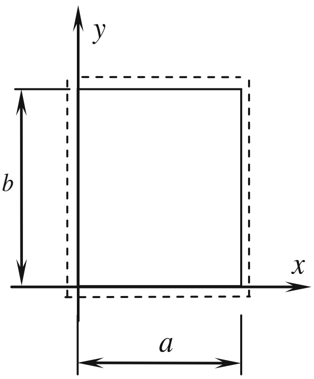
Chapter 2
Thermal Bending of Concrete Rectangular Thin Plate with Four Supported Edges

Abstract The deflection equation and the internal force analytical solution of the rectangular thin plate supported on four sides (four edges simply supported, four edges clamped, three edges clamped and one edge simply supported, one edge clamped and three edges simply supported, two adjacent edges clamped and two adjacent edges simply supported, two opposite edges clamped and two opposite edges simply supported) under temperature difference is systematically introduced in this chapter. In order to facilitate the engineering application, the tables for deflection and internal force coefficient calculation based on concrete material are made.



2.1 Introduction

The rectangular thin plate with four supported edges can be classified into six types: rectangular thin plate with four simply supported edges; rectangular thin plate with four clamped edges; rectangular thin plate with three clamped edges and one simply supported edge; rectangular thin plate with three simply supported edges and one clamped edge; rectangular thin plate with two adjacent simply supported edges and two adjacent clamped edges, rectangular thin plate with two opposite simply supported edges and two clamped opposite edges.

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For the calculation of the temperature effect of the rectangular thin plate, the existing literature gives the analytical solution for the rectangular thin plate with four simply supported edges and the rectangular thin plate with four clamped edges for the isotropic materials. For example, to the rectangular thin plate with four simply supported edges under temperature disparity, the literature [4] gave the calculation formulas of deflection with any temperature change, namely

$$w(x,y) = \frac{1}{(1-\mu)\pi^2 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{mn}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{2.1}$$

where $a_{mn} = \frac{4}{ab}\int_0^a\int_0^b M_T(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{d}y\mathrm{d}x$, $D = \frac{Eh^3}{12(1-\mu^2)}$.

The literature [3] gave the calculation formulas of deflection with temperature change along the thickness change, namely

$$w(x,y) = \frac{16M^*}{(1-\mu)\pi^4 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}(m,n=1,3,5\ldots) \tag{2.2}$$

where, $M^* = E\alpha\int_{-\frac{h}{2}}^{\frac{h}{2}}(\Delta T)z\mathrm{d}z$.

The literature [49] also gave the approximate calculation formulas of bending moment with temperature change along the thickness change, namely

$$M_{xT}(M_{yT}) = k_{xt}(k_{yt})M^T \tag{2.3}$$

in above equation, M^T stands for the distribution moment of rectangular thin plate with four clamped edges caused by lateral temperature disparity; ΔT stands for the lateral temperature disparity; α stands for the thermal expansion coefficient of material; E stands for elastic modulus of material; μ stands for Poisson's ratio of materials; h stands for the thickness of the thin plate; $k_{xt}(k_{yt})$ stands for the coefficient of bending moment; M_{xT} stands for the distribution moment in the x direction caused by lateral temperature disparity; M_{yT} stands for the distribution moment in the y direction caused by lateral temperature disparity.

It is easy to see that (2.2) is the special form of (2.1), and (2.3) is an approximation of (2.2). Therefore, here only (2.1) will be described. By calculating, (2.1) only satisfies the edge condition of $w = 0$, but does not satisfy the edge condition of moment being equal to zero, and it requires further research whether there are other deflection functions.

Although there are solutions of rectangular thin plate with four clamped edges under the action of the temperature and widely used in engineering [18], such as:

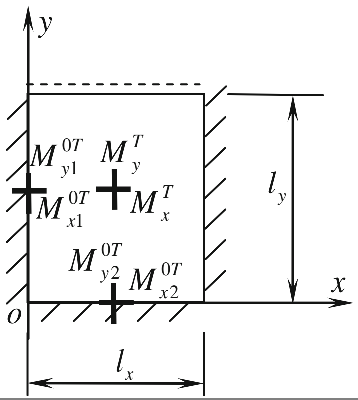
$$M_x = M_y = M^T = \frac{\alpha\Delta T E h^2}{12(1-\mu)} \tag{2.4}$$

(2.4) is obtained by the deflection function satisfying the boundary conditions of $w = 0$. But for statically indeterminate structure under the action of temperature, the lower temperature side is in tension, the higher temperature side is in compression. And it is continuously everywhere on the upper and lower surface. Due to the existence of elastic modulus, whether the deflection function $w = 0$ is appropriate in the existing literature needs further verification, as well as the deflection function should be taken as $w = w\ (x,\ y)$.

Though Liu et al. showed the calculating table of rectangular thin plate with three clamped edges and one simply supported edge (see Table 2.1) in their work [13], transversal moments on the clamped edges were greater than the solution obtained by $w = 0$. For the calculation of temperature effects about other rectangular thin plate with four supported edges, current literatures have not been reported.

Therefore, this chapter is based on the small deflection plate theory. Firstly, through assuming deflection function that meets equilibrium differential equation and some of boundary condition, using the Levy method, the analytical solution of deflection and internal force of isotropic rectangular thin plate with four simply supported edges is derived. Then according to the conclusion of rectangular thin plate with four simply supported edges and boundary conditions of other

Table 2.1 Bending calculation coefficient with three edges clamped and one edge free under temperature disparity



$\mu = \frac{1}{6}$
$M_x^T = k_x^T \alpha \Delta T E h^2 \eta_{re1}$
$M_y^T = k_y^T \alpha \Delta T E h^2 \eta_{re1}$
η_{re1} is reduction factor of considering concrete creep

l_x/l_y	k_{x1}^T	k_{y1}^T	k_{x2}^T	k_{y2}^T	k_x	k_y
0.50	0.1045	0.0987	0.0972	0.1000	0.0973	0.0998
0.75	0.1139	0.0999	0.0982	0.1021	0.0926	0.1003
1.00	0.1233	0.1008	0.0981	0.1094	0.0885	0.0961
1.25	0.1288	0.1011	0.0993	0.1175	0.0869	0.0917
1.50	0.1344	0.1016	0.1008	0.1286	0.0853	0.0873
1.75	0.1329	0.1013	0.1014	0.1344	0.0877	0.0829
2.00	0.1324	0.1008	0.1019	0.1402	0.0901	0.0784

rectangular thin plate with four supported edges, by applying the virtual displacement principle and superposition principle, the deflection equation and analytical solution of internal force of the other rectangular thin plate with four supported edges under the action of lateral temperature disparity is derived, which provides a theoretical basis for later engineering calculation.

2.2 The Basic Equation for the Thermal Elastic Problem of Rectangular Thin Plate

2.2.1 Calculation Assumption

(1) Straight-line which is perpendicular to the mid-plane before deformation is still perpendicular to the deformed mid-plane, and the length has no change;
(2) The stress σ_z, τ_{xz} and τ_{yz} are far less than the other three stresses (σ_x, σ_y and τ_{xy}), so the strain caused by these stress can be neglected;
(3) Each point in mid-plane has not displacement which is parallel to mid-plane, namely $u|_{z=0} = 0$, $v|_{z=0} = 0$.

2.2.2 Basic Equation of Thermal Elasticity

The existing literature [3–5] showed that the geometry equation for the thermal elastic problem of rectangular thin plate is

$$\begin{cases} \varepsilon_x = \frac{\partial u}{\partial x} = -\frac{\partial^2 w}{\partial x^2} z \\ \varepsilon_y = \frac{\partial v}{\partial y} = -\frac{\partial^2 w}{\partial y^2} z \\ \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -2\frac{\partial^2 w}{\partial x \partial y} z \end{cases} \tag{2.5}$$

where u, v stand for the displacements in x, y directions, respectively; w stands for the deflection of any point on the surface of the thin plate; ε_x, ε_y and γ_{xy} stand for the strains of any point on the surface of the thin plate, respectively.

Because of neglecting strain caused by the stress σ_z, physical equation can be written as

$$\begin{cases} \varepsilon_x = \frac{1}{E}\left(\sigma_x - \mu\sigma_y\right) + \alpha T \\ \varepsilon_y = \frac{1}{E}\left(\sigma_y - \mu\sigma_x\right) + \alpha T \\ \gamma_{xy} = \frac{2(1+\mu)}{E}\tau_{xy} \end{cases} \tag{2.6}$$

where $T = T(x, y, z)$ stands for temperature disparity of any point in the thin plate.

Stresses can be written as

$$\begin{cases} \sigma_x = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) - \frac{E\alpha T}{1-\mu} \\ \sigma_y = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) - \frac{E\alpha T}{1-\mu} \\ \tau_{xy} = \tau_{yx} = -\frac{Ez}{1+\mu}\frac{\partial^2 w}{\partial x \partial y} \end{cases} \tag{2.7}$$

As shown in Fig. 2.1, M_x and M_y stand for moments of unit width on the cross section, respectively; M_{xy} stands for torment of unit width on the cross section, hence

$$\begin{cases} M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_x \mathrm{d}z = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) - \frac{1}{1-\mu}\int_{-\frac{h}{2}}^{\frac{h}{2}} E(T)T\alpha(T)z\mathrm{d}z \\ M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_y \mathrm{d}z = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) - \frac{1}{1-\mu}\int_{-\frac{h}{2}}^{\frac{h}{2}} E(T)T\alpha(T)z\mathrm{d}z \\ M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\tau_{xy} \mathrm{d}z = -D(1-\mu)\frac{\partial^2 w}{\partial x \partial y} \end{cases} \tag{2.8}$$

The thickness of the plate compared with the size of the other two directions is very small, for the purpose of engineering application, assuming that the temperature changes along the thickness direction only, namely, we only consider the situation of lateral temperature change. That is to say, in this paper, the analytical solution of lateral deflection and internal forces that is studied is for the specific condition of lateral temperature change; in addition, due to the thin plate, before the plate structure is normally used, the heat release W of concrete condensation sclerosis tends to zero as the change of pouring time. So the original parabolic nonlinear temperature distribution rule becomes the linear situation, namely [17]

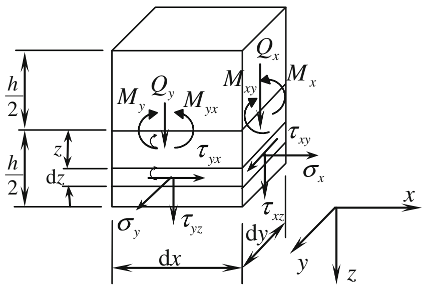


Fig. 2.1 Element forces and stresses sketch map

$$T = \frac{T_2 + T_1}{2} - \frac{(T_2 - T_1)}{h} z \tag{2.9}$$

where T_1 and T_2 stand for the temperature on two surfaces of the thin plate, respectively.

According to the existing literature [15], the relationship between concrete elastic modulus under any temperature normal temperature can be determined by using the following equations:

$$\begin{cases} E(T) = E & T \le 60\,^{\circ}\mathrm{C} \\ E(T) = 0.88E \sim 0.94E & 60\,^{\circ}\mathrm{C} < T \le 100\,^{\circ}\mathrm{C} \\ E(T) = 0.95E \sim 1.08E & 100\,^{\circ}\mathrm{C} < T \le 300\,^{\circ}\mathrm{C} \\ E(T) = \left[1 + 18\left(\frac{T}{1000}\right)^{5.1}\right]^{-1} E & T > 300\,^{\circ}\mathrm{C} \end{cases} \tag{2.10.1}$$

Under the action of temperature, the linear expansion coefficient $\alpha(T)$ is determined by using the following equation:

$$\alpha(T) = 28\left(\frac{T}{1000}\right) \times 10^{-6} \tag{2.10.2}$$

As can be seen, the temperature only slightly affects the elastic modulus of concrete and the linear expansion coefficient under normal temperature. Thus, the approximate values are as follows:

$$E(T) = E,\ \alpha(T) = \alpha \tag{2.11}$$

By substituting (2.9) and (2.11) into (2.8) obtains the following:

$$\begin{cases} M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) - M^T \\ M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right) - M^T \\ M_{xy} = -D(1 - \mu)\frac{\partial^2 w}{\partial x \partial y} \end{cases} \tag{2.12}$$

where $M^T = \frac{E\alpha\Delta T h^2}{12(1-\mu)}$; ΔT stands for the lateral temperature disparity [16, 50].

As it is known that the equilibrium differential equations of elastic surface about thin plate with same thickness is:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

As there is no external load, only temperature, (2.12) is substituted into the above mentioned equation, there is:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 w = 0. \tag{2.13}$$

2.3 Thermal Bending of Rectangular Thin Plate with Four Edges Simply Supported

2.3.1 Boundary Conditions

In Fig. 2.2, the boundary conditions to clamped edges are:

$$w\Big|_{\substack{x=0\\x=a}} = 0,\ M_x\Big|_{\substack{x=0\\x=a}} = 0$$

$$w\Big|_{\substack{y=-\frac{b}{2}\\y=\frac{b}{2}}} = 0,\ M_y\Big|_{\substack{y=-\frac{b}{2}\\y=\frac{b}{2}}} = 0$$

To simply supported edges, due to $w = 0$ on whole edge, according to (2.12), the above formulas are (Fig. 2.2):

$$w\Big|_{\substack{x=0\\x=a}} = 0,\ \frac{\partial^2 w}{\partial x^2}\Big|_{\substack{x=0\\x=a}} = -\frac{M^T}{D} \tag{2.14}$$

$$w\Big|_{\substack{y=-\frac{b}{2}\\y=\frac{b}{2}}} = 0,\ \frac{\partial^2 w}{\partial y^2}\Big|_{\substack{y=\frac{b}{2}\\y=-\frac{b}{2}}} = -\frac{M^T}{D} \tag{2.15}$$

2.3.2 The Analytical Solution of the Thermal Elastic Problem

According to (2.13), ordering

$$w = \sum_{m=1}^{\infty} X_m Y_m - \frac{M^T}{2D}(x-a)x$$

where X_m is only a function about x; Y_m is a function about y only.

According to the edge conditions (2.14), ordering $X_m = \sin\frac{m\pi x}{a}$, so the deflection function w can be written as

$$w = \sum_{m=1}^{\infty} Y_m \sin\frac{m\pi x}{a} - \frac{M^T}{2D}(x-a)x \tag{2.16}$$

By substituting (2.16) into the differential (2.13), hence:

$$Y_m^{(4)} - 2\frac{m^2\pi^2}{a^2}Y''_m + \frac{m^4\pi^4}{a^4}Y_m = 0$$

Solution of this equation can be written as follows [51]:

$$Y_m = A_m \sinh\frac{m\pi y}{a} + B_m \cosh\frac{m\pi y}{a} + C_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + D_m\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}$$

Due to the temperature and the plate are symmetrical about the x-axis, so Y_m must be an even function, thereby $A_m = D_m = 0$, by substituting them into (2.16), hence

$$w = \sum_{m=1}^{\infty}\left(B_m \cosh\frac{m\pi y}{a} + C_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi y}{a} - \frac{M^T}{2D}(x-a)x \tag{2.17}$$

Ordering $\frac{m\pi b}{2a} = \alpha_m$, by substituting (2.17) into the boundary condition (2.15), hence

$$\sum_{m=1}^{\infty}(B_m \cosh\alpha_m + C_m\alpha_m \sinh\alpha_m)\sin\frac{m\pi x}{a} = \frac{M^T}{2D}(x-a)x \tag{2.18}$$

$$\sum_{m=1}^{\infty}\frac{m^2\pi^2}{a^2}[(B_m + 2C_m)\cosh\alpha_m + C_m\alpha_m \sinh\alpha_m]\sin\frac{m\pi x}{a} = -\frac{M^T}{D} \tag{2.19}$$

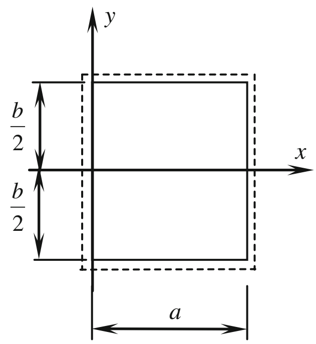


Fig. 2.2 Four edges simply supported

The right side of (2.18) is expanded into a single triangular series, namely

$$\frac{M^T}{2D}(x-a)x = \sum_{m=1}^{\infty}\left[\frac{2}{a}\int_0^a \frac{M^T}{2D}(x-a)x\,\sin\frac{m\pi x}{a}\mathrm{d}x\right]\,\sin\frac{m\pi x}{a}$$
$$= \sum_{m=1}^{\infty}\frac{2a^2M^T}{Dm^3\pi^3}(\cos m\pi-1)\,\sin\frac{m\pi x}{a}$$

Then (2.18) becomes

$$B_m\,\cosh\alpha_m + C_m\alpha_m\,\sinh\alpha_m = \frac{2a^2M^T}{Dm^3\pi^3}(\cos m\pi\,-1) \tag{2.20}$$

Similarly, the right side of (2.19) is expanded into a single triangular series, namely

$$-\frac{M^T}{D} = \frac{2M_T}{\pi D}\sum_{m=1}^{\infty}\frac{\cos m\pi-1}{m}\sin\frac{m\pi x}{a}$$

Then the (2.19) becomes

$$(B_m+2C_m)\,\cosh\alpha_m + C_m\alpha_m\,\sinh\alpha_m = \frac{2a^2M^T}{Dm^3\pi^3}(\cos m\pi-1) \tag{2.21}$$

By (2.20) and (2.21), there is

$$\begin{cases} B_m = \frac{2a^2M^T}{D\pi^3m^3\,\cos\mathrm{h}\alpha_m}(\cos m\pi-1) \\ C_m = 0 \end{cases}$$

By substituting B_m and C_m into (2.17), hence

$$w = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\,\cosh\alpha_m}\cosh\frac{2\alpha_m y}{b}\sin\frac{m\pi x}{a} - \frac{M^T}{2D}(x-a)x \tag{2.22}$$

Since (2.22) is made from satisfying the equilibrium differential Eqs. (2.13) and all boundary condition (2.14) and (2.15), so (2.22) is the deflection function of rectangular thin plate with simply supported edges under transverse temperature disparity. Substituting (2.22) into (2.12), internal force calculation formula is obtained.

Because

$$\begin{cases} \frac{\partial^2 w}{\partial x^2} = \frac{4M^T}{D\pi}\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\,\cosh\alpha_m}\cosh\frac{2\alpha_m y}{b}\sin\frac{m\pi x}{a} - \frac{M^T}{D} \\ \frac{\partial^2 w}{\partial y^2} = -\frac{2M^T}{D\pi}\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\,\cosh\alpha_m}\cosh\frac{2\alpha_m y}{b}\,\sin\frac{m\pi x}{a} \\ \frac{\partial^2 w}{\partial x\partial y} = -\frac{4M^T}{D\pi}\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\,\cosh\alpha_m}\sinh\frac{2\alpha_m y}{b}\,\cos\frac{m\pi x}{a} \end{cases}$$

Therefore, there is

$$\begin{cases} M_x^T = \frac{4M^T}{\pi}(\mu - 1)\sum_{m=1,3,\ldots}^{\infty} \frac{1}{m\,\cosh\alpha_m}\cosh\frac{m\pi y}{a}\sin\frac{m\pi x}{a} \\ M_y^T = \frac{4M^T}{\pi}(1 - \mu)\sum_{m=1,3,\ldots}^{\infty} \frac{1}{m\,\cosh\alpha_m}\cosh\frac{m\pi y}{a}\sin\frac{m\pi x}{a} + (\mu - 1)M^T \\ M_{xy}^T = \frac{4M^T}{\pi}(1 - \mu)\sum_{m=1,3,\ldots}^{\infty} \frac{1}{m\,\cosh\alpha_m}\sinh\frac{m\pi y}{a}\cos\frac{m\pi x}{a} \end{cases} \tag{2.23}$$

(2.23) is internal force analytical solution of rectangular thin plate with simply supported edges under transverse temperature disparity.

2.3.3 Results Analysis

MATLAB software is used to test the accuracy of Eqs. (2.22) and (2.23). The results show that for deflection function w, when taking $m = n = 5$, the result has converged to the exact solution; for the bending moment of unit width, when taking $m = n = 7$, the result has converged to the exact solution; for the bidirectional plate in engineering with any length-width ratio, its internal force solutions are equal to the results with existing literature (Because the existing literature has not given deflection calculation coefficient, it did not make deflection comparison). Because the existing literature has not given the deflection calculation coefficient, so for the convenience and engineering application, supplementing deflection calculation coefficient, thermal bending calculation results of concrete rectangular thin plate with four simply supported sides are made (see Table A.1).

2.3.4 Engineering Design

For a concrete rectangular thin plate with four edges simply supported under temperature variation which is perpendicular to surface, according to (2.10.1) and (2.10.2), E and α are obtained. And then M^T is given, namely $M^T = \frac{E\alpha\Delta Th^2}{12(1-\mu)}$.

According to Table A.1, k_x, k_y and f can be obtained by $\frac{l_x}{l_y}$. Then M_x^T, which is $M_x^T = k_x M^T$, is gotten as well as w_1, and $M_y^T = k_y M^T$. After the bending moment is gotten, the steel bars due to temperature can be designed according to the knowledge of the reinforced concrete. Namely [44−45]

$$\begin{cases} \alpha_{sx} = \frac{M_x^T}{1000\alpha_1 f_c h_0^2} \\ \alpha_{sy} = \frac{M_y^T}{1000\alpha_1 f_c h_0^2} \end{cases} \tag{2.24.1}$$

$$\begin{cases} \gamma_{sx} = 0.5\left(1 + \sqrt{1 - 2\alpha_{sx}}\right) \\ \gamma_{sy} = 0.5\left(1 + \sqrt{1 - 2\alpha_{sy}}\right) \end{cases} \tag{2.24.2}$$

$$\begin{cases} A_{sx1} = \frac{M_x^T}{f_y \gamma_s h_0} \\ A_{sy1} = \frac{M_y^T}{f_y \gamma_s h_0} \end{cases} \tag{2.24.3}$$

where, M_x^T is the bending moment design value in x direction, and $M^T{}_y$ is the bending moment design value in y direction, and A_{sx1} is the steel section area per meter width in x direction, and A_{sy1} is the steel section area per meter width in y direction, and f_y is the tensile strength design value of the steel, and α_{sx} and α_{sy} are the coefficient of section resistance moment in x and y directions, and α_1 is the equivalent rectangular stress diagram coefficient of concrete compressive zone, and γ_s is the internal force arm coefficient of section, and h_0 is the effective height of the section, $h_0 = h - c$ (c is the thickness of the concrete protective layer), and f_c is the compressive strength design value of the concrete.

If the deflection is w_2 and the area of steel bar is $A_{sx2}(A_{sy2})$ per unit width in $x(y)$ direction caused by other factors except for the temperature are known, there are

$$\begin{cases} A_{sx} = A_{sx1} + A_{sx2} \\ A_{sy} = A_{sy1} + A_{sy2} \\ w = w_1 + w_2 \end{cases} \tag{2.25}$$

Pay attention to that, the formulas of this chapter are based on thin plate structure that is homogeneous elastic body, which does not accord with the concrete. Especially, the creep and cracks of concrete reduce component stiffness, and result in thermal stress relaxation. Therefore, according to Table A.1, calculating bending moment should multiply the reduction factor of 0.65, and bearing capacity calculation should multiply the partial coefficient.

2.3.5 Numerical Example

Example: Taking the liquid storage structure with top plate as an example, the numerical analysis is carried out by a reasonable calculation. The length l_x and width l_y of the plate are both 6 m. The thickness h of the plate is 180 mm. The temperature difference ΔT between the upper and lower surface of plate is 60 °C. The live load p is 1 kN/m^2. The bulk density of concrete is 26 kN/m^3. The value of concrete strength is 30 MPa. The value of steel strength is 360 MPa.

Solution: In view of the fact that the stiffness of wallboard is far greater than the stiffness of top plate in general, the top plate can be regarded as the four edges simply supported. According to the literature [45], the linear expansion coefficient α of concrete is 1×10^{-5} °C. The Poisson's ratio μ of concrete is 1/6. The protective layer thickness of concrete is 10 mm. The elastic modulus E of concrete is 3×10^7 kN/m^2. The design value of compressive strength f_c for concrete is 14.3 N/mm^2. The partial coefficients of the dead load and live load are taken as 1.2 and 1.4, respectively.

Dead load: $g = 0.18 \times 26 = 4.68$ kN/m^2
Live load: $p = 1$ kN/m^2
Design load: $q = 1.4p + 1.2g = 7.02$ kN/m^2.

According to the initial assumption that the diameter of the steel is 10 mm, the distance from the center of the steel in x direction to the down surface of concrete plate, $c_x = c + 10/2$, is 15 mm and the distance from the center of the steel in y direction to the down surface of concrete plate, $c_y = c + 10 + 10/2$, is 25 mm. The distance from the center of the steel in x direction to the top surface of concrete plate, $h_{0x} = h - c_x$, is 165 mm and the distance from the center of the steel in y direction to the top surface of concrete plate, $h_{0y} = h - c_y$, is 155 mm.

1. **Temperature Action**

Taking $E = 3 \times 10^7$ kN/m^2, $\alpha = 1 \times 10^{-5}$ °C, $\Delta T = 60$ °C, $h = 180$ mm and $\mu = 1/6$ into the (2.1), $D = \frac{Eh^3}{12(1-\mu^2)}$, and (2.4), the following results can be gotten.

$$D = \frac{Eh^3}{12(1-\mu^2)} = \frac{3 \times 10^7 \times 0.18^3}{12\left(1 - \frac{1}{6^2}\right)} = 14996.57\,\mathrm{kN \cdot m}$$

$$M^T = \frac{\alpha \Delta T E h^2}{12(1-\mu)} = \frac{1 \times 10^{-5} \times 60 \times 3 \times 10^7 \times 0.18^2}{12\left(1 - \frac{1}{6}\right)} = 58.32\,\mathrm{kN}$$

From the Table A.1 in the Appendix A, there are

$$f = 0.0737, k_x = 0.4167, k_y = 0.4167$$

$$w_1 = f\frac{l_x^2 M^T}{D} = 0.0737 \times \frac{6 \times 58.32}{14996.57} = 0.0103\,\mathrm{m}$$

$$M_x^T = k_x M^T = 0.4167 \times 58.42 = 24.30\,\mathrm{kN \cdot m}$$

$$M_y^T = k_y M^T = 0.4167 \times 58.42 = 24.30\,\mathrm{kN \cdot m}$$

According to the literature [45], $\alpha_1 = 1$, assuming that $h_0 = h_{0x}$, and taking M_x^T, f_c, h_0 and α_1 into (2.24.1), there is

$$\alpha_{sx} = \frac{M_x^T}{1000\alpha_1 f_c h_0^2} = \frac{24.3 \times 10^6}{1000 \times 1 \times 14.3 \times 165^2} = 0.0624$$

Assuming that $h_0 = h_{0y}$, taking M_y^T, f_c, h_0, and α_1 into (2.24.1), there is

$$\alpha_{sy} = \frac{M_y^T}{1000\alpha_1 f_c h_0^2} = \frac{24.3 \times 10^6}{1000 \times 1 \times 14.3 \times 155^2} = 0.071$$

Taking α_{sx} and α_{sy} into (2.24.2), there is

$$\begin{cases} \gamma_{sx} = 0.5\left(1 + \sqrt{1 - 2\alpha_{sx}}\right) = 0.5\left(1 + \sqrt{1 - 2 \times 0.0624}\right) = 0.9678 \\ \gamma_{sy} = 0.5\left(1 + \sqrt{1 - 2\alpha_{sy}}\right) = 0.5\left(1 + \sqrt{1 - 2 \times 0.071}\right) = 0.9631 \end{cases}$$

Taking γ_{sx} and γ_{sy} into (2.24.3), there is

$$\begin{cases} A_{sx1} = \dfrac{M_x^T}{f_y \gamma_s h_0} = \dfrac{24.3 \times 10^6}{360 \times 0.9678 \times 165} = 422.7\,\text{mm}^2 \\ A_{sy1} = \dfrac{M_y^T}{f_y \gamma_s h_0} = \dfrac{24.3 \times 10^6}{360 \times 0.9631 \times 155} = 452.2\,\text{mm}^2 \end{cases}$$

2. **Load Action**

From the literature [13, 52], $w_2 = f\frac{ql^4}{D}$, $M_x = k_x ql^2$ and $M_y = k_y ql^2$ can be obtained. The value l is the minimum $[l_x, l_y]$.

According to the literature [13, 52], there are

$$f = 0.00406, k_x = 0.0368 \text{ and } k_y = 0.0368$$

Taking f, q, l and D into $w_2 = f\frac{ql^4}{D}$, there is

$$w_2 = f\frac{ql^4}{D} = 0.00406 \times \frac{7.02 \times 6^4}{14996.57} = 0.0025$$

Taking k_x, k_y into $M_x = k_x ql^2$ and $M_y = k_y ql^2$ respectively, there are

$$M_x = k_x ql^2 = 0.0368 \times 7.02 \times 6^2 = 9.80\,\text{kN} \cdot \text{m}$$

$$M_y = k_y ql^2 = 0.0368 \times 7.02 \times 6^2 = 9.80\,\text{kN} \cdot \text{m}$$

Assuming that $M_x = M_x^T$ and $h_0 = h_{0x}$, and taking M_x, f_c, h_0 and α_1 into (2.24.1), there is

$$\alpha_{sx} = \frac{M_x}{1000\alpha_1 f_c h_0^2} = \frac{9.8 \times 10^6}{1000 \times 1 \times 14.3 \times 165^2} = 0.0252$$

Assuming that $M_y = M_y^T$ and $h_0 = h_{0y}$, and taking M_y, f_c, h_0 and α_1 into (2.24.1), there is

$$\alpha_{sy} = \frac{M_y}{1000\alpha_1 f_c h_0^2} = \frac{9.8 \times 10^6}{1000 \times 1 \times 14.3 \times 155^2} = 0.0285$$

Taking α_{sx} and α_{sy} into (2.24.2), there is

$$\begin{cases} \gamma_{sx} = 0.5\left(1 + \sqrt{1 - 2\alpha_{sx}}\right) = 0.5\left(1 + \sqrt{1 - 2 \times 0.0252}\right) = 0.9872 \\ \gamma_{sy} = 0.5\left(1 + \sqrt{1 - 2\alpha_{sy}}\right) = 0.5\left(1 + \sqrt{1 - 2 \times 0.0285}\right) = 0.9855 \end{cases}$$

Assuming that $M_x = M_x^T$, $\gamma_s = \gamma_{sx}$ and $h_0 = h_{0x}$, and taking M_x, f_y and γ_s into (2.24.3), there is

$$A_{sx2} = \frac{M_x}{f_y \gamma_s h_0} = \frac{9.8 \times 10^6}{360 \times 0.9872 \times 165} = 167.1\,\mathrm{mm}^2$$

Assuming that $M_y = M_y^T$, $\gamma_s = \gamma_{sy}$ and $h_0 = h_{0y}$, and taking M_y, f_y and γ_s into (2.24.3), there is

$$A_{sy2} = \frac{M_y}{f_y \gamma_s h_0} = \frac{9.8 \times 10^6}{360 \times 0.9855 \times 155} = 178.2\,\mathrm{mm}^2$$

In summary, the analysis results can be obtained under the action of temperature and load. That is

$$w = w_1 + w_2 = 0.0103 + 0.0025 = 0.0128\,\mathrm{m}$$

$$A_{sx} = A_{sx1} + A_{sx2} = 422.7 + 167.1 = 589.8\,\mathrm{mm}^2$$

$$A_{sy} = A_{sy1} + A_{sy2} = 452.2 + 178.2 = 630.4\,\mathrm{mm}^2$$

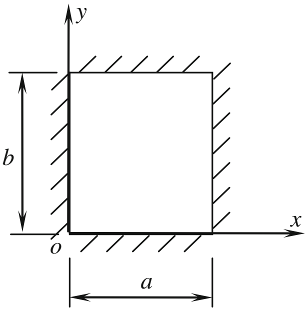
From the above results, the total deflection at the midspan point of the plate is 12.8 mm. The reinforcement area per meter at the center point of the thin plate in the x direction is 589.8 mm^2 and the reinforcement area per meter at the center point of the thin plate in the y direction is 630.4 mm^2.

2.4 Thermal Bending of Rectangular Thin Plate with Four Edges Clamped

2.4.1 Boundary Conditions

In Fig. 2.3, the boundary conditions to the clamped edges are:

Fig. 2.3 Four edges clamped



$$w\Big|_{\substack{x=0\\x=a}}=0,\ \frac{\partial w}{\partial x}\Big|_{\substack{x=0\\x=a}}=0 \tag{2.26}$$

$$w\Big|_{\substack{y=0\\y=b}}=0,\ \frac{\partial w}{\partial x}\Big|_{\substack{y=0\\y=b}}=0 \tag{2.27}$$

2.4.2 Analytical Solution of Thermal Elastic Problem

To satisfy the balance differential Eq. (2.13) and the boundary conditions (2.26) and (2.27), it is apparently that $w = 0$, but on the boundary according to the (2.12), it is known that

$$M_x|_{y=0,b}=M_y|_{x=0,a}=-M^T$$

Rectangular thin plate with four clamped edges under the action of lateral variable temperature disparity is regarded as a superposition of the rectangular thin plate with four simply supported edges under the action of bending moment $M_T^-(M_T^-=-M^T)$ on four edges and rectangular thin plate with four edges simply supported under the action of temperature disparity ΔT.

1. Bending Deformation Energy of Thin Plate

As shown in Fig. 2.1, ignoring the work done by shearing force, $-\frac{1}{2}M_x\frac{\partial^2 w}{\partial x^2}\mathrm{d}x\mathrm{d}y$ is the work done by bending moment $M_x\mathrm{d}y$; $-\frac{1}{2}M_y\frac{\partial^2 w}{\partial y^2}\mathrm{d}x\mathrm{d}y$ is work done by the bending moment $M_y\mathrm{d}x$; $\frac{1}{2}M_{xy}\frac{\partial^2 w}{\partial x\partial y}\mathrm{d}x\mathrm{d}y$ is work done by torque $M_{xy}\mathrm{d}x$; also, because the work done by the torque and the work done by the bending moment are not coupled, deformation energy of differential body is

$$\mathrm{d}V = -\frac{1}{2}\left[M_x\frac{\partial^2 w}{\partial x^2} + M_y\frac{\partial^2 w}{\partial y^2} - 2M_{xy}\frac{\partial^2 w}{\partial x \partial y}\right]\mathrm{d}x\mathrm{d}y$$

Substituting (2.12) into above equation, and letting $M^T = 0$ in (2.12), there is

$$\mathrm{d}V = \frac{1}{2}D\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\}$$

To the whole plate, deformation energy in bending plate is

$$\mathrm{d}V = \frac{1}{2}D\iint\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\}$$

In above equation, the second item of integrand function is transformed using Green's theorem, there is

$$\begin{aligned}\iint\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\mathrm{d}x\mathrm{d}y &= \iint\left[\frac{\partial\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\right)}{\partial x} - \frac{\partial\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\right)}{\partial y}\right]\mathrm{d}x\mathrm{d}y \\ &= \int\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\mathrm{d}x + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\mathrm{d}y\right)\end{aligned} \tag{2.28}$$

The line integral of (2.28) is along the whole edge of rectangular thin plate. Because the thin plate is with four edges simply supported, x is constant on the boundary, $\mathrm{d}x = 0$ and $\frac{\partial^2 w}{\partial y^2} = 0$; on the boundary, y is constant, $\mathrm{d}y = 0$ and $\frac{\partial w}{\partial x} = 0$, so (2.28) is simplified to

$$V = \frac{1}{2}D\iint\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2\mathrm{d}x\mathrm{d}y \tag{2.29}$$

2. The Analytic Solution Under Uniform Bending Moment M_T^- on Four Edges

According to the edge condition of rectangular thin plate with four simply supported edges and equilibrium differential equation for the elastic curved surface of identical thickness plate, deflection function may be supposed as

$$w = \sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty} A_{ij}\sin\frac{i\pi x}{a}\sin\frac{j\pi x}{b} \tag{2.30}$$

By substituting (2.30) into (2.29), deformation energy of plate is

$$V = \frac{\pi^4 abD}{8} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} A_{ij}^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^2 \tag{2.31}$$

The slope of every point along $x = 0$, $x = a$ and $y = 0$ on bending plane of plate is

$$\begin{cases} \left. \dfrac{\partial w}{\partial x} \right|_{\substack{x=0 \\ x=a}} = \dfrac{\pi}{a} \displaystyle\sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} iA_{ij} \sin \dfrac{j\pi y}{b} \\ \left. \dfrac{\partial w}{\partial y} \right|_{\substack{y=0 \\ y=b}} = \pm \dfrac{\pi}{b} \displaystyle\sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} jA_{ij} \sin \dfrac{i\pi x}{a} \end{cases}$$

When A_{ij} increases to $A_{ij} + \delta A_{ij}$, slope increment of every point along $x = 0$, $x = a$ and $y = 0$ on bending plane of plate is

$$\begin{cases} \delta \left. \dfrac{\partial w}{\partial x} \right|_{\substack{x=0 \\ x=a}} = \pm \dfrac{\pi}{a} i \sin \dfrac{j\pi y}{b} \delta A_{ij} \\ \delta \left. \dfrac{\partial w}{\partial y} \right|_{\substack{y=0 \\ y=b}} = \pm \dfrac{\pi}{b} j \sin \dfrac{i\pi x}{a} \delta A_{ij} \end{cases}$$

The work done by bending moment along edges of plate is

$$2 \int_0^a M_T^- \frac{\pi}{b} n \sin \frac{i\pi x}{a} \mathrm{d}x \delta A_{ij} + 2 \int_0^b M_T^- \frac{\pi}{a} i \sin \frac{j\pi y}{b} \mathrm{d}y \delta A_{ij}$$

Because

$$\begin{cases} \int_0^a M_T^- \sin \frac{i\pi x}{a} \mathrm{d}x = \frac{a}{2} E_i \\ \int_0^b M_T^- \sin \frac{j\pi y}{b} \mathrm{d}y = \frac{b}{2} F_j \end{cases}$$

where $E_i = \frac{4M_T^-}{i\pi}$; $F_j = \frac{4M_T^-}{j\pi}$.

The work done of moment is

$$j \frac{\pi a}{b} E_i \delta A_{ij} + i \frac{\pi b}{a} F_j \delta A_{ij}$$

According to (2.31), the increment of deformation energy is

$$\delta V = \frac{\pi^4 abD}{4}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^2 A_{ij}\delta A_{ij}$$

According to the virtual displacement principle, hence

$$A_{ij} = \frac{4}{\pi^4 Dab}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(j\frac{\pi a}{b}E_i + i\frac{\pi b}{a}F_j\right) = \frac{16M_T^-}{\pi^4 D}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}$$

By substituting the above formula into (2.30), hence

$$w(x,y) = \frac{16M_T^-}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \tag{2.32}$$

Substitute (2.32) into (2.12) (where $M^T = 0$), there is

$$\begin{cases} M_x = \frac{16M_T^-}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{i^2b^2+\mu j^2a^2}{ij(i^2b^2+j^2a^2)}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_y = \frac{16M_T^-}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{j^2a^2+\mu i^2b^2}{ij(i^2b^2+j^2a^2)}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{xy} = \frac{16(\mu-1)abM_T^-}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{i^2b^2+j^2a^2}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases} \tag{2.33}$$

3. The Analytical Solution of Rectangular Thin Plate with Four Simply Supported Edges Under Temperature Disparity ΔT

For easy superposition, in (2.22) and (2.23), x axis can be moved to the $y = -\frac{b}{2}$, so there is

$$w = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a} - \frac{M^T}{2D}(x-a)x \tag{2.34}$$

$$\begin{cases} M_x = \frac{4M^T}{\pi}(\mu-1)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\ M_y = \frac{4M^T}{\pi}(1-\mu)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} + (\mu-1)M^T \\ M_{xy} = \frac{4M^T}{\pi}(1-\mu)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\cos\frac{m\pi x}{a} \end{cases} \tag{2.35}$$

4. The Analytical Solution of Rectangular Thin Plate with Four Simply Supported Edges Under Thermal Load

By superposing (2.32) and (2.34), hence

$$\begin{aligned} w = & -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x \\ & -\frac{16M^T}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \end{aligned} \tag{2.36}$$

(2.36) is the deflection formula of rectangular thin plate with four clamped edges under lateral temperature disparity.

Superpose (2.33) and (2.35), hence

$$\begin{cases} M_x^T = \frac{4M^T}{\pi}(\mu-1)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\ \qquad -\frac{16M^T}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\mu\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_y^T = \frac{4M^T}{\pi}(1-\mu)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\ \qquad +(\mu-1)M^T-\frac{16M^T}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\mu\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{xy}^T = \frac{4M^T}{\pi}(1-\mu)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\cos\frac{m\pi x}{a} \\ \qquad +(1-\mu)\frac{16M^T}{\pi^2 ab}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases} \tag{2.37}$$

(2.37) is internal force solution of rectangular thin plate with four clamped edges under lateral temperature disparity.

2.4.3 Result Analysis

To test that formulas (2.36) and (2.37) are correct, the software MATLAB is used to program the formulas. The results show that: for deflection function w, when taking $m = n = 69$, the result has converged to exact solution; for the bending moment M_x of unit width, when taking $m = n = 7999$, result has converged to exact solution; for the bending moment M_y of unit width, when taking $m = n = 10999$, the result basically has converged to the exact solution; for clamped concrete rectangular plate with arbitrary length-width ratio, the internal force can be seen in Table 2, and it is identical to the existing literature.

Engineering application is seen in Sect. 2.3.4.

2.5 Thermal Bending of Rectangular Thin Plate with One Edge Simply Supported and Three Edges Clamped

2.5.1 Boundary Conditions

In Fig. 2.4, the edge conditions for the clamped edge are:

$$w\big|_{y=0} = 0,\ \frac{\partial w}{\partial y}\big|_{y=0} = 0 \tag{2.38}$$

$$w\Big|_{\substack{x=0\\x=a}} = 0,\ \frac{\partial w}{\partial x}\Big|_{\substack{x=0\\x=a}} = 0 \tag{2.39}$$

To simply supported edges, due to the deflection $w = 0$ on the whole boundary, by (2.12), the above equation becomes (Fig. 2.4)

$$w\big|_{y=b} = 0,\ \frac{\partial^2 w}{\partial y^2}\big|_{y=b} = -\frac{M^T}{D} \tag{2.40}$$

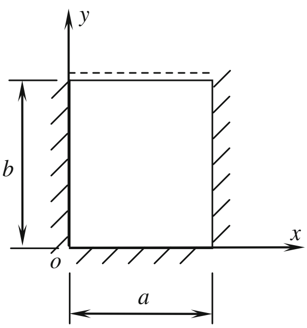
2.5.2 Analytical Solution for Thermal Elastic Problems

On edges, $w = 0$, according to (2.12), (2.38), (2.39) and (2.40), hence

$$M_y\big|_{y=b} = 0,\ M_x\Bigg|_{\substack{x=0\\x=a}} = -M^T,\ M_y\big|_{y=0} = -M^T$$

Now, Rectangular thin plate with three clamped edges and one simply supported edge under temperature disparity along thickness direction is regarded as a superposition of rectangular thin plate with four simply supported edges under the action of the temperature difference ΔT and rectangular thin plate with four simply

Fig. 2.4 Three edges clamped and one edge simply supported



supported edges under the uniform bending moment M_T^- ($M_T^- = -M^T$) on three adjacent edges.

1. The Analytic Solution Under the Uniform Bending Moment M_T^- on Three Adjacent Edges

The deformation energy of plate is equal to (2.31), namely

$$V = \frac{\pi^4 abD}{8} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} A_{ij}^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^2 \tag{2.41}$$

The slope of every point along $x = 0, x = a$ and $y = 0$ on bending plane of plate is

$$\begin{cases} \dfrac{\partial w}{\partial y}\Big|_{y=0} = \pm \dfrac{\pi}{b} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} jA_{ij} \sin \dfrac{i\pi x}{a} \\ \dfrac{\partial w}{\partial x}\Big|_{\substack{x=0 \\ x=a}} = \pm \dfrac{\pi}{a} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} iA_{ij} \sin \dfrac{j\pi y}{b} \end{cases}$$

When A_{ij} increases to $A_{ij} + \delta A_{ij}$, slope increment of every point along $x = 0$, $x = a$ and $y = 0$ on bending plane of plate is

$$\begin{cases} \delta \dfrac{\partial w}{\partial y}\Big|_{y=0} = \pm \dfrac{\pi}{b} j \sin \dfrac{i\pi x}{a} \delta A_{ij} \\ \delta \dfrac{\partial w}{\partial x}\Big|_{\substack{x=0 \\ x=a}} = \pm \dfrac{\pi}{a} i \sin \dfrac{j\pi y}{b} \delta A_{ij} \end{cases}$$

The work done by moment along edges of plate is

$$\int_0^a M_T^- \frac{\pi}{b} j \sin \frac{i\pi x}{a} \mathrm{d}x \delta A_{ij} + 2 \int_0^b M_T^- \frac{\pi}{a} i \sin \frac{j\pi y}{b} \mathrm{d}x \delta A_{ij}$$

Because

$$\begin{cases} \int_0^a M_T^- \sin \frac{i\pi x}{a} \mathrm{d}x = \frac{2aM^T}{i\pi} \\ \int_0^b M_T^- \sin \frac{j\pi y}{b} \mathrm{d}x = \frac{2bM^T}{j\pi} \end{cases}$$

The work done by moment is

$$2 \left(\frac{ja}{ib} + \frac{2ib}{ja} \right) M_T^- \delta A_{ij}$$

According to Eq. (2.41), the increment of deformation energy is

$$\delta V = \frac{\pi^4 abD}{4}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^2 A_{ij}\delta A_{ij}$$

According to the virtual displacement principle, hence

$$A_{ij} = \frac{8M_T^-}{\pi^4 Dij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)$$

By substituting the above formula into the (2.30), hence

$$w(x,y) = \frac{8M_T^-}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \tag{2.42}$$

Substituting (2.42) into (2.12) (where $M^T = 0$), the internal force calculation formula of the thin plate with three simply supported edges and one clamped edges under temperature disparity along thickness is obtained.

$$\begin{cases} M_x = \frac{8M_T^-}{\pi^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\mu\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_y = \frac{8M_T^-}{\pi^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\mu\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{xy} = \frac{8(\mu-1)M_T^-}{\pi^2 ab}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases} \tag{2.43}$$

2. The Analytic Solution of the Rectangular Thin Plate with Three Clamped Edges and One Simply Supported Edge Under Heat Load

By superposing (2.34) and (2.42), hence

$$\begin{aligned} w = & -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x \\ & -\frac{8M^T}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \end{aligned} \tag{2.44}$$

(2.44) is just the deflection formula of rectangular thin plate with three clamped edges and one simply supported edge under temperature disparity along thickness direction.

Superposing (2.35) and (2.43), hence

$$\begin{cases} M_x^T = \frac{4M^T}{\pi}(\mu - 1)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a} - \alpha_m\right)\sin\frac{m\pi x}{a} \\ \qquad - \frac{8M^T}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\frac{i^2}{a^2} + \mu\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2} + \frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_y^T = \frac{4M^T}{\pi}(1 - \mu)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a} - \alpha_m\right)\sin\frac{m\pi x}{a} + (\mu - 1)M^T \\ \qquad - \frac{8M^T}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\mu\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2} + \frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{xy}^T = \frac{4M^T}{\pi}(1 - \mu)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a} - \alpha_m\right)\cos\frac{m\pi x}{a} \\ \qquad + \frac{8(1-\mu)M^T}{\pi^2 ab}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\frac{2i^2}{a^2} + \frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases} \tag{2.45}$$

(2.45) is the internal force calculation formula of the rectangular thin plate with three clamped edges and one simply supported edge under temperature disparity along thickness.

2.5.3 Result Analysis

To test the formulas (2.44) and (2.45) are correct, the software MATLAB is used to program the formulas. The results show that: when taking $m = n = 39$, the result has converged to exact solution; for the bending moment M_x of unit width, when taking $m = n = 7999$, result has converged to exact solution; for the bending moment M_y of unit width, when taking $m = n = 1999$, the result has basically converged to the exact solution at this time, and when taking $m = n = 2001$, the error is only 1/10,000. For the convenience and engineering practical reasons, according to the length-width ratio of the plate, the thermal bending result of the concrete rectangular thin plate with three edges clamped and one simply supported is tabulated (see Table A.3).

Engineering application is seen in Sect. 2.3.4.

2.6 Thermal Bending of Rectangular Thin Plate with Three Edges Simply Supported and One Edge Clamped

2.6.1 Boundary Conditions

In Fig. 2.5, the edge conditions for the clamped edge are:

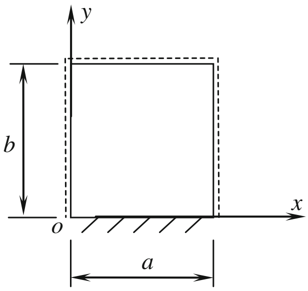


Fig. 2.5 One edge clamped and three edges simply supported

$$w\big|_{y=0} = 0,\ \frac{\partial w}{\partial y}\big|_{y=0} = 0 \tag{2.46}$$

To simply supported edges, due to the deflection w = 0 on the whole boundary, by (2.12), the above equation becomes

$$w\Big|_{\substack{x=0\\x=a}} = 0,\ \frac{\partial^2 w}{\partial x^2}\Big|_{\substack{x=0\\x=a}} = -\frac{M^T}{D} \tag{2.47}$$

$$w\big|_{y=b} = 0,\ \frac{\partial^2 w}{\partial y^2}\big|_{y=b} = -\frac{M^T}{D} \tag{2.48}$$

2.6.2 Analytical Solution for Thermal Elastic Problems

On edges, w = 0, according to (2.12), (2.46), (2.47) and (2.48), it is known that

$$M_x\Bigg|_{\substack{x=0\\x=a}} = 0,\ M_y\big|_{y=0} = -M^T,\ M_y\big|_{y=b} = 0$$

Now, rectangular thin plate with one clamped edges and three simply edge under temperature disparity along thickness is regarded as a superposition of rectangular thin plate with four simply supported edges under the action of the temperature difference ΔT and the rectangular thin plate with four simply supported edges under the action of bending moment M_T^- on edge y = 0.

1. The Analytical Solution of the Uniform Bending Moment M_T^- of One Edge

The deformation energy of plate is equal to (2.31), namely

$$V = \frac{\pi^4 abD}{8} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} A_{ij}^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^2 \tag{2.49}$$

The slope of every point along $y = 0$ on bending plane of plate is

$$\left.\frac{\partial w}{\partial y}\right|_{y=0} = \pm \frac{\pi}{b} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} jA_{ij} \sin\frac{i\pi x}{a}$$

When A_{ij} increases to $A_{ij} + \delta A_{ij}$, slope increment of every point along $y = 0$ on bending plane of plate is

$$\delta \left.\frac{\partial w}{\partial y}\right|_{y=0} = \pm \frac{\pi}{b} j \sin\frac{i\pi x}{a} \delta A_{ij}$$

The work done by bending moment along edges of plate is

$$\int_0^a M_T^- \frac{\pi}{b} j \sin\frac{i\pi x}{a} \mathrm{d}x \delta A_{ij}$$

Because

$$\int_0^a M_T^- \sin\frac{i\pi x}{a} \mathrm{d}x = \frac{2aM_T^-}{i\pi}$$

The work done by moment is

$$2\frac{ja}{ib} M_T^- \delta A_{ij}$$

According to (2.49), the increment of deformation energy is

$$\delta V = \frac{\pi^4 abD}{4} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^2 A_{ij} \delta A_{ij}$$

According to the virtual displacement principle, hence

$$A_{ij} = \frac{8}{\pi^4 Db^2} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \frac{jM_T^-}{i}$$

By substituting the above formula into the (2.30), hence

$$w(x,y) = \frac{8M_T^-}{\pi^4 b^2 D} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \tag{2.50}$$

By substituting (2.50) into (2.12) (where $M^T = 0$) that is the internal force calculation formula of the rectangular thin plate with four simply supported edges under the bending moment M_T^- on one edge.

$$\begin{cases} M_x = \frac{8M_T^-}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{i^2}{a^2} + \frac{\mu j^2}{b^2}\right) \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_y = \frac{8M_T^-}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{\mu i^2}{a^2} + \frac{j^2}{b^2}\right) \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \sin\frac{i\pi x}{a} \sin\frac{i\pi y}{b} \\ M_{xy} = -(1-\mu) \frac{8M_T^-}{\pi^2 a b^3} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} j^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \end{cases} \tag{2.51}$$

2. Analytic Solution of the Rectangular Thin Plate with One Clamped Edge and Three Simply Supported Edges Under Thermal Load

By superposing (2.34) and Eq. (2.50), hence

$$\begin{aligned} w = & -\frac{4a^2 M^T}{D\pi^3} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^3 \cosh\alpha_m} \cosh\left(\frac{2\alpha_m y}{b} - \alpha_m\right) \sin\frac{m\pi x}{a} - \frac{M^T}{2D}(x-a)x \\ & - \frac{8M^T}{\pi^4 D b^2} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \end{aligned} \tag{2.52}$$

(2.52) is just the deflection formula of rectangular thin plate with one clamped edge and three simply supported edges under temperature disparity along thickness.

By superposing (2.35) and (2.43), hence

$$\begin{cases} M_x^T = \frac{4M^T}{\pi}(\mu-1) \sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m\cosh\alpha_m} \cosh\left(\frac{m\pi y}{a} - \alpha_m\right) \sin\frac{m\pi x}{a} \\ \qquad - \frac{8M^T}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \left(\frac{i^2}{a^2} + \frac{\mu j^2}{b^2}\right) \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_y^T = \frac{4M^T}{\pi}(1-\mu) \sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m\cosh\alpha_m} \cosh\left(\frac{m\pi y}{a} + \alpha_m\right) \sin\frac{m\pi x}{a} + (\mu-1)M^T \\ \qquad - \frac{8M^T}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \left(\frac{\mu i^2}{a^2} + \frac{j^2}{b^2}\right) \sin\frac{i\pi x}{a} \sin\frac{i\pi y}{b} \\ M_{xy}^T = \frac{4M^T}{\pi}(1-\mu) \sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m\cosh\alpha_m} \cosh\left(\frac{m\pi y}{a} + \alpha_m\right) \cos\frac{m\pi x}{a} \\ \qquad -(\mu-1) \frac{8M^T}{\pi^2 ab} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} j^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \end{cases} \tag{2.53}$$

(2.53) is the internal force calculation formula of the rectangular thin plate with three simply supported edges and one clamped edge under temperature disparity along thickness.

2.6.3 Results Analysis

To test (2.52) and (2.53) are correct, the software MATLAB is used to program the formulas. The results show that: for the deflection w, when taking $m = n = 39$, the result has converged to exact solution. For the bending moment M_x^T of unit width, when taking $m = n = 1999$, result has basically converged to exact solution, and has an error of only 1/10000 in comparison with the result when taking $m = n = 2001$. For the bending moment M_y^T of unit width, when taking $m = n = 7001$, result has basically converged to exact solution, and has an error of only 1/10000 in comparison with the result when taking $m = n = 7003$. For the convenience and engineering practical reasons, according to the length-width ratio of the rectangular thin plate, the thermal bending result of concrete rectangular thin plate with the one edges clamped and three simply supported is tabulated (see Table A.4).

Engineering application is seen in Sect. 2.3.4.

2.7 Thermal Bending of Rectangular Thin Plate with Two Adjacent Edges Simply Supported and Two Opposite Edges Clamped

2.7.1 Boundary Conditions

In Fig. 2.6, to clamped edges, there is:

$$w|_{y=0} = 0, \left.\frac{\partial w}{\partial y}\right|_{y=0} = 0 \tag{2.54}$$

$$w|_{x=0} = 0, \left.\frac{\partial w}{\partial x}\right|_{x=0} = 0 \tag{2.55}$$

To simply supported edges, due to the deflection w in the whole edge is zero, by (2.12), there is

$$w|_{x=a} = 0, \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=a} = -\frac{M^T}{D} \tag{2.56}$$

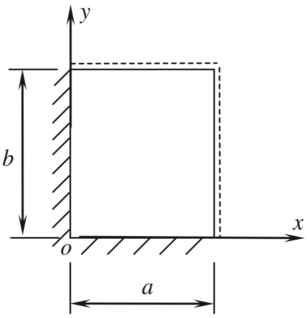


Fig. 2.6 Two adjacent edges clamped and two adjacent edges simply supported

$$w\big|_{y=b} = 0, \ \frac{\partial^2 w}{\partial y^2}\bigg|_{y=b} = -\frac{M^T}{D} \tag{2.57}$$

2.7.2 Analytical Solution for Thermal Elastic Problems

On edges, $w = 0$, according to (2.12), (2.54), (2.55), (2.56) and (2.57), there is

$$M_x\big|_{x=a} = 0, \ M_y\big|_{y=b} = 0, \ M_x\big|_{x=0} = -M^T \ M_y\big|_{y=0} = -M^T$$

Now, Rectangular thin plate with two adjacent edges clamped and two adjacent edges simply supported under temperature disparity along thickness is regarded as a superposition of rectangular thin plate with four simply supported edges under the action of the temperature difference ΔT and the rectangular thin plate with four simply supported edges under uniform bending moment M_T^- on two adjacent edges.

1. The Solution of the Uniform Bending Moment M_T^- on the Two Adjacent Edges

The deformation energy of plate is equal to (2.31), namely

$$V = \frac{\pi^4 abD}{8} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} A_{ij}^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^2 \tag{2.58}$$

The slope of every point along $x = 0$ and $y = 0$ on bending plane of plate is

$$\begin{cases} \dfrac{\partial w}{\partial y}\bigg|_{y=0} = \pm \dfrac{\pi}{b} \displaystyle\sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} jA_{ij} \sin \dfrac{i\pi x}{a} \\ \dfrac{\partial w}{\partial x}\bigg|_{x=0} = \pm \dfrac{\pi}{a} \displaystyle\sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} iA_{ij} \sin \dfrac{j\pi y}{b} \end{cases}$$

When A_{ij} increased to $A_{ij} + \delta A_{ij}$, slope increment of every point along $x = 0$ and $y = 0$ on bending plane of plate is

$$\begin{cases} \delta \left.\dfrac{\partial w}{\partial y}\right|_{y=0} = \pm \dfrac{\pi}{b} j \sin \dfrac{i\pi x}{a} \delta A_{ij} \\ \delta \left.\dfrac{\partial w}{\partial x}\right|_{x=0} = \pm \dfrac{\pi}{a} i \sin \dfrac{j\pi y}{b} \delta A_{ij} \end{cases}$$

The work done by bending moment along edges of plate is

$$\int_0^a M_T^- \frac{\pi}{b} j \sin \frac{i\pi x}{a} \mathrm{d}x \delta A_{ij} + \int_0^b M_T^- \frac{\pi}{a} i \sin \frac{j\pi y}{b} \mathrm{d}x \delta A_{ij}$$

Because

$$\begin{cases} \int_0^a M_T^- \sin \frac{i\pi x}{a} \mathrm{d}x = \frac{2aM^T}{i\pi} \\ \int_0^b M_T^- \sin \frac{j\pi y}{b} \mathrm{d}x = \frac{2bM^T}{j\pi} \end{cases}$$

The work done by bending moment is

$$\frac{2ab}{ij} \left(\frac{j^2}{b^2} + \frac{i^2}{a^2} \right) M_T \delta A_{ij}$$

According to (2.58), the increment of deformation energy is

$$\delta V = \frac{\pi^4 abD}{4} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^2 A_{ij} \delta A_{ij}$$

According to the virtual displacement principle, there is

$$A_{ij} = \frac{8}{\pi^4 D} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-1} \frac{M_T^-}{ij}$$

By substituting the above formula into (2.30), hence

$$w(x, y) = \frac{8M_T^-}{\pi^4 D} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{1}{ij} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-1} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b} \tag{2.59}$$

Substituting (2.59) into (2.12) (where $M^T = 0$), the internal force calculation formula of the plate with four simply supported edges under the bending moment M_T^- on the two adjacent edges is obtained.

$$\begin{cases} M_x = \frac{8M^T}{\pi^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{1}{ij} \left(\frac{i^2}{a^2} + \frac{\mu j^2}{b^2}\right) \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_y = \frac{8M^T}{\pi^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{1}{ij} \left(\frac{\mu i^2}{a^2} + \frac{j^2}{b^2}\right) \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_{xy} = -(1-\mu) \frac{8M^T}{\pi^2 ab} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \end{cases} \tag{2.60}$$

2. Analytic Solution of Rectangular Thin Plate with Two Adjacent Simply Supported Edge and Two Adjacent Clamped Edges

By superposing (2.34) and (2.59), hence

$$\begin{aligned} w = &-\frac{4a^2M^T}{D\pi^3} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^3 \cosh\alpha_m} \cosh\left(\frac{2\alpha_m y}{b} - \alpha_m\right) \sin\frac{m\pi x}{a} \\ &- \frac{M^T}{2D}(x-a)x - \frac{8M^T}{\pi^4 D} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{1}{ij} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \end{aligned} \tag{2.61}$$

(2.61) is just the deflection formula of rectangular thin plate with two adjacent simply supported edge and two adjacent clamped edges under temperature disparity along thickness.

By superposing (2.35) and (2.61), hence

$$\begin{cases} M_x^T = \frac{4M^T}{\pi}(\mu - 1) \sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m \cosh\alpha_m} \cosh\left(\frac{m\pi y}{a} - \alpha_m\right) \sin\frac{m\pi x}{a} \\ \qquad - \frac{8M^T}{\pi^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{1}{ij} \left(\frac{i^2}{a^2} + \frac{\mu j^2}{b^2}\right) \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_y^T = \frac{4M^T}{\pi}(1 - \mu) \sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m \cosh\alpha_m} \cosh\left(\frac{m\pi y}{a} - \alpha_m\right) \sin\frac{m\pi x}{a} \\ \qquad -(\mu - 1)M^T + \frac{8M^T}{\pi^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{1}{ij} \left(\frac{\mu i^2}{a^2} + \frac{j^2}{b^2}\right) \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_{xy}^T = \frac{4M^T}{\pi}(1 - \mu) \sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m \cosh\alpha_m} \cosh\left(\frac{m\pi y}{a} - \alpha_m\right) \cos\frac{m\pi x}{a} \\ \qquad -(\mu - 1) \frac{8M^T}{\pi^2 ab} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \end{cases} \tag{2.62}$$

(2.62) is the internal force calculation formula of rectangular thin plate with two adjacent simply supported edge and two adjacent clamped edges under temperature disparity along thickness.

2.7.3 Results Analysis

To test (2.61) and (2.62) are correct, software MATLAB is used to calculate. The results show that: for deflection function w, when taking $m = n = 69$, the result has converged to exact solution; for the bending moment of M_x^T unit width, when taking $m = n = 5999$, result has converged to exact solution; for the bending moment M_y^T of unit width, when taking $m = n = 5001$, the result has basically converge to the exact solution at this time, and the error is only 1/10000 in comparison with the result when taking $m = n = 4999$. For the convenience and engineering practical reasons, according to the length-width ratio of the rectangular thin plate, the thermal bending results of concrete rectangular plate with two adjacent edges clamped and two adjacent edges simply supported is tabulated (see Table 5).

Engineering application is seen in Sect. 2.3.4.

2.8 Thermal Bending of Rectangular Thin Plate with Two Opposite Edges Simply Supported and Two Opposite Edges Clamped

2.8.1 Boundary Conditions

In Fig. 2.7, to simply supported edges, there is:

$$w\Big|_{\substack{x=0\\x=a}} = 0, \quad \frac{\partial^2 w}{\partial x^2}\Big|_{\substack{x=0\\x=a}} = -\frac{M^T}{D} \tag{2.63}$$

To clamped edges, there is

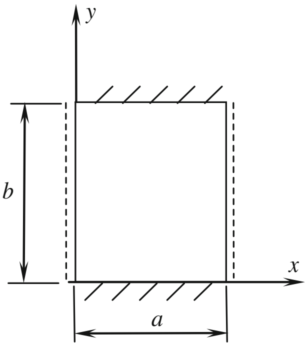
$$w\Big|_{\substack{y=\frac{b}{2}\\y=-\frac{b}{2}}} = 0, \quad \frac{\partial w}{\partial y}\Big|_{\substack{y=\frac{b}{2}\\y=-\frac{b}{2}}} = 0 \tag{2.64}$$

2.8.2 Analytical Solution for Thermal Elastic Problems

On edges, $w = 0$, according to (2.12), (2.63) and (2.64), hence

$$M_x\Big|_{\substack{x=0\\x=a}} = 0, \quad M_y\Big|_{\substack{y=0\\y=b}} = -M^T$$

Fig. 2.7 Two opposite edges clamped and two opposite edges simply supported



Now, rectangular thin plate with two opposite edges clamped and two opposite edges simply supported under temperature disparity along thickness is regarded as a superposition of the rectangular thin plate with four simply supported edges under the action of the temperature difference ΔT and the rectangular thin plate with four simply supported edges under the bending moment M_T^- on two opposite edges.

1. The Solution of the Uniform Bending Moment M_T^- on the Two Opposite Edges

The deformation energy of plate is equal to (2.31), namely

$$V = \frac{\pi^4 abD}{8} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} A_{ij}^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^2 \tag{2.65}$$

The slope of every point along $y = b$ and $y = 0$ on bending plane of plate is

$$\left. \frac{\partial w}{\partial y} \right|_{\substack{y=0 \\ y=b}} = \pm \frac{\pi}{b} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} jA_{ij} \sin \frac{i\pi x}{a}$$

When A_{ij} increased to $A_{ij} + \delta A_{ij}$, slope increment of every point along $y = b$ and $y = 0$ on bending plane of plate is

$$\left. \delta \frac{\partial w}{\partial y} \right|_{\substack{y=0 \\ y=b}} = \pm \frac{\pi}{b} j \sin \frac{i\pi x}{a} \delta A_{ij}$$

The work done by moment along edges of plate is

$$2 \int_0^a M_T^- \frac{\pi}{b} j \sin \frac{i\pi x}{a} \mathrm{d}x \delta A_{ij}$$

Because

$$2\int_0^a M_T^- \sin\frac{i\pi x}{a}\mathrm{d}x = \frac{4aM_T^-}{i\pi}$$

The work done by bending moment is

$$\frac{4aM_T^-}{b}\frac{j}{i}\delta A_{ij}$$

According to (2.65), the increment of deformation energy is

$$\delta V = \frac{\pi^4 abD}{4}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^2 A_{ij}\delta A_{ij}$$

According to the virtual displacement principle, hence

$$A_{ij} = \frac{16M_T^-}{\pi^4 b^2 D}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}$$

By substituting the above formula into (2.30), hence

$$w(x,y) = \frac{16M_T^-}{\pi^4 b^2 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \tag{2.66}$$

By substituting (2.66) into (2.12) (where $M^T = 0$) that is the internal force calculation formula of the plate with four simply supported edges under the bending moment M_T^- on the two opposite edges

$$\begin{cases} M_x = \dfrac{16M_T^-}{\pi^2 b^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_y = \dfrac{16M_T^-}{\pi^2 b^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{\mu i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{xy} = -(1-\mu)\dfrac{16M_T^-}{\pi^2 ab^3}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty} j^2\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases} \tag{2.67}$$

2. Analytic Solution of Rectangular Thin Plate with Two Opposite Edges Simply Supported and Two Opposite Edges Clamped

By superposing (2.34) and (2.66), hence

$$
\begin{aligned}
w = & -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a} \\
& -\frac{M^T}{2D}(x-a)x-\frac{16M^T}{\pi^4b^2D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}
\end{aligned}
\tag{2.68}
$$

(2.68) is just the deflection formula of rectangular thin plate with two opposite edges simply supported and two opposite edges clamped under temperature disparity along thickness.

By superposing (2.35) and (2.67), hence

$$
\begin{cases}
M_x^T = \frac{4M^T}{\pi}(\mu-1)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\
\qquad -\frac{16M^T}{\pi^2b^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
M_y^T = \frac{4M^T}{\pi}(1-\mu)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\
\qquad -(\mu-1)M^T-\frac{16M^T}{\pi^2b^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{\mu i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
M_{xy}^T = \frac{4M^T}{\pi}(1-\mu)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\cos\frac{m\pi x}{a} \\
\qquad -(\mu-1)\frac{16M^T}{\pi^2ab}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}j^2\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b}
\end{cases}
\tag{2.69}
$$

(2.69) is the internal force calculation formula of rectangular thin plate with two opposite edges simply supported and two opposite edges clamped under temperature disparity along thickness.

2.8.3 Results Analysis

To test (2.68) and (2.69) are correct, the software MATLAB is used to program the formulas. The results show that: for the deflection fuction w, when taking m = n = 39, the result has converged to exact solution; for the bending moment M_x^T of unit width, when taking m = n = 5999, result has converged to exact solution; for the bending moment M_y^T of unit width, when taking m = n = 10,001, the result has basically converge to the exact solution at this time, and the error is only 1/10000 in comparison with the result when taking $m = n = 9999$. For convenience and engineering practical reasons, according to the length-width ratio of the rectangular thin plate, the thermal bending results of concrete rectangular plate with two opposite edges clamped and two opposite edges simply supported is tabulated (see Table A.6).

Engineering application is seen in Sect. 2.3.4.

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