

Chapter 2

Self-similarity in Walsh Functions

Abstract A structure which can be divided into smaller and smaller pieces, each piece being an exact replica of the entire structure is called self-similar. The set of Walsh functions can be classified into distinct self-similar groups and subgroups where members of each subgroup exhibit self-similarity. After a brief discussion on the generation of higher order Walsh functions from lower order Walsh functions by an alternating process, a scheme for classification of Walsh functions into self-similar groups and subgroups is presented. Self-similarity in radial and annular Walsh functions and the correspondence between Walsh filters and Walsh functions are also discussed.

Keywords Self-similarity • Fractals • Fractal optics • Self-similar Walsh filters

2.1 Introduction

Self-similarity is a typical property of fractals. The fractal structures play an important role in describing and understanding a large number of phenomena in several areas of science and technology [1]. In optics, the fractal structure of some optical wave fields and the diffraction patterns generated from various fractal apertures are examples of this trend [2, 3]. A family of shapes that are irregular and fragmented and cannot be explained by standard geometry can be put under a class called fractals, a term coined by Mandelbrot in 1975. French-American polymath Benoit Mandelbrot (1924–2010) is often referred to as the ‘Father of Fractals’. In his 1982 book, ‘The Fractal Geometry of Nature’ [4], he noted that the word fractal comes from the Latin word ‘fractus’, meaning broken. A structure may be called fractal if it possesses some or all of the following properties namely, self-similarity, fine-structure, (that is, it preserves details at all scales however small), size of the structure depends on the scale at which it is measured, classical methods of geometry and mathematics are not applicable to the structure, the structure has a natural appearance and the structure has a simple recursive construction.

A structure is said to be self-similar when it can be divided into smaller and smaller pieces, each piece being an exact replica of the entire structure [5]. The Von Koch curve, Sierpinski triangle, Cantor fractals are some of the very simple and common examples of fractals. Fractals with completely different appearance may be constructed by a very similar recursive procedure. The Von Koch curve was obtained by taking a straight line, dividing it into three equal parts, erasing the middle part and replacing it by the other sides of an equilateral triangle on the same base. This gives a chain of four shorter joined up straight line segments and the curve is finally obtained by doing the same thing to each of the four pieces, that is, by repeatedly replacing each straight line segment by a simple figure $\text{—}\wedge\text{—}$ sometimes called a generator or motif of the curve. The curve is self-similar, that is, it is made up of smaller copies of itself. In particular we could cut it into 4 parts each a $1/3$ scale copy of the entire curve. The Cantor fractal is perhaps the most basic self-similar fractal called the middle third Cantor set. Each stage of this construction is obtained by removing the middle third of each interval in the preceding stage, that is, it is made up of 2 scale $1/3$ copies of itself.

The complete set of Walsh functions do not exhibit self-similarity among their individual constituents. But members of the set can be classified into distinct groups, individual members of each group exhibiting self-similarity among themselves. In this chapter we intend to explore the inherent self-similarity present in the various orders of the Walsh functions and classify them into self-similar groups and find analytical expressions to represent members of such self-similar groups. Our study is mostly limited to orders k within the range $(0, 15)$ with occasional foray in Walsh functions of neighbouring higher orders. Discussion on the same is put forward in subsequent sections.

2.2 Generation of Higher Order Walsh Functions from Lower Order Walsh Functions

Unidimensional Walsh functions, $W_k(x)$, for order $k = 0, 1, \dots, 15$ are presented succinctly in Fig. 2.1. The total domain for each of the Walsh functions is $(0, 1)$. For every Walsh function, the total domain consists of a finite number of equal subdomains. The number of subdomains depends on the order of the Walsh function. The minimum number, say P , of subdomains to represent a Walsh function is an integral power of 2. For representing $W_0(x)$, $P = 2^0 = 1$. For other orders of Walsh functions P is given by the relation $P = 2^{\tilde{m}}$, where $\tilde{m}$, a positive integer, satisfies the relation $2^{\tilde{m}-1} < k \leq 2^{\tilde{m}}$. For example, the number of equal parts, P , is 2 for $W_1(x)$, is 4 for each of $W_2(x)$ and $W_3(x)$, and 8 for each of the Walsh functions $W_4(x)$, $W_5(x)$, $W_6(x)$ and $W_7(x)$. Again, P becomes 16 for each of the Walsh functions $W_i(x)$, $i = 8, \dots, 15$.

In Fig. 2.1 each row consists of elements comprised of either ‘+’ or ‘−’ denoting values +1 or −1 respectively. The first row represents $W_0(x)$ that consists of a single

Origin																	Order
	+																0
0	+								-								1
1	+				-				-				+				2
0	+				-				+				-				3
3	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	4
2	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	5
1	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	6
0	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	7
7	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	8
6	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	9
5	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	10
4	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	11
3	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	12
2	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	13
1	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	14
0	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	15

Fig. 2.1 Each row represents unidimensional Walsh functions $W_\ell(x)$ over the interval $(0, 1)$ for $\ell = 0, 1, \dots, 15$ in ascending order of sequence

domain $(0, 1)$ with ‘+’ over it. The second row represents $W_1(x)$. It consists of two subdomains; the first subdomain extending over $(0, \frac{1}{2})$ has ‘+’ over it, and the second subdomain extending over $(\frac{1}{2}, 1)$ has ‘-’ over it. Each of $W_2(x)$ and $W_3(x)$ consists of four subdomains $(0, \frac{1}{4})$, $(\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{4})$ and $(\frac{3}{4}, 1)$, and they are shown in the third and the fourth row respectively. Each of the four Walsh functions $W_i(x)$, $i = 4, \dots, 7$ consists of eight subdomains $(0, \frac{1}{8})$, $(\frac{1}{8}, \frac{2}{8})$, $(\frac{2}{8}, \frac{3}{8})$, $(\frac{3}{8}, \frac{4}{8})$, $(\frac{4}{8}, \frac{5}{8})$, $(\frac{5}{8}, \frac{6}{8})$, $(\frac{6}{8}, \frac{7}{8})$ and $(\frac{7}{8}, 1)$, and they are shown in the corresponding $(i + 1)$ th rows in Fig. 2.1. Walsh functions $W_i(x)$, $i = 8, \dots, 15$ are presented similarly in corresponding $(i + 1)$ th row. Each of them consists of 16 equal subdomains $(0, \frac{1}{16})$, $\dots$, $(\frac{15}{16}, 1)$.

To obtain Walsh functions of higher order from Walsh functions of lower order an ‘alternating process’ can be adopted [6]. The number on the extreme right of each row in Fig. 2.1 gives the order of the Walsh function corresponding to that row, and the number on the extreme left of each row gives the order of the Walsh function from which it is derived by the alternating process. $W_1(x)$ can be derived from $W_0(x)$, by dividing the interval $(0, 1)$ in two equal halves, and writing ‘+’ in the left half and ‘-’ in the right half. $W_2(x)$ can be similarly obtained from $W_1(x)$ by dividing each of the two sub intervals of $W_1(x)$ into equal halves, and writing ‘+ -’ in the sub intervals where it was ‘+’ before, and ‘- +’ where it was ‘-’ before. $W_3(x)$ consists of four equal intervals, and can be derived from $W_0(x)$ by dividing the interval $(0, 1)$ in two equal halves, and then writing ‘+ -’ wherever the value was + in $W_0(x)$. Writing ‘+ -’ over an interval implies that the interval is divided into two equal subintervals, and then ‘+’ is put on the left subinterval, and ‘-’ is put on the right subinterval. Similarly, writing ‘- +’ over an interval implies that the interval is divided into two equal subintervals, and ‘-’ is put on the left subinterval, and ‘+’ is put on the right subinterval. In the same manner, $W_4(x)$ can be derived

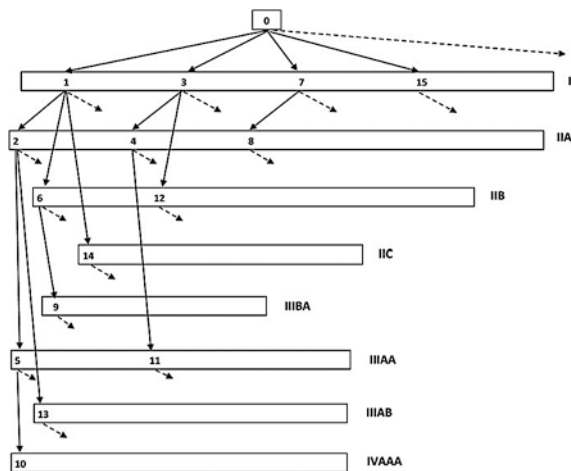
from $W_3(x)$, and $W_5(x)$ can be derived from $W_2(x)$. The choice of appropriate lower order from which a Walsh function of a higher order can be derived is decided by the number and location of zero crossings in the corresponding Walsh functions.

2.3 Classification of Walsh Functions of Various Orders in Self-similar Groups and Subgroups

The set of Walsh functions as a whole does not comprise of self-similar structures. However, these functions may be classified into different groups so that individual constituents of each group demonstrate self-similarity among themselves. Group I consists of Walsh functions of orders 1, 3, 7, 15, ... All of them are derived from $W_0(x)$. Group II consists of Walsh functions of orders 2, 4, 6, 8, 12, 14, ... It may be seen from Fig. 2.1 that each member of this group can be derived from members of group I. Similarly, Group III consists of Walsh functions of orders 5, 9, 11, 13, Figure 2.1 shows that they may be derived from Walsh functions of order 2, 6, 4 and 2 respectively. Each of the latter Walsh functions is a member of group II. Similarly, members of group IV are derived from members of group III, and so on. Figure 2.2 provides a genealogical chart of self-similar groups and subgroups in the set of unidimensional Walsh functions.

Group I is unique. It has no subgroups and members of the group portray self-similarity in their structures. All other groups have subgroups, and members of a particular subgroup exhibit self-similarity in their structures. Members of group II give rise to several subgroups. The order ℓ of the first member of four of these subgroups, namely IIA, IIB, IIC and IID, belonging to group II, is within (0, 15). Each of these subgroups give rise to further subgroups. Taken together, all of them

Fig. 2.2 A genealogical chart of self-similar groups and subgroups in the set of Walsh functions



where b and c are two parameters specific for each subgroup. In general, b is a positive integer or zero, and c is either -1 or zero, or an integer whose value depends on the value of σ . For Group I and the first few subgroups of Group II and Group III, values of b , c , and $Z(\sigma)$ are given in the table. Our study is mostly limited to those subgroups, whose first member has order that falls within the range $(0, 15)$, with occasional foray in Walsh functions of neighboring higher orders.

The list of groups and subgroups given in Table 2.1 is not exhaustive. The results presented here show only those groups and subgroups, for which the order of the first member falls within the range $(0, 15)$. It is obvious that new subgroups will emerge when higher order Walsh functions are taken properly into account. For the groups and subgroups already identified, analytical expression (2.1) for $Z(\sigma)$ can be used to identify members lying among the higher orders of Walsh functions. More details on the self-similar groups and subgroups in the set of Walsh functions can be found in a recent publication [5].

2.4 Self-similarity in Radial Walsh Functions

The set of radial Walsh functions $\varphi_k(r)$ can be classified into distinct self-similar groups and subgroups as described in the previous section for the unidimensional Walsh functions. In $t(=r^2)$ space, the structure of radial Walsh functions $\varphi_k(t)$ becomes identical with unidimensional Walsh functions $W_k(x)$, as shown in Fig. 2.1.

2.5 Self-similarity in Annular Walsh Functions

The set of annular Walsh functions can also be classified into distinct self-similar groups and subgroups where members of each group possess self-similar structures or phase sequences. The classification scheme adopted for unidimensional or radial functions can be utilized directly for this case.

2.6 Radial and Annular Walsh Filters

Radial Walsh filters, derived from radial Walsh functions (as defined in Sect. 1.3.3 of Chap. 1), form a set of orthogonal phase filters that take on values either 0 or π phase, corresponding to $+1$ or -1 values of the radial Walsh functions over pre-specified annular regions of the circular filter [Note: $e^{i0} = +1$; $e^{i\pi} = -1$]. Order of these filters is given by the number of zero-crossings, or equivalently phase transitions within the domain over which the set is defined. In general, radial Walsh

filters are binary phase filters or zone plates, each of them demonstrating distinct focusing characteristics. Annular Walsh filters are derived from the rotationally symmetric annular Walsh functions (as defined in Sect. 1.3.5 of Chap. 1) which form a complete set of orthogonal functions that take on values either $+1$ or -1 over the domain specified by the inner and outer radii of the annulus. The value of any annular Walsh function is taken as zero from the center of the circular aperture to the inner radius of the annulus. The three values 0 , $+1$ and -1 in an annular Walsh function can be realized in a corresponding annular Walsh filter by using transmission values of zero amplitude (i.e. an obscuration), unity amplitude and zero phase and unity amplitude and π phase respectively. Not only does the order of the annular Walsh filter, but also the size of the inner radius of the annulus provides an additional degree of freedom in tailoring of point spread function by using these filters for pupil plane filtering in imaging systems. These filters may be considered as ternary phase filters with values of transmission 0 , $+1$ and -1 .

Since the transmission characteristics of k th order radial Walsh filter is uniquely specified by $\varphi_k(r)$, the k th order radial Walsh function, the k th order radial Walsh filter may be conveniently represented by the same notation. Similarly, the k th order annular Walsh filter with obscuration ratio ε is represented by the notation $\varphi_k^\varepsilon(r)$.

2.7 Self-similarity in Walsh Filters

2.7.1 Self-similarity in Radial Walsh Filters

The set of radial Walsh filters can be classified into distinct self-similar groups and subgroups where members of each subgroup possess self-similar structures or phase sequences, in the same manner as the radial Walsh functions from which they are derived. Group I is unique. It has no subgroup. Groups II, III and IV have subgroups as represented in Table 2.1. Members of each subgroup possess self-similar structures or phase sequences.

2.7.2 Self-similarity in Annular Walsh Filters

Annular Walsh filters may be classified into distinct self-similar groups and subgroups in the same manner as the annular Walsh functions from which they are derived. Group I is unique. It has no subgroups. Groups II, III and IV can be divided into subgroups as represented in Table 2.1. Members of each subgroup possess distinct self-similar structures or phase sequences.

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