

Chapter 2

Sinusoidal Waves as Random Variables

Abstract The probability distribution associated with a random variable is a basic notion on which signal analysis of sounds is based. The probability distribution for a sum of a pair of independent random variables underlies the notion of convolution that is basic to signal analysis from the point of view of the linear system theory. The correlation between a pair of random variables is a standard metric to characterize pairs of random variables as well as the independence of random variables. The difference between the uncorrelated (or orthogonality) and statistically independent pairs of random variables renders typical auditory effects on sound image sensations. The perception of sound image experienced during binaural listening to an independent pair of random noise is called subjective diffuseness. An example of the probability density function for a sinusoidal wave is given to help understand orthogonality as being different from the statistical independence. The square correlation is a measure that represents the energy ratio of the uncorrelated pair of variables. Direct sound and its reverberation is a typical example of an uncorrelated pair in reverberant sound fields.

Keywords Probability · Random variables · Expectation · Convolution · Statistical independence · Orthogonality · Uncorrelation · Correlation function · Probability density function · Probability density function for sinusoidal function · Binaural listening · Subjective diffuseness · Sound image perception

2.1 Random Variables

2.1.1 Probability Distribution and Expectation

The concept of correlation is now introduced into random variable analysis. The expressions for correlation in the previous subsections have seemingly not been developed in terms of non-deterministic events; however, the time average used in the various expressions can be interpreted according to the ensemble average or expectation of random variables. By random variable, here denoted as X , is meant a variable that can have any of a range of values that occur randomly, i.e., occurs

non-deterministically, but can be described with an associated probability. To each random variable individually, this probability, denoted P , is given deterministically; it is a real number in the range $0 \leq P \leq 1$ that provides an estimate of how likely an event of interest occurs [1].

The probability associated with a random variable X taking value x is denoted by $P_X(X = x)$. Suppose that a random variable X takes one of two values, x_0 or x_1 , with probability p and q , respectively, where $p + q = 1$. The expectation of the random variable X is defined as

$$E[X] = p \cdot x_0 + q \cdot x_1. \quad (2.1)$$

Consider a pair of random variables X and Y . One of the main interests in random variable analysis would be how likely the event Y (or X) occurs if X (or Y) occurs. If some statistical relevance can be attached to the pair occurring, then it might be informative and helpful in a probabilistic prediction of two random events. Take a pair of random variables X and Y with probabilities $\mathbf{p}_X$ and $\mathbf{p}_Y$, respectively, such that

$$S_{xy} = \begin{pmatrix} \mathbf{x}^T \\ \mathbf{y}^T \end{pmatrix} = \begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix} \quad (2.2)$$

and

$$P_{XY} = \begin{pmatrix} \mathbf{p}_x^T \\ \mathbf{p}_y^T \end{pmatrix} = \begin{pmatrix} p_{x_0} & p_{x_1} \\ p_{y_0} & p_{y_1} \end{pmatrix} \quad (2.3)$$

where

$$p_{x_0} + p_{x_1} = 1 \quad (2.4)$$

$$p_{y_0} + p_{y_1} = 1. \quad (2.5)$$

The distribution of probabilities for the random events is rewritten as a matrix,

$$P_{XY} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \quad (2.6)$$

where p_{01} denotes the probability that the outcome ($X = x_0, Y = y_1$) occurs, and likewise for the other entries. Recall that to each random variable, there is associated a probability distribution. The probability $p_X(X = x)$ in which the event X takes x is determined unchanged, irrespective of whether the actual random outcome of the event is observed for the random variable. This condition or assumption that the probability assigned to a random variable is independent of the outcome seems unlikely in a practical situation. However, a stationary random process or a series of random events could be fundamental in understanding theoretically the random nature of events or signals from a probabilistic point of view.

In the matrix P_{XY} , if

$$p_{ij} = p_X(X = x_i) \cdot p_Y(Y = y_j) \quad (2.7)$$

holds, then the random variables X and Y are called statistically independent to each other. With this notion of independence

$$\begin{aligned} p_{00} + p_{01} &= p_X(X = x_0) \cdot p_Y(Y = y_0) + p_X(X = x_0) \cdot p_Y(Y = y_1) \\ &= p_X(X = x_0)(p_Y(Y = y_0) + p_Y(Y = y_1)) \\ &= p_X(X = x_0) \end{aligned} \quad (2.8)$$

$$\begin{aligned} p_{10} + p_{11} &= p_X(X = x_1) \cdot p_Y(Y = y_0) + p_X(X = x_1) \cdot p_Y(Y = y_1) \\ &= p_X(X = x_1), \end{aligned} \quad (2.9)$$

where

$$p_Y(Y = y_0) + p_Y(Y = y_1) = 1. \quad (2.10)$$

Similarly,

$$p_{00} + p_{10} = p_Y(Y = y_0) \quad (2.11)$$

$$p_{01} + p_{11} = p_Y(Y = y_1), \quad (2.12)$$

where

$$p_X(X = x_0) + p_X(X = x_1) = 1 \quad (2.13)$$

and $p_X(X = x_0)$ indicates that the random variable X takes $X = x_0$ in the random events solely for the random variable X .

Going one step further now toward a practical situation or an instance with no statistical independence, the probability is rewritten as

$$\begin{aligned} p_{ij} &= p_{XY}(X = x_i, Y = y_j) \\ &= p_{X|Y}(X = x_i|Y = y_j) \cdot p_Y(Y = y_j), \end{aligned} \quad (2.14)$$

where

$$p_{X|Y}(X = x_i|Y = y_j) = \frac{p_{XY}(X = x_i, Y = y_j)}{p_Y(Y = y_j)} \quad (2.15)$$

denotes the conditional probability of the event $X = x_i$, subject to the occurrence of the event $Y = y_j$ with probability $p_Y(Y = y_j)$, which is assigned to the random variable Y . Even if the conditional probability is introduced in regard to random variables, Eqs. 2.10 and 2.13 still hold. Specifically

$$p_{X|Y}(X = x_0|Y = y_0) + p_{X|Y}(X = x_1|Y = y_0) = 1 \quad (2.16)$$

$$p_{X|Y}(X = x_0|Y = y_1) + p_{X|Y}(X = x_1|Y = y_1) = 1 \quad (2.17)$$

$$p_{Y|X}(Y = y_0|X = x_0) + p_{Y|X}(Y = y_1|X = x_0) = 1 \quad (2.18)$$

$$p_{Y|X}(Y = y_0|X = x_1) + p_{Y|X}(Y = y_1|X = x_1) = 1. \quad (2.19)$$

If the random variables X and Y are not statistically independent, the probability associated with both X and Y occurring could be obtained from the probabilities for the joint random events. The probabilities given by the matrix P_{XY} is called the joint probability distribution for a pair of random variables.

One of the main features of importance regarding the joint distribution would be the expectation or ensemble average for a sum of the random variables $Z = X + Y$. The expectation can be formally defined as

$$E[Z] = E[X + Y] \quad (2.20)$$

Following the relations of Eqs. 2.16–2.19

$$E[Z] = E[X] + E[Y], \quad (2.21)$$

which indicates the taking an ensemble average is a linear operation. The relation above holds regardless of the interdependence between the random variables X and Y . It is not necessary to take into account the joint probabilities for random variables for estimating the expectation of the sum of the random variables.

2.1.2 *Sum of Independent Random Variables and Convolution*

Consider again a pair of independent random variables X and Y . The probability distribution for $Z = X + Y$ can be written as

$$p(Z = z_i) = \sum_i p(X = x_i)p(Y = y_i = z_i - x_i) \quad (2.22)$$

where $z_i = x_i + y_i$. This equation defines the operation referred to as convolution.

Consider a pair of identical dice with faces numbered from 1 to 6. The roll of the two dice is assumed independent one from the other. Now sum the numbers showing on the upper surface after each roll. The sum takes random integer values between 2 and 12. The probability distribution associated with the sum can be expressed as

$$\begin{aligned}
p(Z = z) &= \sum_i p_X(X = x_i) p_Y(Y = z - x_i) \\
&= \frac{1}{36} \sum_i N(x_i, z - x_i) \\
&= \frac{1}{36} M(z),
\end{aligned} \tag{2.23}$$

where $M(z)$ denotes the number of the combinations for x_i and y_i that sum to make z . These combinations are given in matrix form by

$$C \cdot \mathbf{d} = \mathbf{m}, \tag{2.24}$$

where C indicates the combination or convolution matrix counting the combinations, $\mathbf{d}$ denotes the 6-dimensional vector for which all entries are unity, and $\mathbf{m}$ is the 11-dimensional vector giving the number of combinations using the entry $M(z)$ for $Z = z$; that is,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}. \tag{2.25}$$

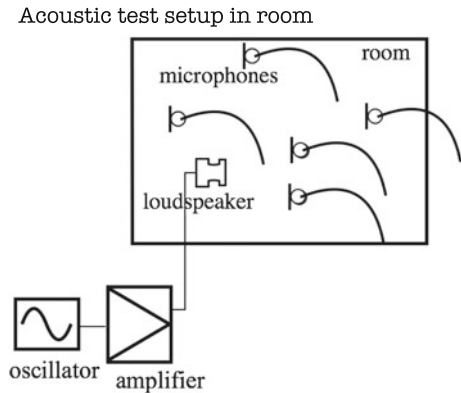
2.2 Correlations

2.2.1 Correlation Functions for Random Variables

According to the definition of the expectation, the cross-correlation coefficient between two random variables X and Y can be rewritten as

$$R_{XY} = \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sqrt{E[X^2]} \cdot \sqrt{E[Y^2]}}, \tag{2.26}$$

Fig. 2.1 Schematic of an acoustic test setup in which an ensemble average is taken



where

$$\mu_X = E[X] \quad (2.27)$$

$$\mu_Y = E[Y]. \quad (2.28)$$

Similarly, the cross-correlation function is defined as

$$r_{XY}(\tau) = E[X(t)Y(t - \tau)] \quad (2.29)$$

and in setting $X = Y$, the auto-correlation function becomes

$$r_{XX}(\tau) = E[X(t)X(t - \tau)]. \quad (2.30)$$

The ensemble average is taken for a set in which many pairs of observed variables X and Y are available at every time and the ensemble average is assumed to be independent of the observation time t . Figure 2.1 presents a sketch of a typical setup in which many pairs of microphones are arranged to record the sounds in a closed space.

This situation exemplifies an event when many people are listening to sounds in the room. Assuming a pair of microphones represents the ears of a listener, taking the ensemble average of pairs of random variables may be the audio signal perceived.

2.2.2 Correlation and Square Correlation

Correlation as mathematically encoded by the cross-correlation coefficients is a fundamental concept in establishing the mutual relationship or connection between pairs of random variables.

Assuming a pair of two data (such as vectors)

$$L = X \quad (2.31)$$

$$R = aX + Y \quad (2.32)$$

$$X \cdot Y = 0. \quad (2.33)$$

The cross-correlation coefficient between the two is formally written as

$$\begin{aligned} R_{LR} &= \frac{X \cdot (aX + Y)}{|X|(aX + Y)|} \\ &= \frac{a|X|^2}{|X|(aX + Y)|} \\ &= \frac{a|X|}{|(aX + Y)|} \rightarrow \pm 1. \quad (Y \rightarrow 0) \end{aligned} \quad (2.34)$$

Then R_{LR}^2 becomes

$$R_{LR}^2 = \frac{|a|^2|X|^2}{|(aX + Y)|^2}. \quad (2.35)$$

Consequently

$$R_{LR}^2|(aX + Y)|^2 = |a|^2|X|^2. \quad (2.36)$$

This expression hints at the meaning of the square correlation. It reflects the ratio of the direct and reverberant sound energy that is very frequently used in describing features involved in the acoustics of a room. Simple examples entail instances when $|X| = |Y| = 1$, then

$$|X + Y| = \sqrt{|X|^2 + |Y|^2} = \sqrt{2}. \quad (2.37)$$

The cross-correlation coefficient becomes $1/\sqrt{2}$ assuming $a = 1$, and square correlation is $1/2$.

2.3 Probability Distribution for Sinusoidal Waves

2.3.1 Probability Density Function

Given the probability distribution P_X for when an arbitrary random variable X takes a value in the range x and $x + dx$, expressly,

$$P_X(x \leq X < x + dx) = p(x)dx, \quad (2.38)$$

$p(x)$ is then called the probability density function and satisfies the normalizing condition [1]

$$\int_{\mathbf{X}} p(x) dx = 1. \quad (2.39)$$

The range of integration is taken over all values associated with random variable $\mathbf{X}$. A graphical display of the probability distribution from a practical view point is customarily called the histogram.

The expectation or the ensemble average for the random variable $\mathbf{X}$ is defined by

$$E[\mathbf{X}] = \int_{\mathbf{X}} x p(x) dx. \quad (2.40)$$

Assuming the observable variable in $\mathbf{X}^2$ instead of $\mathbf{X}$, then the expectation for $\mathbf{X}^2$ is given by

$$E[\mathbf{X}^2] = \int_{\mathbf{X}} x^2 p(x) dx \quad (2.41)$$

and generally

$$E[f(\mathbf{X})] = \int_{\mathbf{X}} f(x) p(x) dx. \quad (2.42)$$

is defined for the observed value $f(\mathbf{X})$ of interest.

The expectation or ensemble average for a random variable $\mathbf{X}$ is a linear operation. Consider now the random variable

$$\mathbf{Z} = \mathbf{X} + \mathbf{Y}, \quad (2.43)$$

where $\mathbf{X}$ and $\mathbf{Y}$ are random variables. The joint probability density function denoting the probability density function for a pair of the random variables $\mathbf{X}$ and $\mathbf{Y}$ is given by

$$p_{(\mathbf{X}, \mathbf{Y})}(x, y) = p_{\mathbf{X}|\mathbf{Y}}(x|y) \cdot p_{(\bar{\mathbf{X}}, \mathbf{Y})}(y), \quad (2.44)$$

where $p_{\mathbf{X}|\mathbf{Y}}(x|y)$ denotes the conditional probability density function, and $p_{(\bar{\mathbf{X}}, \mathbf{Y})}(y)$ is called the marginal probability density function defined by

$$p_{(\bar{\mathbf{X}}, \mathbf{Y})}(y) = \int_{\mathbf{X}} p_{(\mathbf{X}, \mathbf{Y})}(x, y) dx. \quad (2.45)$$

Following the definitions of the density functions, the expectation of the random variable $\mathbf{X} + \mathbf{Y}$ becomes

$$\begin{aligned} E[\mathbf{X} + \mathbf{Y}] &= \int_{\mathbf{X}} x \cdot p_{(\mathbf{X}, \bar{\mathbf{Y}})}(x) + \int_{\mathbf{Y}} y \cdot p_{(\bar{\mathbf{X}}, \mathbf{Y})}(y) \\ &= E_{\bar{\mathbf{Y}}}[\mathbf{X}] + E_{\bar{\mathbf{X}}}[\mathbf{Y}] = E[\mathbf{X}] + E[\mathbf{Y}]. \end{aligned} \quad (2.46)$$

If

$$p_{X|Y}(x|y) = p_X(x) \quad (2.47)$$

$$p_{Y|X}(y|x) = p_Y(y) \quad (2.48)$$

hold, then the random variables X and Y are statistically independent of each other. Consequently, the joint probability density function for a statistically independent pair is written as

$$p_{(X,Y)}(x, y) = p_X(x) \cdot p_Y(y), \quad (2.49)$$

where $p_X(x)$ and $p_Y(y)$ denote the probability density functions for the random variables X and Y , respectively. Hence the expectation of a product of a statistically independent pair becomes

$$E[X \cdot Y] = E[X] \cdot E[Y]. \quad (2.50)$$

2.3.2 Probability Density Function for Sinusoidal Wave

A sinusoidal wave can be viewed as a random event when it is randomly sampled along its time axis. Figure 2.2 presents an example of such a case where a sample shows a random variable distributed between ± 1 .

Consider a probability density for a sinusoidal wave function for which the function is randomly sampled. For that purpose consider the inverse function $x = f^{-1}(y)$ of a function $y = f(x)$, as depicted in Fig. 2.3.

The probability for $y < Y \leq y + dy$ can be interpreted as

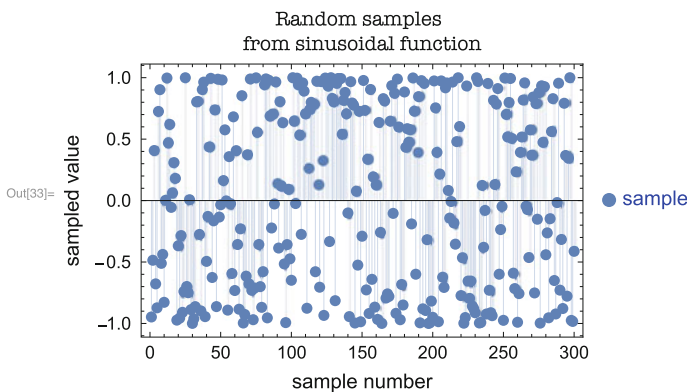
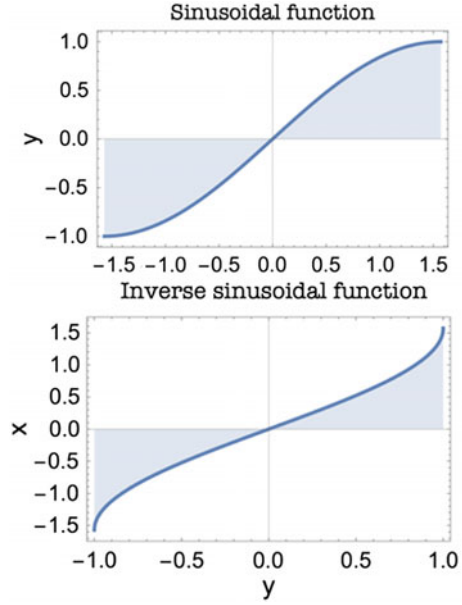


Fig. 2.2 Sinusoidal wave randomly sampled

Fig. 2.3 Schematic of the inverse function



$$\begin{aligned}
 P_Y(y < Y \leq y + dy) & \\
 &= P_X(f^{-1}(y) < X \leq f^{-1}(y + dy)) \\
 &= p_X(x) \cdot |f^{-1}(y + dy) - f^{-1}(y)|,
 \end{aligned} \tag{2.51}$$

where $p_X(x)$ denotes the probability density function for the random variable X . Introducing

$$f^{-1}(y + dy) - f^{-1}(y) = \frac{dx}{dy} dy, \tag{2.52}$$

where $x = f(y)$, then

$$\begin{aligned}
 P_Y(y < Y \leq y + dy) & \\
 &= p_Y(y) dy
 \end{aligned} \tag{2.53}$$

$$= p_X(x) \left| \frac{dx}{dy} \right| dy \tag{2.54}$$

$$p_Y(y) = p_X(x) \left| \frac{dx}{dy} \right|.$$

As an example, take the sinusoidal function

$$y = A \sin \frac{2\pi}{L} x. \tag{2.55}$$

Its inverse function is

$$x = \frac{L}{2\pi} \sin^{-1} \frac{y}{A} \quad (2.56)$$

and therefore its derivative is

$$\left| \frac{dx}{dy} \right| = \frac{L}{2\pi} \frac{1}{\sqrt{A^2 - y^2}}. \quad (2.57)$$

Assuming X is a random variable in the interval $0 < X < L/2$ where L is the period and

$$p_X(x) = \frac{2}{L}, \quad (2.58)$$

then the probability density function for Y becomes

$$p_Y(y) = \frac{1}{\pi \sqrt{A^2 - y^2}}. \quad (2.59)$$

The probability density function for a sinusoidal wave satisfies

$$\int_{-A}^A p_Y(y) dy = 1 \quad (2.60)$$

$$E[Y] = \int_{-A}^A y \cdot p_Y(y) dy = 0 \quad (2.61)$$

$$E[Y^2] = \int_{-A}^A y^2 \cdot p_Y(y) dy = \frac{A^2}{2}. \quad (2.62)$$

The corresponding results for the sinusoidal wave of Eq. 2.55 is

$$E[Y] = \frac{1}{T} \int_0^T A \sin \frac{2\pi}{T} t dt = 0 \quad (2.63)$$

$$E[Y^2] = \frac{1}{T} \int_0^T A^2 \sin^2 \frac{2\pi}{T} t dt = \frac{A^2}{2}. \quad (2.64)$$

A random variable X that depends on time t is called ergodic if its expectation or ensemble average is equal to its cycle average.

2.3.3 Uncorrelation and Independence of Random Variables

Consider a pair of random variables X and Y . When the expectation for $Z = XY$

$$E[Z] = E[XY] = 0 \quad (2.65)$$

the two random variables are referred to as being uncorrelated. In contrast, when the probability density function for $Z = X \cdot Y$ becomes

$$p_Z(z) = p_X(x)p_Y(y) \quad (2.66)$$

the pair of random variables are statistically independent of each other. If two random variables X and Y are independent, then

$$E[Z] = E[XY] = E[X] \cdot E[Y] = 0 \quad (2.67)$$

given that

$$E[X] = E[Y] = 0. \quad (2.68)$$

Conversely, a pair of random variables is not always independent, even if

$$E[Z] = E[XY] = 0 \quad (2.69)$$

holds. Take the pair of random variables

$$X = A \sin \frac{2\pi}{L} U \quad (2.70)$$

$$Y = A \cos \frac{2\pi}{L} U, \quad (2.71)$$

where the probability density functions are identical to each other and the expectations are

$$E[X] = E[Y] = 0. \quad (2.72)$$

Recalling that the product of random variables $Z = XY$ becomes

$$Z = XY \quad (2.73)$$

$$\begin{aligned} &= A^2 \sin \frac{2\pi}{L} U \cdot \cos \frac{2\pi}{L} U \\ &= \frac{A^2}{2} \sin \frac{2\pi}{L/2} U \end{aligned}$$

the probability density function for the random variable Z can be expressed as for the sinusoidal function

$$p_Z(z) = \frac{1}{\pi \sqrt{(A^2/2)^2 - z^2}} \quad (2.74)$$

$$E[Z] = 0. \quad (2.75)$$

The probability density function for the random variable Z is not identical to the product of the probability density functions; that is,

$$p_Z(z) \neq p_X(x) \cdot p_Y(y). \quad (2.76)$$

The pair of random variables may have no cross-correlation, even if the variables are not a statistical independent pair. When the cross-correlation or inner product is zero for a pair of random variables that are not statistically independent, the pair is called orthogonal or $\pi/2$ –phase shifted to each other rather than statistically independent.

2.3.4 Binaural Merit in Listening to Pairs of Signals

Binaural listening refers to the difference between statistical independence and a $\pi/2$ –phase shift for pairs of signals. For example, a binaural pair of independent random noise samples produces a so-called subjective diffuseness [2–5]. In contrast, a pair of $\pi/2$ –phase-shifted random noise stimuli does not produce any diffuseness even under binaural listening conditions, although just a non-focal sound image may be produced instead.

Such differences can be displayed using a parametric plot. Figure 2.4 shows samples of pairs for random variables X and Y where

$$x = \cos \phi, \quad (2.77)$$

$$y = \cos(\phi + \theta), \quad (2.78)$$

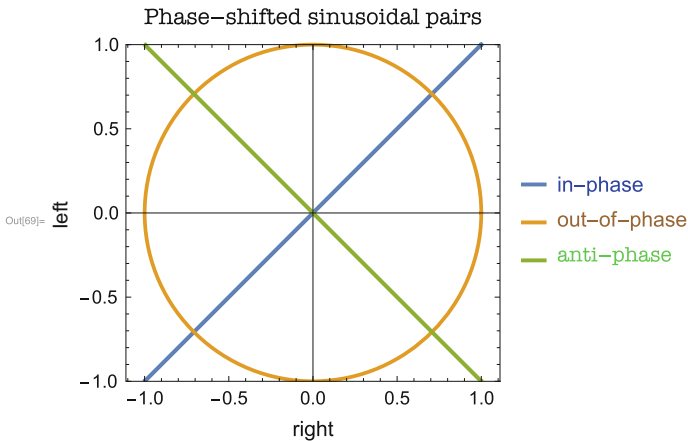


Fig. 2.4 Parametric plots for pairs of phase-shifted sinusoidal functions

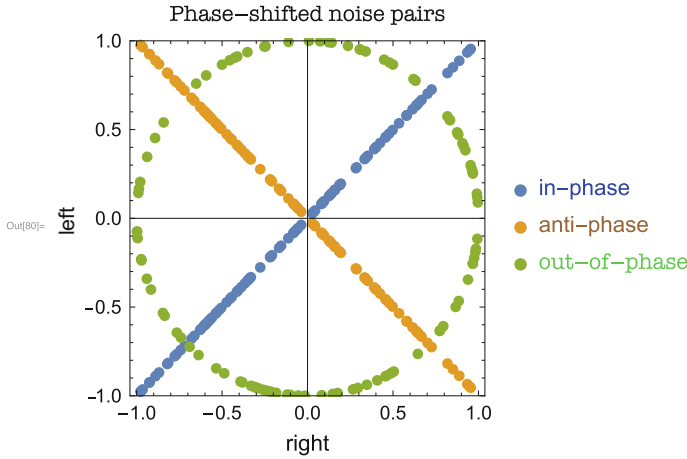


Fig. 2.5 Parametric plots for pairs of out-of-phase random variables

and ϕ ranges between $0 \leq \phi < 2\pi$, and θ is the phase shift. If $\theta = 0$, the pair is said to be in phase; if $\theta = \pi/2$ or $\theta = \pi$, the pair is said to be orthogonal (out-of-phase) or in anti-phase, respectively. For an in-phase pair, the sound image is clearly focused in the head between the pair of ears. In contrast, for an anti-phase pair, the sound image is unnaturally shifted to the back of the head. For the out-of-phase pair no focal sound image is produced. The images are different from that for a pair of independent random noise samples. A subjective diffuseness does not appear for the sinusoidal pair. This sense of subjective diffuseness can be made through the randomness in the time domain of two independent noises.

Figure 2.5 presents examples of random variables X and Y where the random variables are distributed uniformly between -1 and 1 . For the in-phase pair $X = Y$, where $X = -Y$ for the anti-phase pair. The out-of-phase pair was generated using distributions

$$X = \cos \Phi \quad (2.79)$$

$$Y = \sin \Phi, \quad (2.80)$$

where Φ denotes a random variable that is distributed uniformly between $-\pi \leq \Phi < \pi$. The difference between the out-of-phase and independent random variables is clearly displayed. Figure 2.6, presents sample plots for a pair of independent random variables that are uniformly distributed over the interval between -1 and 1 .

A comparison of the out-of-phase samples in Figs. 2.5 and 2.6 emphasizes the difference. The differences between the in-phase and out-of-phase pairs can also be displayed using the cross-correlation functions for the pair of random variables. Take again the pair of sinusoidal functions. The pair of functions

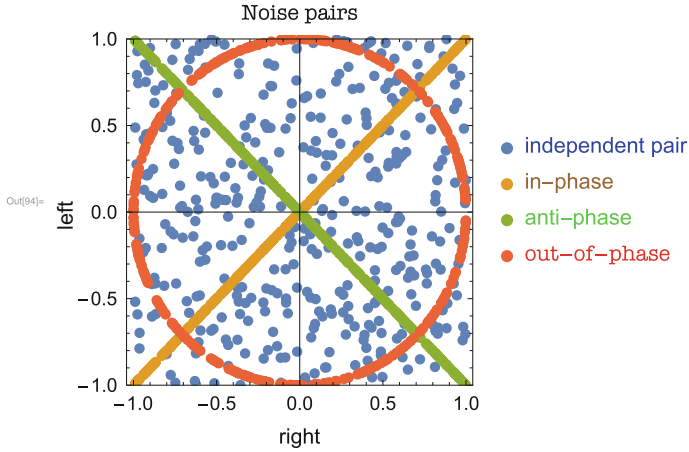


Fig. 2.6 Parametric plots for a pair of independent random variables

$$X = \sin \Phi \quad (2.81)$$

$$Y = \sin \Phi \quad (2.82)$$

yields cross-correlation function

$$\begin{aligned} r(\theta) &= \frac{1}{2\pi} \int_0^{2\pi} \sin \phi \sin(\phi - \theta) d\phi \\ &= \frac{1}{2} \cos \theta. \end{aligned} \quad (2.83)$$

This function is periodic and even with respect to θ ; a peak forms at $r(0)$. In contrast, for the out-of-phase pair

$$X = \sin \Phi \quad (2.84)$$

and

$$Y = \cos \Phi \quad (2.85)$$

the cross-correlation function is given by

$$\begin{aligned} r(\theta) &= \frac{1}{2\pi} \int_0^{2\pi} \sin \phi \cos(\phi - \theta) d\phi \\ &= \frac{1}{2} \sin \theta \end{aligned} \quad (2.86)$$

which is a periodic function that is odd with respect to the variable θ and $r(0) = 0$.

Take a random variable X ; its auto-correlation function is

$$r(n) = E[R(m, n)] = E[E[X(m)X(m - n)]], \quad (2.87)$$

where $X(m)$ denotes the random variable that is sampled at time m . Assuming the random variable X corresponds to white noise, then the cross-correlation function is expected to be

$$r(n) = E[X(m)X(m - n)] = u_0\delta(n), \quad (2.88)$$

where

$$u_0 = E[X^2(m)]. \quad (2.89)$$

The result above indicates that the cross-correlation function becomes

$$r(n) = u_0\delta(n \pm N) \quad (2.90)$$

for a pair of variables such that

$$X = X \quad (2.91)$$

$$Y = X(n \mp N). \quad (2.92)$$

The cross-correlation function is not periodic, and has peaks at $n = \pm N$.

The sound image is clearly focused in binaural listening of the pair of random variables stated above. The focal point is situated just between both ears when $N = 0$. As $|N|$ increases, the image point is shifted to the left or right side depending on the sign of N . Nevertheless, within the some range, which is determined by the interaural distance, the sound image is focused in consequence of the cross-correlation function being even with peaks located at $n = \pm N$.

For an independent pair of variables X and Y , the cross-correlation function becomes

$$\begin{aligned} r(n) &= E[X(m)Y(m - n)] \\ &= E[X(m)]E[Y(m - n)], \\ &= 0, \end{aligned} \quad (2.93)$$

where the peaks have disappeared. The binaural sense from listening to this pair of independent random variables is called subjective diffuseness in which the sound image is not focused but randomly changes with time, provided that no differences between the variables are detected in monaural listening.

Taking an example of an out-of-phase pair described by Eqs. 2.84 and 2.85, the cross-correlation function is interpreted as an instance when the variable Φ is randomly sampled after being uniformly distributed over the interval between $-\pi$ and π . The result highlights the differences in the cross-correlation functions between out-of-phase and independent pairs, where for both instances $r(0) = 0$. However, the

cross-correlation for the out-of-phase pair is an odd function for which a pair of peaks of opposite sign is observed as indicated in Eq. 2.86. In contrast, the cross-correlation for the independent pair is formally an even function and $r(n) = 0$ independent of n . These differences are characteristic of the out-of-phase (or orthogonal) and independent random variable pairs.

2.4 Exercises

1. Consider a probability density function of random variable X given by

$$p(x) = 1/2. \quad -1 \leq x \leq 1$$

Calculate

$$E[X]$$

$$E[X^2]$$

$$E[X^3].$$

2. Consider once more the probability density function of random variable X defined above. Evaluate the cross-correlation coefficient between X and Y for

$$Y = aX + b$$

$$Y = aX^2 + bX + c$$

$$Y = aX^3 + bX^2 + cX + d.$$

3. Suppose

$$E[X] = 0$$

$$E[Y] = 0$$

$$E[XY] = r$$

$$E[X^2] = E[Y^2] = 1.$$

Calculate the expectation value for the random variables

$$Z = X + Y \quad (2.94)$$

$$Z = X - Y$$

$$Z = (X + Y)^2$$

$$Z = (X - Y)^2$$

$$K = \frac{E(X - Y)^2}{E(X + Y)^2}.$$

The cross-correlation coefficient can be obtained using the power ratio between the difference and sum for the pair of random variables.

4. Suppose once more

$$E[X] = 0$$

$$E[Y] = 0$$

$$E[XY] = r$$

$$E[X^2] = E[Y^2] = 1.$$

Obtain the expectation values for random variables

$$Z = X$$

$$Z = rX$$

$$Z = \sqrt{1 - r^2}Y$$

$$Z = rX + \sqrt{1 - r^2}Y$$

$$Z = X(rX + \sqrt{1 - r^2}Y).$$

The cross-correlation coefficient between the pair $Z = rX + \sqrt{1 - r^2}Y$ and X with the same power is r [6].

5. The convolution of a pair of sequences $a(n)$ and $b(n)$ is written as

$$c(n) = a * b(n) = \sum_m a(m)b(n - m).$$

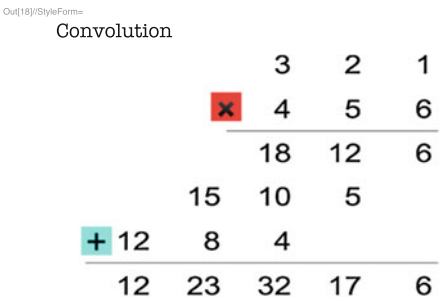
Make a pair of polynomials for $a(n)$ and $b(n)$ such that

$$A(X) = \sum_n a(n)X^n \quad (2.95)$$

$$B(X) = \sum_n b(n)X^n.$$

Suppose the corresponding polynomial for $c(n)$ is defined as

Fig. 2.7 Schematic for the convolution of Exercise 6



$$C(X) = A(X)B(X) = \sum_n \hat{c}(n)X^n.$$

Show

$$\hat{c}(n) = a * b(n) = c(n).$$

6. Derive the matrix form of the convolution for

$$\begin{aligned} a(n) &= 1, 2, 3 \\ b(n) &= 6, 5, 4 \\ c(n) &= a * b(n). \end{aligned} \tag{2.96}$$

Confirm $c(n) = a * b(n) = b * a(n)$. Interestingly, the convolution can be calculated as shown in Fig. 2.7 [7].

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Sound in the Time Domain

Tohyama, M.

2018, XVII, 324 p. 202 illus., 167 illus. in color.,

Hardcover

ISBN: 978-981-10-5887-5