

Chapter 2

Topology Optimization for Unsteady Flows



This chapter presents the topology optimization of unsteady incompressible Navier-Stokes flows, where the density method and level set method are respectively used to implement the implicitly expression of the unsteady flows. In the density method, the optimization problem is formulated by adding the artificial Darcy frictional force into the incompressible Navier-Stokes equations; the continuous adjoint method is used to derive the adjoint derivative; and the method of moving asymptotes is used to evolve the design variable. In the level set method, the fluid velocity is constrained to be zero in the implicitly expressed solid domain; the variational level set method is used to derive the optimization sensitivity; and the level set function is evolved by solving the Hamilton-Jacobian equation with the upwind finite difference method. Furthermore, numerical examples are provided to demonstrate the feasibility and necessity of the topology optimization method for unsteady Navier-Stokes flows, where the dynamic effect and Reynolds number are investigated for the optimal topology.

2.1 Density Method-Based Topology Optimization for Unsteady Flows

In this section, the density method-based topology optimization for unsteady flows is introduced. The key ideas behind this work is on the implicit expression of fluidic flow using artificial Darcy force and the continuous adjoint method-based analysis of the topology optimization problem.

2.1.1 Density Method-Based Optimization Problem

The dynamics of the incompressible fluidic flows can be expressed by the incompressible Navier-Stokes equations as [1]

$$\begin{aligned} \rho \frac{\partial \mathbf{u}}{\partial t} - \eta \nabla \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } Q \\ -\nabla \cdot \mathbf{u} = 0, \quad \text{in } Q \end{aligned} \quad (2.1)$$

where $\mathbf{u}$ is the fluidic velocity; p is the fluidic pressure; ρ is the fluidic density; η is the fluidic viscosity; $\mathbf{f}$ is the body force loaded on the fluid; and t is the time. $Q = (0, T) \times \Omega$ is the time-space, where $(0, T)$ is the computational time interval and Ω is the computational domain. To solve the transient eq. 2.1, the initial condition needs to be enforced

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \text{in } \Omega \quad (2.2)$$

where $\mathbf{u}_0(\mathbf{x})$ satisfies the incompressible condition $\nabla \cdot \mathbf{u}_0 = 0$. The commonly used boundary conditions for incompressible Navier-Stokes equations include the Dirichlet and Neumann type boundary conditions

$$\mathbf{u} = \mathbf{u}_D(t, \mathbf{x}), \quad \text{on } \Sigma_D \quad (2.3)$$

$$[-p\mathbf{I} + \eta(\nabla \mathbf{u} + \nabla \mathbf{u}^T)]\mathbf{n} = \mathbf{g}(t, \mathbf{x}), \quad \text{on } \Sigma_N \quad (2.4)$$

where $\mathbf{u}_D$ and $\mathbf{g}$ are the specified velocity and stress distribution on the boundaries Γ_D and Γ_N ; $\mathbf{n}$ is the outward unitary normal on $\partial\Omega$; $\Sigma_D = (0, T) \times \Gamma_D$ and $\Sigma_N = (0, T) \times \Gamma_N$ are the time-space boundaries. Specifically, the no-slip boundary is a particular Dirichlet type boundary condition with $\mathbf{u}_D = \mathbf{0}$, and the open-boundary on the outlet can be expressed by the Neumann type boundary condition with $\mathbf{g} = \mathbf{0}$. In density method-based topology optimization for the Navier-Stokes flow, the body force is expressed to be [2, 3]

$$\mathbf{f} = -\alpha \mathbf{u} \quad (2.5)$$

where α is the impermeability of an artificial porous medium. Its value depends on the optimization design variable γ , and the functional relation is termed *material interpolation* in topology optimization [2, 3]

$$\alpha(\gamma) = \alpha_{min} + (\alpha_{max} - \alpha_{min}) \frac{q(1 - \gamma)}{q + \gamma} \quad (2.6)$$

where α_{min} and α_{max} are the minimal and maximal values of α , respectively; and q is a real and positive parameter used to tune the convexity of the interpolation function in Eq. 2.6. The value of γ can vary between zero and one, where $\gamma = 0$ corresponds

to an artificial solid phase and $\gamma = 1$ corresponds to the fluid. Usually, α_{min} is chosen to be 0; α_{max} is chosen to be a finite but large number to simultaneously ensure the numerical stability of the optimization procedure and approximate the solid phase with negligible permeability [3, 4]. It is valuable to note that the design variable γ is time independent, because the layout of fluidic domain is kept unchanged during the solving of the transient Navier-Stokes equations.

Then, the topology optimization problem for the unsteady incompressible Navier-Stokes flow can be formulated to be

$$\begin{aligned} &\text{Find } \gamma \in [0, 1] \text{ to minimize } J(\mathbf{u}, p; \gamma), \text{ subject to} \\ &\left\{ \begin{array}{l} \rho \frac{\partial \mathbf{u}}{\partial t} - \eta \nabla \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = -\alpha \mathbf{u}, \text{ in } Q \\ -\nabla \cdot \mathbf{u} = 0, \text{ in } Q \\ \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \text{ in } \Omega \\ \mathbf{u}(t, \mathbf{x}) = \mathbf{u}_D(t, \mathbf{x}), \text{ on } \Sigma_D \\ [-p\mathbf{I} + \eta(\nabla \mathbf{u} + \nabla \mathbf{u}^T)] \mathbf{n} = \mathbf{g}(t, \mathbf{x}), \text{ on } \Sigma_N \\ \int_{\Omega} \gamma d\Omega \leq V_r \cdot V_0 \end{array} \right. \end{aligned} \quad (2.7)$$

where $V_0 = \int_{\Omega} 1 d\Omega$ is the volume of the whole design domain; $V_r \in (0, 1)$ is the upper bound for the volume fraction of the fluid phase. One general optimization objective, which includes both the domain and boundary integrations about the fluidic velocity and pressure, is chosen to be

$$J(\mathbf{u}, p; \gamma) = \int_0^T \int_{\Omega} \beta_1 A(\mathbf{u}, \nabla \mathbf{u}, p; \gamma) d\Omega dt + \int_0^T \int_{\partial\Omega} \beta_2 B(\mathbf{u}, p; \gamma) d\Gamma dt \quad (2.8)$$

where β_1 and β_2 are space-independent parameters.

2.1.2 Continuous Adjoint Analysis

According to the adjoint method for the Navier-Stokes equations in [5–7], the adjoint analysis of the topology optimization problem for unsteady Navier-Stokes flows is implemented as follows. Without considering the inequality constraint on the volume of fluid phase at first, the topology optimization problem into Eq. 2.7 can be rewritten in the following abstract form

$$\min J(\mathbf{u}, p; \gamma), \text{ s.t. } e(\mathbf{u}, p; \gamma) = 0, \gamma \in \mathcal{K} \quad (2.9)$$

where $\mathcal{K}$ is the feasible space of the design variable γ , and $e(\cdot)$ is the weak operator of the Navier-Stokes equations. According to the Karush-Kuhn-Tucker conditions

for partial differential equation constrained optimization problems [5, 8], the optimization problem in Eq. 2.9 can be solved with the solution of the following abstract equations

$$e(\mathbf{u}, p; \gamma) = 0 \quad (2.10)$$

$$\begin{pmatrix} (e_{\mathbf{u}}(\mathbf{u}, p; \gamma))^* & \mathbf{0} \\ \mathbf{0} & (e_p(\mathbf{u}, p; \gamma))^* \end{pmatrix} \begin{pmatrix} \mathbf{u}_a \\ p_a \end{pmatrix} = \begin{pmatrix} -J_{\mathbf{u}}(\mathbf{u}, p; \gamma) \\ -J_p(\mathbf{u}, p; \gamma) \end{pmatrix} \quad (2.11)$$

$$(e_{\gamma}(\mathbf{u}, p; \gamma))^*(\mathbf{u}_a, p_a) + J_{\gamma}(\mathbf{u}, p; \gamma) = 0 \quad (2.12)$$

where $\mathbf{u}_a$ and p_a are the adjoint variables for the fluid velocity $\mathbf{u}$ and pressure p , respectively; $(\cdot)^*$ is the adjoint of the corresponding operator. Based on Eq. 2.11, the adjoint equations of the Navier-Stokes equations can be written as follows (see Appendix 2.3.1 for more details)

$$\begin{aligned} -\rho \frac{\partial \mathbf{u}_a}{\partial t} - \eta \nabla \cdot (\nabla \mathbf{u}_a + \nabla \mathbf{u}_a^T) - \rho (\mathbf{u} \cdot \nabla) \mathbf{u}_a + \rho (\nabla \mathbf{u}) \cdot \mathbf{u}_a + \nabla p_a \\ = -\beta_1 \left(\frac{\partial A}{\partial \mathbf{u}} - \nabla \cdot \frac{\partial A}{\partial \nabla \mathbf{u}} \right) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \mathbf{u}_a, \text{ in } Q \\ -\nabla \cdot \mathbf{u}_a = -\beta_1 \frac{\partial A}{\partial p} + \frac{\partial \mathbf{f}}{\partial p} \cdot \mathbf{u}_a, \text{ in } Q \\ \mathbf{u}_a(T, \mathbf{x}) = \mathbf{0}, \text{ in } \Omega \\ \mathbf{u}_a = -\frac{\partial B}{\partial p} \mathbf{n}, \text{ on } \Sigma_D \\ [-p_a \mathbf{I} + \eta (\nabla \mathbf{u}_a + \nabla \mathbf{u}_a^T)] \mathbf{n} = -\rho (\mathbf{u} \cdot \mathbf{n}) \mathbf{u}_a - \beta_1 \frac{\partial A}{\partial \nabla \mathbf{u}} \mathbf{n} - \beta_2 \frac{\partial B}{\partial \mathbf{u}}, \text{ on } \Sigma_N \end{aligned} \quad (2.13)$$

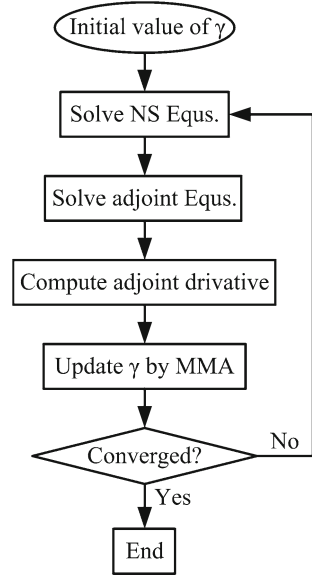
The transient adjoint Eq. 2.13 are terminal value problems, where the value of μ at the terminal time T is specified and the transient solver need to be implemented from time T to 0. Based on Eq. 2.12, the adjoint derivatives of the optimization objective in Eq. 2.8 can be expressed to be (see Appendix 2.3.2 for more details)

$$\begin{aligned} \frac{D\hat{J}}{D\gamma} \Big|_{\Omega} &= \int_0^T \beta_1 \left(\frac{\partial A}{\partial \gamma} + \frac{\partial \alpha}{\partial \gamma} \mathbf{u} \cdot \mathbf{u}_a \right) dt, \text{ in } \Omega \\ \frac{D\hat{J}}{D\gamma} \Big|_{\partial\Omega} &= \int_0^T \beta_2 \frac{\partial B}{\partial \gamma} dt, \text{ on } \partial\Omega \end{aligned} \quad (2.14)$$

2.1.3 Numerical Implementation

The topology optimization for unsteady Navier-Stokes flows is implemented using the gradient based iterative approach. The procedure for an iterative optimization includes the following steps (Fig. 2.1)

Fig. 2.1 The flowchart of the iterative optimization



- the Navier-Stokes equations are solved with the given value of the design variable;
- the adjoint equations are solved based on the numerical solution of the Navier-Stokes equations;
- the adjoint derivatives of the objective function are computed by Eq. 2.14, and the adjoint derivatives of the design constraint are computed similarly;
- the design variable is updated by the method of moving asymptotes (MMA) [9].

The above steps are implemented iteratively until the stopping criteria are satisfied. In the iterative procedure, the transient Navier-Stokes equations and the corresponding adjoint equations are solved by the mixed finite element method using the finite element package *Comsol Multiphysics* [10], where all the numerical implementation is based on the package's basic module: *COMSOL Multiphysics* → *PDE Modes* → *PDE, General Form*.

The PDE modes of *Comsol Multiphysics* can solve partial differential equations of the form

$$\begin{aligned}
 \mathbf{d}_a \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \boldsymbol{\Gamma} &= \mathbf{F}, \text{ in } Q \\
 -\mathbf{n} \cdot \boldsymbol{\Gamma} &= \mathbf{G} + \left(\frac{\partial \mathbf{R}}{\partial \mathbf{u}} \right)^T \lambda, \mathbf{R} = \mathbf{0}, \text{ on } \Sigma
 \end{aligned} \tag{2.15}$$

where $\mathbf{d}_a$ and $\boldsymbol{\Gamma}$ are tensors; $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{R}$ are vectors; $\mathbf{n}$ is the unitary outward normal; $\Sigma = \Sigma_D \cup \Sigma_N$. For the 2D case, the transient Navier-Stokes equations can be solved by setting

$$\begin{aligned}
\mathbf{d}_a &= \begin{pmatrix} \rho & 0 \\ 0 & \rho \\ 0 & 0 \end{pmatrix}; \quad \mathbf{\Gamma} = \begin{pmatrix} -2\eta \frac{\partial u_1}{\partial x} + p & -\eta \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) \\ -\eta \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) & -2\eta \frac{\partial u_2}{\partial y} + p \\ 0 & 0 \end{pmatrix}; \\
\mathbf{F} &= \begin{pmatrix} -\alpha u_1 - \rho \left(u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \right) \\ -\alpha u_2 - \rho \left(u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} \right) \\ - \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \right) \end{pmatrix}; \quad \mathbf{R} = \begin{pmatrix} u_1 - u_{1D} \\ u_2 - u_{2D} \\ 0 \end{pmatrix} \text{ on } \Sigma_D; \\
\mathbf{G} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ on } \Sigma_D; \quad \mathbf{R} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ on } \Sigma_N; \quad \mathbf{G} = \begin{pmatrix} g_1 \\ g_2 \\ 0 \end{pmatrix} \text{ on } \Sigma_N
\end{aligned} \quad (2.16)$$

where $\mathbf{u} = (u_1, u_2)$, $\mathbf{u}_D = (u_{1D}, u_{2D})$ and $\mathbf{g} = (g_1, g_2)$. Similarly, the adjoint equations can be solved by setting

$$\begin{aligned}
\mathbf{d}_a &= \begin{pmatrix} -\rho & 0 \\ 0 & -\rho \\ 0 & 0 \end{pmatrix}; \\
\mathbf{\Gamma} &= \begin{pmatrix} -2\eta \frac{\partial u_{a1}}{\partial x} - \beta_1 \frac{\partial A}{\partial u_{a1x}} + p_a & -\eta \left(\frac{\partial u_{a1}}{\partial y} + \frac{\partial u_{a2}}{\partial x} \right) - \beta_1 \frac{\partial A}{\partial u_{a1y}} \\ -\eta \left(\frac{\partial u_{a1}}{\partial y} + \frac{\partial u_{a2}}{\partial x} \right) - \beta_1 \frac{\partial A}{\partial u_{a2x}} & -2\eta \frac{\partial u_{a2}}{\partial y} - \beta_1 \frac{\partial A}{\partial u_{a2y}} + p_a \\ 0 & 0 \end{pmatrix}; \\
\mathbf{F} &= \begin{pmatrix} -\alpha u_{a1} - \beta_1 \frac{\partial A}{\partial u_{a1}} + \rho \left(u_1 \frac{\partial u_{a1}}{\partial x} + u_2 \frac{\partial u_{a1}}{\partial y} - u_{a1} \frac{\partial u_1}{\partial x} - u_{a2} \frac{\partial u_1}{\partial y} \right) \\ -\alpha u_{a2} - \beta_1 \frac{\partial A}{\partial u_{a2}} + \rho \left(u_1 \frac{\partial u_{a2}}{\partial x} + u_2 \frac{\partial u_{a2}}{\partial y} - u_{a1} \frac{\partial u_2}{\partial x} - u_{a2} \frac{\partial u_2}{\partial y} \right) \\ \beta_1 \frac{\partial A}{\partial p} - \left(\frac{\partial u_{a1}}{\partial x} + \frac{\partial u_{a2}}{\partial y} \right) \end{pmatrix}; \\
\mathbf{R} &= \begin{pmatrix} u_{a1} + n_1 \frac{\partial B}{\partial p} \\ u_{a2} + n_2 \frac{\partial B}{\partial p} \\ 0 \end{pmatrix} \text{ on } \Sigma_D; \quad \mathbf{G} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ on } \Sigma_D; \quad \mathbf{R} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ on } \Sigma_N; \\
\mathbf{G} &= \begin{pmatrix} -\rho u_{a1} (u_1 n_1 + u_2 n_2) - \beta_1 \left(n_1 \frac{\partial A}{\partial u_{1x}} + n_2 \frac{\partial A}{\partial u_{1y}} \right) - \beta_2 \frac{\partial B}{\partial u_1} \\ -\rho u_{a2} (u_1 n_1 + u_2 n_2) - \beta_1 \left(n_1 \frac{\partial A}{\partial u_{2x}} + n_2 \frac{\partial A}{\partial u_{2y}} \right) - \beta_2 \frac{\partial B}{\partial u_2} \\ 0 \end{pmatrix} \text{ on } \Sigma_N
\end{aligned} \quad (2.17)$$

where $\mathbf{u}_a = (u_{a1}, u_{a2})$ and $\mathbf{n} = (n_1, n_2)$. The temporal integrations are numerically implemented by solving a scalar general form equation (Eq. 2.18) in *Comsol*

$$\begin{aligned}
d_a \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{\Gamma} &= F, \text{ in } Q \\
-\mathbf{n} \cdot \mathbf{\Gamma} &= G + \left(\frac{\partial R}{\partial u} \right)^T \lambda, \quad R = 0, \text{ on } \Sigma
\end{aligned} \quad (2.18)$$

By setting

$$d_a = 1, \Gamma = \mathbf{0}, G = 0, R = 0, F = \beta_1 \left(\frac{\partial A}{\partial \gamma} + \frac{\partial \alpha}{\partial \gamma} \mathbf{u} \cdot \mathbf{u}_a \right) \quad (2.19)$$

and solving Eq. 2.18 for u , the adjoint derivative $\frac{D\hat{J}}{D\gamma}|_{\Omega}$ in Eq. 2.14 can be obtained as $u|_{t=T}$. Changing the value of F in Eq. 2.19 to be $\beta_2 \frac{\partial B}{\partial \gamma}$, the adjoint derivative $\frac{D\hat{J}}{D\gamma}|_{\partial\Omega}$ can be solved as $u|_{\partial\Omega, t=T}$. The spatial integrations in the iterative procedure can be performed by the inner function of *Comsol*. For the 3D case, the settings for Eqs. 2.15 and 2.18 are similar to the 2D case. During the optimization procedure, the Navier-Stokes equations and the adjoint equations are solved using Taylor-Hood elements [11], which interpolate the fluidic velocity quadratically and the pressure linearly. The design variable is interpolated linearly based on the corner nodes of the elements (Figs. 2.2 and 2.3). The transient equations are solved using the backward differentiation formula method in the time dependent solver of *Comsol* [12].

The stopping criteria are specified as the change of values of the objective between two consecutive iterations and the residual of the volume constraint satisfying

$$\begin{aligned} |J_k - J_{k-1}|/|J_0| &< 1 \times 10^{-6} \\ |\int_{\Omega} \gamma \, d\Omega / V_0 - V_r|/V_r &< 1 \times 10^{-3} \end{aligned} \quad (2.20)$$

One can note that it is convenient to implement transient optimization problems, when the continuous adjoint method is adopted.

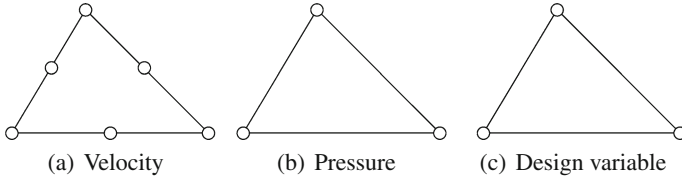


Fig. 2.2 The finite element nodes used to express the velocity, pressure and design variable on a triangular element

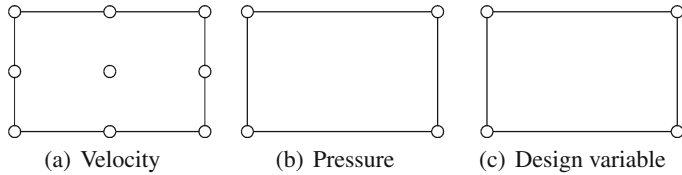


Fig. 2.3 The finite element nodes used to express the velocity, pressure and design variable on a rectangular element

2.1.4 Numerical Examples

In this section, several numerical examples are presented to demonstrate the capability of the topology optimization method for unsteady Navier-Stokes flows. The density and viscosity of the fluid are set to be unitary value, if there is no specification. The values of α_{max} and q are chosen based on numerical experiments. More details are available in [4]. The Reynolds number is calculated as

$$\text{Re} = \rho U_{max} L / \eta \quad (2.21)$$

where L is the width of the inlet, and U_{max} is the maximal value of the velocity on the inlet. The initial value condition of the transient Navier-Stokes equations is $\mathbf{u}_0 = \mathbf{0}$. The objective function for the topology optimization problem in Eq. 2.7 is chosen to be the energy dissipation inside the design domain and the pressure on the inlet

$$\begin{aligned} J(\mathbf{u}, p; \gamma) = & \int_0^T \int_{\Omega} \beta_1 \left[\frac{\eta}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) : (\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \alpha \mathbf{u}^2 \right] d\Omega dt \\ & + \int_0^T \int_{\Gamma_i} \beta_2 p d\Gamma dt \end{aligned} \quad (2.22)$$

where Γ_i is the inlet boundary. The parameters β_1 and β_2 depend on specified examples and can be tuned by the designer based on design necessity or numerical experiments.

2.1.4.1 Double Pipe

One double pipe is used to investigate the feasibility of the proposed optimization method. The design domain is shown in Fig. 2.4 and is discretized by 60×60 rectangular elements. The optimization parameters are listed in Table 2.1. The Neumann boundary condition in Eq. 2.4 is loaded on the outlets Γ_{o1} and Γ_{o2} by setting $\mathbf{g} = \mathbf{0}$. When the transient velocity in Eq. 2.23

$$\begin{aligned} \mathbf{u}_{in1} &= -144(y - 4/6)(5/6 - y)\cos(t)\mathbf{n}, \quad t \in [0, 2\pi] \\ \mathbf{u}_{in2} &= -144(y - 1/6)(2/6 - y)\sin(t)\mathbf{n}, \quad t \in [0, 2\pi] \end{aligned} \quad (2.23)$$

is imposed on the inlets, the optimized result is derived as shown in Fig. 2.5. Snapshots of the optimization procedure are shown in Fig. 2.6. The convergent histories of the objective values and the volume of the fluidic channel is shown in Fig. 2.7. Snapshots of the streamline of the unsteady flow at specified time are shown in Fig. 2.8.

By solving the topology optimization of the double pipe for the steady case, instead of the transient case, one can obtain the optimized channels in Fig. 2.9a, with the injecting velocity in Eq. 2.24 imposed on the inlets

Fig. 2.4 Design domain of the double pipe. The values $\mathbf{u}_{in1}$ and $\mathbf{u}_{in2}$ are the velocity distribution on the inlets

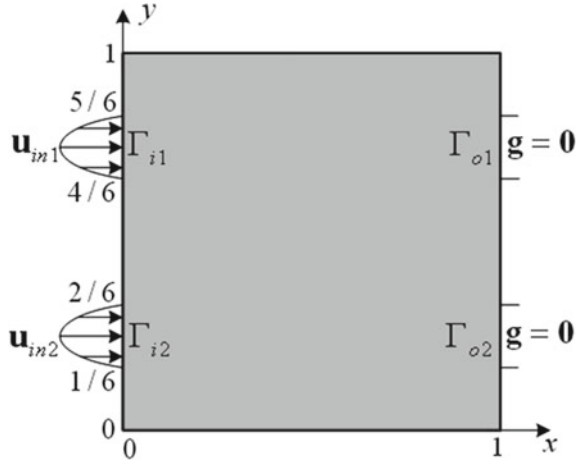


Table 2.1 Parameter settings in the topology optimization of the double pipe

β_1	β_2	V_r	α_{min}	α_{max}	q
1	0	1/3	0	1×10^4	1

Fig. 2.5 Optimal design of the double pipe for unsteady Navier-Stokes flows corresponding to the inlet velocity in Eq. 2.23



$$\begin{aligned}\mathbf{u}_{in1} &= -144(y - 4/6)(5/6 - y)\mathbf{n} \\ \mathbf{u}_{in2} &= -144(y - 1/6)(2/6 - y)\mathbf{n}\end{aligned}\quad (2.24)$$

The optimized result in Fig. 2.9a agrees with the numerical results derived by Borrvall and Petersson in [2] for steady flows. By changing $\mathbf{u}_{in2}$ to be a velocity of suction and maintaining $\mathbf{u}_{in1}$ a velocity of injection (Eq. 2.25),

$$\begin{aligned}\mathbf{u}_{in1} &= -144(y - 4/6)(5/6 - y)\mathbf{n} \\ \mathbf{u}_{in2} &= 144(y - 1/6)(2/6 - y)\mathbf{n}\end{aligned}\quad (2.25)$$

a bend pipe is obtained as shown in Fig. 2.9b, where the fluid flows from Γ_{i1} to Γ_{i2} directly. When both $\mathbf{u}_{in1}$ and $\mathbf{u}_{in2}$ are velocities of suction (Eq. 2.26),

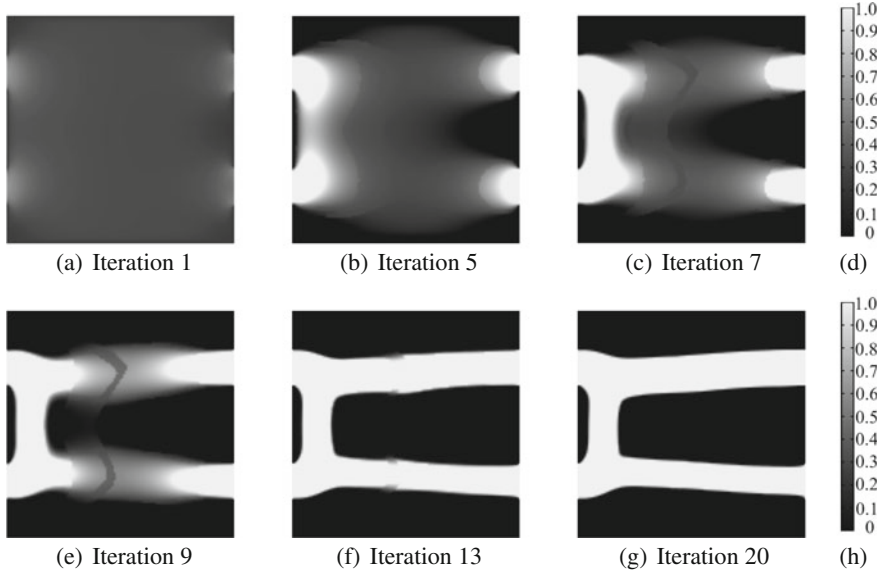


Fig. 2.6 Snapshots of optimization procedure for double pipe example in Fig. 2.5

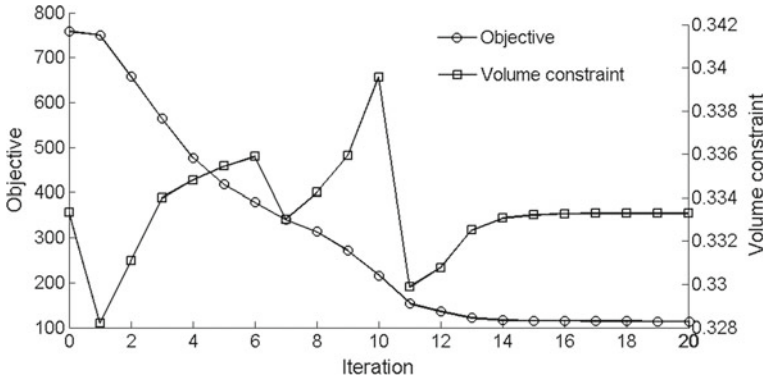


Fig. 2.7 Convergent history of the objective and volume constraint for the optimal design shown in Fig. 2.5

$$\begin{aligned} \mathbf{u}_{in1} &= 144(y - 4/6)(5/6 - y)\mathbf{n} \\ \mathbf{u}_{in2} &= 144(y - 1/6)(2/6 - y)\mathbf{n} \end{aligned} \quad (2.26)$$

the optimized channel is shown in Fig. 2.9c. Figure 2.9a–c show that the optimal design is a double pipe, when $\mathbf{u}_{in1}$ and $\mathbf{u}_{in2}$ are both the velocity of injection or suction; and the optimal design is a bend channel, when $\mathbf{u}_{in1}$ and $\mathbf{u}_{in2}$ are the velocity of suction and injection, respectively. According to Eq. 2.23, $\mathbf{u}_{in1}$ and $\mathbf{u}_{in2}$ are the velocity of injection as $t \in (0, \pi/2)$; the velocity of suction and injection as $t \in (\pi/2, \pi)$; the

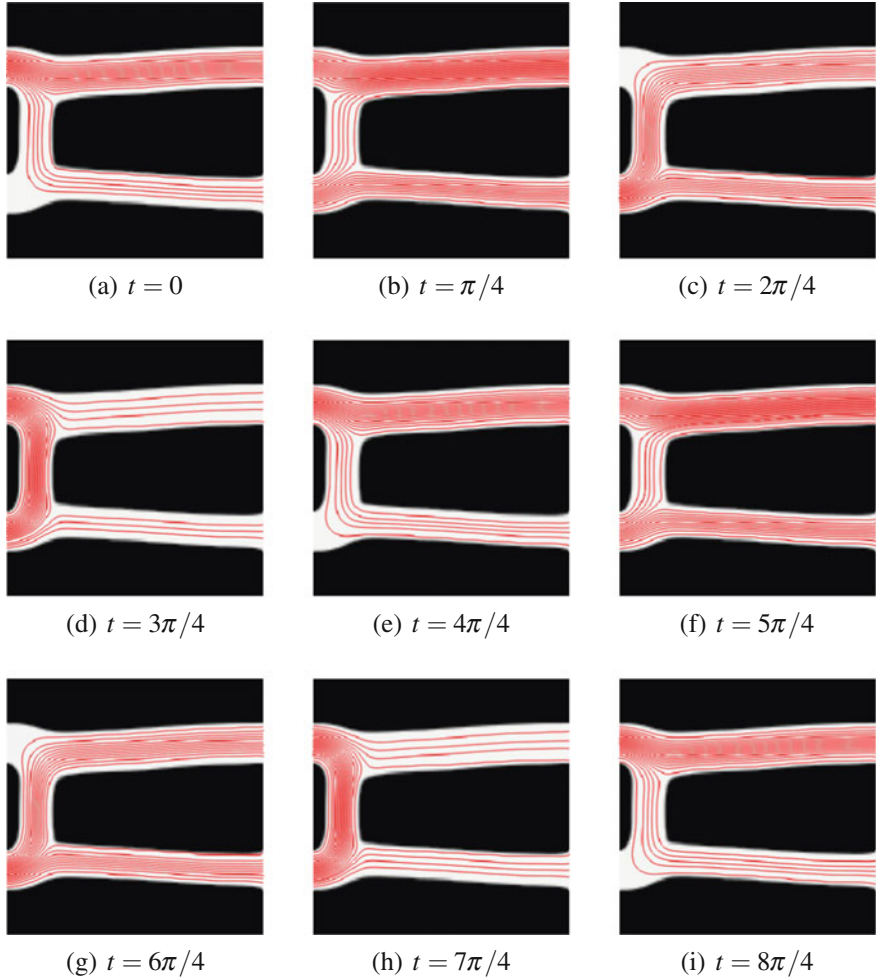


Fig. 2.8 Streamline snapshots of the unsteady double pipe flow corresponding to the optimal design in Fig. 2.5

velocity of suction as $t \in (\pi, 3\pi/2)$; and the velocity of injection and suction as $t \in (3\pi/2, 2\pi)$. Therefore, the fluid could be transported between Γ_{i1} and Γ_{i2} as $t \in (\pi/2, \pi) \cup (3\pi/2, 2\pi)$ and could flow in parallel between the inlets and outlets as $t \in (0, \pi/2) \cup (\pi, 3\pi/2)$. This analysis is consistent with the streamline distribution in Fig. 2.8. By respectively imposing the velocity of inlet in Eqs. 2.23–2.26 for the optimized results in Figs. 2.5 and 2.9, the values of the objective in Eq. 2.22 shown in Table 2.2 are obtained. By cross comparison the data in Table 2.2 and the optimized channels in Figs. 2.5 and 2.9, one can conclude that the optimal topology of a channel is valid only corresponding to the specific inlet velocity that is specified during the

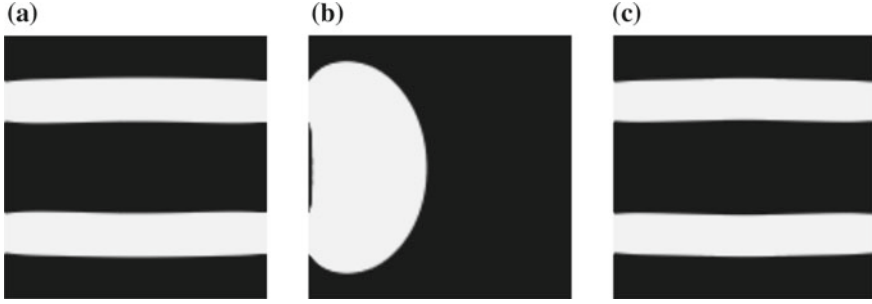


Fig. 2.9 Optimal designs of the double pipe for steady Navier-Stokes flows corresponding to the inlet velocity in Eqs. 2.24–2.26, respectively

Table 2.2 Objective values of the double pipe in the optimal designs of Figs. 2.5 and 2.9, where J_U , J_{Sa} , J_{Sb} and J_{Sc} are the objective values obtained by imposing the velocity in Eqs. 2.23–2.26 on the inlets of the optimal designs in Figs. 2.5 and 2.9, respectively

	Eq. 2.23	Eq. 2.24	Eq. 2.25	Eq. 2.26
J_U	113.8	56.65	16.14	56.65
J_{Sa}	133.6	48.60	37.08	48.73
J_{Sb}	606.6	379.8	6.04	379.8
J_{Sc}	134.8	48.78	48.73	48.71

optimization procedure. Therefore, it is reasonable that the optimal topology for unsteady flow is different from its steady counterparts.

The dynamic effects of inflow can be adjusted by tuning the parameter ω in Eq. 2.27

$$\begin{aligned} \mathbf{u}_{in1} &= -144(y - 4/6)(5/6 - y)\cos(\omega t)\mathbf{n} \\ \mathbf{u}_{in2} &= -144(y - 1/6)(2/6 - y)\sin(\omega t)\mathbf{n} \end{aligned} \quad (2.27)$$

A larger value of ω corresponds to stronger oscillation of inflow. Figure 2.10 shows the optimized double pipes for different values of ω . These results illustrate the necessity of the optimization for unsteady flow.

2.1.4.2 Three-Terminal Device

Fluidic channels with periodic dynamic input on the inlet have been widely used in fluidic devices [13]. In this example, a three-terminal device with periodic transient velocity on the inlet Γ_i , given as the equation

$$\mathbf{u}_{in} = -1 \times 10^4(y - 1)(1.2 - y)\sin(t)\mathbf{n} \quad (2.28)$$

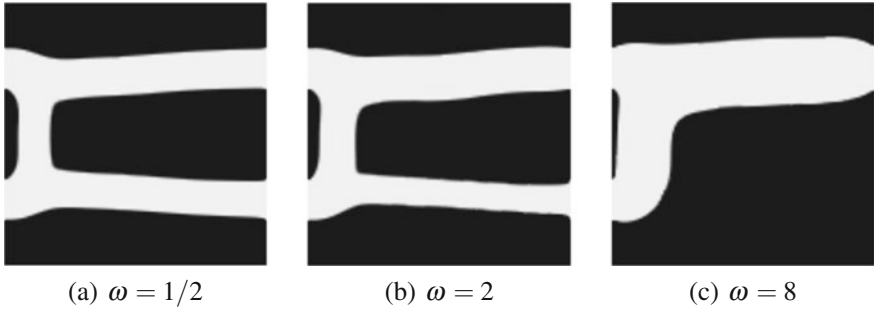


Fig. 2.10 Optimal designs of the double pipe corresponding to different values of ω in Eq. 2.27

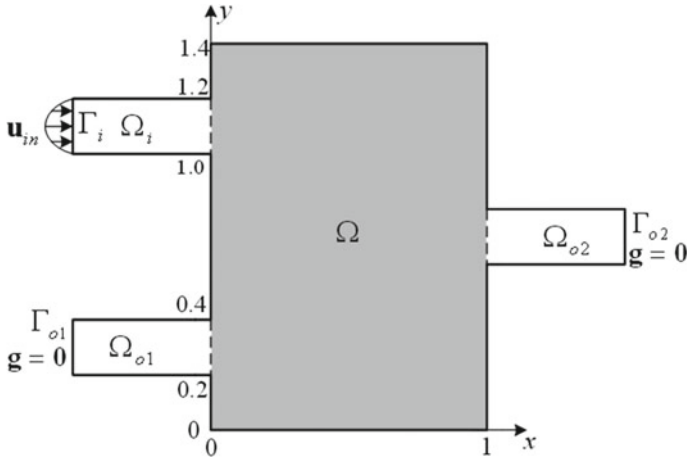


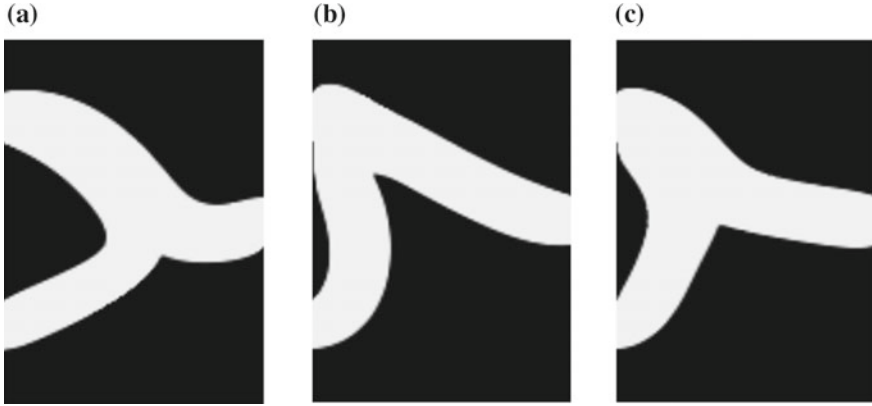
Fig. 2.11 Design domain of the three-terminal device. The value $\mathbf{u}_{in}$ is the velocity distribution on the inlet

is optimized. According to Eq. 2.21, the Reynolds number is 100 in this example. The design domain Ω is shown in Fig. 2.11, where the inlet duct is Ω_i and the outlet ducts are Ω_{o1} and Ω_{o2} . The design domain is discretized by 100×140 rectangular elements. The Neumann boundary condition in Eq. 2.4 is loaded on the outlets Γ_{o1} and Γ_{o2} by setting $\mathbf{g} = \mathbf{0}$. The other boundaries are set as no-slip boundaries. Three time intervals $[0, \pi]$, $[\pi, 2\pi]$ and $[0, 2\pi]$ which correspond to pure injection flow, pure suction flow and periodic injection-suction flow, are considered separately. The optimization parameter values are shown in Table 2.3 and the optimal topologies of the device are shown in Fig. 2.12a–c. Figure 2.13 shows the optimized device where the steady velocity condition is loaded on the inlet Γ_i as

$$\mathbf{u}_{in} = -1 \times 10^4 (y - 1)(1.2 - y) \mathbf{n} \quad (2.29)$$

Table 2.3 Parameter settings in the topology optimization of three-terminal device

β_1	β_2	V_r	α_{min}	α_{max}	q
1	0	0.3	0	1×10^4	1

**Fig. 2.12** Optimal designs of the three-terminal device. Here, **a**, **b** and **c** are optimal designs for unsteady flows corresponding to the time intervals $[0, \pi]$, $[\pi, 2\pi]$ and $[0, 2\pi]$, respectively**Fig. 2.13** Optimal design of the three-terminal device for steady flow

From Fig. 2.12, one can see that the suction flow occurring as $t \in (\pi, 2\pi)$ is a key point in the difference of the shape of the device compared to the steady flow case. By imposing the velocity boundary in Eqs. 2.28 and 2.29 on the inlets of the optimal designs in Figs. 2.12 and 2.13 respectively, the values of objective listed in Table 2.4 can be obtained. The cross comparison of the values of energy dissipation confirms further that the dynamic effect of the unsteady flow can influence the detailed shape of the device. This example illustrates that the optimal design of a unsteady flow is influenced by the dynamic effect induced by the different choice of time intervals.

Table 2.4 Objective values of the optimal designs in Figs. 2.12 and 2.13. J_{Ta} , J_{Tb} , J_{Tc} and J_S are objective values obtained by imposing the velocity in Eqs. 2.28 and 2.29 on the inlets of optimal designs in Figs. 2.12 and 2.13

	Steady	$t \in [0, \pi]$	$t \in [\pi, 2\pi]$	$t \in [0, 2\pi]$
J_{Ta}	1.064×10^5	1.298×10^5	1.305×10^5	2.605×10^5
J_{Tb}	1.347×10^5	1.579×10^5	1.258×10^5	2.841×10^5
J_{Tc}	1.138×10^5	1.302×10^5	1.271×10^5	2.574×10^5
J_S	1.062×10^5	1.299×10^5	1.300×10^5	2.602×10^5

2.1.4.3 Bend Channel

One example of a bend channel with different Reynolds numbers under steady flow has been discussed by Gersborg-Hansen et al. in [3]. The similar example is discussed here for unsteady flows. The computational domain includes the inlet duct Ω_i , the outlet duct Ω_o and the design domain Ω (Fig. 2.14), which are respectively discretized by 40×20 , 40×20 and 100×100 rectangular elements. The transient velocity is set to be

$$\mathbf{u}_{in} = -4U_{max}(y - 3.5)(4.5 - y)t\mathbf{n}, \quad t \in [0, 1] \quad (2.30)$$

where U_{max} is specified to be 1, 50 and 300, respectively corresponding to the Reynolds numbers of 1, 50 and 300. The optimization parameters are listed in Table 2.5. The optimized channels with different Reynolds numbers and the corresponding objective values are shown in Fig. 2.15 and Table 2.6. The results in Fig. 2.15 show that the bend channel has sharp corners for flow with low Reynolds

Fig. 2.14 Design domain of the bend channel. The value $\mathbf{u}_{in}$ is the velocity distribution on the inlet

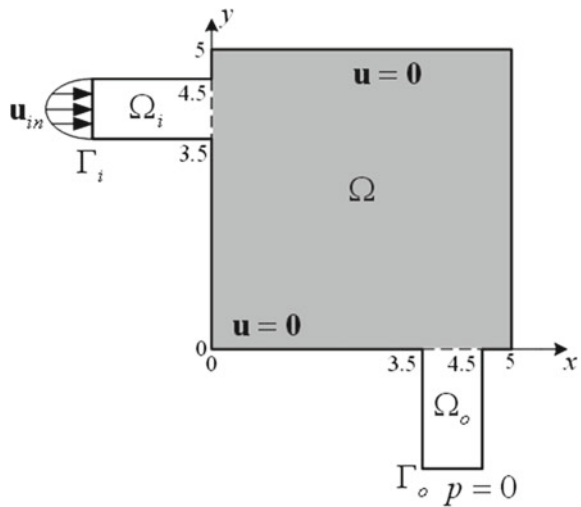
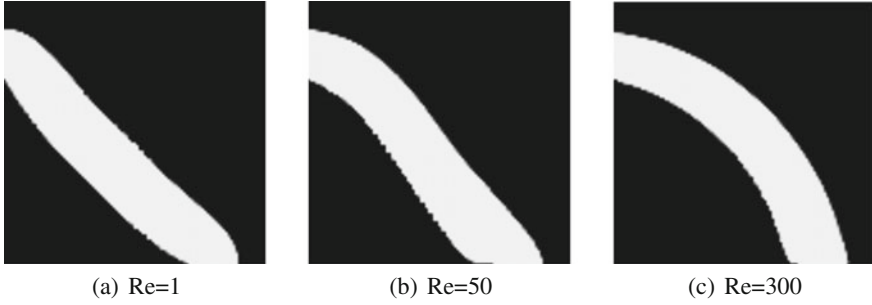


Table 2.5 Parameter settings in the topology optimization of the bend channel

β_1	β_2	V_r	α_{min}	α_{max}	q
1	0.1	0.25	0	1×10^4	1

**Fig. 2.15** Optimal designs of the bend channel corresponding to different Reynolds numbers**Table 2.6** Objective values of the optimal designs in Fig. 2.15

	Re = 1	Re = 50	Re = 300
J	2.094×10^1	5.722×10^4	2.745×10^6

number, and the corners become rounder along with the increase of the Reynolds number. Therefore, the optimized channel is relatively straight for the flow with low Reynolds number, and it becomes bending for the flow with large enough Reynolds number. The results agree with those obtained by Gersborg-Hansen et al. in [3] for steady flows.

2.1.4.4 Target Flux on Outlet

The flow rate on the outlet is an important output performance for fluid flow [13, 14]. The target flux on the outlet can be produced by adding a flux constraint at the outlet of the design domain. According to the flow rate constraint for optimization of steady flows [15, 16], the target flux for a unsteady flow is added into the topology optimization problem in Eq. 2.7 as an inequality constraint

$$\left(\frac{\int_{T_1}^{T_2} \int_{\Gamma_o} \mathbf{u} \cdot \mathbf{n} \, d\Gamma \, dt}{Q_{tar}} - 1 \right)^2 \leq \varepsilon \quad (2.31)$$

where $[T_1, T_2] \subseteq [0, T]$ is the time interval; Γ_o is the outlet boundary; Q_{tar} is the target flux on the corresponding outlet; and ε is the allowable tolerance. The adjoint analysis for the flux constraint follows the same procedure as that for the objective in

Sect. 2.1.2. Additionally, the objective or certain design constraints may be defined on the partial time interval $[T_1, T_2] \subseteq [0, T]$, while the others are still defined on the whole time interval $[0, T]$. Therefore, the adjoint analysis of the corresponding expression is implemented on $[0, T_2]$ instead of $[0, T]$, although the unsteady flow problem is defined on the whole time interval $[0, T]$. Sequentially, the corresponding adjoint equations and adjoint derivatives are transient equations and integrations on $[0, T_2]$ instead of $[0, T]$ (see Sect. 2.3.2.2 for more details).

First, a flux distribution structure is optimized. The design domain is shown in Fig. 2.16a, and is discretized by 100×100 rectangular elements. The velocity loaded on the inlet is (Fig. 2.17c)

$$\begin{aligned} \mathbf{u}_{in} = & -[4(y - 3.5)(4.5 - y)(t \leq 0.5) \\ & + 4(y - 3.5)(4.5 - y)(18t - 8)(0.5 < t \leq 1) \\ & + 40(y - 3.5)(4.5 - y)(1 < t \leq 2) \\ & + 4(y - 3.5)(4.5 - y)(-18t + 46)(2 < t \leq 2.5) \\ & + 4(y - 3.5)(4.5 - y)(t > 2.5)]\mathbf{n}, t \in [0, 3] \end{aligned} \quad (2.32)$$

The optimization parameter values are listed in Table 2.7. The flux constraint expressed in Eq. 2.31 is imposed on the outlet Γ_{o1} to constrain the flux distribution between the two outlets Γ_{o1} and Γ_{o2} . The parameters in Eq. 2.31 are chosen to be 1×10^{-4} for ε and $\frac{1}{3} \int_0^3 \int_{\Gamma_i} -\mathbf{u}_{in} \cdot \mathbf{n} d\Gamma dt$ for Q_{tar} . When $[T_1, T_2]$ is set equal to $[0, T]$, the design constraint is

$$\left(\frac{\int_0^3 \int_{\Gamma_{o1}} \mathbf{u} \cdot \mathbf{n} d\Gamma dt}{\frac{1}{3} \int_0^3 \int_{\Gamma_i} -\mathbf{u}_{in} \cdot \mathbf{n} d\Gamma dt} - 1 \right)^2 \leq 1 \times 10^{-4} \quad (2.33)$$

Then the objective and the design constraint are both defined on the whole time interval. The optimized flux distribution device is shown in Fig. 2.17a. And the corresponding objective value is 1.740×10^3 . When the time interval $[T_1, T_2]$ is set to be $[1, 2]$, the design constraint is modified to be

$$\left(\frac{\int_1^2 \int_{\Gamma_{o1}} \mathbf{u} \cdot \mathbf{n} d\Gamma dt}{\frac{1}{3} \int_0^3 \int_{\Gamma_i} -\mathbf{u}_{in} \cdot \mathbf{n} d\Gamma dt} - 1 \right)^2 \leq 1 \times 10^{-4} \quad (2.34)$$

and the objective is still defined on the whole time interval. According to the derivation of adjoint sensitivity in Appendix 2.3.2.2, the adjoint equations and derivative corresponding to the flux constraint are defined on the time interval $[0, 2]$. By keeping the other parameters unchanged, the optimization problem is solved. The optimized result is derived as shown in Fig. 2.17b. The corresponding objective value is 1.257×10^3 . The outflow rate on the outlet Γ_{o1} corresponding to the above two optimal designs is shown in Fig. 2.17c. Because the fluid considered is incompressible, the left fluid must flow out from the outlet Γ_{o2} . Therefore, the optimized structures

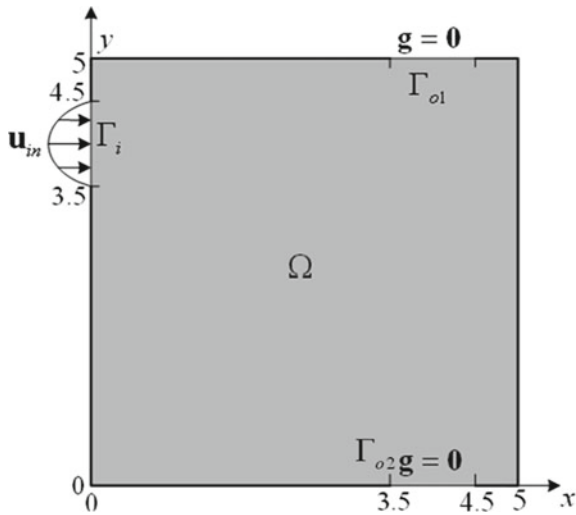


Fig. 2.16 Design domain of the flux distribution structure

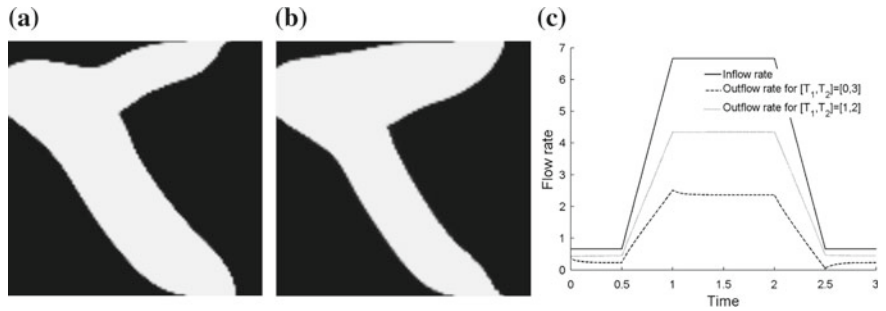


Fig. 2.17 **a** Optimal design of the flux distribution structure with flux constraint defined on the whole time interval $[0, 3]$; **b** optimal design of the flux distribution structure with flux constraint defined on the time interval $[1, 2]$; **c** the absolute inflow rate over time on the inlet Γ_i and the outflow rate over time on the outlet Γ_{o1}

Table 2.7 Parameter settings in the topology optimization of a flux distribution structure with target flux on the outlet

β_1	β_2	V_r	α_{min}	α_{max}	q
1	0	0.4	0	1×10^3	1

in Fig. 2.17a, b have two branches. The flux constraints in Eqs. 2.33 and 2.34 mean that the flux at the outlet Γ_{o1} of the optimal design in Fig. 2.17a is lower than that in Fig. 2.17b. Therefore, the thick branch connected to Γ_{o1} in Fig. 2.17b is helpful for decreasing the velocity gradient and dissipation of the flow.

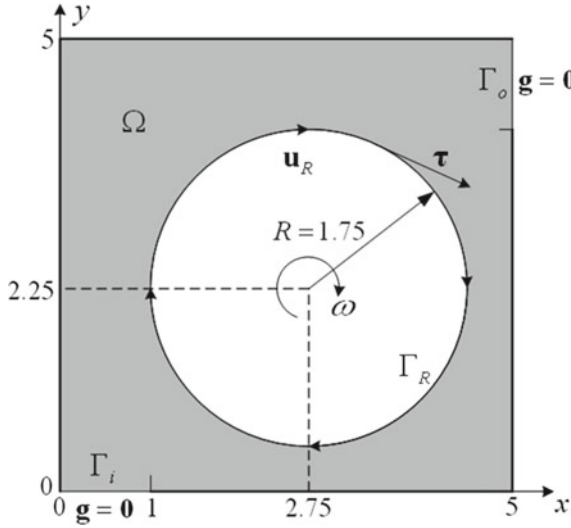


Fig. 2.18 Design domain of the roller-type pump

Second, a roller-type pump, which pumps the liquid using a rotating roller, is optimized. Figure 2.18 shows the design domain, where the rotating roller drives the liquid flowing from the inlet Γ_i to the outlet Γ_o . The fluidic velocity on the surface of the roller is equal to the rotational velocity of the roller surface,

$$\mathbf{u}_r = \omega(t) R \boldsymbol{\tau}, \quad t \in [0, 2] \quad (2.35)$$

where $\omega(t)$ is the transient angular speed of the roller; R is the radius of the roller; and $\boldsymbol{\tau}$ is the unitary tangential vector of the roller surface. The optimization parameters are listed in Table 2.8. By respectively setting Q_{tar} and ε to be 20 and 1×10^{-4} , the flux constraint shown in Eq. 2.31 is imposed on the boundary Γ_o as

$$\left(\frac{\int_0^2 \int_{\Gamma_o} \mathbf{u} \cdot \mathbf{n} d\Gamma dt}{20} - 1 \right)^2 \leq 1 \times 10^{-4} \quad (2.36)$$

The design domain is discretized by 171,519 triangular elements. After the transient angular speed is set to $\omega(t) = 400 \min(t, 1) / 7$ (Fig. 2.19b), the optimized roller-type pump is derived as shown in Fig. 2.19a. The corresponding objective value is 5.866×10^5 . The outflow rate over time is shown in Fig. 2.19b. The streamline distribution of the flow in the optimized pump is shown in Fig. 2.20. Figure 2.20 shows that the streamlines near the surface of the pump's roller is closed, while the others start at the inlet and end at the outlet. Therefore, a net flux between the inlet and outlet of the pump is produced by the rotating roller.

Table 2.8 Parameter settings in the topology optimization of the roller-type pump

β_1	β_2	V_r	α_{min}	α_{max}	q
1	0	0.4	0	1×10^3	1

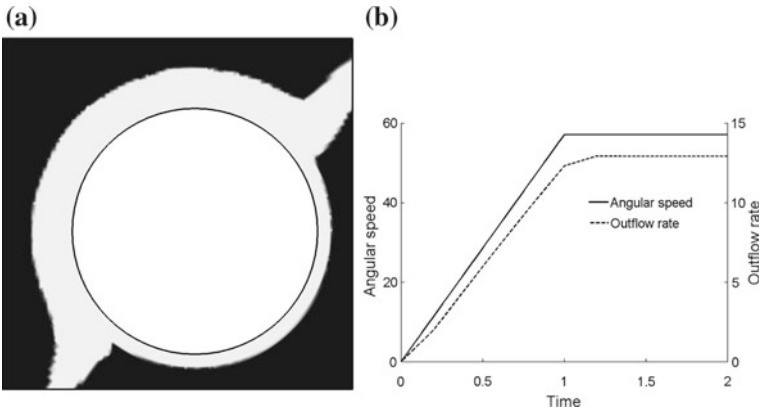
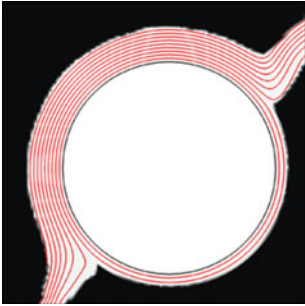


Fig. 2.19 **a** Optimal design of the roller-type pump; **b** the angular velocity over time and the outflow rate over time on the outlet Γ_o

Fig. 2.20 Streamlines of the flow in the optimal design of the roller-type pump at $t = 0.8$



2.1.4.5 Star-Shaped Microchannel Chip

To demonstrate the applications in microfluidics, a popular technique in biochemistry and bioengineering [17, 18], this example involves the design of an infuser, a device that feeds a reactor or a piece of analysis equipment with a specific amount of fluid. Flushing the fluidic channel is an important step to maintain consistent performance and enhance the efficiency of an infuser. The fluid flowing through the infuser is water, with density $1 \times 10^3 \text{ kg/m}^3$ and viscosity $1 \times 10^{-3} \text{ Pa s}$. The design domain is in Fig. 2.21a and it is discretized by 160,866 triangular elements. The transient velocity loaded on the inlets of the Star-shaped microchannel is

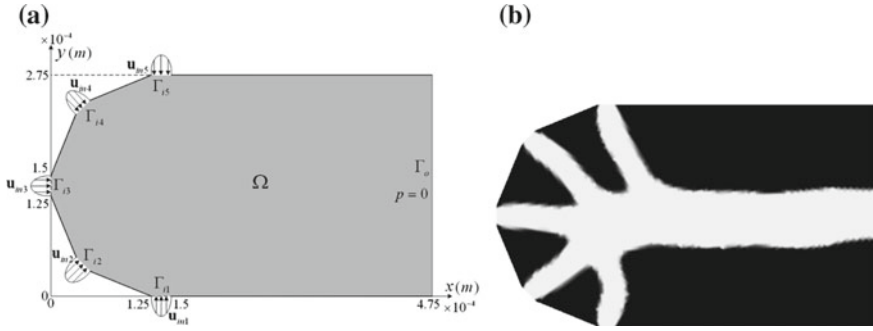


Fig. 2.21 **a** Design domain of the star-shaped microchannel chip; **b** optimal design of the star-shaped microchannel chip

$$\begin{aligned}
 \mathbf{u}_{in1} &= -U (1 + 2\sin(\pi t)) \mathbf{n} \\
 \mathbf{u}_{in2} &= -U (1 + 2\sin(\pi t + \pi/4)) \mathbf{n} \\
 \mathbf{u}_{in3} &= -U (1 + 2\sin(\pi t + \pi/2)) \mathbf{n} \\
 \mathbf{u}_{in4} &= -U (1 + 2\sin(\pi t + 3\pi/4)) \mathbf{n} \\
 \mathbf{u}_{in5} &= -U (1 + 2\sin(\pi t + \pi)) \mathbf{n}
 \end{aligned} \tag{2.37}$$

where $U = 4 \times 10^{-2}s(1-s)$ is a parabolic distribution on the inlets of the design domain, and s is the parametric coordinate on the corresponding inlet. The energy dissipation and pressure distribution at the inlet (Eq. 2.22) are set to be the objective. The time interval is chosen to be $[0, 2]$. The optimization parameters are listed in Table 2.9. The optimized infuser has star-shaped channels (Fig. 2.21b). The corresponding objective value is 795.6. Snapshots for the distribution of the fluidic velocity are shown in Fig. 2.22. It shows that the fluid pulses through the star-shaped chip periodically. This periodic pulsing is helpful for the effective flushing of leftovers in microfluidic chips.

2.1.5 Summary

This topology optimization method can be used to design the unsteady incompressible Navier-Stokes flows at low and moderate Reynolds numbers. It makes topology optimization more useful for practical engineering designs, such as the flow with unsteady

Table 2.9 Parameter settings in the topology optimization of a star-shaped microchannel chip

β_1	β_2	V_r	α_{min}	α_{max}	q
1	0.1	0.35	0	1×10^6	1

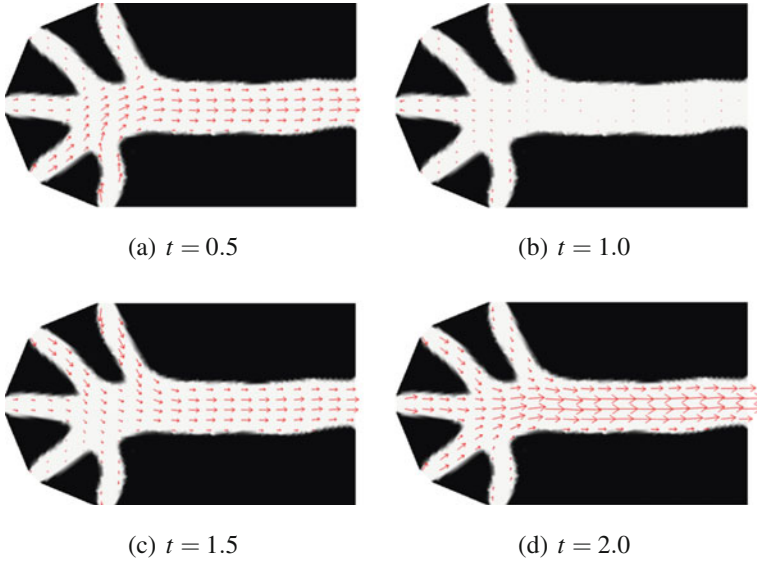


Fig. 2.22 Snapshots of the distribution of fluidic velocity

state or optimization focusing on the dynamic effect of fluid. The topology optimization problem of the unsteady Navier-Stokes flow is analyzed using continuous adjoint method. Then the numerical optimization procedure can be implemented by user-available numerical computational methods, such as finite difference method, finite element method or finite volume method. The numerical examples demonstrated that the optimized design of the unsteady Navier-Stokes flow is influenced by the dynamic effect, the Reynolds number and the constraints on the flux of fluid.

2.2 Level Set Method-Based Topology Optimization for Unsteady Flows

This section introduces the level set method-based topology optimization for unsteady flows. Being different from the density method, the level set method distinguishes the fluid and solid phases by the sign of the level set function defined on the design domain. The key ideas behind this work is on constrain the fluidic velocity to be zero in the implicitly expressed solid phase, and the analysis of the topology optimization problem using variational level set method.

2.2.1 Level Set Method-Based Optimization Problem

In order to express the solid-liquid interface implicitly, the unsteady incompressible Navier-Stokes equations are coupled with the level set function, expressed by a signed distance function ϕ defined on the optimization domain $\Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}$ ($d = 2$ or 3 is the space dimension). The solid and fluid domains is distinguished as

$$\begin{cases} \phi(\mathbf{x}) > 0, & \forall \mathbf{x} \in \Omega_s \\ \phi(\mathbf{x}) < 0, & \forall \mathbf{x} \in \Omega_l \\ \phi(\mathbf{x}) = 0, & \forall \mathbf{x} \in \Gamma \end{cases} \quad (2.38)$$

where Ω_s , Ω_l and Γ are the solid region, fluid region and implicit boundary, respectively. Ω_s and Ω_l satisfy $\Omega_s \cup \Omega_l = \Omega$. Then, the unsteady incompressible Navier-Stokes equations are modified by constraining the fluid velocity to be zero in the solid phase Ω_s ,

$$\begin{aligned} \rho \frac{\partial \mathbf{u}}{\partial t} - \eta \Delta \mathbf{u} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \mathbf{0}, & \text{in } Q \\ -\nabla \cdot \mathbf{u} &= 0, & \text{in } Q \\ H(\phi) \mathbf{u} &= \mathbf{0}, & \text{in } Q \end{aligned} \quad (2.39)$$

where $\mathbf{u}$ is the fluid velocity; p is the pressure; ρ is the fluid density; η is the fluid viscosity; Q is the computational domain $(0, T) \times \Omega$; t is the time and $(0, T)$ is the time interval; $H(\phi)$ is the Heaviside function:

$$H(\phi) = \begin{cases} 1, & \phi \geq 0 \\ 0, & \phi < 0 \end{cases} \quad (2.40)$$

and the derivative of $H(\phi)$ is the Dirac function:

$$\tau(\phi) = \begin{cases} +\infty, & \phi = 0 \\ 0, & \phi \neq 0 \end{cases} \quad (2.41)$$

The implicit no slip boundary, lying between the solid and fluid domains, is represented by the zero level set of the level set function ϕ . Compared with the method in [19], the above coupling method avoids abstracting the fluid domain and changing the computational domain. The initial condition of Navier-Stokes equation 2.39 is

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \text{in } \Omega \quad (2.42)$$

where $\mathbf{u}_0(\mathbf{x})$ satisfies $\nabla \cdot \mathbf{u}_0 = 0$. The boundary conditions contains the Dirichlet type

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{u}_D(t, \mathbf{x}), \quad \text{on } \Sigma_D \quad (2.43)$$

and the Neumann type

$$(-p\mathbf{I} + \eta\nabla\mathbf{u}) \cdot \mathbf{n} = \mathbf{g}, \quad \text{on } \Sigma_N \quad (2.44)$$

where $\mathbf{u}_D$ and $\mathbf{g}$ are the known velocity and stress distribution on Γ_D and Γ_N , respectively; $\mathbf{n}$ is the outward unitary normal on $\partial\Omega$; Σ_D and Σ_N represent $(0, T) \times \Gamma_D$ and $(0, T) \times \Gamma_N$, respectively. Particularly, the no slip boundary condition is Dirichlet type, where $\mathbf{u}_D$ is set to be $\mathbf{0}$; the open boundary condition is Neumann type, where $\mathbf{g}$ is set to be $\mathbf{0}$.

Based on the above description, the level set method-based topology optimization problem for the unsteady Navier-Stokes flow is formulated to be

$$\begin{aligned} &\text{Find } \phi, \text{ to minimize } J(\mathbf{u}, p; \phi), \text{ subject to} \\ &\left\{ \begin{array}{l} \rho \frac{\partial \mathbf{u}}{\partial t} - \eta \Delta \mathbf{u} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{0}, \quad \text{in } Q \\ -\nabla \cdot \mathbf{u} = 0, \quad \text{in } Q \\ H(\phi) \mathbf{u} = \mathbf{0}, \quad \text{in } Q \\ \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \text{in } \Omega \\ \mathbf{u}(t, \mathbf{x}) = \mathbf{u}_D(t, \mathbf{x}), \quad \text{on } \Sigma_D \\ (-p\mathbf{I} + \eta\nabla\mathbf{u}) \mathbf{n} = \mathbf{g}, \quad \text{on } \Sigma_N \\ \int_{\Omega} H(-\phi) d\Omega = V_r V_0 \end{array} \right. \quad (2.45) \end{aligned}$$

where $V_r \in (0, 1)$ is the volume fraction; $V_0 = \int_{\Omega} 1 d\Omega$ is the volume of the optimization domain Ω . The objective is set to be the general form

$$\begin{aligned} J(\mathbf{u}, p; \phi) = & \int_0^T \int_{\Omega} \beta_1 H(-\phi) A(\mathbf{u}, \nabla\mathbf{u}, p) d\Omega dt + \int_0^T \int_{\Gamma} \beta_2 B(\mathbf{u}, \nabla\mathbf{u}, p) d\Gamma dt \\ & + \int_0^T \int_{\partial\Omega} \beta_3 C(\mathbf{u}, p) d\Gamma dt \end{aligned} \quad (2.46)$$

which can be transformed into

$$\begin{aligned} J(\mathbf{u}, p; \phi) = & \int_0^T \int_{\Omega} [\beta_1 H(-\phi) A(\mathbf{u}, \nabla\mathbf{u}, p) + \beta_2 \tau(\phi) \|\nabla\phi\| B(\mathbf{u}, \nabla\mathbf{u}, p)] d\Omega dt \\ & + \int_0^T \int_{\partial\Omega} \beta_3 C(\mathbf{u}, p) d\Gamma dt \end{aligned} \quad (2.47)$$

where $A(\mathbf{u}, \nabla\mathbf{u}, p)$ is a functional defined on Q ; $B(\mathbf{u}, \nabla\mathbf{u}, p)$ is a functional defined on the implicit boundary $(0, T) \times \Gamma$; $C(\mathbf{u}, p)$ is a functional defined on $(0, T) \times \partial\Omega$; β_1, β_2 and β_3 are space-independent coefficients. The definition of the general objective in Eq. 2.46 makes the optimization with the objectives defined on the design domain, the boundary of the design domain or the implicit boundary (i.e. zero level

set) be feasible using this optimization method for unsteady flows. The particular objectives, e.g. the dissipation of the flow, the kinetic energy of the fluid and the pressure drop between the inlet and outlet, can be included in the general formulation in Eq. 2.46.

2.2.2 Sensitivity Analysis

The sensitivity analysis of the optimization problem is implemented using the Lagrangian multiplier based adjoint approach [5–7], and the volume constraint is treated using the Lagrangian multiplier based quadratic penalty method [20]. Then the adjoint equations of the Navier-Stokes equations in the optimization problem (Eq. 2.45) are derived to be (see Appendix 2.4.1 for more details)

$$\begin{aligned}
 -\rho \frac{\partial \mathbf{u}_a}{\partial t} - \eta \Delta \mathbf{u}_a - \rho (\mathbf{u} \cdot \nabla) \mathbf{u}_a + \rho (\nabla \mathbf{u}) \cdot \mathbf{u}_a + \nabla p_a &= - \left(\frac{\partial \tilde{A}}{\partial \mathbf{u}} + \nabla \cdot \frac{\partial \tilde{A}}{\partial \nabla \mathbf{u}} \right), \text{ in } Q \\
 -\nabla \cdot \mathbf{u}_a &= - \frac{\partial \tilde{A}}{\partial p}, \text{ in } Q \\
 \mathbf{u}_a(T, \mathbf{x}) &= \mathbf{0}, \text{ in } \Omega \\
 \mathbf{u}_a &= -\beta_3 \frac{\partial C}{\partial p} \mathbf{n}, \text{ on } \Sigma_D \\
 (-p_a \mathbf{I} + \eta \nabla \mathbf{u}_a) \mathbf{n} &= -\rho (\mathbf{u} \cdot \mathbf{n}) \mathbf{u}_a - \frac{\partial \tilde{A}}{\partial \nabla \mathbf{u}} \mathbf{n} \\
 &\quad - \beta_3 \frac{\partial C}{\partial \mathbf{u}}, \text{ on } \Sigma_N
 \end{aligned} \tag{2.48}$$

where $\tilde{A} = \beta_1 H(-\phi)A + \beta_2 \tau(\phi) \|\nabla \phi\| B$; $\mathbf{u}_a$ and p_a are the adjoint variables of the fluid velocity $\mathbf{u}$ and pressure p , respectively. According to [21], the evolution of the level set function is achieved by solving the Hamilton-Jacobian equation

$$\frac{\partial \phi}{\partial \theta} + v_n \|\nabla \phi\| = 0 \tag{2.49}$$

where θ is the level set evolving time, and it is different from the time in Eq. 2.39; v_n is the normal velocity loaded on the level set function ϕ . Therefore, the variational of ϕ is $\delta \phi = -v_n \|\nabla \phi\| \delta \theta$, where $\delta \theta$ is the evolution of time, and it must be positive [21]. In order to ensure the descent of the objective, the shape sensitivity should satisfy $\delta \hat{J} < 0$, where $\hat{J}$ is the augmented Lagrangian objective functional and $\delta \hat{J}$ is (see Appendix 2.4.2 for more details on the derivation of $\delta \hat{J}$)

$$\begin{aligned} \delta \hat{J} = & - \int_{\Omega} \left[\int_0^T (\beta_1 A + \beta_2 \nabla B \cdot \mathbf{n}_T + \beta_2 B \kappa - \mathbf{u} \cdot \mathbf{u}_a) dt - \lambda \right. \\ & \left. + \Lambda \left(\int_{\Omega} H(-\phi) d\Omega - V_r V_0 \right) \right] \tau(\phi) \delta \phi d\Omega \end{aligned} \quad (2.50)$$

where λ and Λ are the Lagrangian multiplier and the quadratic penalty parameter, respectively. Because $\tau(\phi)$, $\|\nabla \phi\|$ and $\delta \theta$ are all nonnegative, the normal velocity used to evolve the level set function is set to be

$$\begin{aligned} v_n = & - \int_0^T [\beta_1 A + \beta_2 (\nabla B \cdot \mathbf{n}_T + B \kappa) - \mathbf{u} \cdot \mathbf{u}_a] dt \\ & + \lambda - \Lambda \left(\int_{\Omega} H(-\phi) d\Omega - V_r V_0 \right) \end{aligned} \quad (2.51)$$

where $\mathbf{n}_T = \nabla \phi / \|\nabla \phi\|$, $\kappa = \nabla \cdot \mathbf{n}_T$ and $\mathbf{u}_a$ is derived by solving the adjoint Eq. 2.48.

2.2.3 Numerical Implementation

The flowchart of the optimization procedure includes the following steps (Fig. 2.23):

- specify the initial distribution of the level set function ϕ , the initial values of the Lagrangian multiplier λ and the penalty parameter Λ ;
- compute the velocity $\mathbf{u}$ and the pressure p by solving the unsteady incompressible Navier-Stokes eq. 2.39, and the corresponding adjoint variables $\mathbf{u}_a$ and p_a are computed by solving the adjoint Eq. 2.48;
- compute the normal velocity v_n in Eq. 2.51;
- evolve the level set function ϕ by solving the Hamilton-Jacobi eq. 2.49;
- reinitialize the level set function ϕ after several iterations;
- stop the iterative optimization, when the change of the objective value and the volume fraction are less than the user-specified tolerance in five consecutive iterations or the maximum number of iterations is reached.

The level set function is evolved on a grid mesh with ghost elements (the combination of the dash and solid mesh in Fig. 2.24). The mesh for solving the Navier-Stokes equations is a set of elements embedded in the grid mesh for evolving the level set function (solid mesh in Fig. 2.24). The Navier-Stokes equations and the adjoint equations are solved using the finite element package *COMSOL Multiphysics* [10]. One of the advantages of the *COMSOL Multiphysics* package is that one can input the user-defined partial differential equations (PDEs) using the so-called *General PDE* format.

During the optimization procedure, the Navier-Stokes equations and the adjoint equations are solved by the stable Taylor-Hood Q2–Q1 elements [11, 22], which interpolate the fluid velocity quadratically and the fluid pressure linearly (Fig. 2.25a,

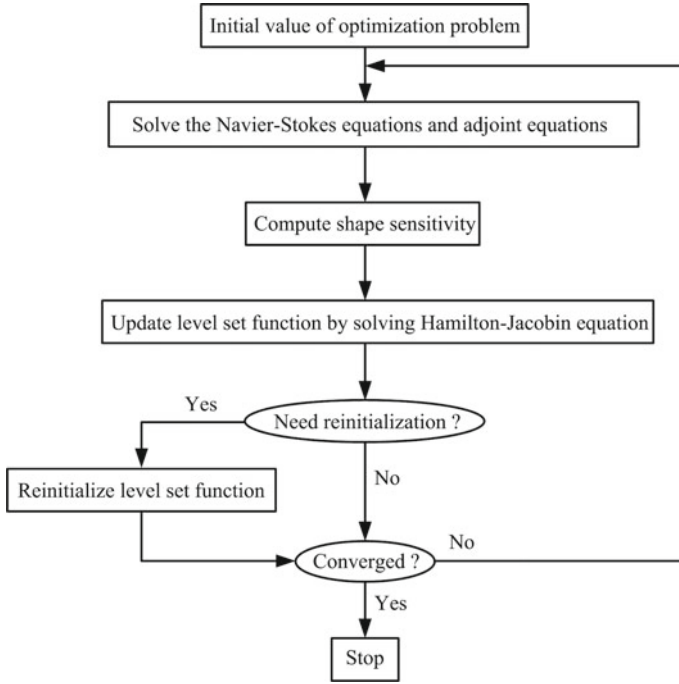


Fig. 2.23 Flowchart of the optimization procedure

b). The level set function is interpolated by the linear Q1 elements (Fig. 2.25c). The unsteady Navier-Stokes equations are solved using the fifth order backward differentiation formula method in the transient solver. The Lagrangian multiplier λ and penalty parameter Λ are updated as [20, 23]

$$\begin{aligned}\lambda_k &= \lambda_{k-1} - \Lambda_{k-1} \left(\int_{\Omega} H(-\phi_{k-1}) \, d\Omega - V_r V_0 \right) \\ \Lambda_k &= \frac{1}{\alpha_{\Lambda}} \Lambda_{k-1}, \quad \alpha_{\Lambda} \in (0, 1)\end{aligned}\tag{2.52}$$

where α_{Λ} is chosen to be 0.9, and the initial values λ_0 and Λ_0 are chosen based on numerical experiments.

In order to evolve the level set function, the Hamilton-Jacobin equation is solved by the upwind finite difference method. The time step for the finite difference scheme is chosen based on the CFL stability condition [24]

$$\Delta\theta \leq \beta_{CFL} \frac{h_E}{\max \{|v_n|\}}, \quad \beta_{CFL} \in (0, 1)\tag{2.53}$$

where h_E is the size of the elements and β_{CFL} is chosen to be 0.1.

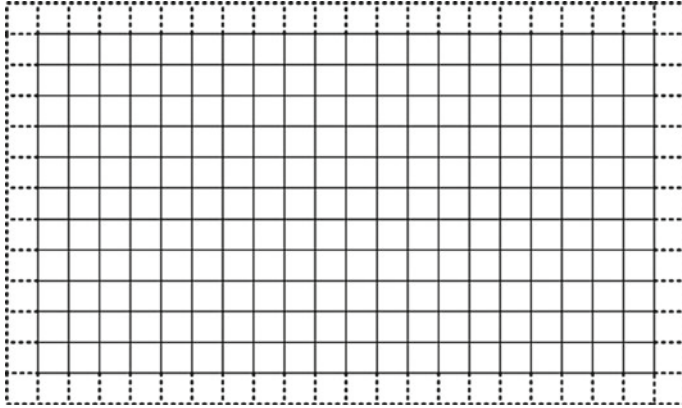


Fig. 2.24 Schematic for the grid mesh used to solve the Navier-Stokes equations (*solid mesh*) and evolve the level set function (the combination of the *solid* and *dash meshes*)

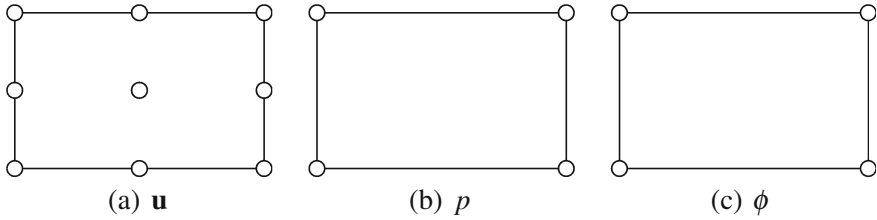


Fig. 2.25 Finite element nodes used to express the velocity, pressure and level set function on a rectangular element

The evolved level set function is reinitialized after several iterations. The reinitialization of the level set function is performed by computing the Euclidean distance transform of the binary value that corresponds to the level set function [25].

The Heaviside function and the Dirac function need to be smoothed in the numerical implementation. The smoothed Heaviside function $H(\phi, h)$ and the smoothed Dirac function $\tau(\phi, h)$ are

$$H(\phi, h) = \begin{cases} 1, & \phi \geq h \\ \frac{1}{2} + \frac{15\phi}{16h} - \frac{5\phi^3}{8h^3} + \frac{3\phi^5}{16h^5}, & |\phi| < h \\ 0, & \phi \leq -h \end{cases} \quad (2.54)$$

$$\tau(\phi, h) = \begin{cases} \frac{15}{16h} \left(1 - \frac{\phi^2}{h^2}\right)^2, & |\phi| < h \\ 0, & |\phi| \geq h \end{cases}$$

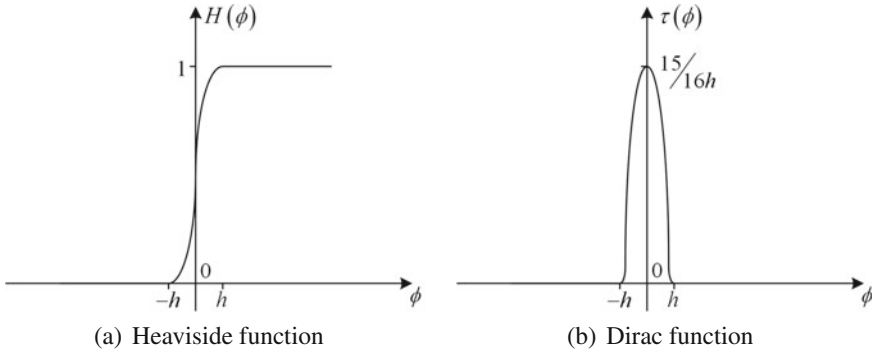


Fig. 2.26 The smoothed Heaviside and Dirac functions, where $h = 5 \times 10^{-5}$

where h is the support size and is chosen to be 5×10^{-5} in the numerical examples (Fig. 2.26).

2.2.4 Numerical Examples

Several numerical examples are presented for the optimization of unsteady Navier-Stokes flows, where the density and viscosity of fluid are set to be unity, the initial condition of Navier-Stokes equations is $\mathbf{u}_0 = \mathbf{0}$, the objective in Eq. 2.46 is fixed by setting

$$A = \frac{1}{2} \eta (\nabla \mathbf{u} + \nabla \mathbf{u}^T) : (\nabla \mathbf{u} + \nabla \mathbf{u}^T); \quad B = 1; \quad C = \begin{cases} p, & \text{on } \Gamma_i \\ 0, & \text{on } \partial\Omega \setminus \Gamma_i \end{cases} \quad (2.55)$$

where A is the dissipation of the flow [4, 26]; B is set to be 1 to limit the perimeter of the zero level set and ensure smoothness of the boundaries in the optimized result; and C is set to be p at the inlet Γ_i to minimize the pressure drop of flows in the case where the pressure is zero at the outlet.

2.2.4.1 Straight Channel

A straight channel, which has been investigated for the unsteady flow in [27], is optimized firstly to show the robustness of the optimization method. The optimization domain is shown in Fig. 2.27a, which is discretized by 100×100 rectangular elements. The initial distribution of the level set function and the corresponding zero level set are shown in Fig. 2.27b, c, respectively. The velocity imposed on the inlet Γ_i of the channel is

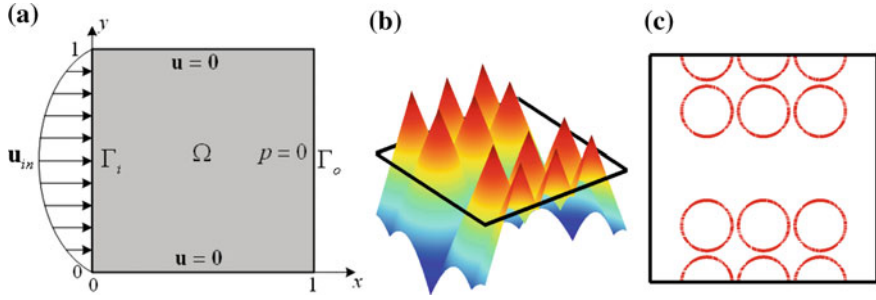


Fig. 2.27 **a** Design domain of the straight channel; **b** initial distribution of the level set function; **c** initial zero level set. $\mathbf{u}_{in}$ is the velocity distribution loaded on the inlet of the domain

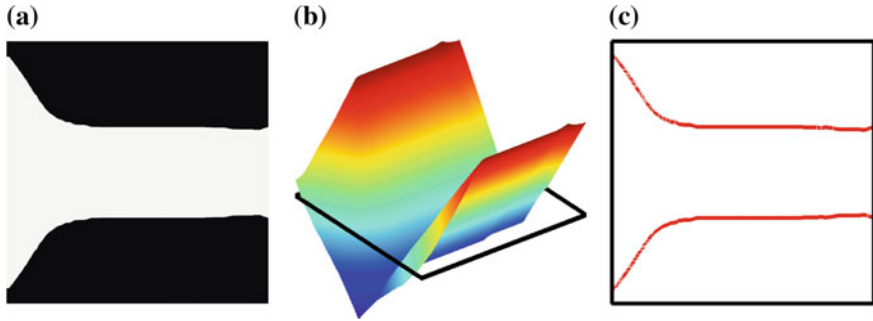


Fig. 2.28 **a** Optimized straight channel for the unsteady incompressible Navier-Stokes flow; **b** distribution of the level set function corresponding to the optimized straight channel; **c** zero level set corresponding to the optimized straight channel

$$\mathbf{u}_{in} = (4y(1-y)t, 0), \quad t \in [0, 1] \quad (2.56)$$

At the outlet Γ_o , the Neumann boundary condition in Eq. 2.44 is imposed by setting $p = 0$ and $\eta \nabla \mathbf{u} \cdot \mathbf{n} = \mathbf{0}$. The other boundaries are set to be no slip type. The volume fraction V_f is set as 0.4. The parameters β_1 , β_2 and β_3 in the Eq. 2.46 are set as 1, 0.1 and 0.1, respectively. By respectively choosing λ_0 and Λ_0 to be -2.25×10^2 and 1×10^{10} , the optimization problem is solved.

The optimized topology, the corresponding distribution of the level set function and zero level set are derived as shown in Fig. 2.28. Snapshots for the evolution of the zero level set are shown in Fig. 2.29. The convergent histories of the objective value and volume fraction are shown in Fig. 2.30. When the initial distribution of the level set function is chosen as shown in Fig. 2.31a, the optimized straight channel and corresponding zero level set are obtained as shown in Fig. 2.31b. And the snapshots for the evolution of the zero level set are shown in Fig. 2.32.

From Figs. 2.28 and 2.31b, one can see that the obtained results have the same topology with the example of the straight channel in [27]. And the results in Figs. 2.28 and 2.31b illustrate that the proposed method has relatively weak dependence on

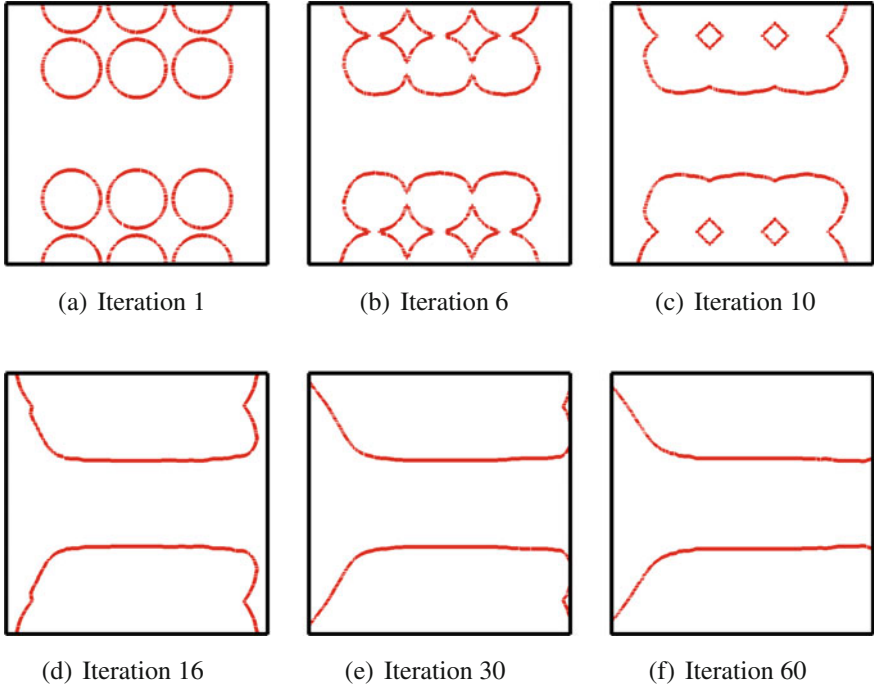


Fig. 2.29 Snapshots for the evolution of the zero level set corresponding to the result shown in Fig. 2.28

the choice of the initial distribution of the level set function. This example on the optimization of the straight channel for the unsteady flow demonstrates that this method can not only change the shape but also the topology of the fluid domain, and the topology optimization of the unsteady Navier-Stokes flows can be achieved using the variational level set method.

2.2.4.2 A Double Pipe

A double pipe is designed in this numerical example. The design domain is shown in Fig. 2.33a, which is discretized by 120×120 rectangular elements. The initial distribution of the level set function and the corresponding zero level set are shown in Figs. 2.33b, c, respectively. The parameters β_1 , β_2 and β_3 in Eq. 2.46 are set to be 1, 0.1 and 0, respectively. The volume fraction V_r is set as $1/3$. The Neumann boundary condition in Eq. 2.44 is imposed on the outlets Γ_{o1} and Γ_{o2} by setting $\mathbf{g} = \mathbf{0}$. By solving the optimization problem with λ_0 and Λ_0 respectively chosen to be -2.75×10^2 and 1×10^{10} , the optimized design of the double pipe is obtained as shown in Fig. 2.34a, where the velocity at the inlets is

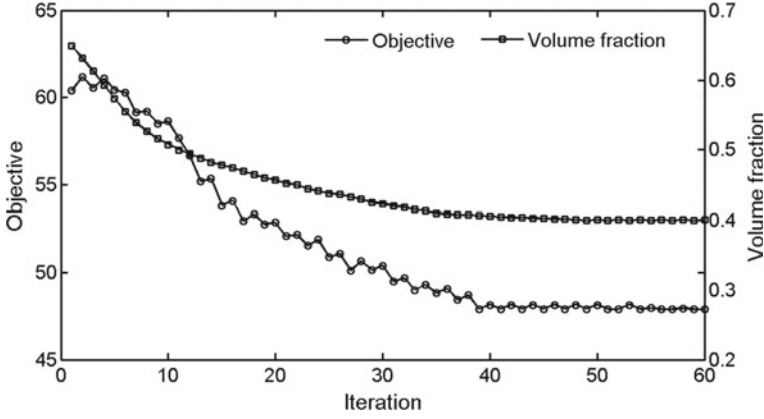


Fig. 2.30 Convergent histories of the objective value and volume fraction corresponding to the optimized straight channel in Fig. 2.28

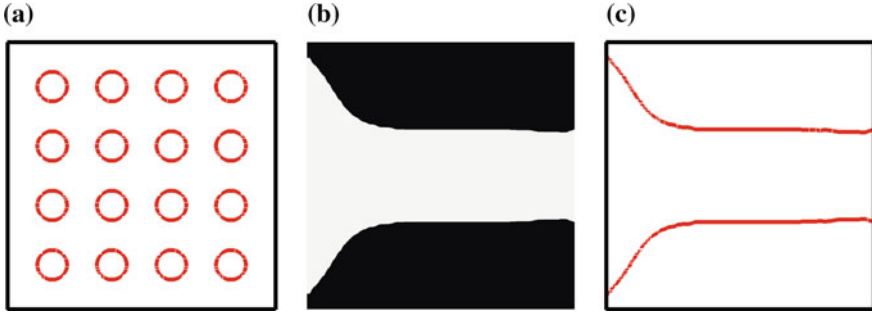


Fig. 2.31 **a** Initial distribution of the level set function; **b** optimized straight channel and the corresponding zero level set

$$\begin{aligned} \mathbf{u}_{in1} &= -144(y - 4/6)(5/6 - y)\mathbf{n} \\ \mathbf{u}_{in2} &= -144(y - 1/6)(2/6 - y)\mathbf{n} \end{aligned} \quad (2.57)$$

This optimized design agrees with the numerical example of a double pipe carried out by Borrvall and Petersson [2] and Challis and Guest in [19] for the steady Stokes flows. When the following time dependent boundary conditions

$$\begin{aligned} \mathbf{u}_{in1} &= -144(y - 4/6)(5/6 - y)\sin(t)\mathbf{n} \\ \mathbf{u}_{in2} &= -144(y - 1/6)(2/6 - y)\cos(t)\mathbf{n} \end{aligned} \quad (2.58)$$

which have the same maximum of the velocity as the corresponding steady case, are imposed on the inlets of the design domain, the optimized design is obtained as shown in Fig. 2.34b, where the time interval is $[0, \pi]$.

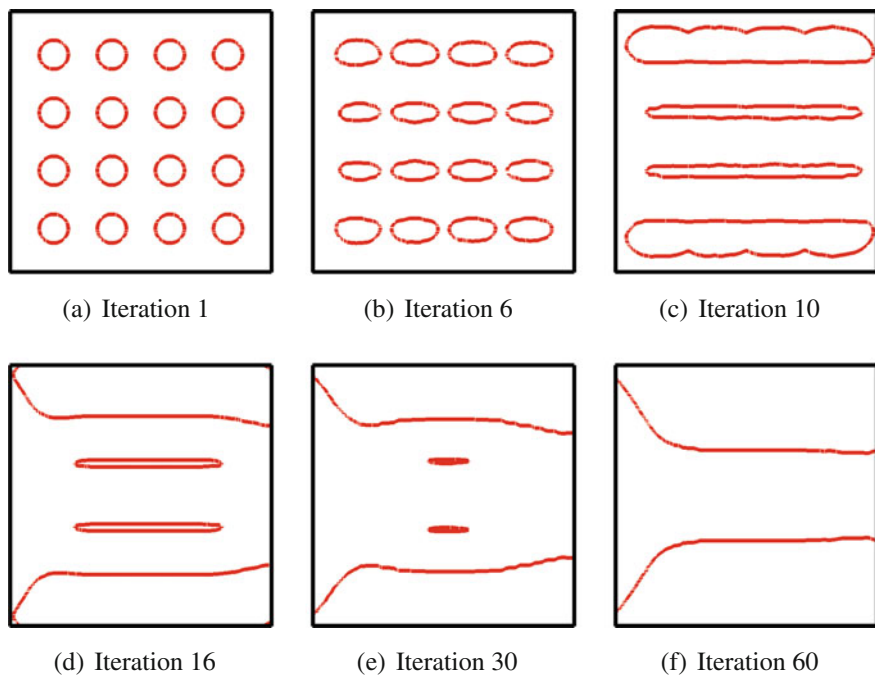


Fig. 2.32 Snapshots for the evolution of the zero level set corresponding to the result shown in Fig. 2.31b

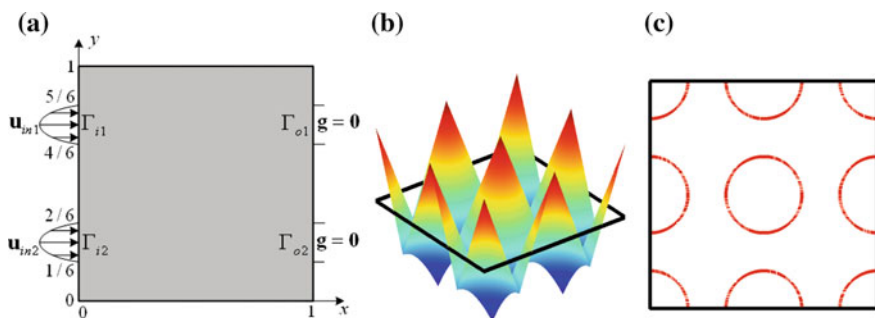


Fig. 2.33 **a** Design domain of the double pipe, where $\mathbf{u}_{in1}$ and $\mathbf{u}_{in2}$ are the velocity distribution at the inlets; **b** initial distribution of the level set function; **c** initial zero level set

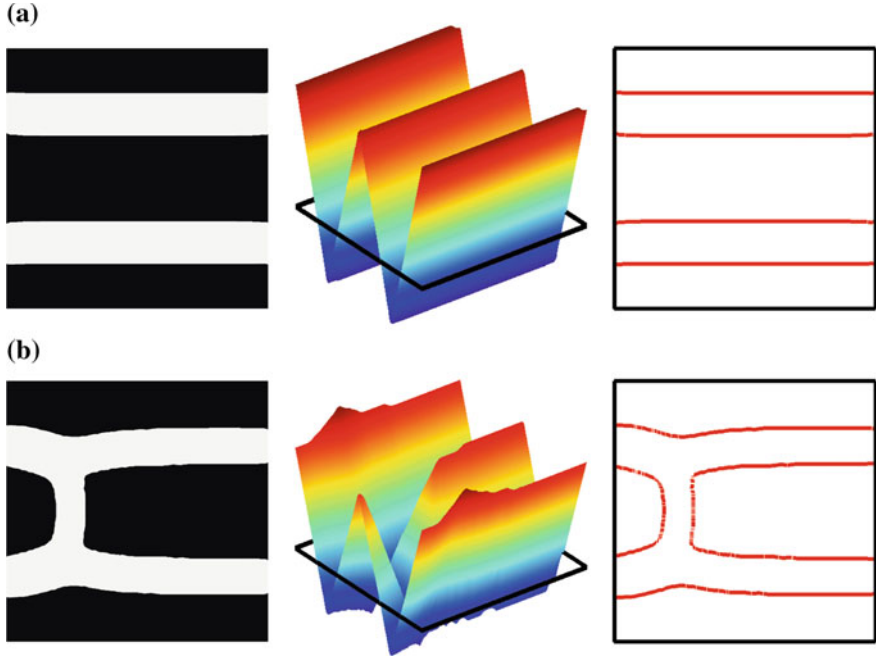


Fig. 2.34 The optimized shape, the level set function and the corresponding zero level set for a double pipe corresponding to the steady flow (a) and unsteady flow (b)

In the obtained optimized double pipe for the unsteady flow, the streamline distribution corresponding to different time is shown in Fig. 2.35. To minimize the dissipation of the flow in the double pipe, the fluid flows between one inlet and two outlets at the time 0 , $\pi/2$ and π , which correspond to $\mathbf{u}_{in1} = \mathbf{0}$ or $\mathbf{u}_{in2} = \mathbf{0}$; the fluid is prone to flow in parallel at the time $3\pi/4$; and the fluid is prone to flow from the port Γ_{i1} to the port Γ_{i2} directly at the time $3\pi/4$. Therefore, the two pipes joint together in the unsteady case.

By imposing the velocity in Eqs. 2.57 and 2.58 on the inlets of the optimized results in Figs. 2.34a, b respectively, the values of objective in Eq. 2.46 are computed (Table 2.10). The cross comparison of the objective values in Table 2.10 and the results in Fig. 2.34 demonstrates that the optimized topologies are different corresponding to the steady and unsteady flows in the double pipe.

Being different from steady flows, unsteady flows have dynamic effect. Figure 2.34 shows that the dynamic effect of the unsteady flow influences the optimized topology of the flow. This further demonstrates the significance on extending the level set method-based topology optimization to the unsteady Navier-Stokes flows.

When the initial distribution of the level set function is chosen as shown in Fig. 2.36a, the optimized double pipe and corresponding zero level set are obtained as shown in Fig. 2.36b. The comparison of the results in Figs. 2.34b and 2.36b

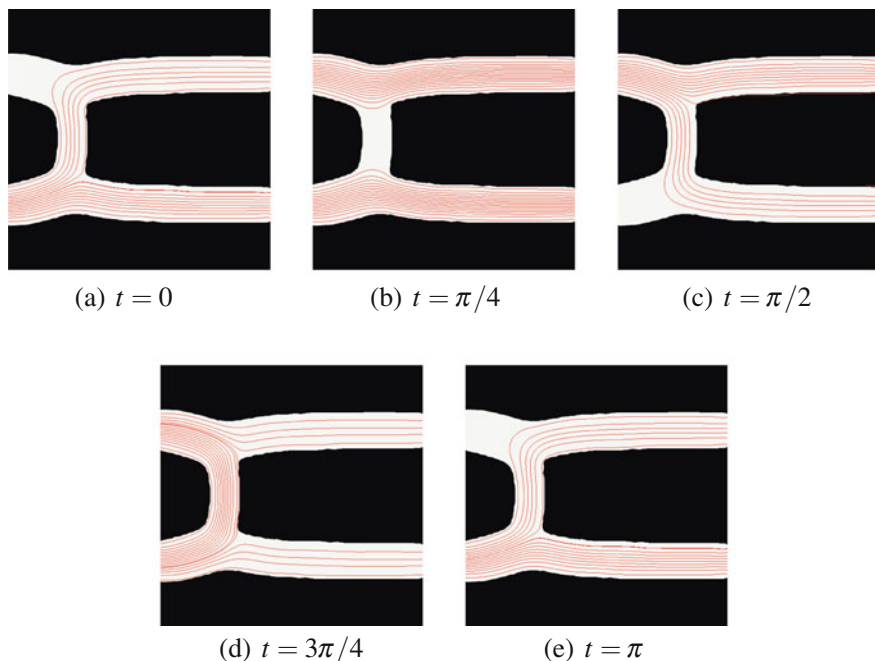


Fig. 2.35 Streamline distribution in the optimized double pipe for the unsteady flow

Table 2.10 Objective values obtained by imposing the velocity in Eqs. 2.57 and 2.58 at the inlets of the optimized results in Fig. 2.34

	Steady	Unsteady
Figure 2.34a	54.77	88.72
Figure 2.34b	59.93	85.49

demonstrates that the proposed method has weak dependence on the initial distribution of the level set function, furthermore.

2.2.5 Summary

The level set method-based topology optimization of the unsteady Navier-Stokes flow can be achieved using the variational level set method. The solid-fluid interface and the zero velocity condition in the solid domain has been expressed implicitly, by coupling the level set method with the unsteady Navier-Stokes equations. The topology optimization problem is analyzed using the Lagrangian-based adjoint approach,

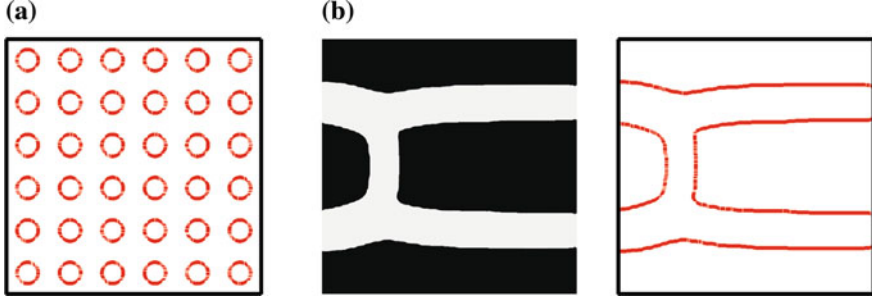


Fig. 2.36 **a** Initial distribution of the level set function; **b** optimized double pipe and the corresponding zero level set

with deriving the continuous adjoint equations and shape sensitivity. The level set function is evolved by solving the Hamilton-Jacobian equation.

2.3 Appendix for Section 2.1

2.3.1 Adjoint Equations in Density Method-Based Topology Optimization

To analyze the topology optimization problem in Eq. 2.7, the functional spaces for variables are chosen to be

$$\begin{aligned} \mathbf{u} &\in \mathcal{U}_{\Omega}^d := (\mathcal{L}^2((0, T); \mathcal{H}(\Omega)))^d; \quad p \in \mathcal{P}_{\Omega} := \mathcal{L}^2((0, T); \mathcal{L}^2(\Omega)) \\ \mathbf{f} &\in \mathcal{F}_{\Omega}^{d*} := (\mathcal{L}^2((0, T); \mathcal{H}^*(\Omega)))^d; \quad \mathbf{u}_0 \in (\mathcal{H}_0^1(\Omega))^d; \quad \gamma \in \mathcal{L}^1(\Omega) \end{aligned} \quad (2.59)$$

where d is the spatial dimension; $\mathcal{L}^1(\Omega)$ and $\mathcal{L}^2(\Omega)$ are the first-order and second-order integrable Lebesgue spaces respectively; $\mathcal{H}^*(\Omega)$ is the dual space of the first order Hilbert space $\mathcal{H}(\Omega)$; and $(\mathcal{H}_0^1(\Omega))^d = \{\mathbf{u} \in (\mathcal{H}(\Omega))^d \mid \nabla \cdot \mathbf{u} = 0\}$. According to the Rietz representation theorem and the Hölder inequality [28], it is known that the Bochner space V_{Ω} is reflexive, i.e., $\mathcal{V}_{\Omega}^* = \mathcal{V}_{\Omega}$. Based on the adjoint analysis method, the sensitivity analysis of the optimization problem in Eq. 2.7 can be derived as follows: based on [29]

$$\left\langle u^*, \int_0^T f(t) dt \right\rangle_{\mathcal{X}^*, \mathcal{X}} = \int_0^T \langle u^*, f(t) \rangle_{\mathcal{X}^*, \mathcal{X}} dt \quad (2.60)$$

where $\mathcal{X}$ is a Bochner space, and its dual is $\mathcal{X}^*$, $u^* \in \mathcal{X}^*$, $f : (0, T) \rightarrow \mathcal{X}$ is Bochner integrable, then the integration on time and on space can change sequence.

Therefore,

$$\begin{aligned}
 e(\mathbf{u}, p; \gamma) = & \int_0^T \int_{\Omega} \left[\rho \frac{\partial \mathbf{u}}{\partial t} - \eta \nabla \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \mathbf{f} \right] \cdot \mathbf{v} \, d\Omega \, dt \\
 & - \int_0^T \int_{\Omega} q \nabla \cdot \mathbf{u} \, d\Omega \, dt + \int_0^T \int_{\Gamma_D} \mathbf{u} \cdot \mathbf{v} \, d\Gamma \, dt \\
 & + \int_0^T \int_{\Gamma_N} \left[-p \mathbf{I} + \eta (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \right] \mathbf{n} \cdot \mathbf{v} \, d\Gamma \, dt + \int_{\Omega} (\mathbf{u} \cdot \mathbf{v})|_{t=0} \, d\Omega
 \end{aligned} \tag{2.61}$$

for $\forall \mathbf{v} \in \mathcal{U}_{\Omega}^d$ and $\forall q \in \mathcal{P}_{\Omega}$. Based on Eq. 2.60, the following transformation of Eq. 2.61 is derived by partial integration on time:

$$\begin{aligned}
 e(\mathbf{u}, p; \gamma) = & \rho \int_{\Omega} \int_0^T \left[\frac{\partial (\mathbf{u} \cdot \mathbf{v})}{\partial t} - \mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial t} \right] \, dt \, d\Omega \\
 & + \eta \int_0^T \int_{\Omega} [(\nabla \mathbf{u} + \nabla \mathbf{u}^T) : \nabla \mathbf{v} - \nabla \cdot ((\nabla \mathbf{u} + \nabla \mathbf{u}^T) \cdot \mathbf{v})] \, d\Omega \, dt \\
 & + \int_0^T \int_{\Omega} \rho (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \, d\Omega \, dt + \int_0^T \int_{\Omega} [\nabla \cdot (p \mathbf{v}) - p \nabla \cdot \mathbf{v}] \, d\Omega \, dt \\
 & - \int_0^T \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\Omega \, dt - \int_0^T \int_{\Omega} q \nabla \cdot \mathbf{u} \, d\Omega \, dt \\
 & + \int_0^T \int_{\Gamma_N} [-p \mathbf{I} + \eta (\nabla \mathbf{u} + \nabla \mathbf{u}^T)] \mathbf{n} \cdot \mathbf{v} \, d\Gamma \, dt + \int_{\Omega} (\mathbf{u} \cdot \mathbf{v})|_{t=0} \, d\Omega
 \end{aligned} \tag{2.62}$$

Based on Gauss theory [5], Eq. 2.62 can be transformed into

$$\begin{aligned}
 e(\mathbf{u}, p; \gamma) = & \rho \int_{\Omega} [(\mathbf{u} \cdot \mathbf{v})|_{t=T} - (\mathbf{u} \cdot \mathbf{v})|_{t=0}] \, d\Omega - \rho \int_0^T \int_{\Omega} \mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial t} \, d\Omega \, dt \\
 & + \eta \int_0^T \int_{\Omega} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) : \nabla \mathbf{v} \, d\Omega \, dt + \int_0^T \int_{\Omega} \rho (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \, d\Omega \, dt \\
 & - \int_0^T \int_{\Omega} p \nabla \cdot \mathbf{v} \, d\Omega \, dt - \int_0^T \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\Omega \, dt \\
 & + \int_0^T \int_{\partial \Omega} [p \mathbf{I} - \eta (\nabla \mathbf{u} + \nabla \mathbf{u}^T)] \mathbf{n} \cdot \mathbf{v} \, d\Gamma \, dt - \int_0^T \int_{\Omega} q \nabla \cdot \mathbf{u} \, d\Omega \, dt \\
 & + \int_0^T \int_{\Gamma_D} \mathbf{u} \cdot \mathbf{v} \, d\Gamma \, dt + \int_0^T \int_{\Gamma_N} [-p \mathbf{I} + \eta (\nabla \mathbf{u} + \nabla \mathbf{u}^T)] \mathbf{n} \cdot \mathbf{v} \, d\Gamma \, dt \\
 & + \int_{\Omega} (\mathbf{u} \cdot \mathbf{v})|_{t=0} \, d\Omega
 \end{aligned} \tag{2.63}$$

By inserting the boundary conditions into Eq. 2.63, the reduced weak operator of transient Navier-Stokes equations is obtained as

$$\begin{aligned}
e(\mathbf{u}, p; \gamma) = & \rho \int_{\Omega} [(\mathbf{u} \cdot \mathbf{v})|_{t=T} - \mathbf{u}_0 \cdot \mathbf{v}|_{t=0}] d\Omega - \rho \int_0^T \int_{\Omega} \mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial t} d\Omega dt \\
& + \eta \int_0^T \int_{\Omega} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) : \nabla \mathbf{v} d\Omega dt + \int_0^T \int_{\Omega} \rho (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} d\Omega dt \\
& - \int_0^T \int_{\Omega} p \nabla \cdot \mathbf{v} d\Omega dt - \int_0^T \int_{\Omega} \mathbf{f} \cdot \mathbf{v} d\Omega dt - \int_0^T \int_{\Omega} q \nabla \cdot \mathbf{u} d\Omega dt \\
& + \int_0^T \int_{\Gamma_D} [p \mathbf{I} - \eta (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \mathbf{n} \cdot \mathbf{v}] \Big|_{\mathbf{u}=\mathbf{u}_D} d\Gamma dt + \int_0^T \int_{\Gamma_D} \mathbf{u}_D \cdot \mathbf{v} d\Gamma dt
\end{aligned} \tag{2.64}$$

According to the definition of the Gâteaux derivative [5], the Gâteaux derivatives of Eq. 2.92 in the direction $(\mathbf{w}, r) \in \mathcal{U}_{\Omega}^d \times \mathcal{P}_{\Omega}$ are

$$\begin{aligned}
& \langle e_{\mathbf{u}}(\mathbf{u}, p; \gamma), \mathbf{w} \rangle_{\mathcal{U}_{\Omega}^{d*}, \mathcal{U}_{\Omega}^d} \\
& = \lim_{l \rightarrow 0^+} \frac{e(\mathbf{u} + l\mathbf{w}, p) - e(\mathbf{u}, p)}{l} \\
& = \rho \int_{\Omega} (\mathbf{w} \cdot \mathbf{v})|_{t=T} d\Omega - \rho \int_0^T \int_{\Omega} \mathbf{w} \cdot \frac{\partial \mathbf{v}}{\partial t} dt d\Omega + \eta \int_0^T \int_{\Omega} (\nabla \mathbf{w} + \nabla \mathbf{w}^T) : \nabla \mathbf{v} d\Omega dt \\
& \quad + \int_0^T \int_{\Omega} \rho (\mathbf{w} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} d\Omega dt + \int_0^T \int_{\Omega} \rho (\mathbf{u} \cdot \nabla) \mathbf{w} \cdot \mathbf{v} d\Omega dt \\
& \quad - \int_0^T \int_{\Omega} \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \mathbf{w} \cdot \mathbf{v} d\Omega dt - \int_0^T \int_{\Omega} q \nabla \cdot \mathbf{w} d\Omega dt
\end{aligned} \tag{2.65}$$

and

$$\begin{aligned}
& \langle e_p(\mathbf{u}, p; \gamma), r \rangle_{\mathcal{P}_{\Omega}^*, \mathcal{P}_{\Omega}} \\
& = \lim_{l \rightarrow 0^+} \frac{e(\mathbf{u}, p + lr) - e(\mathbf{u}, p)}{l} \\
& = - \int_0^T \int_{\Omega} r \nabla \cdot \mathbf{v} d\Omega dt - \int_0^T \int_{\Omega} \frac{\partial \mathbf{f}}{\partial p} r \cdot \mathbf{v} d\Omega dt + \int_0^T \int_{\Gamma_D} r \mathbf{n} \cdot \mathbf{v} d\Gamma dt
\end{aligned} \tag{2.66}$$

Based on the linear property of operators $\langle e_{\mathbf{u}}(\mathbf{u}, p; \gamma), \mathbf{w} \rangle_{\mathcal{U}_{\Omega}^{d*}, \mathcal{U}_{\Omega}^d}$ and $\langle e_p(\mathbf{u}, p; \gamma), r \rangle_{\mathcal{P}_{\Omega}^*, \mathcal{P}_{\Omega}}$, they are bounded based on the Cauchy-Schwarz inequality and the Poincaré's inequality. Therefore, Eqs. 2.65 and 2.66 are the Gâteaux derivatives of $e(\mathbf{u}, p)$. The dual operator of a linear operator is defined as [5]

$$\langle D^* u, v \rangle_{\mathcal{X}^*, \mathcal{X}} = \langle u, Dv \rangle_{\mathcal{Y}^*, \mathcal{Y}}, \quad \forall u \in \mathcal{Y}^*, v \in \mathcal{X} \tag{2.67}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $D \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $D^* \in \mathcal{L}(\mathcal{Y}^*, \mathcal{X}^*)$. By rewriting $e_{\mathbf{u}}(\mathbf{u}, p; \gamma)$ and $e_p(\mathbf{u}, p; \gamma)$ in the formulation of linear operators, one can derive

$$\begin{aligned} \langle e_{\mathbf{u}}(\mathbf{u}, p; \gamma), \mathbf{w} \rangle_{\mathcal{U}_{\Omega}^{d*}, \mathcal{U}_{\Omega}^d} &= \langle A_1 \mathbf{v} + A_2 q, \mathbf{w} \rangle_{\mathcal{U}_{\Omega}^{d*}, \mathcal{U}_{\Omega}^d} \\ \langle e_p(\mathbf{u}, p; \gamma), r \rangle_{\mathcal{P}_{\Omega}^*, \mathcal{P}_{\Omega}} &= \langle A_3 \mathbf{v}, r \rangle_{\mathcal{P}_{\Omega}^*, \mathcal{P}_{\Omega}} \end{aligned} \quad (2.68)$$

and

$$\begin{aligned} \langle e_{\mathbf{u}}^*(\mathbf{u}_a, p_a; \gamma), \mathbf{w} \rangle_{\mathcal{U}_{\Omega}^{d*}, \mathcal{U}_{\Omega}^d} &= \langle A_1 \mathbf{u}_a + A_2 p_a, \mathbf{w} \rangle_{\mathcal{U}_{\Omega}^{d*}, \mathcal{U}_{\Omega}^d} \\ \langle e_p^*(\mathbf{u}_a, p_a; \gamma), r \rangle_{\mathcal{P}_{\Omega}^*, \mathcal{P}_{\Omega}} &= \langle A_3 \mathbf{u}_a, r \rangle_{\mathcal{P}_{\Omega}^*, \mathcal{P}_{\Omega}} \end{aligned} \quad (2.69)$$

where $\mathbf{u}_a \in \mathcal{U}_{\Omega}^d$ and $p_a \in \mathcal{P}_{\Omega}$, A_1 , A_2 and A_3 are linear operators. According to Eqs. 2.11 and 2.69, the weak form of the adjoint equations for the topology optimization problem of unsteady Navier-Stokes flows in Eq. 2.7 can be derived as

$$\begin{aligned} &\rho \int_{\Omega} (\mathbf{w} \cdot \mathbf{u}_a) \big|_{t=T} d\Omega - \rho \int_0^T \int_{\Omega} \mathbf{w} \cdot \frac{\partial \mathbf{u}_a}{\partial t} d\Omega dt + \eta \int_0^T \int_{\Omega} (\nabla \mathbf{w} + \nabla \mathbf{w}^T) : \nabla \mathbf{u}_a d\Omega dt \\ &+ \int_0^T \int_{\Omega} \rho (\mathbf{w} \cdot \nabla) \mathbf{u} \cdot \mathbf{u}_a d\Omega dt + \int_0^T \int_{\Omega} \rho (\mathbf{u} \cdot \nabla) \mathbf{w} \cdot \mathbf{u}_a d\Omega dt - \int_0^T \int_{\Omega} p_a \nabla \cdot \mathbf{w} d\Omega dt \\ &- \int_0^T \int_{\Omega} \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \mathbf{w} \cdot \mathbf{u}_a d\Omega dt = - \langle J_{\mathbf{u}}(\mathbf{u}, \nabla \mathbf{u}, p; \gamma), \mathbf{w} \rangle_{\mathcal{U}_{\Omega}^{d*}, \mathcal{U}_{\Omega}^d} \end{aligned} \quad (2.70)$$

and

$$\begin{aligned} &- \int_0^T \int_{\Omega} r \nabla \cdot \mathbf{u}_a d\Omega dt - \int_0^T \int_{\Omega} \frac{\partial \mathbf{f}}{\partial p} r \cdot \mathbf{u}_a d\Omega dt + \int_0^T \int_{\Gamma_D} r \mathbf{n} \cdot \mathbf{u}_a d\Gamma dt \\ &= - \langle J_p(\mathbf{u}, \nabla \mathbf{u}, p; \gamma), r \rangle_{\mathcal{P}_{\Omega}^*, \mathcal{P}_{\Omega}} \end{aligned} \quad (2.71)$$

Additionally, there are

$$\begin{aligned} &\int_0^T \int_{\Omega} (\nabla \mathbf{w} + \nabla \mathbf{w}^T) : \nabla \mathbf{u}_a d\Omega dt \\ &= - \int_0^T \int_{\Omega} \nabla \cdot (\nabla \mathbf{u}_a + \nabla \mathbf{u}_a^T) \cdot \mathbf{w} d\Omega dt + \int_0^T \int_{\Gamma_N} (\nabla \mathbf{u}_a + \nabla \mathbf{u}_a^T) \mathbf{n} \cdot \mathbf{w} d\Gamma dt \end{aligned} \quad (2.72)$$

$$\int_0^T \int_{\Omega} p_a \nabla \cdot \mathbf{w} d\Omega dt = - \int_0^T \int_{\Omega} \nabla p_a \cdot \mathbf{w} d\Omega dt + \int_0^T \int_{\Gamma_N} p_a \mathbf{n} \cdot \mathbf{w} d\Gamma dt \quad (2.73)$$

and

$$\begin{aligned}
\int_0^T \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{w} \cdot \mathbf{u}_a \, d\Omega \, dt &= - \int_0^T \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u}_a \cdot \mathbf{w} \, d\Omega \, dt \\
&+ \int_0^T \int_{\Gamma_N} (\mathbf{u} \cdot \mathbf{n}) \mathbf{u}_a \cdot \mathbf{w} \, d\Gamma \, dt
\end{aligned} \tag{2.74}$$

Because $\mathbf{u}$ is known on Σ_D and p can be expressed by $\nabla \mathbf{u}$ on Σ_N , according to Eq. 2.4

$$B(\mathbf{u}, p; \gamma) = \begin{cases} B(p; \gamma), & \text{on } \Sigma_D \\ B(\mathbf{u}, \nabla \mathbf{u}; \gamma), & \text{on } \Sigma_N \end{cases} \tag{2.75}$$

By inserting Eqs. 2.72–2.75 into Eqs. 2.70 and 2.71, one can obtain

$$\begin{aligned}
&\langle J_{\mathbf{u}}(\mathbf{u}, p; \gamma), \mathbf{w} \rangle_{\mathcal{U}_{\Omega}^{d*}, \mathcal{U}_{\Omega}^d} \\
&= \int_0^T \int_{\Omega} \beta_1 \left[\frac{\partial A}{\partial \mathbf{u}} \cdot \mathbf{w} - \left(\nabla \cdot \frac{\partial A}{\partial \nabla \mathbf{u}} \right) \cdot \mathbf{w} \right] d\Omega \, dt \\
&+ \int_0^T \int_{\Gamma_N} \beta_1 \frac{\partial A}{\partial \nabla \mathbf{u}} \mathbf{n} \cdot \mathbf{w} \, d\Gamma \, dt + \int_0^T \int_{\Gamma_N} \beta_2 \frac{\partial B}{\partial \mathbf{u}} \cdot \mathbf{w} \, d\Gamma \, dt
\end{aligned} \tag{2.76}$$

and

$$\langle J_p(\mathbf{u}, p; \gamma), r \rangle_{\mathcal{P}_{\Omega}^*, \mathcal{P}_{\Omega}} = \int_0^T \int_{\Omega} \beta_1 \frac{\partial A}{\partial p} r \, d\Omega \, dt + \int_0^T \int_{\Gamma_D} \beta_2 \frac{\partial B}{\partial p} r \, d\Gamma \, dt \tag{2.77}$$

Therefore, the adjoint equations of the Navier-Stokes equations can be written as

$$\begin{aligned}
-\rho \frac{\partial \mathbf{u}_a}{\partial t} - \eta \nabla \cdot (\nabla \mathbf{u}_a + \nabla \mathbf{u}_a^T) - \rho (\mathbf{u} \cdot \nabla) \mathbf{u}_a + \rho (\nabla \mathbf{u}) \cdot \mathbf{u}_a + \nabla p_a \\
&= -\beta_1 \left(\frac{\partial A}{\partial \mathbf{u}} - \nabla \cdot \frac{\partial A}{\partial \nabla \mathbf{u}} \right) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \mathbf{u}_a, \text{ in } Q \\
-\nabla \cdot \mathbf{u}_a &= -\beta_1 \frac{\partial A}{\partial p} + \frac{\partial \mathbf{f}}{\partial p} \cdot \mathbf{u}_a, \text{ in } Q \\
\mathbf{u}_a(T, \mathbf{x}) &= \mathbf{0}, \text{ in } \Omega \\
\mathbf{u}_a &= -\frac{\partial B}{\partial p} \mathbf{n}, \text{ on } \Sigma_D \\
[-p_a \mathbf{I} + \eta (\nabla \mathbf{u}_a + \nabla \mathbf{u}_a^T)] \mathbf{n} &= -\rho (\mathbf{u} \cdot \mathbf{n}) \mathbf{u}_a - \beta_1 \frac{\partial A}{\partial \nabla \mathbf{u}} \mathbf{n} - \beta_2 \frac{\partial B}{\partial \mathbf{u}}, \text{ on } \Sigma_N
\end{aligned} \tag{2.78}$$

2.3.2 Adjoint Sensitivity of Density Method-Based Optimization Problem

2.3.2.1 For Expression Defined on $[0, T]$

Based on the similar analysis of the dual operator of $e_\gamma(\mathbf{u}, p; \gamma)$, the adjoint derivatives of the optimization problem can be obtained as

$$\begin{aligned}
 & \left\langle \frac{D\hat{J}}{D\gamma}, \psi \right\rangle_{\mathcal{L}^\infty(\Omega), \mathcal{L}^1(\Omega)} \\
 &= \int_0^T \int_\Omega \left(\beta_1 \frac{\partial A}{\partial \gamma} - \frac{\partial \mathbf{f}}{\partial \gamma} \cdot \mathbf{u}_a \right) \psi \, d\Omega \, dt \\
 &= \int_\Omega \int_0^T \left(\beta_1 \frac{\partial A}{\partial \gamma} - \frac{\partial \mathbf{f}}{\partial \gamma} \cdot \mathbf{u}_a \right) \psi \, dt \, d\Omega \\
 &= \int_\Omega \left[\int_0^T \left(\beta_1 \frac{\partial A}{\partial \gamma} - \frac{\partial \mathbf{f}}{\partial \gamma} \cdot \mathbf{u}_a \right) dt \right] \psi \, d\Omega
 \end{aligned} \tag{2.79}$$

and

$$\begin{aligned}
 & \left\langle \frac{D\hat{J}}{D\gamma}, \psi \right\rangle_{\mathcal{L}^\infty(\partial\Omega), \mathcal{L}^1(\partial\Omega)} \\
 &= \int_0^T \int_{\partial\Omega} \beta_2 \frac{\partial B}{\partial \gamma} \psi \, d\Gamma \, dt \\
 &= \int_{\partial\Omega} \left[\int_0^T \beta_2 \frac{\partial B}{\partial \gamma} dt \right] \psi \, d\Gamma
 \end{aligned} \tag{2.80}$$

where $\forall \psi \in \mathcal{C}^\infty(\bar{\Omega})$. Therefore, according to Eq. 2.12, the adjoint derivatives can be expressed as

$$\begin{aligned}
 \frac{D\hat{J}}{D\gamma} \Big|_\Omega &= \int_0^T \left(\beta_1 \frac{\partial A}{\partial \gamma} + \frac{\partial \alpha}{\partial \gamma} \mathbf{u} \cdot \mathbf{u}_a \right) dt, \quad \text{in } \Omega \\
 \frac{D\hat{J}}{D\gamma} \Big|_{\partial\Omega} &= \int_0^T \beta_2 \frac{\partial B}{\partial \gamma} dt, \quad \text{on } \partial\Omega
 \end{aligned} \tag{2.81}$$

2.3.2.2 For Expression Defined on $[T_1, T_2] \subseteq [0, T]$

In some cases, the expression in Eq. 2.8 is defined on the subset $[T_1, T_2]$ instead of $[0, T]$:

$$\begin{aligned}
J(\mathbf{u}, p; \gamma) &= \int_{T_1}^{T_2} \int_{\Omega} \beta_1 A(\mathbf{u}, \nabla \mathbf{u}, p; \gamma) \, d\Omega \, dt \\
&\quad + \int_{T_1}^{T_2} \int_{\partial\Omega} \beta_2 B(\mathbf{u}, p; \gamma) \, d\Gamma \, dt
\end{aligned} \tag{2.82}$$

where $[T_1, T_2]$ is a subset of $[0, T]$. Then Eq. 2.82 can be rewritten as

$$\begin{aligned}
J(\mathbf{u}, p; \gamma) &= \int_0^{T_1} \int_{\Omega} 0 \, d\Omega \, dt + \int_0^{T_1} \int_{\partial\Omega} 0 \, d\Gamma \, dt \\
&\quad + \int_{T_1}^{T_2} \int_{\Omega} \beta_1 A(\mathbf{u}, \nabla \mathbf{u}, p; \gamma) \, d\Omega \, dt + \int_{T_1}^{T_2} \int_{\partial\Omega} \beta_2 B(\mathbf{u}, p; \gamma) \, d\Gamma \, dt \\
&\quad + \int_{T_2}^T \int_{\Omega} 0 \, d\Omega \, dt + \int_{T_2}^T \int_{\partial\Omega} 0 \, d\Gamma \, dt
\end{aligned} \tag{2.83}$$

where A and B are set to 0 in $(0, T_1)$ and (T_1, T_2) , respectively. Then Eqs. 2.76 and 2.77 can be rewritten as

$$\begin{aligned}
&\langle J_{\mathbf{u}}(\mathbf{u}, p; \gamma), \mathbf{w} \rangle_{\mathcal{U}_{\Omega}^{d*}, \mathcal{U}_{\Omega}^d} \\
&= \int_0^{T_1} \int_{\Omega} 0 \, d\Omega \, dt + \int_0^{T_1} \int_{\Gamma_N} 0 \, d\Gamma \, dt + \int_0^{T_1} \int_{\Gamma_N} 0 \, d\Gamma \, dt \\
&\quad + \int_{T_1}^{T_2} \int_{\Omega} \beta_1 \left[\frac{\partial A}{\partial \mathbf{u}} \cdot \mathbf{w} - \left(\nabla \cdot \frac{\partial A}{\partial \nabla \mathbf{u}} \right) \cdot \mathbf{w} \right] \, d\Omega \, dt + \int_{T_1}^{T_2} \int_{\Gamma_N} \beta_1 \frac{\partial A}{\partial \nabla \mathbf{u}} \mathbf{n} \cdot \mathbf{w} \, d\Gamma \, dt \\
&\quad + \int_{T_1}^{T_2} \int_{\Gamma_N} \beta_2 \frac{\partial B}{\partial \mathbf{u}} \cdot \mathbf{w} \, d\Gamma \, dt + \int_{T_2}^T \int_{\Omega} 0 \, d\Omega \, dt + \int_{T_2}^T \int_{\Gamma_N} 0 \, d\Gamma \, dt + \int_{T_2}^T \int_{\Gamma_N} 0 \, d\Gamma \, dt
\end{aligned} \tag{2.84}$$

and

$$\begin{aligned}
&\langle J_p(\mathbf{u}, p; \gamma), r \rangle_{\mathcal{P}_{\Omega}^*, \mathcal{P}_{\Omega}} \\
&= \int_0^{T_1} \int_{\Omega} 0 \, d\Omega \, dt + \int_{T_1}^{T_2} \int_{\Omega} \beta_1 \frac{\partial A}{\partial p} r \, d\Omega \, dt + \int_{T_2}^T \int_{\Omega} 0 \, d\Omega \, dt \\
&\quad + \int_0^{T_1} \int_{\Gamma_D} 0 \, d\Gamma \, dt + \int_{T_1}^{T_2} \int_{\Gamma_D} \beta_2 \frac{\partial B}{\partial p} r \, d\Gamma \, dt + \int_{T_2}^T \int_{\Gamma_D} 0 \, d\Gamma \, dt
\end{aligned} \tag{2.85}$$

Therefore, the adjoint equations of the Navier-Stokes equations can be written as
When $t \in (T_2, T)$

$$\begin{aligned}
& -\rho \frac{\partial \mathbf{u}_a}{\partial t} - \eta \nabla \cdot (\nabla \mathbf{u}_a + \nabla \mathbf{u}_a^T) - \rho (\mathbf{u} \cdot \nabla) \mathbf{u}_a + \rho (\nabla \mathbf{u}) \cdot \mathbf{u}_a \\
& \quad + \nabla p_a = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \mathbf{u}_a, \text{ in } (T_2, T) \times \Omega \\
& \quad - \nabla \cdot \mathbf{u}_a = \frac{\partial \mathbf{f}}{\partial p} \cdot \mathbf{u}_a, \text{ in } (T_2, T) \times \Omega \\
& \quad \mathbf{u}_a(T, \mathbf{x}) = \mathbf{0}, \text{ in } \Omega \\
& \quad \mathbf{u}_a = \mathbf{0}, \text{ on } (T_2, T) \times \Gamma_D \\
& \quad [-p_a \mathbf{I} + \eta (\nabla \mathbf{u}_a + \nabla \mathbf{u}_a^T)] \mathbf{n} = -\rho (\mathbf{u} \cdot \mathbf{n}) \mathbf{u}_a, \text{ on } (T_2, T) \times \Gamma_N
\end{aligned} \tag{2.86}$$

When $t \in (T_1, T_2)$

$$\begin{aligned}
& -\rho \frac{\partial \mathbf{u}_a}{\partial t} - \eta \nabla \cdot (\nabla \mathbf{u}_a + \nabla \mathbf{u}_a^T) - \rho (\mathbf{u} \cdot \nabla) \mathbf{u}_a + \rho (\nabla \mathbf{u}) \cdot \mathbf{u}_a \\
& \quad + \nabla p_a = -\beta_1 \left(\frac{\partial A}{\partial \mathbf{u}} - \nabla \cdot \frac{\partial A}{\partial \nabla \mathbf{u}} \right) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \mathbf{u}_a, \text{ in } (T_1, T_2) \times \Omega \\
& \quad - \nabla \cdot \mathbf{u}_a = -\beta_1 \frac{\partial A}{\partial p} + \frac{\partial \mathbf{f}}{\partial p} \cdot \mathbf{u}_a, \text{ in } (T_1, T_2) \times \Omega \\
& \quad \mathbf{u}_a = -\frac{\partial B}{\partial p} \mathbf{n}, \text{ on } (T_1, T_2) \times \Gamma_D \\
& \quad [-p_a \mathbf{I} + \eta (\nabla \mathbf{u}_a + \nabla \mathbf{u}_a^T)] \mathbf{n} = -\rho (\mathbf{u} \cdot \mathbf{n}) \mathbf{u}_a - \beta_1 \frac{\partial A}{\partial \nabla \mathbf{u}} \mathbf{n} - \beta_2 \frac{\partial B}{\partial \mathbf{u}}, \text{ on } (T_1, T_2) \times \Gamma_N
\end{aligned} \tag{2.87}$$

When $t \in (0, T_1)$

$$\begin{aligned}
& -\rho \frac{\partial \mathbf{u}_a}{\partial t} - \eta \nabla \cdot (\nabla \mathbf{u}_a + \nabla \mathbf{u}_a^T) - \rho (\mathbf{u} \cdot \nabla) \mathbf{u}_a + \rho (\nabla \mathbf{u}) \cdot \mathbf{u}_a \\
& \quad + \nabla p_a = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \mathbf{u}_a, \text{ in } (0, T_1) \times \Omega \\
& \quad - \nabla \cdot \mathbf{u}_a = \frac{\partial \mathbf{f}}{\partial p} \cdot \mathbf{u}_a, \text{ in } (0, T_1) \times \Omega \\
& \quad \mathbf{u}_a = \mathbf{0}, \text{ on } (0, T_1) \times \Gamma_D \\
& \quad [-p_a \mathbf{I} + \eta (\nabla \mathbf{u}_a + \nabla \mathbf{u}_a^T)] \mathbf{n} = -\rho (\mathbf{u} \cdot \mathbf{n}) \mathbf{u}_a, \text{ on } (0, T_1) \times \Gamma_N
\end{aligned} \tag{2.88}$$

By solving Eqs. 2.86, 2.87 and 2.88 sequentially, the adjoint variables of $\mathbf{u}$ and p can be obtained. Because the solution of Eq. 2.86 is $\mathbf{u}_a = \mathbf{0}$ and $p_a = 0$, one only needs to solved Eqs. 2.87 and 2.88 with the initial condition $\mathbf{u}_a(T_2, \mathbf{x}) = \mathbf{0}$. The adjoint derivatives corresponding to expression 2.82 can be expressed as

$$\begin{aligned}
\frac{D\hat{J}}{D\gamma} \Big|_{\Omega} &= \int_{T_1}^{T_2} \left(\beta_1 \frac{\partial A}{\partial \gamma} + \frac{\partial \alpha}{\partial \gamma} \mathbf{u} \cdot \mathbf{u}_a \right) dt + \int_0^{T_1} \frac{\partial \alpha}{\partial \gamma} \mathbf{u} \cdot \mathbf{u}_a dt, \text{ in } \Omega \\
\frac{D\hat{J}}{D\gamma} \Big|_{\partial\Omega} &= \int_{T_1}^{T_2} \beta_2 \frac{\partial B}{\partial \gamma} dt, \text{ on } \partial\Omega
\end{aligned} \tag{2.89}$$

Therefore, the adjoint analysis of the objective or design constraints defined on the subinterval $[T_1, T_2] \subseteq [0, T]$ is implemented on $[0, T_2]$ instead of $[0, T]$, although the unsteady flow problem is defined on the time interval $[0, T]$. Subsequently, the corresponding adjoint equations and adjoint derivatives are transient equations and integrations on $[0, T_2]$ instead of $[0, T]$, respectively.

2.4 Appendix for Section 2.2

2.4.1 Adjoint Equations for Level Set Method-Based Topology Optimization

The optimization problem (Eq. 2.45) constrained by the unsteady Navier-Stokes equations can be written into the following abstract form:

$$\min J(\mathbf{u}, p; \phi) \quad \text{s.t.} \quad e(\mathbf{u}, p; \phi) = 0, \quad \phi \in \mathcal{K} \quad (2.90)$$

where $\mathcal{K}$ is the set of feasible level set function ϕ ; $e(\cdot)$ is the operator corresponding to the weak form of the Navier-Stokes equations. The weak form of the Navier-Stokes equations in Eq. 2.39 is

$$\begin{aligned} & e(\mathbf{u}, p; \phi) \\ &= \rho \int_{\Omega} (\mathbf{u} \cdot \mathbf{u}_a) \big|_{t=T} d\Omega - \rho \int_{\Omega} \mathbf{u}_0 \cdot \mathbf{u}_a \big|_{t=0} d\Omega - \rho \int_0^T \int_{\Omega} \mathbf{u} \cdot \frac{\partial \mathbf{u}_a}{\partial t} d\Omega dt \\ & \quad + \eta \int_0^T \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{u}_a d\Omega dt + \int_0^T \int_{\Omega} \rho (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{u}_a d\Omega dt \\ & \quad - \int_0^T \int_{\Omega} p \nabla \cdot \mathbf{u}_a d\Omega dt - \int_0^T \int_{\Omega} \mathbf{f} \cdot \mathbf{u}_a d\Omega dt - \int_0^T \int_{\Omega} p_a \nabla \cdot \mathbf{u} d\Omega dt \\ & \quad + \int_0^T \int_{\Gamma_D} [(p\mathbf{I} - \eta \nabla \mathbf{u}) \mathbf{n} \cdot \mathbf{u}_a] \big|_{\mathbf{u}=\mathbf{u}_D} d\Gamma dt + \int_0^T \int_{\Omega} H(\phi) \mathbf{u} \cdot \mathbf{u}_a d\Omega dt \\ &= 0, \quad \forall \mathbf{u}_a \in \mathcal{U}_{\Omega}^d \quad \forall p_a \in \mathcal{P}_{\Omega} \end{aligned} \quad (2.91)$$

where $\mathcal{U}_{\Omega}^d := (\mathcal{L}^2((0, T); \mathcal{H}(\Omega)))^d$, $\mathcal{P}_{\Omega} := \mathcal{L}^2((0, T); \mathcal{L}^2(\Omega))$; $\mathbf{u} \in \mathcal{U}_{\Omega}^d$, $p \in \mathcal{P}_{\Omega}$, $\mathbf{f} \in \mathcal{U}_{\Omega}^{d*} := (\mathcal{L}^2((0, T); \mathcal{H}^*(\Omega)))^d$, $\mathbf{u}_0 \in (\mathcal{H}_0^1(\Omega))^d$ and $\phi \in \mathcal{H}(\Omega)$; d is the spatial dimension; $\mathcal{L}^2(\Omega)$ is second order integrable Lebesgue space; $\mathcal{H}^*(\Omega)$ is the dual space of Hilbert space $\mathcal{H}(\Omega)$; $\mathcal{H}_0^1(\Omega) = \{\mathbf{u} \in \mathcal{H}(\Omega) \mid \nabla \cdot \mathbf{u} = 0\}$; $\mathcal{L}^1(\Omega)$ is the first order integrable Lebesgue space. On the further treatment of the integral on Γ_D in Eq. 2.91, one can refer to [30].

According to the Lagrangian multiplier method [20], the augmented Lagrangian objective functional $\hat{J}(\mathbf{u}, p; \phi)$ for the optimization problem in Eq. 2.45 can be expressed as

$$\begin{aligned} \hat{J}(\mathbf{u}, p; \phi) = & J(\mathbf{u}, p; \phi) + e(\mathbf{u}, p; \phi) - \lambda \left(\int_{\Omega} H(-\phi) \, d\Omega - V_r V_0 \right) \\ & + \frac{\Lambda}{2} \left(\int_{\Omega} H(-\phi) \, d\Omega - V_r V_0 \right)^2 \end{aligned} \quad (2.92)$$

where the volume constraint is treated with the Lagrangian multiplier based quadratic penalty method [20]; λ and Λ are the Lagrangian multiplier and the quadratic penalty parameter, respectively. According to Karush-Kuhn-Tucker conditions [5], the variational of $\hat{J}(\mathbf{u}, p; \phi)$

$$\delta \hat{J} = \frac{\partial \hat{J}}{\partial \mathbf{u}} \cdot \delta \mathbf{u} + \frac{\partial \hat{J}}{\partial p} \delta p + \frac{\partial \hat{J}}{\partial \phi} \delta \phi \quad (2.93)$$

should be zero corresponding to the optimal distribution of the level set function. Therefore, there are

$$\frac{\partial \hat{J}}{\partial \mathbf{u}} \cdot \delta \mathbf{u} = \mathbf{0}; \quad \frac{\partial \hat{J}}{\partial p} \delta p = 0; \quad \frac{\partial \hat{J}}{\partial \phi} \delta \phi = 0 \quad (2.94)$$

By setting $\tilde{A} = \beta_1 H(-\phi) A + \beta_2 \tau(\phi) \|\nabla \phi\| B$ for Eq. 2.47, one can obtain the variational of J to $\mathbf{u}$:

$$\begin{aligned} \frac{\partial J}{\partial \mathbf{u}} \cdot \delta \mathbf{u} = & \int_0^T \int_{\Omega} \frac{\partial \tilde{A}}{\partial \mathbf{u}} \cdot \delta \mathbf{u} \, d\Omega \, dt + \int_0^T \int_{\partial\Omega} \beta_3 \frac{\partial C}{\partial \mathbf{u}} \cdot \delta \mathbf{u} \, d\Omega \, dt \\ & + \int_0^T \int_{\Gamma_N} \frac{\partial \tilde{A}}{\partial \nabla \mathbf{u}} \mathbf{n} \cdot \delta \mathbf{u} \, d\Gamma \, dt - \int_0^T \int_{\Omega} \left(\nabla \cdot \frac{\partial \tilde{A}}{\partial \nabla \mathbf{u}} \right) \cdot \delta \mathbf{u} \, d\Omega \, dt \end{aligned} \quad (2.95)$$

and the variational of J to p :

$$\frac{\partial J}{\partial p} \delta p = \int_0^T \int_{\Omega} \frac{\partial \tilde{A}}{\partial p} \delta p \, d\Omega \, dt + \int_0^T \int_{\partial\Omega} \beta_3 \frac{\partial C}{\partial p} \delta p \, d\Gamma \, dt \quad (2.96)$$

The Lebesgue measure of $(0, T) \times \Gamma$ in Q is zero, then

$$\int_0^T \int_{\Omega} H(\phi) \delta \mathbf{u} \cdot \mathbf{u}_a \, d\Omega \, dt = 0 \quad (2.97)$$

Based on the partial integration approach, the Gauss theory [5] and $\delta \mathbf{u} = \mathbf{0}$ on Γ_D , the following transformations are obtained:

$$\begin{aligned} \int_0^T \int_{\Omega} \nabla \delta \mathbf{u} : \nabla \mathbf{u}_a \, d\Omega \, dt &= \int_0^T \int_{\Omega} \nabla \cdot (\nabla \mathbf{v} \cdot \delta \mathbf{u}) \, d\Omega \, dt - \int_0^T \int_{\Omega} \Delta \mathbf{u}_a \cdot \delta \mathbf{u} \, d\Omega \, dt \\ &= \int_0^T \int_{\Gamma_N} (\nabla \mathbf{u}_a) \mathbf{n} \cdot \delta \mathbf{u} \, d\Gamma \, dt - \int_0^T \int_{\Omega} \Delta \mathbf{u}_a \cdot \delta \mathbf{u} \, d\Omega \, dt \end{aligned} \quad (2.98)$$

$$\begin{aligned} - \int_0^T \int_{\Omega} p_a \nabla \cdot \delta \mathbf{u} \, d\Omega \, dt &= - \int_0^T \int_{\Omega} \nabla \cdot (p_a \delta \mathbf{u}) \, d\Omega \, dt + \int_0^T \int_{\Omega} \nabla p_a \cdot \delta \mathbf{u} \, d\Omega \, dt \\ &= - \int_0^T \int_{\partial \Omega} p_a \mathbf{n} \cdot \delta \mathbf{u} \, d\Gamma \, dt + \int_0^T \int_{\Omega} \nabla p_a \cdot \delta \mathbf{u} \, d\Omega \, dt \\ &= - \int_0^T \int_{\Gamma_N} p_a \mathbf{n} \cdot \delta \mathbf{u} \, d\Gamma \, dt + \int_0^T \int_{\Omega} \nabla p_a \cdot \delta \mathbf{u} \, d\Omega \, dt \end{aligned} \quad (2.99)$$

$$\begin{aligned} \int_0^T \int_{\Omega} (\mathbf{u} \cdot \nabla) \delta \mathbf{u} \cdot \mathbf{u}_a \, d\Omega \, dt &= \int_0^T \int_{\Omega} \nabla \cdot [\mathbf{u} \cdot (\mathbf{v} \cdot \delta \mathbf{u})] \, d\Omega \, dt - \int_0^T \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u}_a \cdot \delta \mathbf{u} \, d\Omega \, dt \\ &= \int_0^T \int_{\partial \Omega} (\mathbf{u} \cdot \mathbf{n}) \mathbf{v} \cdot \delta \mathbf{u} \, d\Gamma \, dt - \int_0^T \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u}_a \cdot \delta \mathbf{u} \, d\Omega \, dt \\ &= \int_0^T \int_{\Gamma_N} (\mathbf{u} \cdot \mathbf{n}) \mathbf{u}_a \cdot \delta \mathbf{u} \, d\Gamma \, dt - \int_0^T \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u}_a \cdot \delta \mathbf{u} \, d\Omega \, dt \end{aligned} \quad (2.100)$$

$$\int_0^T \int_{\Omega} (\delta \mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{u}_a \, d\Omega \, dt = \int_0^T \int_{\Omega} (\nabla \mathbf{u}) \mathbf{u}_a \cdot \delta \mathbf{u} \, d\Omega \, dt \quad (2.101)$$

Because $\mathbf{u}$ is known on Σ_D , and p can be expressed by $\nabla \mathbf{u}$ on Σ_N according to Eq. 2.44, the functional C in Eq. 2.46 can be expressed as

$$C(\mathbf{u}, p; \gamma) = \begin{cases} C(p; \gamma), & \text{on } \Sigma_D \\ C(\mathbf{u}, \nabla \mathbf{u}; \gamma), & \text{on } \Sigma_N \end{cases} \quad (2.102)$$

Substituting Eqs. 2.95–2.102 into the first two formulas of Eq. 2.94, the following equations are obtained:

$$\int_{\Omega} \rho (\delta \mathbf{u} \cdot \mathbf{v}) \big|_{t=T} \, d\Omega = 0 \quad (2.103)$$

$$\begin{aligned}
& \int_0^T \int_{\Omega} \left[-\rho \frac{\partial \mathbf{v}}{\partial t} - \eta \Delta \mathbf{v} - \rho (\mathbf{u} \cdot \nabla) \mathbf{v} + \rho (\nabla \mathbf{u}) \cdot \mathbf{v} + \nabla p_a - \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \cdot \mathbf{v} \right. \\
& \quad \left. + \frac{\partial \tilde{A}}{\partial \mathbf{u}} - \nabla \cdot \frac{\partial \tilde{A}}{\partial \nabla \mathbf{u}} \right] \cdot \delta \mathbf{u} \, d\Omega \, dt + \int_0^T \int_{\Gamma_N} \left[(-p_a \mathbf{I} + \eta \nabla \mathbf{u}_a) \mathbf{n} \right. \\
& \quad \left. + \rho (\mathbf{u} \cdot \mathbf{n}) \mathbf{u}_a + \frac{\partial \tilde{A}}{\partial \nabla \mathbf{u}} \mathbf{n} + \beta_3 \frac{\partial C}{\partial \mathbf{u}} \right] \cdot \delta \mathbf{u} \, d\Gamma \, dt = 0 \quad (2.104)
\end{aligned}$$

$$\begin{aligned}
& \int_0^T \int_{\Omega} \left(-\nabla \cdot \mathbf{v} + \frac{\partial \tilde{A}}{\partial p} - \frac{\partial \mathbf{f}}{\partial p} \cdot \mathbf{v} \right) \delta p \, d\Omega \, dt + \int_0^T \int_{\Gamma_D} \left(\mathbf{v} + \beta_3 \frac{\partial C}{\partial p} \mathbf{n} \right) \delta p \, d\Gamma \, dt = 0 \\
& \quad (2.105)
\end{aligned}$$

Based on Eqs. 2.103–2.105 and the arbitrariness of $\delta \mathbf{u}$ and δp , the adjoint equations of the Navier-Stokes equations for the optimization problem in Eq. 2.45 can be obtained as in Eq. 2.48.

2.4.2 Variational of Augmented Lagrangian Objective Functional to Level Set Function

According to Eq. 2.92 and the third formula of Eq. 2.94, the variational of the augmented Lagrangian objective functional $\hat{J}$ to the level set function ϕ can be obtained as

$$\begin{aligned}
\delta \hat{J} &= \frac{\partial \hat{J}}{\partial \phi} \delta \phi \\
&= \frac{\partial J}{\partial \phi} \delta \phi + \lambda \int_{\Omega} \tau(\phi) \delta \phi \, d\Omega + \int_0^T \int_{\Omega} \tau(\phi) \mathbf{u} \cdot \mathbf{u}_a \delta \phi \, d\Omega \, dt \\
&\quad - \Lambda \int_{\Omega} \tau(\phi) \delta \phi \, d\Omega \left(\int_{\Omega} H(-\phi) \, d\Omega - V_r V_0 \right) \quad (2.106)
\end{aligned}$$

In addition,

$$\begin{aligned}
\delta \|\nabla \phi\| &= \delta \left(\frac{(\nabla \phi)^2}{\|\nabla \phi\|} \right) \\
&= \frac{2 \nabla \phi \cdot \nabla (\delta \phi)}{\|\nabla \phi\|} - \frac{(\nabla \phi)^2 \cdot \delta \|\nabla \phi\|}{\|\nabla \phi\|^2} \\
&= \frac{2 \nabla \phi \cdot \nabla (\delta \phi)}{\|\nabla \phi\|} - \delta \|\nabla \phi\| \quad (2.107)
\end{aligned}$$

Therefore,

$$\delta \|\nabla \phi\| = \frac{\nabla \phi \cdot \nabla(\delta \phi)}{\|\nabla \phi\|} \quad (2.108)$$

Based on the similar transformation in [31] and Eq. 2.108, the following transformation can be obtained:

$$\begin{aligned} & \frac{\partial J}{\partial \phi} \delta \phi \\ &= - \int_0^T \int_{\Omega} \tau(\phi) A \delta \phi \, d\Omega \, dt + \int_0^T \int_{\Omega} \delta(\tau(\phi) \|\nabla \phi\|) B \, d\Omega \, dt \\ &= - \int_0^T \int_{\Omega} \tau(\phi) A \delta \phi \, d\Omega \, dt + \int_0^T \int_{\Omega} [\delta(\tau(\phi)) \|\nabla \phi\| + \tau(\phi) \delta \|\nabla \phi\|] B \, d\Omega \, dt \\ &= - \int_0^T \int_{\Omega} \tau(\phi) A \delta \phi \, d\Omega \, dt + \int_0^T \int_{\Omega} \left[\delta(\tau(\phi)) \|\nabla \phi\| + \tau(\phi) \frac{\nabla \phi \cdot \nabla(\delta \phi)}{\|\nabla \phi\|} \right] B \, d\Omega \, dt \\ &= - \int_0^T \int_{\Omega} \tau(\phi) A \delta \phi \, d\Omega \, dt + \int_0^T \int_{\Omega} \left[\delta(\tau(\phi)) \|\nabla \phi\| \right. \\ &\quad \left. + \nabla \cdot \left(\tau(\phi) \frac{\nabla \phi}{\|\nabla \phi\|} \delta \phi \right) - \nabla \cdot \left(\tau(\phi) \frac{\nabla \phi}{\|\nabla \phi\|} \right) \delta \phi \right] B \, d\Omega \, dt \\ &= - \int_0^T \int_{\Omega} \tau(\phi) A \delta \phi \, d\Omega \, dt + \int_0^T \int_{\Omega} [\delta(\tau(\phi)) \|\nabla \phi\| + \nabla \cdot (\tau(\phi) \mathbf{n}_{\Gamma} \delta \phi) \\ &\quad - \delta \phi \tau(\phi) \nabla \cdot \mathbf{n}_{\Gamma} - \delta(\tau(\phi)) \|\nabla \phi\|] B \, d\Omega \, dt \quad (2.109) \\ &= - \int_0^T \int_{\Omega} \tau(\phi) A \delta \phi \, d\Omega \, dt + \int_0^T \int_{\Omega} [\nabla \cdot (\tau(\phi) \mathbf{n}_{\Gamma} \delta \phi) - \delta \phi \tau(\phi) \nabla \cdot \mathbf{n}_{\Gamma}] B \, d\Omega \, dt \\ &= - \int_0^T \int_{\Omega} \tau(\phi) A \delta \phi \, d\Omega \, dt - \int_0^T \int_{\Omega} \tau(\phi) B \nabla \cdot \mathbf{n}_{\Gamma} \delta \phi \, d\Omega \, dt \\ &\quad + \int_0^T \int_{\Omega} [\nabla \cdot (\tau(\phi) \mathbf{n}_{\Gamma} B \delta \phi) - \nabla B \cdot \mathbf{n}_{\Gamma} \tau(\phi) \delta \phi] \, d\Omega \, dt \\ &= - \int_0^T \int_{\Omega} \tau(\phi) A \delta \phi \, d\Omega \, dt + \int_0^T \int_{\partial \Omega} \tau(\phi) B \delta \phi \mathbf{n}_{\Gamma} \cdot \mathbf{n} \, d\Gamma \, dt \\ &\quad - \int_0^T \int_{\Omega} \nabla B \cdot \mathbf{n}_{\Gamma} \tau(\phi) \delta \phi \, d\Omega \, dt - \int_0^T \int_{\Omega} \tau(\phi) B \nabla \cdot \mathbf{n}_{\Gamma} \delta \phi \, d\Omega \, dt \\ &= - \int_0^T \int_{\Omega} \tau(\phi) A \delta \phi \, d\Omega \, dt - \int_0^T \int_{\Omega} \nabla B \cdot \mathbf{n}_{\Gamma} \tau(\phi) \delta \phi \, d\Omega \, dt \\ &\quad - \int_0^T \int_{\Omega} \tau(\phi) B \kappa \delta \phi \, d\Omega \, dt \end{aligned}$$

where $\mathbf{n}_{\Gamma} = \nabla \phi / \|\nabla \phi\|$ and $\kappa = \nabla \cdot \mathbf{n}_{\Gamma}$. By substituting Eq. 2.109 into Eq. 2.106, the adjoint sensitivity of the optimization problem in Eq. 2.45 is obtained as:

$$\begin{aligned}
\delta \hat{J} = & - \int_0^T \int_{\Omega} \tau(\phi) A \delta \phi \, d\Omega \, dt - \int_0^T \int_{\Omega} \nabla B \cdot \mathbf{n}_\Gamma \tau(\phi) \delta \phi \, d\Omega \, dt \\
& - \int_0^T \int_{\Omega} \tau(\phi) B \kappa \delta \phi \, d\Omega \, dt + \int_0^T \int_{\Omega} \tau(\phi) \mathbf{u} \cdot \mathbf{w} \delta \phi \, d\Omega \, dt \\
& + \lambda \int_{\Omega} \tau(\phi) \delta \phi \, d\Omega - \Lambda \left(\int_{\Omega} H(-\phi) \, d\Omega - V_r V_{\Omega} \right) \int_{\Omega} \tau(\phi) \delta \phi \, d\Omega \\
= & - \int_{\Omega} \left[\int_0^T (\beta_1 A + \beta_2 \nabla B \cdot \mathbf{n}_\Gamma + \beta_2 B \kappa - \mathbf{u} \cdot \mathbf{u}_a) \, dt - \lambda \right. \\
& \left. + \Lambda \left(\int_{\Omega} H(-\phi) \, d\Omega - V_r V_0 \right) \right] \tau(\phi) \delta \phi \, d\Omega
\end{aligned} \tag{2.110}$$

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