it is necessary to simplify the equations still further. The first and most important approximation is to drop the nonlinear terms. There are analytical theories that include nonlinear effects, but these are perturbative calculations in which the starting point is the linearized set of equations. Hence, linearize-and-separate-variables is the fundamental first step in all analytical calculations up to the present.

Assumption 2.6 Drop *all* nonlinear terms in the momentum equations.

McCreary rates this QUESTIONABLE; I would add the qualifier NECESSARY to obtain linear, analytically solvable equations. This assumption will be relaxed in Chaps. 16 and 17 on instability and nonlinear waves.

Assumption 2.7 Drop all nonlinear terms in the heat equation *except* for wdT_0/dz (or $wd\rho_0/dz$) where $T_0(z\text{ only})$ and $\rho_0(z\text{ only})$ represent the mean temperature /density stratification of the ocean, which is assumed to be a function only of the vertical coordinate.

This also is QUESTIONABLE but NECESSARY. This one nonlinear term cannot be neglected, even in a lowest approximation, because this term is *buoyancy*. If it is missing, then the linearized equations have no resistance to vertical motions and cannot respond to thermal driving. In reality, the density stratification of both the ocean and atmosphere strongly resists vertical motion, which is why the synoptic high and low pressure systems on a daily weather map are typically a hundred times larger in the horizontal dimension than the vertical. Thus, we cannot even come close to the right answer unless density stratification is included.

Fortunately, the mean density/temperature profile is known in both air and sea. It does vary with altitude and longitude, too, but rather slowly; in particular, the ocean and atmosphere are stably stratified over the whole globe. Consequently, inserting a mean temperature or density that is a function only of depth is a satisfactory first approximation.

It should be noted that the mean buoyancy term must also be inserted into the fully nonlinear heat equation, (2.5) or (2.5*), since numerical models usually are not sufficiently sophisticated and sufficiently high resolution to parameterize the small-scale convection, radiative transfer, etc. well enough to correctly generate the observed stratification from first principles. Thus, even for numerical models, the mean stratification is usually specified rather than internally generated.

Assumption 2.8 Assume special forms for mixing coefficients:

$$\nu = \daleth = A/N^2 \tag{2.8}$$

where A is a constant and N is the mean Brunt–Vaisala frequency defined by

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho_0}{dz} \tag{2.9}$$

and modify the form of the eddy mixing of heat to

$$(\daleth\rho_z)_z \longrightarrow (\daleth\rho)_{zz} \tag{2.10}$$

McCreary does not rate this assumption, but I would describe it as CONVENIENT and AS REASONABLE AS ANYTHING ELSE. The reason that it is convenient is that these forms for the mixing are the only known forms that permit separation of variables. Whatever their lack of realism, these approximations give analytical models with damping, and with a damping that increases with decreasing vertical scale as is true of almost any physically reasonable dissipation.

In point of fact, we need not feel too guilty about using these approximations. For laminar laboratory flow, the viscosities are molecular and have known, measured values, and off-the-wall assumptions are unneeded. In the atmosphere and ocean, however, the dominant mixing is by small-scale turbulence. Since these are known only to within an order-of-magnitude anyway — questionable approximation — (2.8)–(2.10) really are reasonable as any plausible alternative.

The Assumptions (2.8)–(2.10) were first used by Fjelstad and Mork to study internal waves and by McCreary (1980) to explore equatorial dynamics. When small-scale turbulence provides the mixing, it is reasonable that heat and momentum will be mixed with equal efficiency even though the *molecular* viscosity and diffusivity are different. The greater the static stability, the larger the Brunt–Vaisala frequency N; we expect that greater static stability would, by resisting vertical motion, reduce the vertical eddy mixing, and the forms (2.8) explicitly allow for this. However, there is no particularly good physical reason to suppose that ν and $\daleth$ must decrease precisely as the square of N; the argument merely implies that bigger N will make ν and $\daleth$ smaller.

Assumption 2.9 The pressure is replaced by a new variable ϕ such that

$$\phi_x = (1/\rho)p_x; \qquad \phi_y = (1/\rho)p_y \tag{2.11}$$

Equation (2.11) is both a definition and an approximation. The definition is the introduction of the new variable ϕ. This is a meteorologist's notation rather than an oceanographer's; the "geopotential" ϕ always replaces the pressure when pressure or its logarithm is used as the vertical coordinate, and no further approximations to justify (2.11) are needed in the atmospheric case except the hydrostatic approximation, which we have already made. In the ocean, taking

$$\phi \equiv p/\rho_M \tag{2.12}$$

where ρ_M is a constant equal to the vertical average of the basic state density, $\rho_0(z)$, is necessary to obtain a simple relation between p and its replacement, ϕ. This approximation of ignoring density variations in the *momentum* equations while leaving ρ *untouched* in the *heat* and *hydrostatic* equations is known as the Boussinesq approximation. Since $(\rho - \rho_0)/\rho$ is $O(10^{-2})$ or smaller in the ocean, the Boussinesq approximation is VERY GOOD.

The replacement of p by ϕ is very convenient. McCreary [1] simply drops the factor of ρ_0 and uses p as the symbol for both the pressure and for the pressure divided by the mean density. It seems less confusing to borrow the meteorologist's symbol.

All these simplifications finally give the linearized set of equations

$$u_t - fv + \phi_x = [(A/N^2)u_z]_z \tag{2.13a}$$

$$v_t + fu + \phi_y = [(A/N^2)v_z]_z \tag{2.13b}$$

$$\rho_M \phi_z = -\rho g \tag{2.13c}$$

$$\rho_t - (\rho_M/g)N^2 w = [(A/N^2)\rho]_z z \tag{2.13d}$$

$$u_x + v_y + w_z = 0 \tag{2.13e}$$

where we have ignored the distinction between ρ_M and $\rho_0(z)$ in the denominator of N^2 in obtaining (2.13d).

2.2 Boundary Conditions

The boundary conditions of the separable model are four at the top and four at the bottom; since the system is eighth order in z (i.e., 8 z-derivatives appear in these equations), one unknown (ϕ, or in other words, the pressure) must be left unconstrained on the boundaries. The imposed conditions are:

SURFACE: (at $z = 0$)

$$\nu u_z = \tau^x \tag{2.14a}$$

$$\nu v_z = \tau^y \tag{2.14b}$$

$$w = 0 \tag{2.14c}$$

$$\rho = \rho_0(z = 0) \tag{2.14d}$$

where τ^x and τ^y are the surface wind stresses on the ocean in the x and y directions.

The first two boundary conditions are physically correct and are used even in nonlinear numerical models. The difficulty is that wind stress is created by turbulent air-sea interaction which is parameterized by formulas relating wind stress to wind speed. Once the wind stress has been calculated by some formula/incantation/voodoo, the boundary conditions are mathematically correct and induce no further error.

Assumption 2.10 Rigid lid.

Taking $w = 0$ at the top of the ocean is physically equivalent to imposing a rigid lid at $z = 0$ and is obviously unrealistic; an ocean voyager sees only an endless field of waves. However, when we separate variables and analyze the structure of individual vertical modes, we find that setting $w = 0$ at $z = 0$ has a major effect only on the barotropic mode, that is to say, the mode in which the horizontal currents are independent of depth. To be sure, all the baroclinic modes have very small but non-zero vertical velocities at the sea surface. However, setting $w = 0$ at the top for the baroclinic modes creates an error of $O(\Delta\rho/\rho)$ where $\Delta\rho$ is the variation of the density with depth. Since this is $O(1/250)$ for the equatorial ocean, we make only a negligible error for the baroclinic modes by making this approximation.

But what about the barotropic mode? As we shall see below, only the baroclinic modes support equatorially trapped motions, so we must largely exclude the barotropic mode from consideration anyway. McCreary therefore rates this approximation as GOOD.

Assumption 2.11 Density = mean density at surface; equivalently, temperature = mean temperature at the surface.

This assumption is "UNPLEASANT" in McCreary's words because it demands that the atmosphere act as a constant-temperature source of heat, and does not allow the sea surface to change with time, latitude, or anything else. A coupled ocean-atmosphere model would *never* make this assumption because in reality, the situation is the other way around. The ocean surface temperature does change, and this provides a constant-temperature source of heat for the atmosphere. Namias' method of long-range weather forecasting, for example, is based on looking at sea surface temperatures and their effect on the atmosphere. A more realistic condition would be to specify the heat flux at the ocean surface, or better yet, some kind of radiative balance equation involving such complications as evaporation and precipitation, albedo, and so on.

Nonetheless, this assumption is essential to separating variables, so for analytical theory, McCreary rightly labels it as UNFORTUNATE but NECESSARY.

The bottom boundary conditions are

BOTTOM: (at $z = -D$ where D is depth of the ocean, a constant)

$$\nu u_z = 0 \tag{2.15a}$$

$$\nu v_z = 0 \tag{2.15b}$$

$$w = 0 \tag{2.15c}$$

$$\rho = \rho_0(z = 0) \tag{2.15d}$$

Assumption 2.12 Constant bottom depth D

In reality, the bottom topography of the sea is highly irregular and this can give rise to all kinds of complicated effects: topographic Rossby waves, jets that follow the depth contours, topographic scattering of internal waves, etc., etc. However, the important equatorial motions – at least the sort that have been studied up to now – do not extend to the bottom of the abyss. The equatorial ocean is sufficiently deep in most places so that topographic effects are not important; topography is most important in very shallow seas and along the continental shelves. I rate this assumption as NECESSARY, but NOT BAD for the class of motions we are interested in. This assumption of a flat bottom then forces us to impose (2.15c); note over *sloping* topography, the vertical velocity w does not have to be 0.

Assumption 2.13 No stress at the bottom.

A numerical model would use the normal viscous condition of no horizontal velocities at the bottom; the condition of no stress filters out a bottom Ekman layer driven by the geostrophic current just above the bottom. However, for surface trapped motions, there is no bottom Ekman layer anyway. While Ekman spin-down could in principle be an important dissipative mechanism, the turbulent eddy viscosities are so low in the deep ocean in comparison to the surface layer that the bottom Ekman layer can often be ignored when there are bottom currents. McCreary rates this assumption as NECESSARY and NOT BAD.

Assumption 2.14 No temperature changes at the sea bottom.

In view of the approximate equation of state (2.6), requiring that the density equal the mean density at the bottom has multiple implications.It is equivalent to compelling the bottom temperature to equal the mean temperature. This assumption also forces the sea-bottom to act as a constant temperature source of heat to the ocean, i.e., a reservoir of infinite heat capacity and conductivity. This is a little unrealistic, but the lack of realism is not serious for surface-trapped motions. McCreary rates this NECESSARY and NOT BAD.

For clarity, the full list of assumptions is collected in Table 2.1.

As noted by McCreary, the single most restrictive assumption implicit in the boundary conditions is the constant temperature condition at the surface of the sea.

Table 2.1 The assumptions of the basic model and their ratings

Equatorial beta-plane	EXCELLENT
Neglect of vertical Coriolis	GOOD
No horizontal viscosity	SENSIBLE
Hydrostatic approximation	GOOD
Incompressibility (Nondivergence)	VERY GOOD
Drop all nonlinear terms in momentum equation	QUESTIONABLE
Drop all nonlinear terms except wdT_0/dz in heat equation	QUESTIONABLE
Assume special forms for mixing coefficients: $\nu = \daleth = A/N^2$	NECESSARY
Replace of p by ϕ with Boussinesq approximation	VERY GOOD
Rigid lid upper boundary	GOOD (for baroclinic modes)
Density and temperature equal their mean values at surface	(UNFORTUNATE but NECESSARY)
Constant bottom depth D	NECESSARY but NOT BAD (for surface-trapped motion)
No stress at the sea bottom	NECESSARY and NOT BAD
No temperature changes at sea bottom	NECESSARY but NOT BAD

This is a real problem because a coupled ocean-atmosphere model of the Southern Oscillation/El Niño phenomenon, for example, must have sea surface temperature changes to drive the atmospheric changes of the cycle. The other surface conditions are either correct or very accurate approximations, and we need not worry about them. None of the bottom boundary conditions, in contrast, is realistic, but equatorial flows are not sensitive to the bottom boundary conditions.

2.3 Separation of Variables

In a nonlinear, numerical model, all five of (2.13) would be retained with the local time derivatives in the momentum and heat equations being replaced by total time derivatives, and perhaps the special, stability-dependent viscosity coefficients with more realistic forms. In a linearized treatment, however, we can reduce the number of equations in stages.

The first step is to rewrite the heat equation, using the hydrostatic equation to replace ρ by ϕ_z, to give an expression for the vertical velocity:

$$w = -(1/N^2)[\phi_t - A(\phi_z/N^2)_z]_z \tag{2.16a}$$

The density is given in terms of ϕ by the hydrostatic equation,

$$\rho = -(\rho_M/g)\phi_z \tag{2.16b}$$

We are then left with a system of 3 equations in the three other unknowns:

$$u_t - fv + \phi_x = A\Lambda u \tag{2.16c}$$

$$v_t + fu + \phi_y = A\Lambda v \tag{2.16d}$$

$$-\Lambda\phi_t + u_x + v_y = -A\Lambda^2\phi \tag{2.16e}$$

where Λ is the linear operator defined by

$$\Lambda = \partial_z \frac{1}{N^2}\partial_z \tag{2.17}$$

The key point is that as a result of the careful choice of the form of the viscosity, conductivity, and boundary conditions, the vertical derivatives in (2.16) appear solely in the form of the operator Λ, the resulting spectral equations will uncouple and we will have separated variables. The eigenfunction problem is

$$\frac{d}{dz}\left[\frac{1}{N^2}\frac{d}{dz}\mathfrak{z}_m\right] = -\lambda_m \mathfrak{z}_m \tag{2.18}$$

with boundary conditions

$$\frac{d}{dz}\mathfrak{z}_m = 0 \qquad \text{at } z = 0, -D \tag{2.19}$$

This is a classic Sturm–Liouville eigenproblem – no singularities, no complications of any kind since $N(z)$ is positive definite – but a little care isneeded because different unknowns have different dependence on $\mathfrak{z}_m$. The proper expansions are

$$u = \sum_{m=0}^{\infty} u_m(x, y, t)\mathfrak{z}_m(z) \tag{2.20a}$$

$$v = \sum_{m=0}^{\infty} v_m(x, y, t)\mathfrak{z}_m(z) \tag{2.20b}$$

$$\phi = \sum_{m=0}^{\infty} \phi_m(x, y, t)\mathfrak{z}_m(z) \tag{2.20c}$$

$$\rho = \rho_0(z) - \frac{\rho_M}{g}\sum_{m=0}^{\infty} \phi_m(x, y, t)\frac{d\mathfrak{z}_m}{dz} \tag{2.20d}$$

$$w = \sum_{m=0}^{\infty} w_m(x, y, t) \int_{-D}^{z} \mathfrak{z}_m(z')dz' \tag{2.20e}$$

The three independent variables (u, v, and ϕ) which solve the closed set of equations, (2.16a)–(2.16c), thus depend on the vertical eigenfunctions directly, but ρ and w are different. That ρ depends on the derivatives of the eigenfunctions rather than on the $\mathfrak{z}_m$ themselves is an obvious consequence of the hydrostatic equation – ρ is on one side while the *z-derivative* of ϕ is on the other – which also gives the simple relationship of the expansion coefficients of ρ to those of ϕ which is explicitly displayed in (2.20d).

The vertical dependence of w can be deduce from the equation of continuity in its original form

$$u_x + v_y + w_z = 0 \tag{2.21}$$

The vertical differentiation in (2.21) eliminates the integral in (2.20e) so that u, v, and the z-derivative of w are all proportional to $\mathfrak{z}_m$ for a given vertical mode.

The assumed expansions are all consistent with the boundary conditions, too. One can see why it is necessary to impose somewhat artificial conditions, such as no stress at the bottom, rather than the more natural no-slip: there are four boundary conditions at each surface, but only a single vertical structure function $\mathfrak{z}_m$. The sole boundary condition on $\mathfrak{z}_m$ at $z = 0$ is imposed, like it or not, on all four fields, and similarly at the bottom boundary.

The mechanics of this consistency are a little tricky. The integrals in (2.20e), for example, automatically impose the boundary condition $w = 0$ at the bottom because the lower limit of integration is the bottom, $z = -D$. The top boundary condition of $w = 0$ at $z = 0$ can only be satisfied if

$$\int_{-D}^{0} \mathfrak{z}_m(z')dz' = 0 \tag{2.22}$$

Extra boundary conditions like this cannot be satisfied in general, but (2.22) is fact a direct consequence of the eigenvalue problem which defines the $\mathfrak{z}_m$ as can be seen by rewriting (2.18) and (2.19) in integral equation form. By integrating both sides of (2.18) with respect to z, one obtains

$$\mathfrak{z}_{mz} = -N^2\lambda \int_{-D}^{z} \mathfrak{z}_m(z')dz' \tag{2.23}$$

This is valid for all z. Evaluating (2.23) at $z = 0$ and imposing the boundary condition (2.19), which makes the L.H.S. of (2.23) equal to 0, then gives (2.22).

The top boundary conditions on u and v are more complex: the vanishing of the derivatives of $\mathfrak{z}_m$ implies that each term in the z-derivative of the sums for u and v, (2.20a) and (2.20b), must individually vanish at $z = 0$ — and yet the boundary conditions there demand that u_x and v_x be non-zero to balance the wind stress. This issue is complex enough to deserve further discussion in Appendix B. Here, we will simply go ahead to derive the equations for each baroclinic mode and then invoke the integration-by-parts argument of McCreary [1].

Since (2.18) and (2.19) constitute a regular Sturm–Liouville problem, the eigenfunctions $\mathfrak{z}_m$ are a complete orthogonal set. This justifies the expansions (2.20). It also implies that we can derive spectral equations to compute the coefficients $u_m(x, y, t)$, $v_m(x, y, t)$, etc. through the usual procedure of multiplying the equations of motion by each basis function $\mathfrak{z}_m(z)$ to reduce each infinite sum to a single term.

The x-momentum equation is typical. Since none of the terms involve z-differentiation, the expansion of the L.H.S. of (2.16c) gives only terms proportional to $\mathfrak{z}_m(z)$ for various m. Multiplication by a particular basis function, $\mathfrak{z}_m(z)$, followed by integration then eliminates all terms in the infinite sum except $m = n$. The spectral projection of the R.H.S. of (2.16c) is

$$\int_{-D}^{0} \mathfrak{z}_n A[u_z/N^2]_z dz = A\mathfrak{z}_n (u_z/N^2)|_{-D}^{0} - \int_{-D}^{0} A(\mathfrak{z}_{nz}/N^2)u_z\, dz \tag{2.24}$$

$$= A\mathfrak{z}_n (u_z/N^2)|_{-D}^{0} - Au(\mathfrak{z}_{nz}/N^2)|_{-D}^{0} + \int_{-D}^{0} Au[\mathfrak{z}_{nz}/N^2]_z\, dz \tag{2.25}$$

by twice integrating-by-parts. To evaluate the integral, use the physical boundary conditions on $\partial u/\partial z$ to evaluate the first surface term, the boundary conditions on $\mathfrak{z}_m$ to evaluate the second, and the expansion of u into eigenfunctions and the orthogonality of the eigenfunctions. This gives

$$\int_{-D}^{0} \mathfrak{z}_n A[u_z/N^2]_z dz = \tau^x \mathfrak{z}_n(0) - 0 - A\lambda_n u_n(x, y, t) \int_{-D}^{0} \mathfrak{z}_n^2 dz \tag{2.26}$$

We see from (2.26) that the convenient normalizations for the vertical eigenfunctions are either

$$\mathfrak{z}_n(0) = 1 \qquad \text{[McCreary normalization]}, \tag{2.27}$$

which simplifies the first term in (2.26) and is the choice made by McCreary [1], or

$$\int_{-D}^{0} \mathfrak{z}_n^2 dz = 1, \qquad \text{[orthonormal]} \tag{2.28}$$

which simplifies the last term in (2.26). Either way, the spectral equivalent of (2.2) becomes

$$(\partial_t + A\lambda_n)u_n - f v_n + \phi_{nx} = F_n \tag{2.29a}$$
$$(\partial_t + A\lambda_n)v_n - f u_n + \phi_{ny} = G_n \tag{2.29b}$$
$$\lambda(\partial_t + A\lambda_n)\phi_n + u_{nx} + v_{ny} = 0 \tag{2.29c}$$

and the two dependent equations

$$\rho_n = \phi_n(\rho_M/g) \tag{2.30a}$$
$$w_n = \lambda_n(\partial_t + A\lambda_n)\phi_n \tag{2.30b}$$

where

$$F_n = \tau^x \mathfrak{z}_n(0) / \int_{-D}^{0} \mathfrak{z}_n^2 dz \tag{2.31a}$$
$$G_n = \tau^y \mathfrak{z}_n(0) / \int_{-D}^{0} \mathfrak{z}_n^2 da \tag{2.31b}$$

The closed set of three equations, (2.29), is identical in form with the linearized shallow water wave equations, and will therefore sometimes be referred to as the "shallow water wave equations". The two dependent equations, (2.30), will be needed only rarely.

2.4 Lamb's Parameter, Equivalent Depths, Kelvin Phase Speeds and All that

The reason for the rather lengthy section title is that there are several popular conventions for expressing the eigenvalues of the vertical structure equation. One of the older conventions is based on the analogue between (2.29) and the shallow water equations: the form is identical provided that we write

$$\lambda_n = \frac{1}{(gH_{eq})} \qquad \text{["equivalent depth"]} \tag{2.32}$$

where $H_{eq}(n)$ is the so-called "equivalent depth" for the n-th baroclinic mode, which replaces the actual depth H in the shallow water equations. Because ocean tides were first studied using the model of a single layer of homogeneous fluid, i.e., the shallow water equations, this "equivalent depth" convention is popular in both oceanic and atmospheric tidal theory, among others.

The second convention is based on the fact that $(gH)^{1/2}$ is the speed of a (non-rotating) gravity wave in the shallow water model. This would seem to have little relevance to the large-scale motions that are our primary interest here since these are strongly affected by rotation. However, the equatorial Kelvin wave (and its mid-latitude counterpart, the coastal Kelvin wave) are in geostrophic balance in one direction while having the structure of a non-rotating gravity wave in the other horizontal coordinate and z; its phase speed is always equal to $(gH)^{1/2}$ or to $(gH_{eq})^{1/2}$ for the baroclinic Kelvin waves that will be discussed at great length in later chapters. Because the Kelvin wave is of such physical importance in its own right as well as providing a convenient scale for other waves–the lowest meridional mode Rossby wave always travels, for long wavelength, at exactly 1/3 the phase speed of the Kelvin wave of the same vertical mode, for example — some equatorial dynamicists use the "Kelvin wave convention":

$$\lambda_n = \frac{1}{c_n^2} \qquad \text{[“Kelvin convention”]} \tag{2.33}$$

where c_n is the phase speed of the Kelvin wave for the n-th baroclinic mode. This is the way of writing the eigenvalue which is used in McCreary [1].

The third convention is to define a quantity, usually called "Lamb's parameter", to be the non-dimensional equivalent of the eigenvalue. Letting a = radius of the earth and Ω = angular frequency of the earth's rotation ($= 2\pi/86,400$s),

$$\varepsilon = (4\Omega^2 a^2)\lambda_n = \frac{(4\Omega^2 a^2)}{gH_{eq}} \qquad \text{[“Lamb's parameter”]} \tag{2.34}$$

Alas, there is no standard symbol for Lamb's parameter; the famous treatise of Longuet–Higgins [17] used ε. The nonlinear papers of Boyd used E so as to reserve ε for the perturbation parameter.

Fortunately, the issue is not of paramount importance because it is conventional in equatorial oceanography (although there a few diehards) to nondimensionalize the shallow water equations (2.29) in an ε-dependent way so that (nondimensionally) $\lambda_m = 1$. Note that because we have separated variables, we can study each baroclinic mode in isolation and are free to adopt a different nondimensionalization, i.e., one that depends on λ_m, if this is convenient.

The actual eigenvalues, expressed as equivalent depths, for a set of observations and for an idealized numerical model are given in Table 2.2.

Table 2.2 Properties of baroclinic modes for a simple analytical $T(z)$ [top] and observations [bottom]

Equivalent			Length	Time
Vertical mode n	Depth H_0 cm	$(gH_0)^{1/2}$ (cm s^{-1})	Scale L (km)	Scale T (days)
1	100	312	395	1.46
2	22	146	271	2.15
3	9.4	96	219	2.64
4	5.1	71	189	3.09
5	3.3	57	169	3.44
6	2.3	47	154	3.80
7	1.7	41	143	4.04
8	1.3	36	134	4.31
9	1.1	32	126	4.55
10	0.9	29	120	4.80
	H' (cm)	$\sqrt{gH'}$ (cm/s)	L (km)	T (days)
1	60	240	325	1.5
2	20	140	247	2.0
3	8	88	197	2.6
4	4	63	165	3.1
5	2	44	139	3.6

(a) (Model) $T(z) = 4. + 20.e^{z/500\text{m}}$ (in degrees $C°$). Data from Semtner and Holland [16]
(b) (Observed) [Atlantic] E.J. Katz, private communication, [18]

2.5 Vertical Modes and Layer Models

The weak flow below the thermocline in either the layered or continuously stratified case has led to a further idealization popular in equatorial oceanography: the "one-and-a-half layer" model. The vertically integrated horizontal velocities are always zero in the two-layer model. When the lower layer is much deeper than the top, as is always true in the real ocean — the best two-layer fit to the observed density is a bottom layer roughly *40* times thicker (4 km vs 100 m) than the upper — the motion in the lower layer is therefore extremely weak. It is then useful to simplify the model still further by pretending that the depth of the lower layer is infinite, instead of merely very large, so that the *lower layer* is at *rest*. Symbolically,

$$1\frac{1}{2}\text{layer model} \equiv 2\text{ layer model with lower layer of infinite depth and no motion}$$

This "one-and-a-half layer" model distorts the linear dynamics of the baroclinic mode only a little, but it profoundly alters the nonlinear dynamics by eliminating the coupling between the baroclinic and barotropic modes. The reason is that when

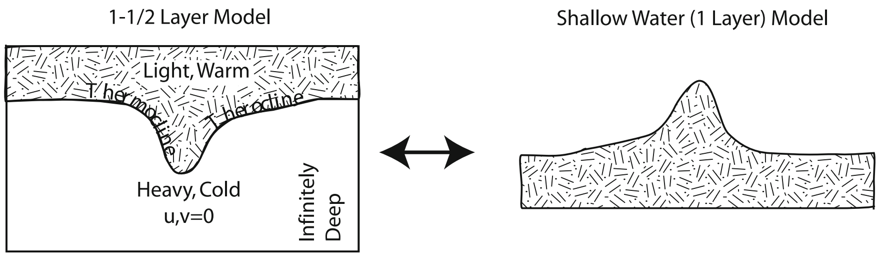

Fig. 2.1 The 1-1/2 layer model (*left*) is mathematically equivalent to the shallow water equations (*right*). Note that a *trough* in the thermocline, which is the boundary between the warm, dynamically-active *upper* layer and the cold, dynamically-passive layer translates into a *crest* in the shallow water model. Note also that the mean depth of the shallow water is the *equivalent* depth (typically half a meter) rather than the actual depth of the layer above the thermocline, which is O(100) m

a horizontal velocity like u is zero in the lower layer, a nonlinear product like uu_x is also zero in the lower layer. Therefore, the nonlinear terms involving the self-interaction of a motion in the baroclinic mode project back only onto the depth-varying baroclinic mode rather than onto the barotropic mode, which is independent of depth. The special cases of (i) equal depth or (ii) an infinite lower layer depth as here are the only circumstances in which the nonlinear barotropic mode/baroclinic mode coupling in the two-layer model is 0.

In reality, however, the coupling to the second baroclinic mode – which does not even exist in the two-layer model – can be as strong as the barotropic mode in the continuously stratified model. Furthermore, the second baroclinic mode motion is more interesting than the barotropic mode because only the baroclinic modes are equatorially trapped. Consequently, it is not possible to do a satisfactory treatment of nonlinear mode-coupling in the two-layer model even when the depth of the lower layer is kept finite, so most equatorial oceanographers have been quite shameless in retreating to the one-and-a-half layer models for nonlinear calculations. (Note that for purely *linear* calculations, the baroclinic and barotropic modes are completely uncoupled, and the difference between the two-layer and one-and-a-half layer models is irrelevant.)

The virtue of the one-and-a-half layer model is that it is identical in form with the *nonlinear* shallow water wave equations except that (i) the actual depth of the upper layer is a factor of a hundred or more larger than the *equivalent* depth, which is (O[0.6 m]) and (ii) the model must be turned upside down (Fig. 2.1). What is meant by this is that the height ϕ is proportional to the total *thickness* of the layer that is in motion (Fig. 2.1). Since the density difference across the thermocline is two orders of magnitude smaller than the difference across the air-water interface, it is the thermocline that moves, and not the sea surface. When the thermocline moves *down*, the total thickness of the upper layer increases. Thus, when the shallow water wave equations are being used to represent the baroclinic mode of the one-and-a-half layer model, a positive value of ϕ corresponds to downward movement of the thermocline, an increased thickness of the upper layer, and downwelling at the thermocline, i.e., negative vertical velocity. The numerical values in the shallow water model must be multiplied by the ratio of thermocline depth/equivalent depth to obtain the actual thermocline changes: ϕ equivalent to a 30 cm displacement in a shallow water model with an equivalent depth of only 60 cm corresponds to a 50 m change in the thermocline depth.

The one-and-a-half layer model has been described here in some detail because it has been surprisingly effective in explaining many features of the large-scale ocean circulation when supplied with realistic boundaries and wind stresses.

Nonetheless, the formalism of continuously stratified models is well understood, and they are clearly the wave of the future. Figure 2.2 compares the first four baroclinic modes as calculated by Semtner and Holland [16] for the idealized stratification

$$T(z) = 4 + 20e^{z/500m} \quad {}^\circ C \tag{2.35}$$

(the eigenvalues are given in Table 2.2) with the lone baroclinic mode of the two-layer model for a case in which the bottom layer is much deeper than the top layer. The structure of the first baroclinic mode for the exponential stratification is similar to that of the baroclinic mode of the two-layer model except for the smearing out of the step-function jump in the corresponding eigenfunction. However, both first baroclinic modes have strong flow in the upper ocean and much weaker flow – in the opposite direction – in the deep ocean. Thus, the layer model gives a reasonable, albeit crude, approximation to the lowest baroclinic mode of a more realistic, continuously stratified model. While the baroclinic modes shown in Fig. 2.2 are intended to be representative only—a more realistic stratification would give less amplitude at the bottom—it is easy to calculate the vertical modes for any specified stratification.

2.6 Nondimensionalization

Letting a = radius of the earth, Ω = angular frequency of the earth's rotation, and

$$\varepsilon = 4\Omega^2 a^2 / g H_{equivalent} \qquad \text{[“Lamb's parameter”]} \tag{2.34bis}$$

as before, one can *eliminate* a, Ω, and ε as explicit parameters by choosing vertical mode-dependent scalings. Letting asterisks denote the non-dimensional quantities,

$$x^* = \varepsilon^{1/4} x/a \tag{2.36a}$$

$$y^* = \varepsilon^{1/4} y/a \tag{2.36b}$$

$$t^* = \varepsilon^{-1/4} (2\Omega) t \tag{2.36c}$$

$$u^* = \varepsilon^{1/2} u/(2\Omega a) \tag{2.36d}$$

$$v^* = \varepsilon^{1/2} v/(2\Omega a) \tag{2.36e}$$

$$\phi^* = \varepsilon \phi/(2\Omega a)^2 \tag{2.36f}$$

$$A^* = \varepsilon^{5/4} A/[(2\Omega a)^2 2\Omega] \tag{2.36g}$$

will reduce the shallow water equations to a form with *no explicit parameters* except the viscosity coefficient, which we shall usually set equal to 0 (Fig. 2.2).

Planetary wave theory and middle atmosphere dynamics often uses a nondimensional frequency σ = dimensional frequency$/(2\Omega)$ and the zonal wavenumber is an integer s. The corresponding nondimensional phase speed is $c_{planetary}$ = phase speed$/(2\Omega a)$. These are connected to the nondimensional quantities used in equatorial oceanography by

$$k^* = s/\varepsilon^{1/4} \tag{2.37}$$

$$\omega^* = \varepsilon^{1/4}\sigma \tag{2.38}$$

$$c^* = \varepsilon^{1/2}\, c_{planetary} \tag{2.39}$$

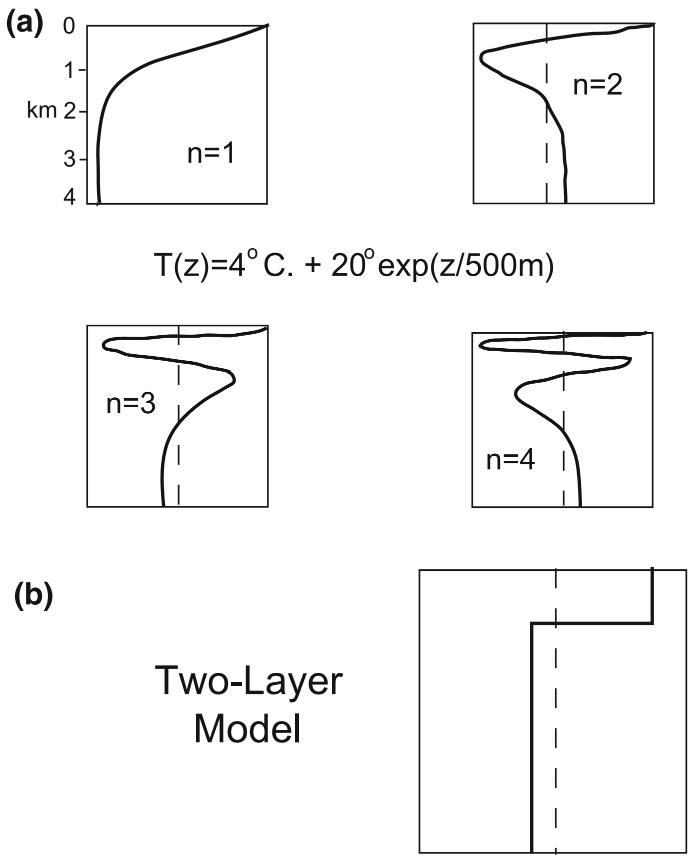

Fig. 2.2 Baroclinic modes versus depth where $\mathfrak{z}$ is the horizontal axis and depth z is the vertical axis. The *dashed vertical lines* are the axis $\mathfrak{z} = 0$. **a** First four modes for an exponential variation of mean temperature with depth. **b** First (and only) baroclinic mode of a two-layer model

$$\varepsilon\sigma^3 - [\varepsilon^{1/2}(2n+1) + s^2]\sigma - s = 0 \qquad \leftrightarrow \qquad \omega^3 - [(2n+1) + k^2]\omega - k = 0$$

The scalings in (2.36) can be summarized by stating that the nondimensional length scale L, time scale T and velocity scale U are given by

$$L = \varepsilon^{-1/4} a \tag{2.40a}$$

$$T = \varepsilon^{1/4}/(2\Omega) \tag{2.40b}$$

$$U = \varepsilon^{1/2}\, 2\Omega\, a \tag{2.40c}$$

where typically $L \sim O(300\text{km})$, $T \sim O(2\text{days})$ and U is $O(2\text{m/s})$.

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