Chapter 2
Finite Element Analysis in Space

2.1 Introduction

In this chapter, finite element in space is discussed in detail, and it is typically the first step in the solution of the elastic rotor problem as it yields the rotating natural frequencies. Bar, beam, and rotating beam finite element formulation are explained.

2.2 Finite Element in Space

Finite element method is a numerical method for getting approximate solution of differential equations. Analytical solution of the equation gives us the exact solution at any point in that domain, while finite element method gives us the approximate solution at discrete number of points in that domain. In finite element method, we divide the full domain into a number of elements, which are connected through nodal points. Then, we write equations for each element and combine them to get the solution. Finite element approach is a weak formulation of the physical problem.

2.3 Strong Form of the Equation

Consider Fig. 2.1, a case of an elastic bar subjected to uniform load.

The governing differential equation (2.1) along with boundary conditions (2.2) and (2.3) gives the strong form of the problem

$$EA\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = f_0 \tag{2.1}$$

R. Ganguli and V. Panchore, *The Rotating Beam Problem in Helicopter Dynamics*,
Foundations of Engineering Mechanics, https://doi.org/10.1007/978-981-10-6098-4_2

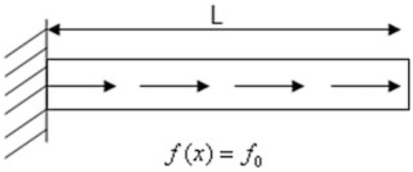

Fig. 2.1 Elastic bar subjected to uniform load

$$u(0) = 0 \tag{2.2}$$

$$EA\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{(x=L)} = 0 \tag{2.3}$$

where u is the axial displacement; Young's modulus (E) and cross sectional area (A) are constant over the length of the bar.

2.4 Weak Form of the Equation

The weak form is a variational statement of the equation, where we multiply the differential equation by a test function (v) and integrate it over the domain.

$$\int_0^L v\left(EA\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - f_0\right)\mathrm{d}x = 0 \tag{2.4}$$

We choose test function (v) such that it satisfies the homogeneous boundary conditions. Equation (2.4) can be written after integration by parts as

$$-\int_0^L EA\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x} + \left[EAv\frac{\mathrm{d}u}{\mathrm{d}x}\right]_0^L = \int_0^L f_0 v\mathrm{d}x$$

or

$$-\int_0^L EA\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x} + EAv(L)\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=L} - EAv(0)\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=0} = \int_0^L f_0 v\mathrm{d}x$$

We apply boundary condition (2.2), (2.3), and $v(0) = 0$, to get the weak form of the equation

$$-\int_0^L EA\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x} = \int_0^L f_0 v\mathrm{d}x \tag{2.5}$$

The order of the derivatives in the equation is reduced in the weak form.

2.5 Galerkin's Method

In Galerkin's method, we find the solution $u = \tilde{u}$, such that

$$\int_0^L v\left(EA\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}x^2} - f_0\right)\mathrm{d}x = 0 \tag{2.6}$$

$$\tilde{u}(0) = 0 \tag{2.7}$$

$$EA\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}x^2} = f_0 \tag{2.8}$$

where we choose $\tilde{u}(x) = \sum_{j=1}^{N} c_j\phi_j(x)$ and $v(x) = \sum_{i=1}^{N} b_j\phi_j(x)$. Here, c_j is unknown and b_j is arbitrarily chosen. The interpolation or basis function $\phi_j(x)$ must satisfy all the boundary conditions for the problem. A good solution is obtained by taking many terms of the series. While Galerkin's method uses global interpolation function, the key idea in finite element method is to interpolate locally.

2.6 Shape Function in 1 Dimension

In finite element, we get the solution at nodal points of an element. Interpolation within the element is achieved by shape function.

Here, we take a bar element for illustration. Governing equation of linear-elastic bar element is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) = 0 \tag{2.9}$$

Consider Fig. 2.2, two-node bar element. Here q_1 and q_2 are the displacements at the two nodes and are called degrees of freedom.

Fig. 2.2 Bar element for shape function formulation

Linear displacement along the x-axis is assumed as

$$u(x) = a_0 + a_1 x \tag{2.10}$$

At $x = 0$

$$u(0) = a_0 \Rightarrow q_1 = a_0 \tag{2.11}$$

At $x = L$

$$u(L) = a_0 + a_1 L \Rightarrow q_2 = a_0 + a_1 L \tag{2.12}$$

From (2.11) and (2.12), we get $a_0 = q_1, a_1 = \frac{q_2 - q_1}{L}$

We write the linear displacement within the finite element as $u(x) = q_1 + \left(\frac{q_2 - q_1}{L}\right)x \Rightarrow u(x) = q_1\left(1 - \frac{x}{L}\right) + q_2\frac{x}{L} \Rightarrow u(x) = H_1 q_1 + H_2 q_2$

or

$$u(x) = \begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{2.13}$$

where $H_1 = \left(1 - \frac{x}{L}\right)$ and $H_2 = \frac{x}{L}$ are shape functions for the bar elements. Typically, polynomials are used as shape functions in finite element methods.

2.7 Shape Function Formulation for Beam Element

The static governing differential equation of an Euler–Bernoulli beam is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right) = 0 \tag{2.14}$$

Here, in Fig. 2.3, we consider a beam element

Each element has two nodes; each node has vertical displacement and rotation.

Fig. 2.3 Beam element for shape function formulation

Total degree of freedom per element is 4. Assume the transverse displacement $v(x)$ to be

$$v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{2.15}$$

At $x = 0$

$$v(0) = a_0 \Rightarrow q_1 = a_0 \tag{2.16}$$

$$\frac{\mathrm{d}v(0)}{\mathrm{d}x} = a_1 \Rightarrow q_2 = a_1 \tag{2.17}$$

At $x = L$

$$v(L) = a_0 + a_1 L + a_2 L^2 + a_3 L^3 \Rightarrow q_3 = a_0 + a_1 L + a_2 L^2 + a_3 L^3 \tag{2.18}$$

$$\frac{\mathrm{d}v(L)}{\mathrm{d}x} = a_1 + 2a_2 L + 3a_3 L^2 \Rightarrow q_4 = a_1 + 2a_2 L + 3a_3 L^2 \tag{2.19}$$

From Eqs. (2.16)–(2.19), we write

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{2.20}$$

or

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{L^2} & \frac{-2}{L} & \frac{3}{L^2} & \frac{-1}{L} \\ \frac{2}{L^3} & \frac{1}{L^2} & \frac{-2}{L^3} & \frac{1}{L^2} \end{bmatrix} \begin{bmatrix} q_1 \\ q_1 \\ q_3 \\ q_4 \end{bmatrix} \tag{2.21}$$

From Eq. (2.15), we write

$$v(x) = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{2.22}$$

or

$$v(x) = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{L^2} & \frac{-2}{L} & \frac{3}{L^2} & \frac{-1}{L} \\ \frac{2}{L^3} & \frac{1}{L^2} & \frac{-2}{L^3} & \frac{1}{L^2} \end{bmatrix} \begin{bmatrix} q_1 \\ q_1 \\ q_3 \\ q_4 \end{bmatrix} \tag{2.23}$$

or

$$v(x) = \begin{bmatrix} 2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 + 1 & \frac{x^3}{L^2} - 2\frac{x^2}{L} + x & 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 & \frac{x^3}{L^2} - \frac{x^2}{L} \end{bmatrix} \begin{bmatrix} q_1 \\ q_1 \\ q_3 \\ q_4 \end{bmatrix} \tag{2.24}$$

or

$$v(x) = \begin{bmatrix} H_1 & H_2 & H_3 & H_4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = [H][q] \tag{2.25}$$

where H_1, H_2, H_3 and H_4 are shape functions for the beam finite element.

$$H_1 = 2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 + 1, H_2 = \frac{x^3}{L^2} - 2\frac{x^2}{L} + x, H_3 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3, H_4 = \frac{x^3}{L^2} - \frac{x^2}{L}.$$

2.8 Properties of Shape Function in 1D

In the rotor blade problem, we focus on 1D structures. The properties of the shape functions are discussed next.

1. Kronecker delta property

Shape function of a node has value equal to one on that node and zero at all the other nodes.

Consider Fig. 2.2.

Node 1 $(x = 0)$

$$H_1 = 1 - \frac{x}{L} \Rightarrow H_1 = 1 \quad \text{and} \quad H_2 = \frac{x}{L} \Rightarrow H_2 = 0$$

Node 2 $(x = L)$

$$H_2 = \frac{x}{L} \Rightarrow H_2 = 1 \quad \text{and} \quad H_1 = 1 - \frac{x}{L} \Rightarrow H_1 = 0.$$

2. Compatibility condition

Displacement approximation is continuous across element boundaries.

Consider Fig. 2.4, where two elements are taken in a bar.

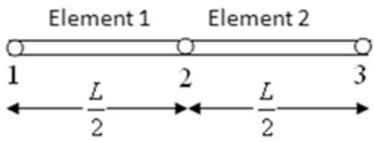

Fig. 2.4 Two elements in a bar for FEM in space

For formulation of shape function, we follow Eqs. (2.10)–(2.13). We substitute $x = 0$ and $x = L/2$ and get the shape function for first element $\left[H_1^{(1)} \quad H_2^{(1)} \right]$, and we substitute $x = L/2$ and $x = L$ and get the shape function for second element $\left[H_1^{(2)} \quad H_2^{(2)} \right]$.

$$u(x) = H_1^{(1)} q_1 + H_2^{(1)} q_2 = \left(1 - \frac{2x}{L}\right) q_1 + \frac{2x}{L} q_2 \tag{2.26}$$

For second element, we write

$$u(x) = H_1^{(2)} q_2 + H_2^{(2)} q_3 = \left(2 - \frac{2x}{L}\right) q_2 + \left(\frac{2x}{L} - 1\right) q_3 \tag{2.27}$$

Put $x = \frac{L}{2}$ into Eqs. (2.26) and (2.27) to get

$$u(x) = q_2.$$

3. Completeness

(a) Rigid body mode

$$H_1 + H_2 = 1$$

If the element moves by an unit displacement ($q_1, q_2 = 1$), displacement at any point in the element should be one.

$$u(x) = H_1 q_1 + H_2 q_2 = 1 \quad (\text{for } q_1, q_2 = 1)$$

(b) Constant strain state

Consider Fig. 2.5, if $q_1 = L/2$ and $q_2 = L$, then

$$\varepsilon(x) = \frac{q_1 - q_2}{\frac{L}{2}} = \frac{L - \frac{L}{2}}{\frac{L}{2}} = 1$$

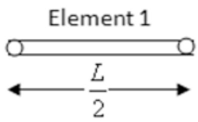

Fig. 2.5 Bar element (shape function properties)

Check of strain state with displacement approximation

$$u(x) = H_1 q_1 + H_2 q_2 = \left(1 - \frac{2x}{L}\right) q_1 + \frac{2x}{L} q_2$$

$$u(x) = \left(1 - \frac{2x}{L}\right)\frac{L}{2} + \frac{2x}{L} L$$

or

$$u(x) = x + \frac{L}{2}$$

or

$$\varepsilon(x) = 1$$

A brief outline of finite element has been provided. We are now ready to apply the finite element method for the rotating beam problem.

2.9 Finite Element Formulation of Rotating Beam

Finite element formulation in space for the rotating beam is done using Hamilton's energy principle. (Complete derivation is given in [14].)

Potential energy is given by

$$V = \frac{1}{2}\int_0^R EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x + \frac{1}{2}\int_0^R G\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x \tag{2.28}$$

Kinetic energy is given by

$$T = \frac{1}{2}\int_0^R m\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \tag{2.29}$$

where w is the transverse displacement, G is the centrifugal force, and m is the mass per unit length.

From Eq. (2.25), we have shape function of the beam element

$$w(x) = \begin{bmatrix} H_1 & H_2 & H_3 & H_4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = [H][q]$$

or

$$\frac{\partial w}{\partial x} = \begin{bmatrix} H_1' & H_2' & H_3' & H_4' \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = [H'][q]$$

or

$$\left(\frac{\partial^2 w}{\partial x^2}\right) = \begin{bmatrix} H_1'' & H_2'' & H_3'' & H_4'' \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = [H''][q]$$

$$\left(\frac{\partial w}{\partial x}\right)^2 = [q]^{\mathrm{T}}[H']^{\mathrm{T}}[H'][q]$$

We write Eq. (2.28) as

$$V = \frac{1}{2}\int_0^R EI[q]^{\mathrm{T}}[H'']^{\mathrm{T}}[H''][q]\mathrm{d}x + \frac{1}{2}\int_0^R G[q]^{\mathrm{T}}[H']^{\mathrm{T}}[H'][q]\mathrm{d}x \tag{2.30}$$

or

$$V = \frac{1}{2}[q]^{\mathrm{T}}\left(\int_0^R EI[H'']^{\mathrm{T}}[H'']\mathrm{d}x + \int_0^R G[H']^{\mathrm{T}}[H']\mathrm{d}x\right)[q] \tag{2.31}$$

or

$$V = \frac{1}{2}[q]^{\mathrm{T}}_{1*4}[K]_{4*4}[q]_{4*1} \tag{2.32}$$

We write Eq. (2.29) as

$$T = \frac{1}{2}\int_0^R m[\dot{q}]^{\mathrm{T}}[H]^{\mathrm{T}}[H][q]\mathrm{d}x \tag{2.33}$$

or

$$T = [\dot{q}]^{\mathrm{T}}\left(\frac{1}{2}\int_0^R m[H]^{\mathrm{T}}[H]\mathrm{d}x\right)[\dot{q}] \tag{2.34}$$

or

$$T = [\dot{q}]^{\mathrm{T}}_{1*4}[M]_{4*4}[\dot{q}]_{4*1} \tag{2.35}$$

Rewriting the Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{1.47}$$

where $L = T - V$. From Eqs. (2.32), (2.35), and (1.35), we get the free vibration problem

$$[M][\ddot{q}] + [K][q] = 0 \tag{2.36}$$

where

$$[M] = \int_0^R m[H]^{\mathrm{T}}[H]\mathrm{d}x \tag{2.37}$$

$$[K] = \left(\int_0^R EI[H'']^{\mathrm{T}}[H'']\mathrm{d}x + \int_0^R G[H']^{\mathrm{T}}[H']\mathrm{d}x\right) \tag{2.38}$$

or

$$[K] = [K_1] + [K_2] \tag{2.39}$$

Assuming EI and m constant over the length of the blade:
where

$$[K_1] = EI\begin{bmatrix} \int_0^R (H_1'')^2\mathrm{d}x & \int_0^R H_1''H_2''\mathrm{d}x & \int_0^R H_1''H_3''\mathrm{d}x & \int_0^R H_1''H_4''\mathrm{d}x \\ & \int_0^R (H_2'')^2\mathrm{d}x & \int_0^R H_2''H_3''\mathrm{d}x & \int_0^R H_2''H_4''dx \\ & & \int_0^R (H_3'')^2\mathrm{d}x & \int_0^R H_3''H_4''\mathrm{d}x \\ & & & \int_0^R (H_4'')^2\mathrm{d}x \end{bmatrix},$$

$$[K_2] = \begin{bmatrix} \int_0^R G(H_1')^2\mathrm{d}x & \int_0^R GH_1'H_2'\mathrm{d}x & \int_0^R GH_1'H_3'\mathrm{d}x & \int_0^R GH_1'H_4'\mathrm{d}x \\ & \int_0^R G(H_2')^2\mathrm{d}x & \int_0^R GH_2'H_3'\mathrm{d}x & \int_0^R GH_2'H_4'\mathrm{d}x \\ & & \int_0^R G(H_3')^2\mathrm{d}x & \int_0^R GH_3'H_4'\mathrm{d}x \\ & & & \int_0^R G(H_4')^2\mathrm{d}x \end{bmatrix},$$

$$[M] = m\begin{bmatrix} \int_0^R (H_1)^2\mathrm{d}x & \int_0^R H_1H_2\mathrm{d}x & \int_0^R H_1H_3\mathrm{d}x & \int_0^R H_1H_4\mathrm{d}x \\ & \int_0^R (H_2)^2\mathrm{d}x & \int_0^R H_2H_3\mathrm{d}x & \int_0^R H_2H_4\mathrm{d}x \\ & & \int_0^R (H_3)^2\mathrm{d}x & \int_0^R H_3H_4\mathrm{d}x \\ & & & \int_0^R (H_4)^2\mathrm{d}x \end{bmatrix}$$

We solve the free vibration problem with the above matrices. The formulation is valid for any general axial force $G(x)$. For a rotating beam, we are interested in the centrifugal force.

2.10 Centrifugal Force

Centrifugal force is the additional term to beam equation in rotating beam equation. In the formulation, it is $[K_2]$ (Fig. 2.6).

Fig. 2.6 Centrifugal force on the rotating beam

$$G = \int_{x}^{R} m\Omega^2 x \mathrm{d}x \tag{2.40}$$

For constant mass per unit length

$$G = m\Omega^2 \left(\frac{R^2}{2} - \frac{x^2}{2} \right). \tag{2.41}$$

2.11 Shape Function Formulation for Two Elements

Consider Fig. 2.7, shape function for both the elements will be different.

For formulation of shape function, we follow Eqs. (2.15)–(2.25). For first element, we evaluate value of x at 0 and $L/2$. For second element, we evaluate value of x at $L/2$ and L.

Shape function for element 1 ($[H]_1$)	Shape function for element 2 ($[H]_2$)
$16\frac{x^3}{L^3} - 12\frac{x^2}{L^2} + 1,$	$24\frac{x}{L} - 36\frac{x^2}{L^2} + 16\frac{x^3}{L^3} - 4,$
$x - 4\frac{x^2}{L} + 4\frac{x^3}{L^2},$	$8x - 2L - 10\frac{x^2}{L} + 4\frac{x^3}{L^2},$
$12\frac{x^2}{L^2} - 16\frac{x^3}{L^3},$	$36\frac{x^2}{L^2} - 24\frac{x}{L} - 16\frac{x^3}{L^3} + 5,$
$4\frac{x^3}{L^2} - 2\frac{x^2}{L}$	$5x - L - 8\frac{x^2}{L} + 4\frac{x^3}{L^2}$

We integrate shape functions over the domain

Element 1 $\left(\int_0^{L/2} [H]_1^{\mathrm{T}} \mathrm{d}x\right)$	Element 2 $\left(\int_{L/2}^{L} [H]_2^{\mathrm{T}} \mathrm{d}x\right)$
$\frac{L}{4},$	$\frac{L}{4}$
$\frac{L^2}{48},$	$\frac{L^2}{48}$
$\frac{L}{4}$	$\frac{L}{4}$
$-\frac{L^2}{48}$	$-\frac{L^2}{48}$

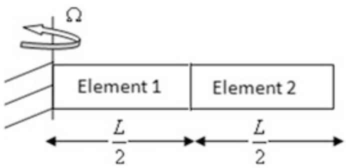

Fig. 2.7 Shape function formulation of two elements

We see that

$$\int_0^{L/2} [H]_1^{\mathrm{T}} \mathrm{d}x = \int_{L/2}^{L} [H]_2^{\mathrm{T}} \mathrm{d}x \tag{2.42}$$

We write $[K_1]$ matrix

$$[K_1] = EI \int_0^R [H'']^{\mathrm{T}} [H''] \mathrm{d}x$$

We notice that

$$[K_1]1\text{st(element)} = [K_1]2\text{nd(element)}$$

or

$$EI \int_0^{L/2} [H'']_1^{\mathrm{T}} [H'']_1 \mathrm{d}x = EI \int_{L/2}^{L} [H'']_2^{\mathrm{T}} [H'']_2 \mathrm{d}x \tag{2.43}$$

So, we should calculate the stiffness matrix $[K_1]$ for only one element, and it will be the same for all the elements.

Now we integrate $\int_0^{L/2} [H]_1^{\mathrm{T}} f(x) \mathrm{d}x$ and $\int_{L/2}^{L} [H]_2^{\mathrm{T}} f(x) \mathrm{d}x$

where $f(x) = x$

Element 1 $\left(\int_0^{L/2} [H]_1^{\mathrm{T}} x \mathrm{d}x\right)$	Element 2 $\left(\int_{L/2}^{L} [H]_2^{\mathrm{T}} x \mathrm{d}x\right)$
$\frac{3L^2}{80}$	$\frac{13L^2}{80}$
$\frac{L^3}{240}$	$\frac{7L^3}{480}$
$\frac{7L^2}{80}$	$\frac{17L^2}{80}$
$-\frac{L^3}{160}$	$-\frac{L^3}{60}$

We see that

$$\int_{0}^{L/2} [H]_1^{\mathrm{T}} x \mathrm{d}x \neq \int_{L/2}^{L} [H]_2^{\mathrm{T}} x \mathrm{d}x \tag{2.44}$$

We write $[K_2]$ matrix

$$[K_2] = \int_{0}^{R} G[H']^{\mathrm{T}}[H']\mathrm{d}x$$

We notice that

$$[K_2]\mathrm{1st(element)} \neq [K_2]\mathrm{2nd(element)}$$

or

$$\int_{0}^{L/2} G(x)[H']_1^{\mathrm{T}}[H']_1 \mathrm{d}x \neq \int_{L/2}^{L} G(x)[H']_2^{\mathrm{T}}[H']_2 \mathrm{d}x \tag{2.45}$$

So, we should calculate the stiffness matrix $[K_2]$ for each element.

2.12 FEM Formulation of Rotating Beam with Only One Shape Function (for Free Vibration)

Formulation of $[K_2]$ matrix with the shape function of the first element

Considering Fig. 2.8, we write the FEM formulation for third element using the shape function of one element. Here, L is the length of each element, R is the radius of rotor, x is the length along the element, x_i is the distance of element from the starting point and will depend on the element we have taken, Ω is the angular velocity, N is the number of elements, ρ is the global coordinate system, and x is the local coordinate system.

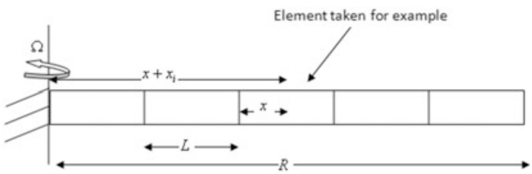

Fig. 2.8 FEM formulation of rotating beam with the shape function of one element (for free vibration)

EI is stiffness, and m_i is mass of particular element.

Rewriting $[K_2]$ matrix

$$[K_2] = \int_0^l G(x)[H']_1^{\mathrm{T}}[H']_1 \mathrm{d}x$$

From Fig. 3.9, we write the centrifugal force for the element as

$$G(x) = \int_{x_i+x}^{R} m\Omega^2 \rho \mathrm{d}\rho \Rightarrow G(x) = \int_{x_i}^{R} m\Omega^2 \rho \mathrm{d}\rho - \int_{x_i}^{x+x_i} m\Omega^2 \rho \mathrm{d}\rho$$

or

$$G(x) = \mathrm{Term1} - \mathrm{Term2} \tag{2.46}$$

$$\mathrm{Term1} = \int_{x_i}^{R} m\Omega^2 \rho \mathrm{d}\rho \Rightarrow \mathrm{Term1} = \sum_{j=i}^{N} \int_{x_j}^{x_{j+1}} m\Omega^2 x \mathrm{d}x$$

or

$$\mathrm{Term1} = \sum_{j=i}^{N} m_j \Omega^2 \frac{(x_{j+1}^2 - x_j^2)}{2} \Rightarrow \mathrm{Term1} = \Omega^2 \frac{A_i}{2}$$

where $A_i = \sum_{j=i}^{N} m_j \frac{(x_{j+1}^2 - x_j^2)}{2}$ (It will be a constant term)

We write Term2

$$\mathrm{Term2} = \int_{x_i}^{x+x_i} m\Omega^2 \rho \mathrm{d}\rho \Rightarrow \mathrm{Term2} = \frac{m_i \Omega^2}{2} \{(x_i + x)^2 - x_i^2\}$$

or

$\mathrm{Term2} = \frac{m_i\Omega^2}{2}(2xx_i + x^2)$ (It will be a varying term because of x)

We write Eq. (2.46) as

$$G(x) = \frac{\Omega^2 A_i}{2} - \frac{m_i \Omega^2}{2}(2xx_i + x^2) \tag{2.47}$$

Now we rewrite $[K_2]$

$$[K_2] = \int_0^l G(x)[H']_1^{\mathrm{T}}[H']_1 \mathrm{d}x$$

$$[K_2]_i = \int_0^l \left\{ \frac{\Omega^2 A_i}{2} - \frac{m_i \Omega^2}{2}(2xx_i + x^2) \right\} [H']_1^{\mathrm{T}}[H']_1 \mathrm{d}x \tag{2.48}$$

where $i = 1, 2, 3$ (for different elements).

From this formulation, we get

$$[M] = m \begin{bmatrix} \frac{13l}{35} & \frac{11l^2}{210} & \frac{9l}{70} & \frac{-13l^2}{420} \\ & \frac{l^3}{105} & \frac{13l^2}{420} & \frac{-l^3}{140} \\ & & \frac{13l}{35} & \frac{-11l^2}{210} \\ & & & \frac{l^3}{105} \end{bmatrix},$$

$$[K_1] = EI \begin{bmatrix} \frac{12}{l^3} & \frac{6}{l^2} & \frac{-12}{l^3} & \frac{6}{l^2} \\ & \frac{4}{l} & \frac{-6}{l^2} & \frac{2}{l} \\ & & \frac{12}{l^3} & \frac{-6}{l^2} \\ & & & \frac{4}{l} \end{bmatrix}$$

$$[K_2]_i = \frac{\Omega^2 A_i}{2} \begin{bmatrix} \frac{6}{5l} & \frac{1}{10} & \frac{-6}{5l} & \frac{1}{10} \\ & \frac{2l}{25} & \frac{-1}{10} & \frac{-l}{30} \\ & & \frac{6}{5l} & \frac{-1}{10} \\ & & & \frac{2l}{15} \end{bmatrix}$$

$$- m_i \Omega^2 \begin{bmatrix} \frac{3x_i}{5} + \frac{6l}{35} & \frac{x_i l}{10} + \frac{l^2}{28} & \frac{-3x_i}{5} - \frac{6l}{35} & -\frac{l^2}{70} \\ & \frac{l^2 x_i}{30} + \frac{l^3}{105} & \frac{-lx_i}{10} - \frac{l^2}{28} & \frac{-l^2 x_i}{60} - \frac{l^3}{70} \\ & & \frac{3x_i}{5} + \frac{6l}{35} & \frac{l^2}{70} \\ & & & \frac{l^2 x_i}{10} + \frac{3l^2}{70} \end{bmatrix}$$

When $\Omega = 0, [K_2] = [0]$ and the formulation reduces to the well-known non-rotating beam element found in [15]. When $\Omega \neq 0$, the centrifugal stiffening causes a spatially varying stiffness matrix $[K_2]$.

2.13 Calculation of Mode Shapes and Frequencies

We rewrite Eq. (2.36)

$$[M][\ddot{q}] + [K][q] = 0 \text{ (Free vibration problem)}$$

The natural frequency f and the respective mode shape V of a rotating beam can be obtained from the Jacobian matrix $[A]$.

$$[A] = [M]^{-1}[K] \tag{2.49}$$

$$f = \sqrt{\mathrm{eigval}([A])} \tag{2.50}$$

$$V = \mathrm{eigvec}([A]) \tag{2.51}$$

Non-dimensional rotating frequency η and non-dimensional rotating speed s are given by

$$s = \Omega\sqrt{\frac{mR^4}{EI}}, \eta = f\sqrt{\frac{mR^4}{EI}}$$

We have solved the free vibration problem, so we consider the case of forced vibration.

$$[M][\ddot{q}] + [K][q] = \int_0^L [H]^{\mathrm{T}} F(x,\psi)\mathrm{d}x \tag{2.52}$$

where $\dot{q} = \mathrm{d}q/\mathrm{d}t$.

2.14 FEM Formulation of Aerodynamic Force for Rotor Problem

We have done finite element formulation for the free vibration. We develop the finite element formulation for the forced vibration in this section.

Finite element formulation of the right-hand side of Eq. (1.105) yields

$$[Q] = \int_0^L [H]^{\mathrm{T}} F(x,\psi)\mathrm{d}x \text{ (Element load vector)} \tag{2.53}$$

We rewrite Eq. (1.108)

$$F(x,\psi) = a_{1x}a_{1\psi} + a_{2x}a_{2\psi} + a_{3x}a_{3\psi} + a_{4x}a_{4\psi} + a_{5x}a_{5\psi} + a_{6x}a_{6\psi} + a_{7x}a_{7\psi} + (b_{1x}b_{1\psi} + b_{2x}b_{2\psi})\dot{w} + (c_{1x}c_{1\psi} + c_{2x}c_{2\psi})w' \tag{1.120}$$

Element load vector is given by

$$[Q_F] = \int_0^L [H]^{\mathrm{T}}\{a_{1x}a_{1\psi} + a_{2x}a_{2\psi} + a_{3x}a_{3\psi} + a_{4x}a_{4\psi} + a_{5x}a_{5\psi} + a_{6x}a_{6\psi} + a_{7x}a_{7\psi} + (b_{1x}b_{1\psi} + b_{2x}b_{2\psi})\dot{w} + (c_{1x}c_{1\psi} + c_{2x}c_{2\psi})w'\}\mathrm{d}x \tag{2.54}$$

Here, $w = [H][q]$ and $\dot{w} = \frac{\partial w}{\partial \psi}$.

We write Eq. (2.54) as

$$[Q_F] = a_{1\psi}[Q_{a1}] + a_{2\psi}[Q_{a2}] + a_{3\psi}[Q_{a3}] + a_{4\psi}[Q_{a4}] + a_{5\psi}[Q_{a5}] + a_{6\psi}[Q_{a6}] + a_{7\psi}[Q_{a7}] + (b_{1\psi}[C_{a1}] + b_{2\psi}[C_{a2}])[\dot{q}] + (c_{1\psi}[D_{a1}] + c_{2\psi}[D_{a2}])[q] \tag{2.55}$$

where

$$[Q_{a1}] = \int_0^L a_{1x}[H]^{\mathrm{T}}\mathrm{d}x, [Q_{a2}] = \int_0^L a_{2x}[H]^{\mathrm{T}}\mathrm{d}x, [Q_{a3}] = \int_0^L a_{3x}[H]^{\mathrm{T}}\mathrm{d}x, [Q_{a4}] = \int_0^L a_{4x}[H]^{\mathrm{T}}\mathrm{d}x,$$

$$[Q_{a5}] = \int_0^L a_{5x}[H]^{\mathrm{T}}\mathrm{d}x, [Q_{a6}] = \int_0^L a_{6x}[H]^{\mathrm{T}}\mathrm{d}x, [Q_{a7}] = \int_0^L a_{7x}[H]^{\mathrm{T}}\mathrm{d}x,$$

$$[C_{a1}] = \int_0^L b_{1x}[H]^{\mathrm{T}}[H]\mathrm{d}x, [C_{a2}] = \int_0^L b_{2x}[H]^{\mathrm{T}}[H]\mathrm{d}x,$$

$$[D_{a1}] = \int_0^L c_{1x}[H]^T[H']\mathrm{d}x, \text{ and } [D_{a2}] = \int_0^L c_{2x}[H]^T[H']\mathrm{d}x.$$

From Eqs. (2.52) and (2.55), we write

$$\Omega^2[M][\ddot{q}] + [K][q] = a_{1\psi}[Q_{a1}] + a_{2\psi}[Q_{a2}] + a_{3\psi}[Q_{a3}] + a_{4\psi}[Q_{a4}] + a_{5\psi}[Q_{a5}] + a_{6\psi}[Q_{a6}] + a_{7\psi}[Q_{a7}] + (b_{1\psi}[C_{a1}] + b_{2\psi}[C_{a2}])[\dot{q}] + (c_{1\psi}[D_{a1}] + c_{2\psi}[D_{a2}])[q] \tag{2.56}$$

We can see the presence of term involving $[q]$ and $[\dot{q}]$ on the right-hand side of this equation. These are motion-dependent forces. The presence of three forces changes the problem from a structural dynamic problem to an aeroelastic problem.

$$\Omega^2[M][\ddot{q}] + [K][q] = [Q] + [C][\dot{q}] + [D][q] \tag{2.57}$$

where

$$[Q] = a_{1\psi}[Q_{a1}] + a_{2\psi}[Q_{a2}] + a_{3\psi}[Q_{a3}] + a_{4\psi}[Q_{a4}] + a_{5\psi}[Q_{a5}] + a_{6\psi}[Q_{a6}] + a_{7\psi}[Q_{a7}]$$

$$[C] = b_{1\psi}[C_{a1}] + b_{2\psi}[C_{a2}]$$

$$[D] = c_{1\psi}[D_{a1}] + c_{2\psi}[D_{a2}]$$

After transformation of coordinate, we write Eq. (2.57) as

$$\Omega^2[M][\phi][\ddot{\zeta}] + [K][\phi][\zeta] = [Q] + [C][\phi][\dot{\zeta}] + [D][\phi][\zeta] \tag{2.58}$$

where $[q] = [\phi][\zeta]$, and $[\phi]$ being the eigenvectors. We can then write

$$\Omega^2[\phi]^{\mathrm{T}}[M][\phi][\ddot{\zeta}] + [\phi]^{\mathrm{T}}[K][\phi][\zeta] = [\phi]^{\mathrm{T}}[Q] + [\phi]^{\mathrm{T}}[C][\phi][\dot{\zeta}] + [\phi]^{\mathrm{T}}[D][\phi][\zeta] \tag{2.59}$$

or

$$\Omega^2[M_1][\ddot{\zeta}] + [K_1][\zeta] = [Q_1] + [C_1][\dot{\zeta}] + [D_1][\zeta] \tag{2.60}$$

where $[M_1] = [\phi]^{\mathrm{T}}[M][\phi], [K_1] = [\phi]^{\mathrm{T}}[K][\phi], [Q_1] = [\phi]^{\mathrm{T}}[Q], [C_1] = [\phi]^{\mathrm{T}}[C][\phi]$, and $[D_1] = [\phi]^{\mathrm{T}}[D][\phi]$. We notice that Eq. (2.60) is an ordinary equation having periodic coefficients and motion-dependent forcing.

We write Eq. (2.60) as

$$[A(\psi)][\ddot{\zeta}] + [B(\psi)][\dot{\zeta}] + [C(\psi)][\zeta] = [D(\psi)] \tag{2.61}$$

where $[A], [B], [C]$, and $[D]$ contain periodic functions. Thus, all the motion-dependent forces are moved to the left-hand side. This is important for solving the equation. The motion-dependent forces change the stiffness and damping terms and thus the behavior of the system. At this point, the spatial coordinate has been removed from the equation. The resulting set of ordinary differential equations now need to be solved. Chapter 3 will address the solutions of the rotor dynamics problem for the time response.

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