Chapter 2
Pinning Control Strategies for Synchronization of CRDNNs

2.1 Introduction

As one of the most significant and interesting dynamical properties of the complex networks, synchronization has been extensively studied by the researchers [41, 42, 45, 47, 53, 57, 63, 66, 119–121]. To our knowledge, in most existing works on the synchronization of complex networks, three kinds of coupling forms (named respectively state coupling, output coupling and derivative coupling) were considered in network models. So far, a great many important results on synchronization have been obtained for various complex networks with state coupling, see ([41, 45, 47, 53, 66, 119] and relevant references therein). In [57], Chen proposed a complex dynamical network model, in which nodes are connected by measured outputs experiencing the random sensor delay. Synchronization in the proposed network model was analyzed by the stochastic stability theory. Considering that the node state in complex networks is difficult to be observed or measured, some researchers investigated the output synchronization of complex delayed dynamical networks with output coupling [42, 63]. In [120, 121], the synchronization was studied for complex dynamical networks with non-derivative and derivative coupling.

It should be noticed that the above mentioned works are based on the network models with time-varying state variables. However, in reality, the node state is not only dependent on the time, but also intensively dependent on space variable in many circumstances. As a special class of complex networks, coupled neural networks have attracted much attention in recent years. Especially, the synchronization problem of coupled neural networks has stirred much research interest due to its fruitful applications in various fields [76–79]. It is well known that the diffusion phenomena can not be ignored in neural networks and electric circuits once electrons transport in a nonuniform electromagnetic field [28–30]. Therefore, we must consider the diffusion effects in neural networks. Obviously, in CRDNNs, the state variable of node is seriously dependent on the time and space.

More recently, researchers have investigated the synchronization problem of coupled neural networks with reaction-diffusion terms, and some interesting results

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have also been established, e.g., see also [97–99, 101]. Yang et al. [101] studied the global exponential synchronization of a class of CRDNNs with time-varying delay by adding impulsive controller to a small fraction of nodes. In [99], Wang et al. respectively investigated the synchronization problem of two kinds of linearly coupled neural networks with reaction-diffusion terms using edge-based adaptive strategy. Unfortunately, most of the existing results of synchronization for CRDNNs are concerned about state coupling. But, in reaction-diffusion networks, different diffusion of node may cause different changes of other nodes [122]. For example, as is well known, different diffusion of species may cause different movements of other species in food webs [123, 124]. Therefore, it is also interesting to study the coupled neural networks with reaction-diffusion terms and spatial diffusion coupling. To our knowledge, very few researchers have investigated the synchronization of CRDNNs with spatial diffusion coupling [100]. In [100], the authors proposed a general model of an array of N linearly coupled RDNNs with spatial diffusion coupling, and respectively investigated the synchronization and $\mathcal{H}_\infty$ synchronization of the proposed network model.

Motivated by the above discussions, in this chapter, we propose two kinds of CRDNNs. In the first one, the nodes are coupled through their states. In the second one, the nodes are coupled through the spatial diffusion terms. In many circumstances, CRDNNs can not be synchronized by themselves, thus some control strategies should be adopted to achieve synchronization. Considering that it is difficult to apply control actions to all nodes in a large-scale network, some authors developed several pinning control schemes for complex networks [125–127]. For instance, Tang et al. [125] investigated the pinning distributed synchronization problem of a class of nonlinear dynamical networks with multiple stochastic disturbances using fixed pinning and switching pinning schemes. In [126], distributed robust pinning synchronization was investigated for a class of complex networks with parameter uncertainties and stochastic coupling. In these existing works [125–127], the node state is only dependent on the time. Obviously, it is also beneficial to apply the pinning control technique to study the synchronization problem of the CRDNNs. To our knowledge, very few researchers have investigated the pinning control of the CRDNNs [97, 101]. Therefore, the objective of this chapter is to design some pinning control strategies such that all nodes in the CRDNNs can synchronize onto a desired state. It is well known that the topological structure and the coupling strength are two key factors impacting the synchronization in CRDNNs. Therefore, it is natural to raise the following problem: Does the coupling form play an important role in the synchronization of the CRDNNs? This chapter also analyzes the relationship among pinning synchronization, the coupling form, the coupling strength, and the topological structure in CRDNNs.

The main contributions of this chapter are as follows. First, several sufficient conditions are established to guarantee the synchronization of the CRDNNs with state coupling by using the designed pinning controllers. Second, an effective adaptive strategy to adjust the coupling strength of the CRDNNs with state coupling is designed. Third, a sufficient condition ensuring synchronization of the CRDNNs with spatial diffusion coupling is obtained by using the designed pinning controllers, and an adaptive strategy is proposed to obtain appropriate pinning feedback gains for achieving network synchronization.

2.2 Pinning Control of CRDNNs with State Coupling

In this section, we consider a CRDNNs consisting of N identical nodes with state coupling, in which each node is an n-dimensional reaction-diffusion neural network. By using Lyapunov functional method and pinning control technique, some sufficient conditions are established to ensure that the CRDNNs is synchronized. In addition, an adaptive strategy to tune the coupling strength is proposed, and a general criterion for synchronization is obtained by using the designed adaptive law.

2.2.1 Network Model

To facilitate the readers, the CRDNNs model is presented in a step-by-step format.

A single reaction-diffusion neural network with Dirichlet boundary conditions is described by the following PDEs:

$$\frac{\partial w_i(x,t)}{\partial t} = d_i \triangle w_i(x,t) - a_i w_i(x,t) + J_i + \sum_{j=1}^{n} b_{ij} f_j(w_j(x,t)), \tag{2.1}$$

where $i = 1, 2, \dots, n$, n is the number of neurons in the network; $x = (x_1, x_2, \dots, x_q)^T \in \Omega \subset \mathbb{R}^q$; $w_i(x,t) \in \mathbb{R}$ is the state of the ith neuron at time t and in space x; $\triangle = \sum_{k=1}^{q} \frac{\partial^2}{\partial x_k^2}$ is the Laplace diffusion operator on Ω; $d_i > 0$ represents the transmission diffusion coefficient along the ith neuron; $a_i > 0$ represents the rate with which the ith neuron will reset its potential to the resting state when disconnected from the network and external input; b_{ij} denotes the strength of the jth neuron on the ith neuron; $f_j(\cdot)$ denotes the activation function of the jth neuron; J_i is a constant external input.

The initial value and boundary value conditions associated with system (2.1) are given in the form

$$w_i(x,0) = \phi_i(x), \qquad x \in \Omega, \tag{2.2}$$

$$w_i(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0, +\infty), \tag{2.3}$$

where $\phi_i(x)(i = 1, 2, \dots, n)$ is bounded and continuous on Ω.

We can rewrite system (2.1) in a compact form as follows:

$$\frac{\partial w(x,t)}{\partial t} = D\triangle w(x,t) - Aw(x,t) + J + Bf(w(x,t)), \tag{2.4}$$

where $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, $B = (b_{ij})_{n\times n}$, $J = (J_1, J_2, \ldots, J_n)^T$, $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$, $f(w(x,t)) = (f_1(w_1(x,t)), f_2(w_2(x,t)), \ldots, f_n(w_n(x,t)))^T$, $w(x,t) = (w_1(x,t), w_2(x,t), \ldots, w_n(x,t))^T$.

N mutually coupled RDNNs (2.4) can result in a CRDNNs, which is described by

$$\begin{aligned}\frac{\partial z_i(x,t)}{\partial t} &= D\triangle z_i(x,t) - Az_i(x,t) + J + Bf(z_i(x,t))\\ &\quad + c\sum_{j=1}^{N} G_{ij}\Gamma z_j(x,t), \end{aligned}\tag{2.5}$$

where $i = 1, 2, \ldots, N$, N is the number of nodes in the network; $z_i(x,t) = (z_{i1}(x,t), z_{i2}(x,t), \ldots, z_{in}(x,t))^T \in \mathbb{R}^n$ is the state vector of node i; c is a positive real number, which represents the overall coupling strength; $\Gamma \in \mathbb{R}^{n\times n} > 0$ is inner coupling matrix; $G = (G_{ij})_{N\times N}$ is the coupling configuration matrix representing the topological structure of the network, where G_{ij} is defined as follows: if there exists a connection from node i to node j, then $G_{ij} > 0$; otherwise, $G_{ij} = 0 (i \neq j)$; and the diagonal elements of matrix G are defined by

$$G_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{N} G_{ij}, i = 1, 2, \ldots, N.$$

In this section, we always assume that CRDNNs (2.5) is strongly connected. The initial value and boundary value conditions associated with network (2.5) are given in the form

$$z_i(x,0) = \Phi_i(x) \in \mathbb{R}^n, \qquad x \in \Omega, \tag{2.6}$$

$$z_i(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0, +\infty), \tag{2.7}$$

where $\Phi_i(x)$ is bounded and continuous on Ω.

Suppose $w^*(x,t) = (w_1^*(x,t), w_2^*(x,t), \ldots, w_n^*(x,t))^T$ is an arbitrary desired solution of the system (2.1), then it satisfies (2.3) and

$$\frac{\partial w^*(x,t)}{\partial t} = D\triangle w^*(x,t) - Aw^*(x,t) + J + Bf(w^*(x,t)). \tag{2.8}$$

The objective of this section is to design some pinning control strategies such that the solution of the controlled network (2.5) can achieve synchronization in the sense that

$$\lim_{t\to+\infty} \|z_i(\cdot,t) - w^*(\cdot,t)\|_2 = 0,\ i = 1, 2, \ldots, N.$$

2.2.2 Pinning Synchronization of CRDNNs

Without loss of generality, rearrange the order of all nodes and let the first $l(1 \leqslant l < N)$ nodes be selected to be pinned. Thus, the pinning controlled network can be described by

$$\begin{aligned}
\frac{\partial z_i(x,t)}{\partial t} &= D\triangle z_i(x,t) - Az_i(x,t) + J + Bf(z_i(x,t)) + c\sum_{j=1}^{N} G_{ij}\Gamma z_j(x,t) \\
&\quad + u_i,\ i = 1, 2, \ldots, l, \\
\frac{\partial z_i(x,t)}{\partial t} &= D\triangle z_i(x,t) - Az_i(x,t) + J + Bf(z_i(x,t)) + c\sum_{j=1}^{N} G_{ij}\Gamma z_j(x,t), \\
&\quad i = l+1, \ldots, N,
\end{aligned} \tag{2.9}$$

where

$$u_i = -ck_i\Gamma(z_i(x,t) - w^*(x,t)),\ i = 1, 2, \ldots, l \tag{2.10}$$

are n-dimensional linear feedback controllers with all the control gains $k_i > 0$. Defining $e_i(x,t) = z_i(x,t) - w^*(x,t)$, then the dynamics of the error vector $e_i(x,t)$ is governed by the following equation:

$$\begin{aligned}
\frac{\partial e_i(x,t)}{\partial t} &= D\triangle e_i(x,t) - Ae_i(x,t) + Bf(z_i(x,t)) - Bf(w^*(x,t)) \\
&\quad + c\sum_{j=1}^{N} G_{ij}\Gamma e_j(x,t) - ck_i\Gamma e_i(x,t),
\end{aligned} \tag{2.11}$$

where $i = 1, 2, \ldots, N$, and $k_i = 0$ for $i = l+1, l+2, \ldots, N$.

For the convenience, we denote

$$\begin{aligned}
&\tilde{D} = \sum_{k=1}^{q} \frac{D}{l_k^2}, \quad \Theta = \operatorname{diag}(\rho_1^2, \rho_2^2, \ldots, \rho_n^2), \\
&K = \operatorname{diag}(k_1, k_2, \ldots, k_N), \\
&\Upsilon = -\tilde{D} - A + \frac{\Theta}{2} + \frac{BB^T}{2}.
\end{aligned}$$

Theorem 2.1 *If there exists a positive definite diagonal matrix* $\Xi = \operatorname{diag}(\eta_1, \eta_2, \ldots, \eta_N) \in \mathbb{R}^{N\times N}$ *such that*

$$\Xi \otimes \Upsilon - c[(\Xi K)\otimes \Gamma] + c\left(\frac{\Xi G + G^T\Xi}{2} \otimes \Gamma\right) < 0, \tag{2.12}$$

then the pinning controlled network (2.9) *is synchronized.*

Proof Define the following Lyapunov functional for the system (2.11):

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{N}\eta_i \int_{\Omega} e_i^T(x,t)e_i(x,t)dx. \tag{2.13}$$

In the following, we calculate the time derivative $\dot{V}_1(t)$ along the trajectory of system (2.11)

$$\begin{aligned}\dot{V}_1(t) &= \sum_{i=1}^{N}\eta_i \int_{\Omega} e_i^T(x,t)\frac{\partial e_i(x,t)}{\partial t}dx \\ &= \sum_{i=1}^{N}\int_{\Omega}\eta_i e_i^T(x,t)\big(D\triangle e_i(x,t) - Ae_i(x,t) + Bf(z_i(x,t)) \\ &\quad + c\sum_{j=1}^{N}G_{ij}\Gamma e_j(x,t) - Bf(w^*(x,t)) - ck_i\Gamma e_i(x,t)\big)dx.\end{aligned} \tag{2.14}$$

From Green's formula and the boundary condition, we have

$$\int_{\Omega} e_{is}(x,t)\triangle e_{is}(x,t)dx = -\sum_{k=1}^{q}\int_{\Omega}\left(\frac{\partial e_{is}(x,t)}{\partial x_k}\right)^2 dx,$$

where $e_i(x,t) = (e_{i1}(x,t), e_{i2}(x,t), \ldots, e_{in}(x,t))^T$, $s = 1, 2, \ldots, n$. According to Lemma 1.22, we can obtain

$$\begin{aligned}\int_{\Omega} e_i^T(x,t)D\triangle e_i(x,t)dx &= \sum_{s=1}^{n}\int_{\Omega} d_s e_{is}(x,t)\triangle e_{is}(x,t)dx \\ &= -\sum_{k=1}^{q}\sum_{s=1}^{n}\int_{\Omega} d_s\left(\frac{\partial e_{is}(x,t)}{\partial x_k}\right)^2 dx \\ &\leqslant -\sum_{k=1}^{q}\frac{1}{l_k^2}\sum_{s=1}^{n}\int_{\Omega} d_s e_{is}^2(x,t)dx \\ &= -\int_{\Omega} e_i^T(x,t)\tilde{D}e_i(x,t)dx.\end{aligned} \tag{2.15}$$

Furthermore, we can easily derive

$$\begin{aligned}&e_i^T(x,t)B[f(z_i(x,t)) - f(w^*(x,t))] \\ &\leqslant \frac{1}{2}e_i^T(x,t)BB^Te_i(x,t) + \frac{1}{2}e_i^T(x,t)\Theta e_i(x,t).\end{aligned} \tag{2.16}$$

It follows from (2.14) to (2.16) that

$$
\begin{aligned}
\dot{V}_1(t) &\leqslant \sum_{i=1}^{N} \int_{\Omega} \eta_i e_i^T(x,t)(-\tilde{D} - A + \frac{BB^T}{2} + \frac{\Theta}{2} - ck_i\Gamma)e_i(x,t)dx \\
&\quad + c\sum_{i=1}^{N}\sum_{j=1}^{N} \int_{\Omega} \eta_i G_{ij} e_i^T(x,t)\Gamma e_j(x,t)dx \\
&= \int_{\Omega} e^T(x,t)\left[\Xi \otimes \Upsilon + c\left(\frac{\Xi G + G^T\Xi}{2} \otimes \Gamma\right) - c((\Xi K) \otimes \Gamma)\right] e(x,t)dx \\
&\leqslant \gamma_1 \|e(\cdot,t)\|_2^2,
\end{aligned} \tag{2.17}
$$

where $e(x,t) = (e_1^T(x,t), e_2^T(x,t), \ldots, e_N^T(x,t))^T$, $\gamma_1 = \lambda_M(\Xi \otimes \Upsilon + c(\frac{\Xi G + G^T\Xi}{2} \otimes \Gamma) - c(\Xi K \otimes \Gamma)) < 0$. By the definition of $V_1(t)$, we have

$$
\gamma_2\|e(\cdot,t)\|_2^2 \leqslant V_1(t) \leqslant \gamma_3\|e(\cdot,t)\|_2^2, \tag{2.18}
$$

where $\gamma_2 = \min_{i=1,2,\ldots,N}\{\frac{\eta_i}{2}\}$, $\gamma_3 = \max_{i=1,2,\ldots,N}\{\frac{\eta_i}{2}\}$. Thus, by (2.17) and (2.18), we can get

$$
\dot{V}_1(t) \leqslant \frac{\gamma_1}{\gamma_3} V_1(t). \tag{2.19}
$$

Then, we can derive from (2.18) and (2.19) that

$$
\|e(\cdot,t)\|_2 \leqslant \sqrt{\frac{\gamma_3}{\gamma_2}} e^{\frac{\gamma_1}{2\gamma_3}t} \|e(\cdot,0)\|_2.
$$

Obviously, the pinning controlled network (2.9) is synchronized. The proof is completed.

According to Lemmas 1.11, 1.16 and 1.19, there obviously exists a positive definite diagonal matrix $\Xi = \mathrm{diag}(\eta_1, \eta_2, \ldots, \eta_N)$ such that

$$
c\left(\frac{\Xi G + G^T\Xi}{2} \otimes \Gamma\right) \leqslant 0. \tag{2.20}
$$

When $\Upsilon < 0$, we can derive from (2.20) that

$$
\Xi \otimes \Upsilon + c\left(\frac{\Xi G + G^T\Xi}{2} \otimes \Gamma\right) < 0.
$$

From Theorem 2.1, CRDNNs (2.5) can synchronize by itself. Therefore, in this section, we always assume $\lambda_M(\Upsilon) \geqslant 0$.

By applying Theorem 2.1, we can easily obtain the following conclusion.

Corollary 2.2 *If there exists a positive definite diagonal matrix* $\varXi = \mathrm{diag}(\eta_1, \eta_2, \ldots, \eta_N) \in \mathbb{R}^{N\times N}$ *such that*

$$c > \frac{-\lambda_M(\varXi \otimes \varUpsilon)}{\lambda_M\left(\frac{\varXi(G-K)+(G-K)^T\varXi}{2} \otimes \varGamma\right)}, \tag{2.21}$$

where $\varXi(G-K) + (G-K)^T\varXi < 0$*, then the pinning controlled network* (2.9) *is synchronized.*

Remark 2.3 According to Corollary 2.2, there exists a critical coupling strength c^* for given $\varXi$ such that the pinning controlled network (2.9) will synchronize if $c > c^*$. Therefore, if

$$\varXi(G-K) + (G-K)^T\varXi < 0$$

is satisfied, then the pinning controlled network (2.9) can synchronize as long as the coupling strength c is large enough.

Remark 2.4 From Lemma 1.19, there is a positive definite diagonal matrix $\varXi = \mathrm{diag}(\eta_1, \eta_2, \ldots, \eta_N) \in \mathbb{R}^{N\times N}$ such that the sum of the entries in each row of matrix $\varXi G + G^T\varXi$ is zero. In addition, it is obvious that matrix $\varXi G + G^T\varXi$ is symmetric and irreducible. Then, by utilizing Lemma 1.16, we can easily obtain the eigenvalues of $\varXi G + G^T\varXi$ are real-valued and strictly negative except an eigenvalue 0 with multiplicity 1. Thus, we have that for any $y = (y_1, y_2, \ldots, y_N)^T \neq 0 \in \mathbb{R}^N$,

$$y^T(\varXi G + G^T\varXi)y = 0 \iff y_1 = y_2 = \cdots = y_N \neq 0.$$

Then, we can get

$$y^T[\varXi(G-K) + (G-K)^T\varXi]y < 0$$

for any $y = (y_1, y_2, \ldots, y_N)^T \neq 0 \in \mathbb{R}^N$, namely,

$$\varXi(G-K) + (G-K)^T\varXi < 0.$$

Therefore, for any given matrices G and K, we can always find the positive definite diagonal matrix $\varXi$ satisfying

$$\varXi(G-K) + (G-K)^T\varXi < 0.$$

2.2.3 Pinning Synchronization of CRDNNs with Adaptive Coupling Strength

The coupling strength c given in (2.21) is very conservative, usually much larger than the needed value. Therefore, an adaptive strategy to tune the coupling strength c is designed in this subsection.

The pinning controlled network (2.9) with adaptive coupling strength can be described by

$$\begin{aligned}\frac{\partial z_i(x,t)}{\partial t} &= D\triangle z_i(x,t) - Az_i(x,t) + J + Bf(z_i(x,t)) + c(t)\sum_{j=1}^{N} G_{ij}\Gamma z_j(x,t)\\ &\quad + u_i,\ i = 1, 2, \ldots, l,\\ \frac{\partial z_i(x,t)}{\partial t} &= D\triangle z_i(x,t) - Az_i(x,t) + J + Bf(z_i(x,t)) + c(t)\sum_{j=1}^{N} G_{ij}\Gamma z_j(x,t),\\ &\quad i = l+1, \ldots, N,\end{aligned} \tag{2.22}$$

where

$$u_i = -c(t)k_i\Gamma(z_i(x,t) - w^*(x,t)),\ i = 1, 2, \ldots, l \tag{2.23}$$

are n-dimensional linear feedback controllers with all the control gains $k_i > 0$. Let $e_i(x,t) = (e_{i1}(x,t), e_{i2}(x,t), \ldots, e_{in}(x,t))^T = z_i(x,t) - w^*(x,t)$. Then, the dynamics of the error vector $e_i(x,t)$ is governed by the following equation:

$$\begin{aligned}\frac{\partial e_i(x,t)}{\partial t} &= D\triangle e_i(x,t) - Ae_i(x,t) + Bf(z_i(x,t)) - Bf(w^*(x,t))\\ &\quad + c(t)\sum_{j=1}^{N} G_{ij}\Gamma e_j(x,t) - c(t)k_i\Gamma e_i(x,t),\end{aligned} \tag{2.24}$$

where $i = 1, 2, \ldots, N$, and $k_i = 0$ for $i = l+1, l+2, \ldots, N$.

Theorem 2.5 *If there exists a positive definite diagonal matrix* $\Xi = \mathrm{diag}(\eta_1, \eta_2, \ldots, \eta_N) \in \mathbb{R}^{N\times N}$ *such that*

$$\Xi(G-K) + (G-K)^T\Xi < 0, \tag{2.25}$$

where $K = \mathrm{diag}(k_1, k_2, \ldots, k_N)$*, then the pinning controlled network* (2.9) *is synchronized under the following adaptive law:*

$$\dot{c}(t) = \beta\sum_{i=1}^{N}\eta_i\int_{\Omega}(z_i(x,t) - w^*(x,t))^T\Gamma(z_i(x,t) - w^*(x,t))dx, \tag{2.26}$$

where $c(0) > 0$, β *is a positive real number.*

Proof Firstly, according to (2.25), there obviously exists a positive constant r_1 such that

$$\Xi(G-K)+(G-K)^T\Xi+2r_1\Xi<0.$$

Construct a Lyapunov functional for the system (2.24) as follows:

$$V_2(t)=\frac{1}{2}\sum_{i=1}^{N}\eta_i\int_{\Omega}e_i^T(x,t)e_i(x,t)dx+\frac{r_1}{2\beta}(c(t)-\tilde{c})^2, \tag{2.27}$$

where $\tilde{c}$ is a positive real number.

Calculating the time derivative of $V_2(t)$ along the trajectory of system (2.24), we can get

$$\begin{aligned}
\dot{V}_2(t)=&\sum_{i=1}^{N}\eta_i\int_{\Omega}e_i^T(x,t)\frac{\partial e_i(x,t)}{\partial t}dx+r_1(c(t)-\tilde{c})\sum_{i=1}^{N}\eta_i\int_{\Omega}e_i^T(x,t)\Gamma e_i(x,t)dx\\
=&\sum_{i=1}^{N}\int_{\Omega}\eta_i e_i^T(x,t)\Big[D\triangle e_i(x,t)-Ae_i(x,t)+Bf(z_i(x,t))\\
&+c(t)\sum_{j=1}^{N}G_{ij}\Gamma e_j(x,t)-Bf(w^*(x,t))-c(t)k_i\Gamma e_i(x,t)\Big]dx\\
&+r_1(c(t)-\tilde{c})\sum_{i=1}^{N}\eta_i\int_{\Omega}e_i^T(x,t)\Gamma e_i(x,t)dx\\
\leqslant&\int_{\Omega}e^T(x,t)[\Xi\otimes(-\tilde{D}-A+\frac{BB^T}{2}+\frac{\Theta}{2}-r_1\tilde{c}\Gamma)]e(x,t)dx\\
&+\frac{c(t)}{2}\int_{\Omega}e^T(x,t)\Big\{\big[\Xi(G-K)+(G-K)^T\Xi\\
&+2r_1\Xi\big]\otimes\Gamma\Big\}e(x,t)dx\\
\leqslant&\int_{\Omega}e^T(x,t)\big[\Xi\otimes(-\tilde{D}-A+\frac{BB^T}{2}+\frac{\Theta}{2}-r_1\tilde{c}\Gamma)\big]e(x,t)dx,
\end{aligned} \tag{2.28}$$

where $e(x,t)=(e_1^T(x,t),e_2^T(x,t),\ldots,e_N^T(x,t))^T$, $\Theta=\mathrm{diag}(\rho_1^2,\rho_2^2,\ldots,\rho_n^2)$, $\tilde{D}=\sum_{k=1}^{q}\frac{D}{l_k^2}$.

By selecting $\tilde{c}$ sufficiently large such that

$$-\tilde{D}-A+\frac{BB^T}{2}+\frac{\Theta}{2}-r_1\tilde{c}\Gamma<0,$$

one obtains

$$\dot{V}_2(t) \leqslant -\varrho \|e(\cdot, t)\|_2^2, \tag{2.29}$$

where $\varrho = \lambda_m(\Xi \otimes (\tilde{D} + A - \frac{BB^T}{2} - \frac{\Theta}{2} + r_1 \tilde{c} \Gamma)) > 0$.

Obviously, $V_2(t)$ is non-increasing, and each term of $V_2(t)$ is bounded. Consequently, $c(t)$ is bounded, and $\lim_{t \to +\infty} V_2(t)$ exists and is a non-negative real number. Since $c(t)$ is monotonically increasing, one can conclude that $c(t)$ asymptotically converges to a finite positive value. Therefore, by the definition of $V_2(t)$, we can derive that

$$\lim_{t \to +\infty} \sum_{i=1}^{N} \eta_i \int_{\Omega} e_i^T(x, t) e_i(x, t) dx$$

exists and is a non-negative real number. In what follows, we shall prove that

$$\lim_{t \to +\infty} \sum_{i=1}^{N} \eta_i \int_{\Omega} e_i^T(x, t) e_i(x, t) dx = 0.$$

If this is not true, we have

$$\lim_{t \to +\infty} \sum_{i=1}^{N} \eta_i \int_{\Omega} e_i^T(x, t) e_i(x, t) dx = \mu > 0.$$

Then, there obviously exists a real number $M > 0$ such that

$$\sum_{i=1}^{N} \eta_i \int_{\Omega} e_i^T(x, t) e_i(x, t) dx > \frac{\mu}{2}$$

for $t \geqslant M$. Therefore,

$$\|e(\cdot, t)\|_2^2 > \frac{\mu}{2\delta}, \quad t \geqslant M, \tag{2.30}$$

where $\delta = \max_{i=1,2,\ldots,N}\{\eta_i\}$. From (2.29) and (2.30), we can get

$$\dot{V}_2(t) < -\frac{\varrho\mu}{2\delta}, \quad t \geqslant M. \tag{2.31}$$

By integrating (2.31) with respect to t over the time period M to $+\infty$, we can obtain

$$\begin{aligned} -V_2(M) \leqslant V_2(+\infty) - V_2(M) &= \int_M^{+\infty} \dot{V}_2(t) dt \\ &< -\int_M^{+\infty} \frac{\varrho\mu}{2\delta} dt \\ &= -\infty. \end{aligned}$$

This yields a contradiction, and so

$$\lim_{t\to+\infty}\sum_{i=1}^{N}\eta_i\int_{\Omega}e_i^T(x,t)e_i(x,t)dx=0.$$

Then, we can easily obtain

$$\lim_{t\to+\infty}\|e(\cdot,t)\|_2=0.$$

Therefore, the pinning controlled network (2.9) is synchronized under the designed adaptive law. The proof is completed.

Remark 2.6 To our knowledge, in many real-world networks, the coupling strength is adaptively adjusted according to changes in the environment or the network itself (for instance, wireless sensor networks, biological networks, neural networks) [128]. Therefore, it is important and interesting to study the pinning synchronization of CRDNNs with adaptive coupling strength. In Theorem 2.5, a sufficient condition is obtained to guarantee the synchronization of the pinning controlled network (2.9) by using the designed adaptive law.

Remark 2.7 Obviously, for any given matrices G and K, we can always find the positive definite diagonal matrix Ξ satisfying

$$\Xi(G-K)+(G-K)^T\Xi<0.$$

Therefore, the CRDNNs (2.5) can realize synchronization under any pinning controllers in the form of (2.10) if the coupling strength c is adjusted according to the designed adaptive law.

Remark 2.8 It follows from Corollary 2.2 that the CRDNNs (2.5) can realize synchronization by controlling only one node. But it requires a very large coupling strength c, which may not be very practical [47]. According to Theorem 2.5, the CRDNNs (2.5) is synchronized by pinning only one node if the coupling strength is adjusted according to the adaptive law (2.26). In this case, $c(t)$ may also converge to a very large positive real number. Therefore, we need to find a good balance between the number of pinned nodes and the coupling strength such that they are as small as possible and acceptable for practical use. Obviously, this is an important and interesting problem and will become our future investigative direction.

2.3 Pinning Control of CRDNNs with Spatial Diffusion Coupling

In this section, we consider a CRDNNs consisting of N identical nodes with spatial diffusion coupling, in which each node is an n-dimensional reaction-diffusion neural network. Based on the Lyapunov functional method and the pinning control technique, a sufficient condition is obtained to guarantee the synchronization of the CRDNNs. In addition, we also investigate the pinning adaptive synchronization of the CRDNNs, and a general criterion for ensuring network synchronization is established.

2.3.1 Network Model

In this section, we consider a CRDNNs consisting of N identical RDNNs (2.4) with spatial diffusion coupling. The mathematical model of the CRDNNs can be described as follows:

$$\begin{aligned}\frac{\partial z_i(x,t)}{\partial t} &= D\triangle z_i(x,t) - Az_i(x,t) + J + Bf(z_i(x,t)) \\ &\quad + \hat{c}\sum_{j=1}^{N}\hat{G}_{ij}\hat{\Gamma}\triangle z_j(x,t),\end{aligned} \tag{2.32}$$

where $i = 1, 2, \ldots, N$, N is the number of nodes in the network; $z_i(x,t) = (z_{i1}(x,t), z_{i2}(x,t), \ldots, z_{in}(x,t))^T \in \mathbb{R}^n$ is the state vector of node i; $\hat{c}$ is a positive real number, which represents the overall coupling strength; $\hat{\Gamma} = (\hat{\Gamma}_{ij})_{n\times n} \in \mathbb{R}^{n\times n} > 0$ is inner coupling matrix; $\hat{G} = (\hat{G}_{ij})_{N\times N}$ is the coupling configuration matrix representing the topological structure of the network, where $\hat{G}_{ij}$ is defined as follows: if there exists a connection from node i to node j, then $\hat{G}_{ij} > 0$; otherwise, $\hat{G}_{ij} = 0 (i \neq j)$; and the diagonal elements of matrix $\hat{G}$ are defined by

$$\hat{G}_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{N}\hat{G}_{ij}, i = 1, 2, \ldots, N.$$

In this section, we always assume that CRDNNs (2.32) is strongly connected. The initial value and boundary value conditions associated with network (2.32) are given in the form

$$z_i(x,0) = \hat{\Phi}_i(x) \in \mathbb{R}^n, \quad x \in \Omega, \tag{2.33}$$

$$z_i(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0, +\infty), \tag{2.34}$$

where $\hat{\Phi}_i(x)$ is bounded and continuous on Ω.

The objective of this section is to design some pinning control strategies such that the solutions of the controlled network (2.32) can achieve synchronization in the sense that

$$\lim_{t\to+\infty} \|z_i(\cdot,t) - w^*(\cdot,t)\|_2 = 0,\ i = 1, 2, \ldots, N,$$

where $w^*(x,t) = (w_1^*(x,t), w_2^*(x,t), \ldots, w_n^*(x,t))^T$ is an arbitrary desired solution of the system (2.1).

2.3.2 Pinning Synchronization of CRDNNs

Without loss of generality, rearrange the order of all nodes and let the first $l(1 \leqslant l < N)$ nodes be selected to be pinned. Thus, the pinning controlled network can be described by

$$\begin{aligned}
\frac{\partial z_i(x,t)}{\partial t} &= D\triangle z_i(x,t) - Az_i(x,t) + J + Bf(z_i(x,t)) + \hat{c}\sum_{j=1}^{N} \hat{G}_{ij}\hat{\Gamma}\triangle z_j(x,t) \\
&\quad + u_i,\ i = 1, 2, \ldots, l, \\
\frac{\partial z_i(x,t)}{\partial t} &= D\triangle z_i(x,t) - Az_i(x,t) + J + Bf(z_i(x,t)) + \hat{c}\sum_{j=1}^{N} \hat{G}_{ij}\hat{\Gamma}\triangle z_j(x,t), \\
&\quad i = l+1, \ldots, N,
\end{aligned} \tag{2.35}$$

where

$$u_i = -\hat{c}k_i\hat{\Gamma}(z_i(x,t) - w^*(x,t)),\ i = 1, 2, \ldots, l \tag{2.36}$$

are n-dimensional linear feedback controllers with all the control gains $k_i > 0$. Defining $e_i(x,t) = z_i(x,t) - w^*(x,t)$, then the dynamics of the error vector $e_i(x,t)$ is governed by the following equation:

$$\begin{aligned}
\frac{\partial e_i(x,t)}{\partial t} &= D\triangle e_i(x,t) - Ae_i(x,t) + Bf(z_i(x,t)) - Bf(w^*(x,t)) \\
&\quad + \hat{c}\sum_{j=1}^{N} \hat{G}_{ij}\hat{\Gamma}\triangle e_j(x,t) - \hat{c}k_i\hat{\Gamma}e_i(x,t),
\end{aligned} \tag{2.37}$$

where $i = 1, 2, \ldots, N$, and $k_i = 0$ for $i = l+1, l+2, \ldots, N$.

For the convenience, we denote

$$\begin{aligned}
&\tilde{D} = \sum_{k=1}^{q} \frac{D}{l_k^2},\ \Theta = \mathrm{diag}(\rho_1^2, \rho_2^2, \ldots, \rho_n^2), \\
&K = \mathrm{diag}(k_1, k_2, \ldots, k_N), \\
&\Upsilon = -\tilde{D} - A + \frac{\Theta}{2} + \frac{BB^T}{2}.
\end{aligned}$$

Theorem 2.9 *If there exists a positive definite diagonal matrix* $\Xi = \mathrm{diag}(\eta_1, \eta_2, \ldots, \eta_N) \in \mathbb{R}^{N\times N}$ *such that*

$$\Xi \otimes D + \frac{\hat{c}\Xi\hat{G} + \hat{c}\hat{G}^T\Xi}{2} \otimes \hat{\Gamma} \geqslant 0, \tag{2.38}$$

$$\Xi \otimes \Upsilon - (\hat{c}\Xi K) \otimes \hat{\Gamma} - \sum_{k=1}^{q} \frac{\hat{c}}{l_k^2}\left(\frac{\Xi\hat{G} + \hat{G}^T\Xi}{2} \otimes \hat{\Gamma}\right) < 0, \tag{2.39}$$

then the pinning controlled network (2.35) *is synchronized.*

Proof Take the same Lyapunov functional $V_1(t)$ as in Theorem 2.1, that is,

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{N}\eta_i\int_{\Omega} e_i^T(x,t)e_i(x,t)dx.$$

In the following, we calculate the time derivative $\dot{V}_1(t)$ along the trajectory of system (2.37)

$$\begin{aligned}
\dot{V}_1(t) &= \sum_{i=1}^{N}\eta_i\int_{\Omega} e_i^T(x,t)\frac{\partial e_i(x,t)}{\partial t}dx \\
&= \sum_{i=1}^{N}\eta_i\int_{\Omega} e_i^T(x,t)\big(D\triangle e_i(x,t) - Ae_i(x,t) + Bf(z_i(x,t)) - Bf(w^*(x,t)) \\
&\quad - \hat{c}k_i\hat{\Gamma}e_i(x,t) + \hat{c}\sum_{j=1}^{N}\hat{G}_{ij}\hat{\Gamma}\triangle e_j(x,t)\big)dx \\
&\leqslant \int_{\Omega} e^T(x,t)[\Xi \otimes D + (\hat{c}\Xi\hat{G}) \otimes \hat{\Gamma}]\triangle e(x,t)dx \\
&\quad + \int_{\Omega} e^T(x,t)\left[\Xi \otimes \left(-A + \frac{\Theta}{2} + \frac{BB^T}{2}\right) - (\hat{c}\Xi K) \otimes \hat{\Gamma}\right]e(x,t)dx,
\end{aligned}$$

where $e(x,t) = (e_1^T(x,t), e_2^T(x,t), \ldots, e_N^T(x,t))^T$.

From Green's formula and the boundary condition, we then have

$$\begin{aligned}
&\int_{\Omega} e^T(x,t)(\Xi \otimes D)\triangle e(x,t)dx \\
&= \sum_{i=1}^{N}\eta_i\int_{\Omega} e_i^T(x,t)D\triangle e_i(x,t)dx \\
&= \sum_{i=1}^{N}\eta_i\sum_{l=1}^{n}d_l\int_{\Omega} e_{il}(x,t)\triangle e_{il}(x,t)dx
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{k=1}^{q}\sum_{i=1}^{N}\eta_i\sum_{l=1}^{n}d_l\int_{\Omega}\left(\frac{\partial e_{il}(x,t)}{\partial x_k}\right)^2 dx\\
&= -\sum_{k=1}^{q}\int_{\Omega}\left(\frac{\partial e(x,t)}{\partial x_k}\right)^T(\Xi\otimes D)\frac{\partial e(x,t)}{\partial x_k}dx,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\Omega}e^T(x,t)[(\hat{c}\Xi\hat{G})\otimes\hat{\Gamma}]\triangle e(x,t)dx\\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\hat{c}\eta_i\hat{G}_{ij}\int_{\Omega}e_i^T(x,t)\hat{\Gamma}\triangle e_j(x,t)dx\\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\hat{c}\eta_i\hat{G}_{ij}\sum_{l=1}^{n}\sum_{s=1}^{n}\hat{\Gamma}_{ls}\int_{\Omega}e_{il}(x,t)\triangle e_{js}(x,t)dx\\
&= -\sum_{k=1}^{q}\sum_{i=1}^{N}\sum_{j=1}^{N}\hat{c}\eta_i\hat{G}_{ij}\sum_{l=1}^{n}\sum_{s=1}^{n}\hat{\Gamma}_{ls}\int_{\Omega}\frac{\partial e_{il}(x,t)}{\partial x_k}\frac{\partial e_{js}(x,t)}{\partial x_k}dx\\
&= -\sum_{k=1}^{q}\int_{\Omega}\left(\frac{\partial e(x,t)}{\partial x_k}\right)^T[(\hat{c}\Xi\hat{G})\otimes\hat{\Gamma}]\frac{\partial e(x,t)}{\partial x_k}dx,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\Omega}(\triangle e(x,t))^T[(\hat{c}\hat{G}^T\Xi)\otimes\hat{\Gamma}]e(x,t)dx\\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\hat{c}\eta_j\hat{G}_{ji}\int_{\Omega}(\triangle e_i(x,t))^T\,\hat{\Gamma}e_j(x,t)dx\\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\hat{c}\eta_j\hat{G}_{ji}\sum_{l=1}^{n}\sum_{s=1}^{n}\hat{\Gamma}_{ls}\int_{\Omega}\triangle e_{il}(x,t)e_{js}(x,t)dx\\
&= -\sum_{k=1}^{q}\int_{\Omega}\left(\frac{\partial e(x,t)}{\partial x_k}\right)^T[(\hat{c}\hat{G}^T\Xi)\otimes\hat{\Gamma}]\frac{\partial e(x,t)}{\partial x_k}dx. \qquad (2.40)
\end{aligned}
$$

By (2.40), we can get

$$
\begin{aligned}
&\int_{\Omega}e^T(x,t)[\Xi\otimes D+(\hat{c}\Xi\hat{G})\otimes\hat{\Gamma}]\triangle e(x,t)dx\\
&= \int_{\Omega}e^T(x,t)\left(\Xi\otimes D+\frac{\hat{c}\Xi\hat{G}}{2}\otimes\hat{\Gamma}\right)\triangle e(x,t)dx\\
&\quad+\int_{\Omega}(\triangle e(x,t))^T\left(\frac{\hat{c}\hat{G}^T\Xi}{2}\otimes\hat{\Gamma}\right)e(x,t)dx\\
&= -\sum_{k=1}^{q}\int_{\Omega}\left(\frac{\partial e(x,t)}{\partial x_k}\right)^T\left(\Xi\otimes D+\frac{\hat{c}\Xi\hat{G}+\hat{c}\hat{G}^T\Xi}{2}\otimes\hat{\Gamma}\right)\frac{\partial e(x,t)}{\partial x_k}dx.
\end{aligned}
$$

According to (2.38), there exists a real square matrix Q such that

$$\varXi \otimes D + \frac{\hat{c}\varXi\hat{G} + \hat{c}\hat{G}^T\varXi}{2} \otimes \hat{\varGamma} = Q^T Q.$$

Then, we can easily derive

$$\begin{aligned}
&\left(\frac{\partial e(x,t)}{\partial x_k}\right)^T \left(\varXi \otimes D + \frac{\hat{c}\varXi\hat{G} + \hat{c}\hat{G}^T\varXi}{2} \otimes \hat{\varGamma}\right) \frac{\partial e(x,t)}{\partial x_k} \\
&= \left(\frac{\partial (Qe(x,t))}{\partial x_k}\right)^T \frac{\partial (Qe(x,t))}{\partial x_k}.
\end{aligned}$$

Let $y(x,t) = Qe(x,t)$, for $(x,t) \in \partial\varOmega \times [0,+\infty)$ from the boundary condition (2.34), we have $y(x,t) = Qe(x,t) = 0$. In view of Lemma 1.22, one has

$$\begin{aligned}
&\sum_{k=1}^{q} \int_{\varOmega} \left(\frac{\partial y(x,t)}{\partial x_k}\right)^T \frac{\partial y(x,t)}{\partial x_k} dx \\
&\geqslant \sum_{k=1}^{q} \frac{1}{l_k^2} \int_{\varOmega} e^T(x,t) \left(\varXi \otimes D + \frac{\hat{c}\varXi\hat{G} + \hat{c}\hat{G}^T\varXi}{2} \otimes \hat{\varGamma}\right) e(x,t) dx. \qquad (2.41)
\end{aligned}$$

Therefore,

$$\dot{V}_1(t) \leqslant \int_{\varOmega} e^T(x,t) \left[\varXi \otimes \varUpsilon - \sum_{k=1}^{q} \frac{\hat{c}}{l_k^2} \left(\frac{\varXi\hat{G} + \hat{G}^T\varXi}{2} \otimes \hat{\varGamma}\right) - (\hat{c}\varXi K) \otimes \hat{\varGamma}\right] e(x,t) dx.$$

Then, following similar arguments as in the proof of Theorem 2.1, we can obtain the desired result immediately.

Remark 2.10 By using the properties of Kronecker product, we can get

$$\varXi \otimes D + \frac{\hat{c}\varXi\hat{G} + \hat{c}\hat{G}^T\varXi}{2} \otimes \hat{\varGamma} < \left(\kappa\varXi + \frac{\hat{c}\varXi\hat{G} + \hat{c}\hat{G}^T\varXi}{2}\right) \otimes \hat{\varGamma},$$

where $\kappa = \frac{2\lambda_M(D)}{\lambda_m(\hat{\varGamma})}$. When $\hat{c} > \max_{i=1,2,\ldots,N} \left\{\frac{\kappa}{|\hat{G}_{ii}|}\right\}$, the diagonal elements of matrix $\kappa\varXi + \frac{\hat{c}\varXi\hat{G}+\hat{c}\hat{G}^T\varXi}{2}$ are strictly negative. In this case, $\kappa\varXi + \frac{\hat{c}\varXi\hat{G}+\hat{c}\hat{G}^T\varXi}{2}$ has at least one negative eigenvalue. Therefore, if

$$\hat{c} > \max_{i=1,2,\ldots,N} \left\{\frac{\kappa}{|\hat{G}_{ii}|}\right\},$$

$\varXi \otimes D + \frac{\hat{c}\varXi\hat{G}+\hat{c}\hat{G}^T\varXi}{2} \otimes \hat{\varGamma}$ always has a negative eigenvalue for any matrix $\varXi$.

Remark 2.11 In the past few years, some pinning control schemes for the CRDNNs with state coupling have been developed [97, 101]. Obviously, it is also beneficial to apply the pinning control technique to study the synchronization of the CRDNNs with spatial diffusion coupling. To the best of our knowledge, this is the first to consider the pinning control problem of the CRDNNs with spatial diffusion coupling, which is a very important and challenging problem.

Remark 2.12 In Sect. 2.2, we consider a CRDNNs consisting of N identical RDNNs (2.4) with state coupling, and prove that the CRDNNs (2.5) under the pinning controllers (2.10) can synchronize as long as the coupling strength c is large enough [see Corollary 2.2]. In this section, we investigate the pinning synchronization of a CRDNNs with spatial diffusion coupling, and a criterion for reaching synchronization is established by using the designed pinning controllers (2.36) [see Theorem 2.9]. If the coupling strength $\hat{c}$ is very large, it is impossible to find matrix Ξ to satisfy the condition (2.38) in Theorem 2.9 [see Remark 2.10]. In such a case, the CRDNNs (2.32) under the pinning controllers (2.36) may not be synchronized. These results show that coupling form has a strong influence on the dynamic behavior of the CRDNNs.

2.3.3 Pinning Adaptive Synchronization of CRDNNs

Obviously, it is desirable to make the pinning feedback gains $k_i (i = 1, 2, \ldots, l)$ as small as possible. Therefore, an effective adaptive strategy to tune the feedback gains k_i is designed in this subsection.

The pinning controlled network (2.35) with adaptive feedback gains can be described by

$$
\begin{aligned}
\frac{\partial z_i(x,t)}{\partial t} &= D\triangle z_i(x,t) - Az_i(x,t) + J + Bf(z_i(x,t)) + \hat{c}\sum_{j=1}^{N}\hat{G}_{ij}\hat{\Gamma}\triangle z_j(x,t) \\
&\quad - \hat{c}k_i(t)\hat{\Gamma}(z_i(x,t) - w^*(x,t)),\ i = 1,2,\ldots,l, \\
\frac{\partial z_i(x,t)}{\partial t} &= D\triangle z_i(x,t) - Az_i(x,t) + J + Bf(z_i(x,t)) + \hat{c}\sum_{j=1}^{N}\hat{G}_{ij}\hat{\Gamma}\triangle z_j(x,t), \\
&\quad i = l+1,\ldots,N, \qquad (2.42)
\end{aligned}
$$

$$
\begin{aligned}
\dot{k}_i(t) &= \hat{\beta}_i \int_{\Omega} (z_i(x,t) - w^*(x,t))^T \hat{\Gamma}(z_i(x,t) - w^*(x,t))dx, \\
&\quad i = 1,2,\ldots,l, \qquad (2.43)
\end{aligned}
$$

where $\hat{\beta}_i$ and $k_i(0)$ are positive constants. Let $e_i(x,t) = z_i(x,t) - w^*(x,t)$. Then, the dynamics of the error vector $e_i(x,t)$ is governed by the following equation:

$$
\begin{aligned}
\frac{\partial e_i(x,t)}{\partial t} &= D\triangle e_i(x,t) - Ae_i(x,t) + Bf(z_i(x,t)) - Bf(w^*(x,t)) \\
&\quad + \hat{c}\sum_{j=1}^{N}\hat{G}_{ij}\hat{\Gamma}\triangle e_j(x,t) - \hat{c}k_i(t)\hat{\Gamma}e_i(x,t), \\
\dot{k}_i(t) &= \hat{\beta}_i\int_{\Omega} e_i^T(x,t)\hat{\Gamma}e_i(x,t)dx, i = 1,\dots,l,
\end{aligned} \tag{2.44}
$$

where $i = 1, 2, \dots, N$, and $k_i(t) \equiv 0$ for $i = l+1, l+2, \dots, N$.

Theorem 2.13 *If there exist matrices* $\Xi = \mathrm{diag}(\eta_1, \eta_2, \dots, \eta_N) \in \mathbb{R}^{N\times N} > 0$ *and* $\hat{K} = \mathrm{diag}(\underbrace{\hat{k}_1, \hat{k}_2, \dots, \hat{k}_l}_{l}, \underbrace{0, \dots, 0}_{N-l}) \in \mathbb{R}^{N\times N}$ *such that*

$$
\Xi \otimes D + \frac{\hat{c}\Xi\hat{G} + \hat{c}\hat{G}^T\Xi}{2} \otimes \hat{\Gamma} \geqslant 0, \tag{2.45}
$$

$$
\Xi \otimes \Upsilon - (\hat{c}\Xi\hat{K}) \otimes \hat{\Gamma} - \sum_{k=1}^{q}\frac{\hat{c}}{l_k^2}\left(\frac{\Xi\hat{G} + \hat{G}^T\Xi}{2} \otimes \hat{\Gamma}\right) < 0, \tag{2.46}
$$

where $\hat{k}_i > 0, i = 1, 2, \dots, l$, $\tilde{D} = \sum_{k=1}^{q}\frac{D}{l_k^2}$, $\Theta = \mathrm{diag}(\rho_1^2, \rho_2^2, \dots, \rho_n^2)$, $\Upsilon = -\tilde{D} - A + \frac{\Theta}{2} + \frac{BB^T}{2}$, *then the controlled network* (2.42) *is synchronized under the adaptive law* (2.43).

Proof Construct a Lyapunov functional for the system (2.44) as follows:

$$
V_3(t) = \frac{1}{2}\sum_{i=1}^{N}\eta_i\int_{\Omega} e_i^T(x,t)e_i(x,t)dx + \sum_{i=1}^{l}\frac{\hat{c}\eta_i}{2\hat{\beta}_i}(k_i(t) - \hat{k}_i)^2. \tag{2.47}
$$

Calculating the time derivative of $V_3(t)$ along the trajectory of system (2.44), we can get

$$
\begin{aligned}
\dot{V}_3(t) &= \sum_{i=1}^{N}\eta_i\int_{\Omega} e_i^T(x,t)\frac{\partial e_i(x,t)}{\partial t}dx \\
&\quad + \sum_{i=1}^{l}\hat{c}\eta_i(k_i(t) - \hat{k}_i)\int_{\Omega} e_i^T(x,t)\hat{\Gamma}e_i(x,t)dx \\
&= \sum_{i=1}^{N}\eta_i\int_{\Omega} e_i^T(x,t)\Big[D\triangle e_i(x,t) - Ae_i(x,t) - \hat{c}k_i(t)\hat{\Gamma}e_i(x,t) \\
&\quad + Bf(z_i(x,t)) - Bf(w^*(x,t)) + \hat{c}\sum_{j=1}^{N}\hat{G}_{ij}\hat{\Gamma}\triangle e_j(x,t)\Big]dx
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{l}\hat{c}\eta_i(k_i(t)-\hat{k}_i)\int_{\Omega}e_i^T(x,t)\hat{\Gamma}e_i(x,t)dx\\
&\leqslant \int_{\Omega}e^T(x,t)\big[\varXi\otimes D+(\hat{c}\varXi\hat{G})\otimes\hat{\Gamma}\big]\triangle e(x,t)dx\\
&+\int_{\Omega}e^T(x,t)\left[\varXi\otimes\left(-A+\frac{BB^T}{2}+\frac{\Theta}{2}\right)-(\hat{c}\varXi\hat{K})\otimes\hat{\Gamma}\right]e(x,t)dx\\
&\leqslant \int_{\Omega}e^T(x,t)\Bigg[\varXi\otimes\left(-\tilde{D}-A+\frac{BB^T}{2}+\frac{\Theta}{2}\right)-(\hat{c}\varXi\hat{K})\otimes\hat{\Gamma}\\
&-\sum_{k=1}^{q}\frac{\hat{c}}{l_k^2}\left(\frac{\varXi\hat{G}+\hat{G}^T\varXi}{2}\otimes\hat{\Gamma}\right)\Bigg]e(x,t)dx\\
&\leqslant \hat{\varrho}\|e(\cdot,t)\|_2^2,
\end{aligned}
\tag{2.48}
$$

where $e(x,t)=(e_1^T(x,t),e_2^T(x,t),\ldots,e_N^T(x,t))^T$, $\hat{\varrho}=\lambda_M(\varXi\otimes\Upsilon-(\hat{c}\varXi\hat{K})\otimes\hat{\Gamma}-\sum_{k=1}^{q}\frac{\hat{c}}{l_k^2}(\frac{\varXi\hat{G}+\hat{G}^T\varXi}{2}\otimes\hat{\Gamma}))<0$.

Obviously, $V_3(t)$ is non-increasing, and each term of $V_3(t)$ is bounded. Therefore, $k_i(t)$, $i=1,2,\ldots,l$, are bounded, and $\lim_{t\to+\infty}V_3(t)$ exists and is a non-negative real number. Because $k_i(t)$ is monotonically increasing [see (2.43)], one can conclude that $k_i(t)(i=1,2,\ldots,l)$ asymptotically converges to a finite positive value. Therefore, by the definition of $V_3(t)$, we can derive that

$$
\lim_{t\to+\infty}\sum_{i=1}^{N}\eta_i\int_{\Omega}e_i^T(x,t)e_i(x,t)dx
$$

exists and is a non-negative real number. Then, by the similar proof of Theorem 2.5, we can obtain

$$
\lim_{t\to+\infty}\|e(\cdot,t)\|_2=0.
$$

Therefore, the controlled network (2.42) is synchronized under the adaptive law (2.43). The proof is completed.

Remark 2.14 To our knowledge, very few researchers have discussed the adaptive synchronization of the CRDNNs [98–100]. In [98], adaptive method was applied to design controller feedback gains, and some sufficient conditions for adaptive synchronization were obtained. It should be noticed that a network model with state coupling was considered in [98]. In [99, 100], the authors investigated the synchronization problem of the CRDNNs using edge-based adaptive strategy, and some adaptive strategies to tune all (or a small fraction of) the coupling weights were designed. In this subsection, we study the synchronization of the CRDNNs with spatial diffusion coupling by pinning a small fraction of nodes with adaptive feedback controllers, and a criterion is obtained to guarantee the synchronization of the network (2.32).

Remark 2.15 In this chapter, some adaptive strategies to tune the coupling strength c and pinning feedback gains k_i are designed [see (2.26) and (2.43)]. Adaptive technique is a very effective method to tune the coupling strength and pinning feedback gains. Firstly, coupling strength and pinning feedback gains can be adjusted very quickly by utilizing the adaptive laws defined by the synchronization errors. Another important advantage of adaptive technique is to adjust coupling strength and pinning feedback gains for achieving suitable values in practice [129].

2.4 Numerical Examples

As an application of the above obtained theoretical results, two representative examples are given in this section.

Example 2.16 Consider the following 3-dimensional reaction-diffusion neural network

$$\begin{aligned} \frac{\partial w_i(x,t)}{\partial t} &= d_i \frac{\partial^2 w_i(x,t)}{\partial x^2} - a_i w_i(x,t) + J_i + \sum_{j=1}^{3} b_{ij} f_j(w_j(x,t)), \\ w_i(x,t) &= 0, \ (x,t) \in \partial\Omega \times [0,+\infty), \end{aligned} \tag{2.49}$$

where $i = 1, 2, 3$, $\Omega = \{x \mid -0.5 < x < 0.5\}$, $f_j(\xi) = \frac{|\xi+1|-|\xi-1|}{2}$, $d_1 = 0.2$, $d_2 = 0.2$, $d_3 = 0.3$, $a_1 = 0.3$, $a_2 = 0.2$, $a_3 = 0.3$, $J_1 = J_2 = J_3 = 0$, and the matrix $B = (b_{ij})_{3\times 3}$ is chosen as

$$B = \begin{pmatrix} 2 & -0.3 & -0.2 \\ -2.5 & 3 & -0.6 \\ -3 & -2 & 4 \end{pmatrix}.$$

Obviously, $(0, 0, 0)^T \in \mathbb{R}^3$ is an equilibrium solution of the network (2.49), and $f_j(\cdot)(j = 1, 2, 3)$ satisfies the Lipschitz condition with $\rho_j = 1$.

Now we consider a CRDNNs consisting of five linearly coupled identical model (2.49) with state coupling. The state equations of the entire network are

$$\begin{aligned} \frac{\partial z_i(x,t)}{\partial t} &= D \frac{\partial^2 z_i(x,t)}{\partial x^2} - A z_i(x,t) + B f(z_i(x,t)) + c \sum_{j=1}^{5} G_{ij} \Gamma z_j(x,t), \\ & i = 1, 2, \ldots, 5, \end{aligned} \tag{2.50}$$

where

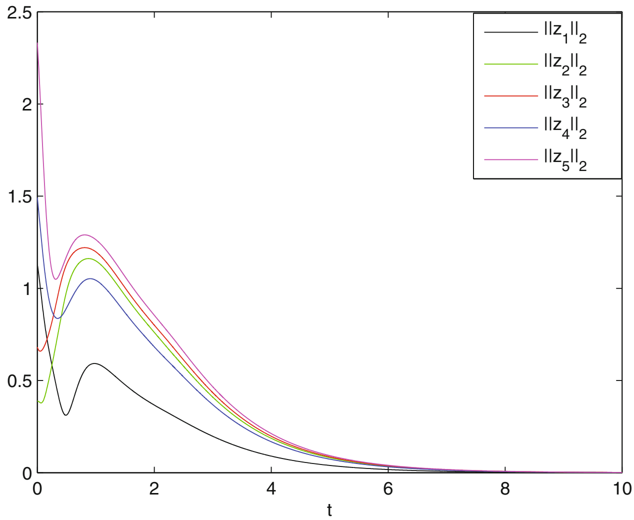

Fig. 2.1 The change processes of $\|z_i(\cdot,t)\|_2, i = 1, 2, \ldots, 5(\|z_1(\cdot,0)\|_2 = 1.1336, \|z_2(\cdot,0)\|_2 = 0.3873, \|z_3(\cdot,0)\|_2 = 0.6819, \|z_4(\cdot,0)\|_2 = 1.4933, \|z_5(\cdot,0)\|_2 = 2.3335)$

$$\Gamma = \mathrm{diag}(0.6, 0.8, 0.5),$$

$$G = \begin{pmatrix} -0.2 & 0 & 0.2 & 0 & 0 \\ 0 & -0.6 & 0 & 0.6 & 0 \\ 0 & 0 & -0.4 & 0.4 & 0 \\ 0.3 & 0 & 0 & -0.6 & 0.3 \\ 0 & 0.5 & 0 & 0 & -0.5 \end{pmatrix}.$$

The control objective here is to design appropriate control strategies such that all nodes in the CRDNNs (2.50) can synchronize onto $(0, 0, 0)^T \in \mathbb{R}^3$.

We choose the node 1 as pinned node. Select the parameters as follows: $k_1 = 0.3, \eta_1 = 0.3, \eta_2 = 0.1, \eta_3 = 0.15, \eta_4 = 0.2, \eta_5 = 0.12$. It is easy to verify that the condition (2.25) in Theorem 2.5 is satisfied. According to Theorem 2.5, the CRDNNs (2.50) under pinning control and adaptive law is synchronized. The simulation results are shown in Figs. 2.1 and 2.2.

Example 2.17 Consider the following 3-dimensional reaction-diffusion neural network

$$\begin{aligned} \frac{\partial w_i(x,t)}{\partial t} &= d_i \frac{\partial^2 w_i(x,t)}{\partial x^2} - a_i w_i(x,t) + J_i + \sum_{j=1}^{3} b_{ij} f_j(w_j(x,t)), \\ w_i(x,t) &= 0, \ (x,t) \in \partial\Omega \times [0, +\infty), \end{aligned} \tag{2.51}$$

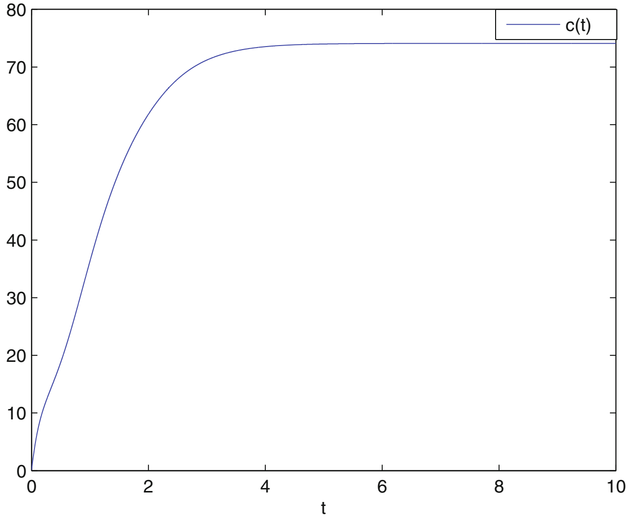

Fig. 2.2 Adaptive coupling strength ($c(0) = 0.1$)

where $i = 1, 2, 3$, $\Omega = \{x \mid -0.5 < x < 0.5\}$, $f_j(\xi) = \frac{|\xi+1|-|\xi-1|}{2}$, $d_1 = 0.6$, $d_2 = 0.7$, $d_3 = 0.5$, $a_1 = 0.6$, $a_2 = 0.8$, $a_3 = 0.4$, $J_1 = J_2 = J_3 = 0$, and the matrix $B = (b_{ij})_{3\times 3}$ is chosen as

$$B = \begin{pmatrix} 0.7\ 0.3\ 0.4 \\ 0.6\ 0.4\ 0.2 \\ 0.5\ 0.6\ 0.3 \end{pmatrix}.$$

Obviously, $(0, 0, 0)^T \in \mathbb{R}^3$ is an equilibrium solution of the network (2.51), and $f_j(\cdot)(j = 1, 2, 3)$ satisfies the Lipschitz condition with $\rho_j = 1$.

Now we consider a CRDNNs consisting of five linearly coupled identical model (2.51) with spatial diffusion coupling. The state equations of the entire network are

$$\frac{\partial z_i(x,t)}{\partial t} = D\frac{\partial^2 z_i(x,t)}{\partial x^2} - Az_i(x,t) + Bf(z_i(x,t)) + \hat{c}\sum_{j=1}^{5}\hat{G}_{ij}\hat{\Gamma}\frac{\partial^2 z_j(x,t)}{\partial x^2},$$
$$i = 1, 2, \ldots, 5, \tag{2.52}$$

where

$$\hat{\Gamma} = \mathrm{diag}(0.5, 0.7, 0.6), \;\; \hat{c} = 0.5,$$
$$\hat{G} = \begin{pmatrix} -0.1 & 0 & 0 & 0.1 & 0 \\ 0.2 & -0.2 & 0 & 0 & 0 \\ 0 & 0.3 & -0.5 & 0 & 0.2 \\ 0 & 0 & 0.4 & -0.4 & 0 \\ 0.1 & 0 & 0 & 0 & -0.1 \end{pmatrix}.$$

The control objective here is to design appropriate pinning adaptive controllers such that all nodes in the CRDNNs (2.52) can synchronize onto $(0, 0, 0)^T \in \mathbb{R}^3$.

We can find the following matrices

$$\hat{K} = \mathrm{diag}(0.6, 0.7, 0, 0, 0.6),$$
$$\Xi = \mathrm{diag}(0.4, 0.3, 0.5, 0.6, 0.4)$$

satisfying (2.45) and (2.46). That is, we only need to control the nodes 1, 2 and 5 for realizing network synchronization. From Theorem 2.13, the CRDNNs (2.52) can realize synchronization by using the designed pinning adaptive controllers. The simulation results are shown in Figs. 2.3 and 2.4.

According to the change processes of $\|z_i(\cdot, t)\|_2, i = 1, 2, \ldots, 5$, in Figs. 2.1 and 2.3, we clearly see that the CRDNNs is synchronized. Figures 2.2 and 2.4 visualize the change processes of $c(t)$ and $k_i(t)(i = 1, 2, 5)$ in time interval [0, 10]. The numerical results clearly show that $c(t)$ and $k_i(t)(i = 1, 2, 5)$ asymptotically converge to some positive real numbers.

Remark 2.18 Finite-difference method [130], as an effective numerical method, in the past ten years, has been widely used to simulate the reaction-diffusion systems. In this section, we plot the curves of the synchronization errors, adaptive coupling strength and adaptive feedback gains by employing the finite difference method. For instance, by utilizing the finite-difference method, $\frac{\partial^2 z_i(x,t)}{\partial x^2}$ and $\frac{\partial z_i(x,t)}{\partial t}$ can be approximated by

$$\frac{\partial^2 z_i(x_k, t_j)}{\partial x^2} \approx \frac{z_i(x_k, t_j) - 2z_i(x_{k-1}, t_j) + z_i(x_{k-2}, t_j)}{x_{\text{sample}}^2},$$
$$\frac{\partial z_i(x_k, t_j)}{\partial t} \approx \frac{z_i(x_k, t_j) - z_i(x_k, t_{j-1})}{t_{\text{sample}}},$$

where $x_0 = -0.5, x_{N_x} = 0.5, t_0 = 0, t_{N_t} = 10, k \in \mathcal{N}_x, j \in \mathcal{N}_t, x_k - x_{k-1} = x_{\text{sample}} = \frac{x_{N_x} - x_0}{N_x}, \mathcal{N}_x = \{0, 1, \ldots, N_x\}, t_j - t_{j-1} = t_{\text{sample}} = \frac{t_{N_t} - t_0}{N_t}, \mathcal{N}_t = \{0, 1, \ldots, N_t\}$. Similarly, $\dot{c}(t)$ and $\dot{k}_i(t)$ can be discretized.

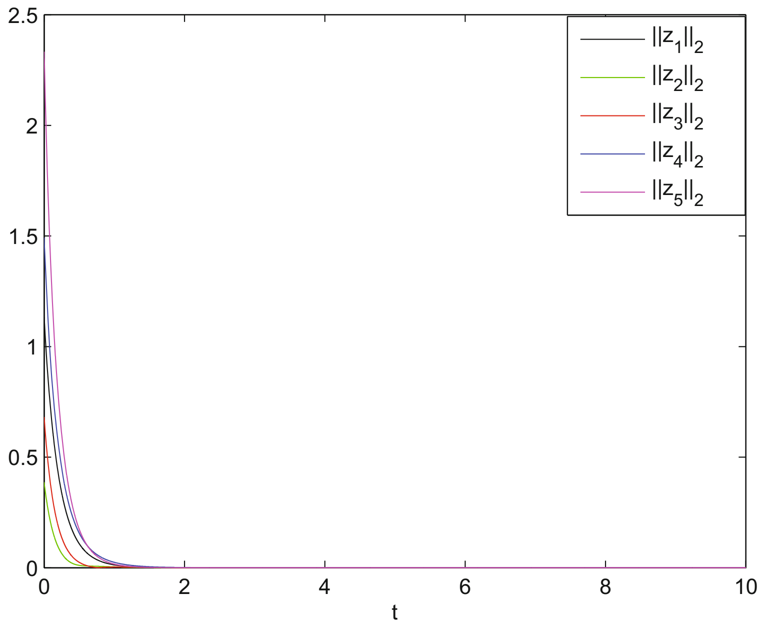

Fig. 2.3 The change processes of $\|z_i(\cdot,t)\|_2$, $i = 1, 2, \ldots, 5$($\|z_1(\cdot,0)\|_2 = 1.1336$, $\|z_2(\cdot,0)\|_2 = 0.3873$, $\|z_3(\cdot,0)\|_2 = 0.6819$, $\|z_4(\cdot,0)\|_2 = 1.4933$, $\|z_5(\cdot,0)\|_2 = 2.3335$)

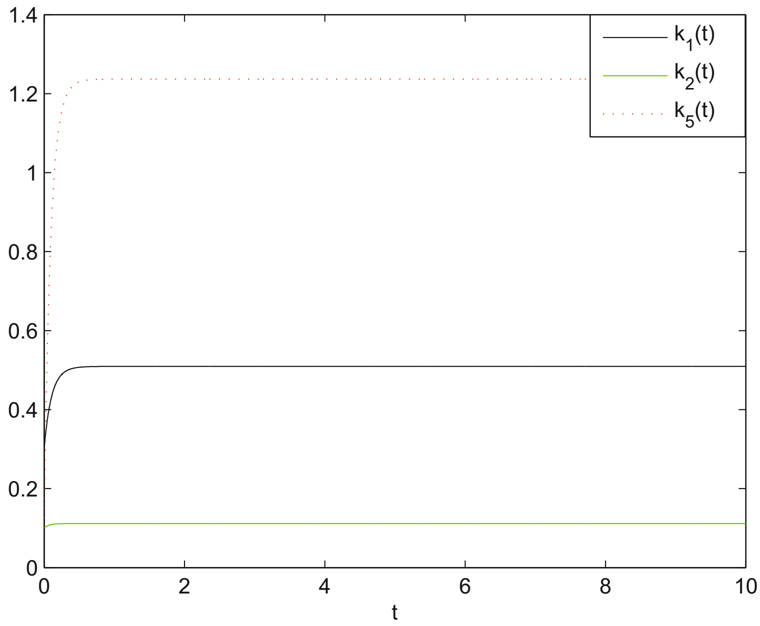

Fig. 2.4 Adaptive feedback gains ($k_1(0) = 0.3$, $k_2(0) = 0.1$, $k_5(0) = 0.1$)

2.5 Conclusions

Two kinds of linearly coupled neural networks with reaction-diffusion terms have been introduced, which have different coupling forms. We have investigated the pinning synchronization of the proposed network models. Some sufficient conditions have been established to ensure that the proposed network models are synchronized. It has been shown that, if the network is strongly connected, then the CRDNNs (2.5) under any pinning controllers in the form of (2.10) can synchronize as long as the coupling strength is large enough. On the other hand, when the coupling strength is very large, the CRDNNs (2.32) under the pinning controllers (2.36) may not be synchronized. Finally, two numerical examples have been provided to verify the correctness and effectiveness of the obtained results.

This chapter is only a first step toward the pinning control of the CRDNNs with state coupling and spatial diffusion coupling, and there are still some interesting and challenging problems deserving further investigation. For example, (1) what kind of nodes should be pinned? (2) how many nodes are needed to be pinned? In this chapter, some sufficient conditions ensuring the synchronization of the CRDNNs with spatial diffusion coupling have been established by pinning a small fraction of nodes with state feedback controllers. Practically, it may be more interesting to consider the case that the external control u_i be defined also by spatial diffusion coupling.

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