

Chapter 2

Rigid Perfectly Plastic Material

2.1 Plane Strain Deformation

The (q, s) coordinate system illustrated in Fig. 1.1 will be used. The system of equations consisting of Eqs. (1.1), (1.2), (1.15), and (1.16) is hyperbolic [3]. Therefore, the maximum friction law in the form of Eq. (1.29) may be adopted. The yield criterion (1.15) is satisfied by the following standard substitution [3]

$$\sigma_{qq} = \sigma + k_0 \cos 2\psi, \quad \sigma_{ss} = \sigma - k_0 \cos 2\psi, \quad \sigma_{qs} = k_0 \sin 2\psi. \quad (2.1)$$

Here σ is the stress invariant introduced in Eq. (1.7). It is worthy of note that σ is equal to the hydrostatic stress for the model under consideration. Substituting Eq. (2.1) into Eqs. (1.1) and (1.16) results in

$$\begin{aligned} \frac{\partial \sigma}{\partial q} - 2k_0 \sin 2\psi \frac{\partial \psi}{\partial q} + 2Hk_0 \cos 2\psi \frac{\partial \psi}{\partial s} + 2k_0 \sin 2\psi \frac{\partial H}{\partial s} &= 0, \\ H \frac{\partial \sigma}{\partial s} + 2Hk_0 \sin 2\psi \frac{\partial \psi}{\partial s} + 2k_0 \cos 2\psi \frac{\partial \psi}{\partial q} - 2k_0 \cos 2\psi \frac{\partial H}{\partial s} &= 0. \end{aligned} \quad (2.2)$$

and

$$\xi_{qq} = \lambda_1 \cos 2\psi, \quad \xi_{ss} = -\lambda_1 \cos 2\psi, \quad \xi_{qs} = \lambda_1 \sin 2\psi, \quad (2.3)$$

respectively. In Eq. (2.3), $\lambda_1 = 2k_0\lambda > 0$. It is seen from Eqs. (1.29) and (2.3) that $\xi_{qq} = \xi_{ss} = 0$ along the friction surface unless

$$\lambda_1 \rightarrow \infty \quad (2.4)$$

as $s \rightarrow 0$. The condition $\xi_{qq} = 0$ means that $s = 0$ is a characteristic curve [3] (i.e. the friction surface coincides with this characteristic curve). In this case the solution

is not singular. Therefore, this case is not considered in the present monograph. It is assumed that

$$\xi_{qq} \neq 0 \quad (2.5)$$

along the friction surface and Eq. (2.4) is satisfied. This means that the friction surface coincides with an envelope of characteristics. It follows from Eqs. (1.27), (1.29), (2.3) and (2.4) that

$$\xi_{qs} \rightarrow \infty \quad (2.6)$$

as $s \rightarrow 0$. Using assumption (ii) of Assumptions 1.1 (see p.5) it is possible to conclude that the derivative $\partial u_s / \partial q$ is bounded at $s = 0$. Then, it follows from Eqs. (1.2) and (2.6) that

$$\frac{\partial u_q}{\partial s} \rightarrow \infty \quad (2.7)$$

as $s \rightarrow 0$. Using assumption (iii) of Assumptions 1.1 (see p.5) it is possible to represent the angle ψ as

$$\psi = \frac{\pi}{4} + \psi_0 s^\beta + o(s^\beta) \quad (2.8)$$

as $s \rightarrow 0$. Here ψ_0 is independent of s and

$$\beta > 0. \quad (2.9)$$

It follows from Eq. (2.8) that

$$\sin 2\psi = 1 - 2\psi_0^2 s^{2\beta} + o(s^{2\beta}), \quad \cos 2\psi = -2\psi_0 s^\beta + o(s^\beta) \quad (2.10)$$

as $s \rightarrow 0$. Substituting Eq. (2.10) into Eq. (2.3) gives

$$\xi_{qq} = -\xi_{ss} = -2\psi_0 \lambda_1 s^\beta + o(s^\beta) \quad (2.11)$$

as $s \rightarrow 0$. Using assumption (ii) of Assumptions 1.1 (see p.5) and Eq. (1.2) it is possible to conclude that ξ_{qq} is bounded at $s = 0$. Therefore, it follows from Eqs. (2.5) and (2.11) that

$$\lambda_1 = \lambda_0 s^{-\beta} + o(s^{-\beta}) \quad (2.12)$$

as $s \rightarrow 0$. Here λ_0 is independent of s . Substituting Eq. (2.12) into Eq. (2.3) for ξ_{qs} and taking into account Eq. (2.10) results in

$$\xi_{qs} = \lambda_0 s^{-\beta} + o(s^{-\beta}) \quad (2.13)$$

as $s \rightarrow 0$. Since $\sigma_{qs} \neq 0$ and the strain rate components ξ_{qq} and ξ_{ss} are bounded at $s = 0$, it follows from Eqs. (1.4) and (2.13) that $O(W) = O(\xi_{qs}) = O(s^{-\beta})$

as $s \rightarrow 0$. Therefore, the inequality (1.5) is satisfied if $\beta < 1$. This inequality and Eq. (2.9) combine to give

$$0 < \beta < 1. \quad (2.14)$$

Substituting Eqs. (2.8) and (2.10) into Eq. (2.2)¹ leads to

$$\left[\frac{\partial \sigma}{\partial q} + 2k_0 \frac{\partial H}{\partial s} + o(1) \right] - \left[2k_0 \frac{d\psi_0}{dq} s^\beta + o(s^\beta) \right] - \{ 4k_0 H \psi_0^2 \beta s^{(2\beta-1)} + o[s^{(2\beta-1)}] \} = 0 \quad (2.15)$$

as $s \rightarrow 0$. In this equation, H and $\partial H / \partial s$ are understood to be calculated at $s = 0$. Using assumption (ii) of Assumptions 1.1 (see p. 5) it is possible to conclude that the derivative $\partial \sigma / \partial q$ is bounded at $s = 0$. Therefore, it follows from Eq. (2.15) that it is necessary to examine two cases, namely $2\beta - 1 = \beta$ and $2\beta - 1 = 0$. Assume that $2\beta - 1 = \beta$. Then, $\beta = 1$. This contradicts Eq. (2.14). Therefore, $2\beta - 1 = 0$ or

$$\beta = \frac{1}{2}. \quad (2.16)$$

Then, it follows from Eq. (2.15) that

$$\sigma = \Sigma_0 + \Sigma_1 \sqrt{s} + o(\sqrt{s}) \quad (2.17)$$

as $s \rightarrow 0$. Here Σ_0 and Σ_1 are independent of s . Substituting Eqs. (2.8), (2.10), (2.16) and (2.17) into Eq. (2.2)² yields

$$\frac{H \Sigma_1}{2} s^{-1/2} + H k_0 \psi_0 s^{-1/2} + o(s^{-1/2}) = 0$$

as $s \rightarrow 0$. Therefore,

$$\Sigma_1 = -2k_0 \psi_0. \quad (2.18)$$

Substituting this equation into Eq. (2.17) leads to

$$\sigma = \Sigma_0 - 2k_0 \psi_0 \sqrt{s} + o(\sqrt{s}) \quad (2.19)$$

as $s \rightarrow 0$. Eliminating σ in Eq. (2.15) by means of Eq. (2.19) shows that Eq. (2.15) contains the term $-4k_0 (d\psi_0/dq) \sqrt{s}$. This is the only term of the order \sqrt{s} as $s \rightarrow 0$ involved in this equation. Therefore, it is necessary to assume that ψ_0 is constant. Since the strain components ξ_{qq} and ξ_{ss} are bounded at $s = 0$, it follows from Eqs. (1.3), (2.13) and (2.16) that

$$\xi_{eq} = \frac{D}{\sqrt{s}} + o\left(\frac{1}{\sqrt{s}}\right) \quad (2.20)$$

as $s \rightarrow 0$. Here D is the strain rate intensity factor [2]. The distribution of stresses near the maximum friction surface is found from Eqs. (2.1), (2.10), (2.16), and (2.19) as

$$\begin{aligned}\sigma_{qq} &= \Sigma_0 - 4k_0\psi_0\sqrt{s} + o(\sqrt{s}), \\ \sigma_{ss} &= \Sigma_0 + o(\sqrt{s}), \quad \sigma_{qs} = k_0(1 - 2\psi_0^2s) + o(s)\end{aligned}\tag{2.21}$$

as $s \rightarrow 0$. The distribution of velocity follows from Eqs. (2.11) and (2.13). In particular, substituting Eq. (2.11) for ξ_{ss} into Eq. (1.2), eliminating λ_1 by means of Eq. (2.12) and integrating result in

$$u_s = 2\psi_0\lambda_0s + o(s)\tag{2.22}$$

as $s \rightarrow 0$. Here it has been taken into account that this velocity component should satisfy the boundary condition (1.8). Taking into account Eqs. (2.7), (2.13) and (2.16) the equation for the shear strain rate in Eq. (1.2) can be represented as $\partial u_q / \partial s = 2\lambda_0s^{-1/2} + o(s^{-1/2})$ as $s \rightarrow 0$. Integrating leads to

$$u_q = u_0 + 4\lambda_0\sqrt{s} + o(\sqrt{s})\tag{2.23}$$

as $s \rightarrow 0$. Here u_0 is independent of s .

2.2 Axisymmetric Deformation

The (q, θ, s) coordinate system illustrated in Fig. 1.2 will be used. The system of equations consisting of Eqs. (1.9), (1.10), (1.18), and (1.19) is not hyperbolic [3]. Therefore, the maximum friction law in the form of Eq. (1.28) should be adopted. Equations (1.14), (1.18) and (1.28) combine to give

$$\sigma_{qq} = \sigma_{ss} = \sigma_{\theta\theta} = \sigma_h\tag{2.24}$$

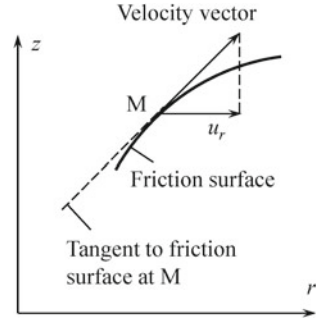
at $s = 0$. Substituting this equation into Eq. (1.19) for $\xi_{\theta\theta}$ shows that $\xi_{\theta\theta} = 0$ unless

$$\lambda \rightarrow \infty\tag{2.25}$$

as $s \rightarrow 0$. The condition $\xi_{\theta\theta} = 0$ contradicts Eq. (1.10) for $\xi_{\theta\theta}$ if $u_r \neq 0$ and $u_r \neq 0$ if the regime of sliding occurs (the velocity vector is not equal to zero) and the tangent to the friction surface is not parallel to the z -axis, which is the axis of symmetry of the process, at a generic point M of the friction surface (Fig. 2.1). Therefore, Eq. (2.25) is in general valid. Then, it follows from Eqs. (1.19) and (1.28) that

$$\xi_{qs} \rightarrow \infty\tag{2.26}$$

Fig. 2.1 Illustration of the condition $\xi_{\theta\theta} \neq 0$ at friction surface



as $s \rightarrow 0$. Using assumption (ii) of Assumptions 1.1 (see p. 5) it is possible to conclude that the derivative $\partial u_s / \partial q$ is bounded at $s = 0$. Then, it follows from Eqs. (1.10) and (2.26) that

$$\frac{\partial u_q}{\partial s} \rightarrow \infty \quad (2.27)$$

as $s \rightarrow 0$. Using assumption (iii) of Assumptions 1.1 (see p. 5) it is possible to represent u_q as

$$u_q = u_0 + u_1 s^\beta + o(s^\beta) \quad (2.28)$$

as $s \rightarrow 0$. Here u_0 and u_1 are independent of s . It follows from assumption (i) of Assumptions 1.1 (see p. 5) that $\beta > 0$. On the other hand, it is seen from Eqs. (2.27) and (2.28) that $\beta < 1$. Hence,

$$0 < \beta < 1. \quad (2.29)$$

Using assumptions (i) and (ii) of Assumptions 1.1 (see p. 5) it is possible to find from Eq. (1.10) that the strain rate components ξ_{qq} and $\xi_{\theta\theta}$ are bounded as $s \rightarrow 0$. Then, it follows from Eqs. (1.10) and (1.13) that the strain rate component ξ_{ss} is bounded as $s \rightarrow 0$. Therefore,

$$\xi_{qq} = O(1), \quad \xi_{ss} = O(1) \quad \text{and} \quad \xi_{\theta\theta} = O(1) \quad (2.30)$$

as $s \rightarrow 0$. Substituting Eq. (2.28) into Eq. (1.10) leads to

$$\xi_{qs} = O(s^{\beta-1}) \quad (2.31)$$

as $s \rightarrow 0$. It is seen from Eqs. (1.12), (2.29), (2.30) and (2.31) that the inequality (1.5) is satisfied. Equations (1.19) for ξ_{qs} , (1.28) and (2.31) combine to give $\lambda = O(s^{\beta-1})$ as $s \rightarrow 0$. This equation can be rewritten as

$$\lambda = \lambda_0 s^{\beta-1} + o(s^{\beta-1}) \quad (2.32)$$

as $s \rightarrow 0$. Here λ_0 is independent of s . Substituting Eq. (2.32) into Eq. (1.19) leads to

$$\begin{aligned}
\xi_{qq} &= [\lambda_0 s^{\beta-1} + o(s^{\beta-1})] (2\sigma_{qq} - \sigma_{ss} - \sigma_{\theta\theta}), \\
\xi_{ss} &= [\lambda_0 s^{\beta-1} + o(s^{\beta-1})] (2\sigma_{ss} - \sigma_{\theta\theta} - \sigma_{qq}), \\
\xi_{\theta\theta} &= [\lambda_0 s^{\beta-1} + o(s^{\beta-1})] (2\sigma_{\theta\theta} - \sigma_{ss} - \sigma_{qq}).
\end{aligned} \tag{2.33}$$

as $s \rightarrow 0$. Then, it follows from Eq. (2.30) that

$$\begin{aligned}
2\sigma_{qq} - \sigma_{ss} - \sigma_{\theta\theta} &= A_q s^{1-\beta} + o(s^{1-\beta}), \\
2\sigma_{ss} - \sigma_{\theta\theta} - \sigma_{qq} &= A_s s^{1-\beta} + o(s^{1-\beta}), \\
2\sigma_{\theta\theta} - \sigma_{qq} - \sigma_{ss} &= A_\theta s^{1-\beta} + o(s^{1-\beta})
\end{aligned} \tag{2.34}$$

as $s \rightarrow 0$. Here A_q , A_s and A_θ are independent of s . Equation (2.34) can be transformed to

$$\begin{aligned}
\sigma_{qq} - \sigma_{ss} &= \frac{(A_q - A_s)}{3} s^{1-\beta} + o(s^{1-\beta}), \\
\sigma_{ss} - \sigma_{\theta\theta} &= \frac{(A_s - A_\theta)}{3} s^{1-\beta} + o(s^{1-\beta}), \\
\sigma_{\theta\theta} - \sigma_{qq} &= \frac{(A_\theta - A_q)}{3} s^{1-\beta} + o(s^{1-\beta})
\end{aligned} \tag{2.35}$$

as $s \rightarrow 0$. Using assumption (iii) of Assumptions 1.1 (see p. 5) and taking into account Eq. (1.28) it is possible to represent σ_{qs} as

$$\sigma_{qs} = k_0 + O(s^\omega) \tag{2.36}$$

as $s \rightarrow 0$. Here $\omega > 0$. Substituting Eqs. (2.35) and (2.36) into Eq. (1.18) results in

$$\frac{[(A_q - A_s)^2 + (A_s - A_\theta)^2 + (A_\theta - A_q)^2]}{9} s^{2(1-\beta)} = O(s^\omega). \tag{2.37}$$

as $s \rightarrow 0$. It is worthy of note that the coefficient of $s^{2(1-\beta)}$ never vanishes. Then, it follows from Eq. (2.37) that $\omega = 2(1 - \beta)$ and Eq. (2.36) can be rewritten as

$$\sigma_{qs} = k_0 + k_1 s^{2(1-\beta)} + o[s^{2(1-\beta)}]. \tag{2.38}$$

as $s \rightarrow 0$. Here k_1 is independent of s . Using Eq. (1.14) it is possible to rewrite Eq. (2.34) as

$$\begin{aligned}
\sigma_{qq} - \sigma_h &= \frac{A_q}{3} s^{1-\beta} + o(s^{1-\beta}), \\
\sigma_{ss} - \sigma_h &= \frac{A_s}{3} s^{1-\beta} + o(s^{1-\beta}), \quad \sigma_{\theta\theta} - \sigma_h = \frac{A_\theta}{3} s^{1-\beta} + o(s^{1-\beta})
\end{aligned} \tag{2.39}$$

as $s \rightarrow 0$. The derivative $\partial\sigma_{ss}/\partial s$ is the first term of Eq. (1.9)². The other terms of this equation are bounded by assumptions (i) and (ii) of Assumptions 1.1 (see p. 5). Therefore, the first term must also be bounded. Using Eq. (2.39) this term can be represented as

$$\frac{\partial\sigma_{ss}}{\partial s} = \frac{\partial\sigma_h}{\partial s} + \frac{A_s(1-\beta)}{3}s^{-\beta} + o(s^{-\beta}) \quad (2.40)$$

as $s \rightarrow 0$. The second term on the right hand side of this equation approaches infinity as $s \rightarrow 0$. In order to cancel this term, it is necessary to assume that

$$\sigma_h = \sigma_0 - \frac{A_s s^{(1-\beta)}}{3} + o[s^{(1-\beta)}] \quad (2.41)$$

as $s \rightarrow 0$. Here σ_0 is independent of s . Equations (2.39) and (2.41) combine to give

$$\begin{aligned} \sigma_{qq} &= \sigma_0 + \frac{(A_q - A_s)}{3}s^{(1-\beta)} + o[s^{(1-\beta)}], \quad \sigma_{ss} = \sigma_0 + o[s^{(1-\beta)}], \\ \sigma_{\theta\theta} &= \sigma_0 + \frac{(A_\theta - A_s)}{3}s^{(1-\beta)} + o[s^{(1-\beta)}]. \end{aligned} \quad (2.42)$$

Substituting this equation and Eq. (2.38) into Eq. (1.9)¹ leads to

$$\begin{aligned} &\left\{ \frac{\partial\sigma_0}{\partial q} + k_0 \left(2 \frac{\partial H}{\partial s} + \frac{H}{r} \frac{\partial r}{\partial s} \right) + o(1) \right\} + \\ &\left\{ \frac{1}{3} \left[\frac{d(A_q - A_s)}{dq} + \frac{(A_q - A_\theta)}{r} \frac{\partial r}{\partial q} \right] s^{(1-\beta)} + o[s^{(1-\beta)}] \right\} + \\ &\left\{ 2Hk_1(1-\beta)s^{(1-2\beta)} + o[s^{(1-2\beta)}] \right\} = 0 \end{aligned} \quad (2.43)$$

as $s \rightarrow 0$. If $1 - \beta = 1 - 2\beta$ then $\beta = 0$. This contradicts Eq. (2.29). Therefore, it follows from Eq. (2.43) that $1 - 2\beta = 0$ or

$$\beta = \frac{1}{2} \quad (2.44)$$

and

$$\frac{d(A_q - A_s)}{dq} + \frac{(A_q - A_\theta)}{r} \frac{\partial r}{\partial q} = 0$$

at $s = 0$. Substituting Eqs. (2.30) and (2.31) into Eq. (1.11) and using Eq. (2.44) it is possible to arrive at Eq. (2.20). The distribution of stresses near the maximum friction surface is found from Eqs. (2.38), (2.42), and (2.44) as

$$\begin{aligned} \sigma_{qq} &= \sigma_0 + \frac{(A_q - A_s)}{3}\sqrt{s} + o(\sqrt{s}), \quad \sigma_{ss} = \sigma_0 + o(\sqrt{s}), \\ \sigma_{\theta\theta} &= \sigma_0 + \frac{(A_\theta - A_s)}{3}\sqrt{s} + o(\sqrt{s}), \quad \sigma_{qs} = k_0 + k_1 s + o(s) \end{aligned} \quad (2.45)$$

as $s \rightarrow 0$. The distribution of the velocity component u_q follows from Eqs. (2.28) and (2.44) in the form

$$u_q = u_0 + u_1 \sqrt{s} + o(\sqrt{s}) \quad (2.46)$$

as $s \rightarrow 0$. Using this representation of u_q it is possible to determine the shear strain rate from Eq. (1.10) as

$$\xi_{qs} = \frac{u_1}{4\sqrt{s}} + o\left(\frac{1}{\sqrt{s}}\right) \quad (2.47)$$

as $s \rightarrow 0$. Equations (2.32) and (2.44) combine to give

$$\lambda = \lambda_0 s^{-1/2} + o(s^{-1/2}) \quad (2.48)$$

as $s \rightarrow 0$. Substituting this equation along with Eqs. (1.28) and (2.47) into Eq. (1.19) yields

$$\lambda_0 = \frac{u_1}{12k_0}. \quad (2.49)$$

Equations (2.33) and (2.34) combine to give $\xi_{ss} = \lambda_0 A_s + o(1)$ as $s \rightarrow 0$. Substituting Eq. (2.49) into this equation and integrating results in

$$u_s = \frac{u_1 A_s}{12k_0} s + o(s) \quad (2.50)$$

as $s \rightarrow 0$. It has been taken into account here that u_s should satisfy the boundary condition (1.8).

In contrast to plane strain deformation, the constitutive equations for axisymmetric deformation depend on the yield criterion. The analysis presented in this section is based on the von Mises yield criterion. However, it has been shown in [2] that Eq. (2.20) is valid for quite a general smooth yield criterion and in [1] that this equation is valid for Tresca yield criterion.

2.3 Compression of a Layer Between Rough Plates

As an illustrative example of singular solutions, an approximate solution for compression of a rigid plastic layer between parallel plates is given in this section. This solution can be found in any monograph on plasticity theory, for example [3].

2.3.1 Statement of the Problem

Consider a rigid perfectly plastic layer of thickness $2h$ and width $2w$. The layer is compressed between two parallel plates. The speed of each plate is V . The Cartesian coordinate system (x, y) is chosen as shown in Fig. 2.2. The process has two axes of symmetry, $x = w$ and $y = 0$. Therefore, it is sufficient to consider the domain $0 \leq x \leq w$ and $0 \leq y \leq h$.

Let u_x and u_y be the velocity components referred to the Cartesian coordinate system. The exact velocity boundary conditions are

$$u_y = 0 \quad (2.51)$$

for $y = 0$ and

$$u_y = -V \quad (2.52)$$

for $y = h$. The exact velocity boundary condition at $x = w$ is replaced with the following approximate condition [3]

$$\int_0^h u_x dy = 0. \quad (2.53)$$

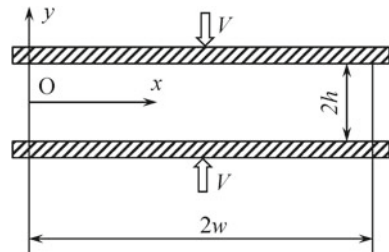
It is understood here that the velocity component u_x involved in the integrand is calculated at $x = w$.

Let σ_{xx} , σ_{yy} and σ_{xy} be the stress components referred to the Cartesian coordinate system. The exact stress boundary conditions are

$$\sigma_{xy} = 0 \quad (2.54)$$

for $y = 0$ and the maximum friction law at $y = h$. The exact stress boundary conditions at $x = 0$ are replaced with the following approximate condition [3]

Fig. 2.2 Compression of a plastic layer between two parallel plates notation



$$\int_0^h \sigma_{xx} dy = 0. \quad (2.55)$$

It is understood here that the stress component σ_{xx} involved in the integrand is calculated at $x = 0$.

In the Cartesian coordinate system, Eqs. (1.1) and (1.2) become

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \quad (2.56)$$

and

$$\xi_{xx} = \frac{\partial u_x}{\partial x}, \quad \xi_{yy} = \frac{\partial u_y}{\partial y}, \quad \xi_{xy} = \frac{1}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right), \quad (2.57)$$

respectively.

It is evident that in the case under consideration the normal distance from the maximum friction surface is

$$s = h - y. \quad (2.58)$$

The coordinate line $s = 0$ (Fig. 1.1) corresponds to $y = h$ and the angle ψ shown in Fig. 1.3 is also the angle which the principal stress direction corresponding to σ_1 makes with the x -axis.

2.3.2 Solution

The yield criterion (1.15) and the associate flow rule (1.16) in the Cartesian coordinate system read

$$(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2 = 4k_0^2. \quad (2.59)$$

and

$$\xi_{xx} = \lambda (\sigma_{xx} - \sigma_{yy}), \quad \xi_{yy} = \lambda (\sigma_{yy} - \sigma_{xx}), \quad \xi_{xy} = 2\lambda \sigma_{xy}, \quad (2.60)$$

respectively. By analogy to Eq. (2.1) the yield criterion (2.59) is satisfied by the following substitution

$$\sigma_{xx} = \sigma + k_0 \cos 2\psi, \quad \sigma_{yy} = \sigma - k_0 \cos 2\psi, \quad \sigma_{xy} = k_0 \sin 2\psi. \quad (2.61)$$

Equations (2.60) and (2.61) combine to give

$$\xi_{xx} = 2k_0\lambda \cos 2\psi, \quad \xi_{yy} = -2k_0\lambda \cos 2\psi, \quad \xi_{xy} = 2k_0\lambda \sin 2\psi. \quad (2.62)$$

Eliminating λ between these equations results in

$$\xi_{xx} + \xi_{yy} = 0, \quad \frac{2\xi_{xy}}{\xi_{xx} - \xi_{yy}} = \tan 2\psi \quad (2.63)$$

The direction of flow (Fig. 2.2) dictates that $\sigma_{xy} \geq 0$ at $y = h$ in the range $0 \leq x \leq w$. The maximum friction law (1.29) becomes

$$\psi = \pi/4 \quad (2.64)$$

for $y = h$. Substituting Eq. (2.61) into Eq. (2.56) yields

$$\begin{aligned} \frac{\partial \sigma}{\partial x} - 2k_0 \sin 2\psi \frac{\partial \psi}{\partial x} + 2k_0 \cos 2\psi \frac{\partial \psi}{\partial y} &= 0, \\ \frac{\partial \sigma}{\partial y} + 2k_0 \sin 2\psi \frac{\partial \psi}{\partial y} + 2k_0 \cos 2\psi \frac{\partial \psi}{\partial x} &= 0. \end{aligned} \quad (2.65)$$

In the case of $h/w \ll 1$ it is reasonable to assume that ψ is independent of x [3]. Then, Eq. (2.65) becomes

$$\frac{\partial \sigma}{\partial x} + 2k_0 \cos 2\psi \frac{d\psi}{dy} = 0, \quad \frac{\partial \sigma}{\partial y} + 2k_0 \sin 2\psi \frac{d\psi}{dy} = 0. \quad (2.66)$$

Equation (2.66)² can be immediately integrated to give $\sigma = k_0 \cos 2\psi + \Phi_1(x)$ where $\Phi_1(x)$ is an arbitrary function of x . Using this solution to eliminate σ in Eq. (2.66)¹ and taking into account that the second term of this equation is independent of x it is possible to find that

$$\frac{\sigma}{k_0} = B - Ax + \cos 2\psi \quad (2.67)$$

and

$$2 \cos 2\psi \frac{d\psi}{dy} = A. \quad (2.68)$$

Here A and B are constant. It follows from the boundary condition (2.54) and Eq. (2.61) that $\psi = 0$ at $y = 0$. The solution of Eq. (2.68) satisfying this boundary condition is $\sin 2\psi = Ay$. Using the boundary condition (2.64) it is possible to determine that $A = 1/h$. Then,

$$\sin 2\psi = \frac{y}{h} \quad (2.69)$$

and Eq. (2.67) becomes

$$\frac{\sigma}{k_0} = B - \frac{x}{h} + \cos 2\psi. \quad (2.70)$$

The value of B can be found from the boundary condition (2.55) using Eqs. (2.61), (2.69) and (2.70). However, it is not necessary to determine this value for demonstrating that the solution is singular. Equations (2.57) and (2.63) combine to give

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \quad \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} \right) \tan 2\psi. \quad (2.71)$$

In the case of $h/w \ll 1$ it is reasonable to assume that u_y is independent of x [3]. Then, Eq. (2.71) becomes

$$\frac{\partial u_x}{\partial x} = -\frac{du_y}{dy}, \quad \frac{\partial u_x}{\partial y} = -2\frac{du_y}{dy} \tan 2\psi. \quad (2.72)$$

Differentiating Eq. (2.72)¹ with respect to y and Eq. (2.72)² with respect to x leads to

$$\frac{\partial^2 u_x}{\partial x \partial y} = -\frac{d^2 u_y}{dy^2}, \quad \frac{\partial^2 u_x}{\partial y \partial x} = 0.$$

Then, using the equation

$$\frac{\partial^2 u_x}{\partial x \partial y} = \frac{\partial^2 u_x}{\partial y \partial x}$$

it is possible to find that

$$\frac{d^2 u_y}{dy^2} = 0.$$

The solution of this equation satisfying the boundary conditions (2.51) and (2.52) is

$$\frac{u_y}{V} = -\frac{y}{h}. \quad (2.73)$$

Equations (2.72)² and (2.73) combine to give

$$\frac{\partial u_x}{\partial y} = \frac{2V}{h} \tan 2\psi. \quad (2.74)$$

Replacing here differentiation with respect to y with differentiation with respect to ψ by means of Eq. (2.69) yields

$$\frac{\partial u_x}{\partial \psi} = 4V \sin 2\psi. \quad (2.75)$$

Integrating this equation results in

$$\frac{u_x}{V} = u_0 - 2 \cos 2\psi. \quad (2.76)$$

Here u_0 is an arbitrary function of x . It is seen from Eq. (2.72)¹ that the derivative $\partial u_x / \partial x$ is independent of x . It is therefore evident that u_0 is a linear function of x and Eq. (2.76) becomes

$$\frac{u_x}{V} = a + bx - 2 \cos 2\psi \quad (2.77)$$

where a and b are constant. Substituting Eqs. (2.73) and (2.77) into Eq. (2.72)¹ shows that $b = 1/h$. Then, Eq. (2.77) becomes

$$\frac{u_x}{V} = a + \frac{x}{h} - 2 \cos 2\psi. \quad (2.78)$$

The value of a can be found from the boundary condition (2.53) using Eqs. (2.69) and (2.78). However, it is not necessary to determine this value for demonstrating that the solution is singular. In particular, it is seen from Eq. (2.73) that $\partial u_y / \partial x = 0$. Therefore, it follows from Eq. (2.57) that $2\xi_{xy} = \partial u_x / \partial y$. Then, using Eq. (2.74)

$$\xi_{xy} = \frac{V}{h} \tan 2\psi. \quad (2.79)$$

Expanding the left hand side of Eq. (2.69) in a series in the vicinity of $\psi = \pi/4$ and using Eq. (2.58) yield

$$s = 2h \left(\psi - \frac{\pi}{4} \right)^2 + o \left[\left(\psi - \frac{\pi}{4} \right)^2 \right] \quad (2.80)$$

as $\psi \rightarrow \pi/4$. Expanding the right hand side of Eq. (2.79) in a series in the vicinity of $\psi = \pi/4$ and using Eq. (2.80) yield

$$\xi_{xy} = \frac{V}{\sqrt{2hs}} + o \left(\frac{1}{\sqrt{s}} \right) \quad (2.81)$$

as $s \rightarrow 0$. Since the strain rate components ξ_{xx} and ξ_{yy} are bounded, Eq. (2.81) coincides with Eq. (2.20).

References

1. Alexandrov S, Richmond O (1998) Asymptotic behavior of the velocity field in the case of axially symmetric flow of a material obeying the Tresca condition. Dokl Phys 43:362–364
2. Alexandrov S, Richmond O (2001) Singular plastic flow fields near surfaces of maximum friction stress. Int J Non Linear Mech 36:1–11
3. Hill R (1950) The mathematical theory of plasticity. Clarendon Press, Oxford

Singular Solutions in Plasticity

Alexandrov, S.

2018, XI, 107 p. 40 illus., Softcover

ISBN: 978-981-10-5226-2