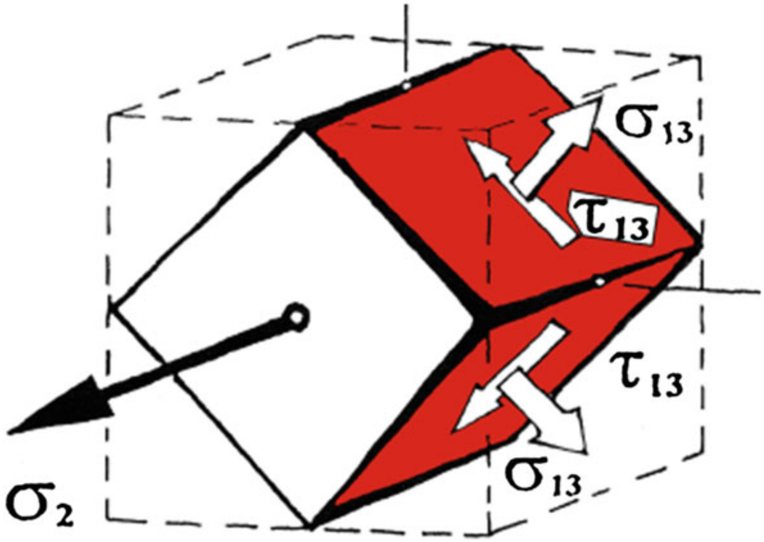
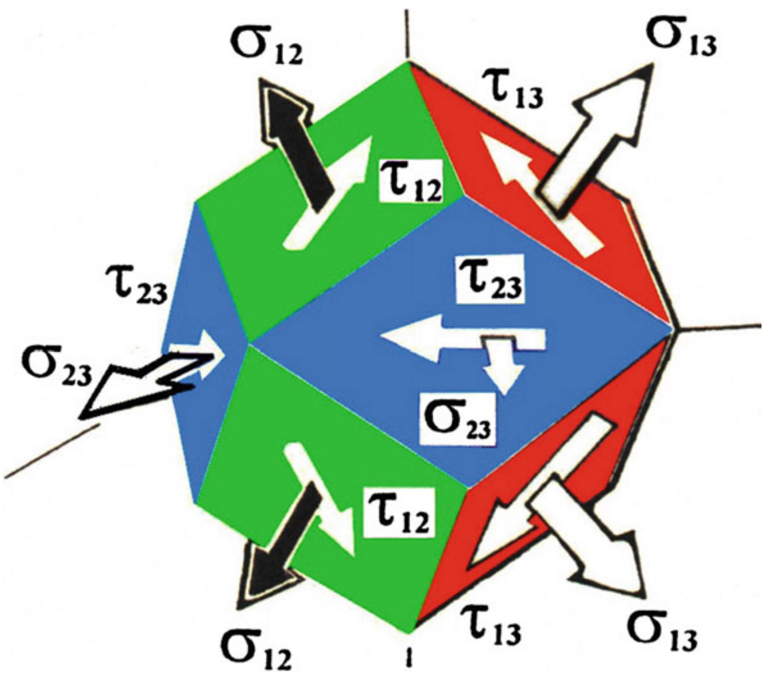
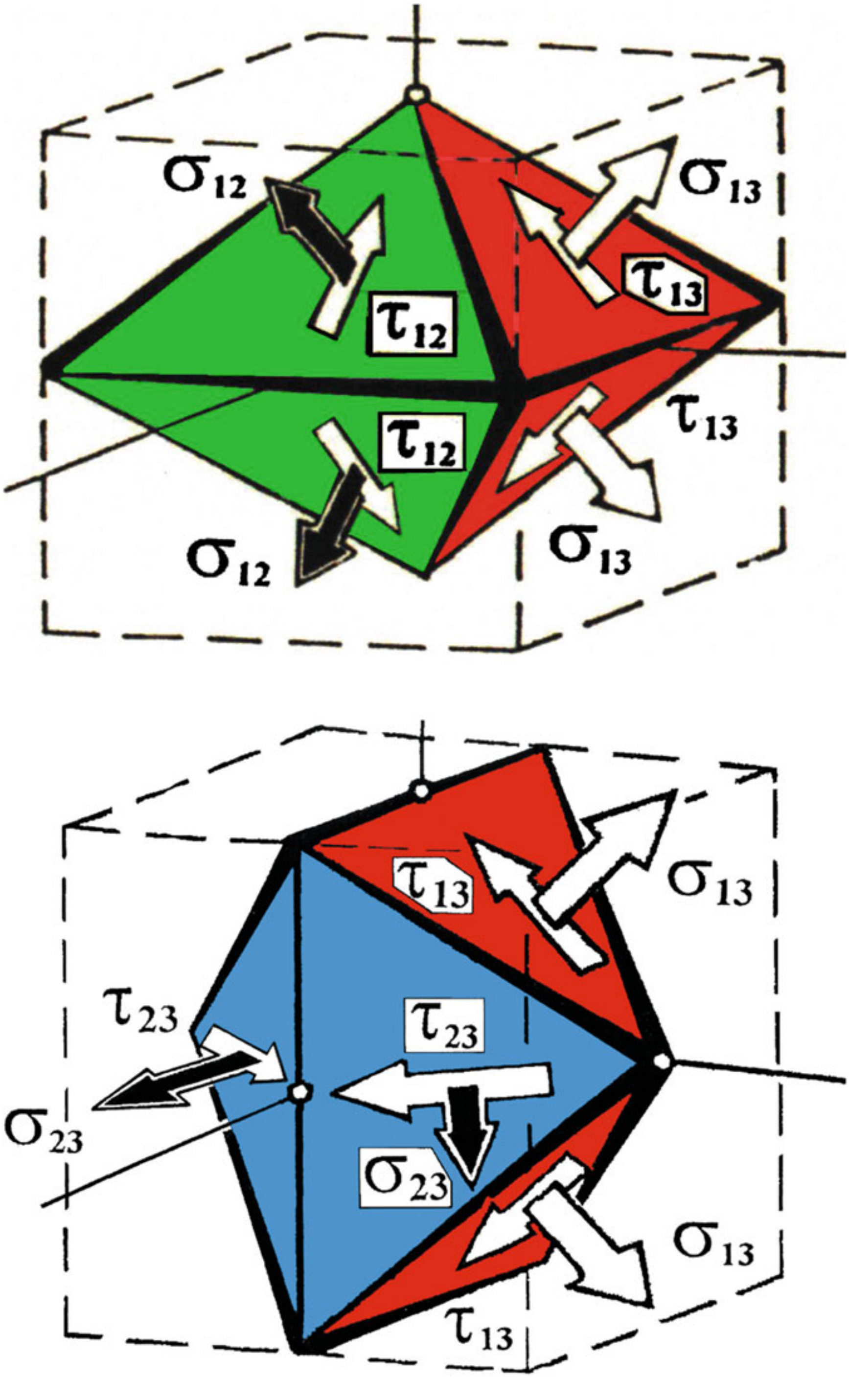


Single-shear element



Tri-shear element



Twin-shear element

Chapter 2
Stress State and Elements

2.1 Elements

The strength theory is concerned with yield or failure of material under complex stress states. It is used to provide strength design criteria for analysis and design of engineering structures. Element and stress state is a key concept of mechanics, so it will be discussed in this Chapter. The stress state theory can be also seen in mechanics of materials, elasticity, plasticity, rock mechanics, soil mechanics, computational mechanics, etc. This chapter gives a less detailed description of the stress state and the results we need only, and focuses on the contents rarely talked about in general.

In applied mechanics and engineering, materials and structures are generally regarded as continua. This permits us to describe the behavior and consequences of materials and structures by means of continuous functions. A material element can be regarded as a point and a structure can be treated as a body. The structure may be considered as a body filled with a partly ordered set of material elements (points). The cubic element is often used. An element that can fill a space without gaps and overlapping is called the spatial equipartition.

It is worth noticing that the deformed orthogonal octahedral model remains a parallel hedron, which can fill the space without leaving any gaps or overlaps. The combination of many orthogonal octahedral models can be used as a continuous body, as shown in Fig. 2.1.

M.-H. Yu, *Unified Strength Theory and Its Applications*,
https://doi.org/10.1007/978-981-10-6247-6_2

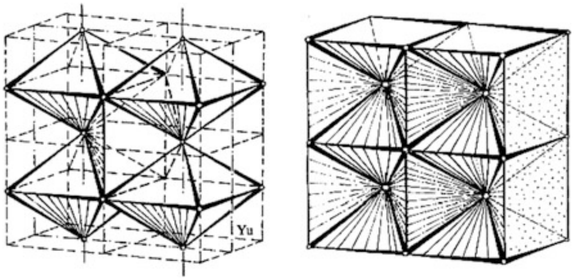


Fig. 2.1 Spatial equipartition of orthogonal octahedral element

The various polyhedra shown in Fig. 2.2 are spatial equipartitions. They are the cubic element (a), regular hexagonal element (b), isoclinal octahedron element (c), dodecahedron element (d), orthogonal octahedron element (e) and pentahedron element (f).

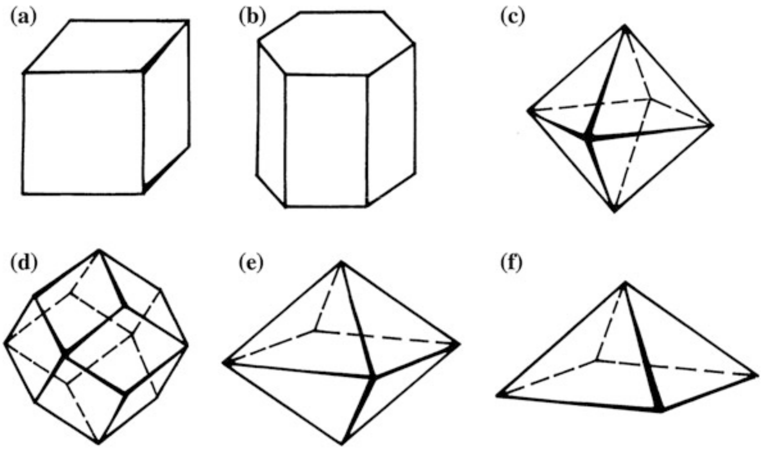


Fig. 2.2 a, b, c, d, e and **f** Spatial equipartition elements

2.2 Stress at a Points: Stress Invariants

A general state of stress at a point can be determined by a stress tensor σ_{ij}, which stands for nine components:

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{2.1}$$

It can be seen in any course of mechanics of materials, elasticity, mechanics of solids or plasticity, by three-dimensional transformations, that there exists a coordinate system σ_1, σ_2, σ_3 where the state of stress at the same point can be described by the following:

$$\sigma_i = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{2.2}$$

The stresses σ_1, σ_2, σ_3 are referred to as the principal stresses.

An element of material subjected to principal stresses σ_1, σ_2 and σ_3 acting in mutually perpendicular directions (Fig. 2.3) is said to be in a state of triaxial stress or three-dimensional stress. If one of the principal stresses equals zero, this is referred to as the plane stress state or biaxial stress state. The triaxial stress and biaxial stress are called the polyaxial stresses, multiaxial stresses or complex stress. The principal planes are the planes on which the principal stresses occur on mutually perpendicular planes.

The principal stresses are the three roots of the equation:

$$\begin{aligned} &\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\sigma \\ &\quad - (\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2) = 0 \end{aligned} \tag{2.3}$$

which can be rewritten as

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \tag{2.4}$$

where I_1, I_2, I_3 are

$$\begin{aligned} I_1 &= \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - (\sigma_x\tau_{yz}^2 + \sigma_y\tau_{zx}^2 + \sigma_z\tau_{xy}^2) = \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{2.5}$$

The quantities I_1, I_2 and I_3 are independent of the direction of the axes chosen; they are called the first, second, and third invariants, respectively, of the stress tensor at a point.

2.3 Deviatoric Stress Tensor, Deviatoric Tensor Invariants

It is convenient in the study of strength theory and plasticity to split the stress tensor into two parts, one called the deviatoric stress tensor S_{ij} and the other the spherical stress tensor p_{ij}. The relation is

$$\sigma_{ij} = S_{ij} + p_{ij} = S_{ij} + \sigma_m \delta_{ij} \tag{2.6}$$

The spherical stress tensor is the tensor whose components are $\sigma_m \delta_{ij}$, where σ_m is the mean stress, i.e.,

$$p_{ij} = \sigma_m \delta_{ij} = \sigma_m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

where

$$\sigma_m = (\sigma_x + \sigma_y + \sigma_z)/3 = (\sigma_1 + \sigma_2 + \sigma_3)/3 = I_1/3 \tag{2.7}$$

It is apparent that σ_m is the same for all possible orientations of the axes; hence the name spherical stress. Also, since σ_m is the same in all directions, it can be considered to act as a hydrostatic stress.

The deviatoric stress tensor S_{ij} can be determined as follows

$$S_{ij} = \sigma_{ij} - p_{ij} = \sigma_{ij} - \sigma_m \delta_{ij} = \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_m \end{bmatrix} \tag{2.8}$$

The invariants of the deviatoric stress tensor are denoted by J_1, J_2, J_3 and can be obtained as follows

$$J_1 = S_1 + S_2 + S_3 = 0 \tag{2.9}$$

$$J_2 = \frac{1}{2} S_{ij} S_{ij} = \frac{2}{3}\left(\tau_{13}^2 + \tau_{12}^2 + \tau_{23}^2\right) = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \tag{2.10}$$

$$J_3 = \left|S_{ij}\right| = S_1 S_2 S_3 = \frac{1}{27}(\tau_{13} + \tau_{12})(\tau_{21} + \tau_{23})(\tau_{31} + \tau_{32}) \tag{2.11}$$

2.4 Stresses on the Oblique Plane

If the three principal stresses σ_1, σ_2, σ_3 acting on three principal planes, respectively, at a point are given, we can determine the stresses acting on any plane through this point. This can be done by consideration of the static equilibrium of an infinitesimal tetrahedron formed by this plane and the principal planes, as shown in Fig. 2.3. In this figure, we have shown the principal stresses acting on the three principal planes. These stresses are assumed to be known. We wish to find the stresses σ_α, τ_α acting on the oblique plane whose normal has direction cosines *l, m* and *n*.

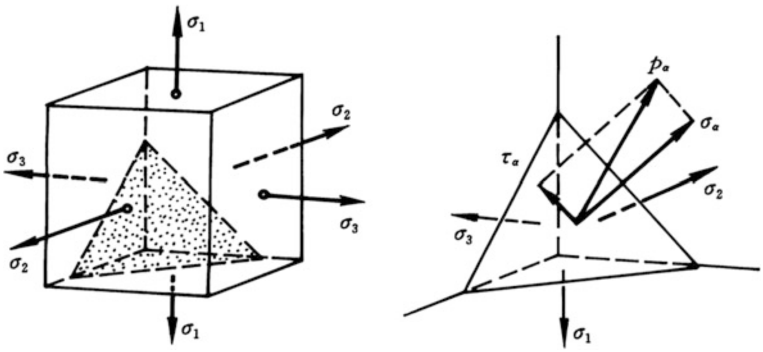


Fig. 2.3 Stress on an infinitesimal tetrahedron

2.4.1 Stresses on the Oblique Plane

The normal stress σ_α and shear stress τ_α acting on this plane can be determined as follows:

$$\sigma_\alpha = \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2 \tag{2.12}$$

$$\tau_\alpha = \sigma_1^2 l^2 + \sigma_2^2 m^2 + \sigma_3^2 n^2 - (\sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2) \tag{2.13}$$

$$\vec{p}_\alpha = \vec{\sigma}_\alpha + \vec{\tau}_\alpha \tag{2.14}$$

2.4.2 Principal Shear Stresses

The three principal shear stresses τ_{13}, τ_{12} and τ_{23} can be obtained as follows:

$$\tau_{13} = \frac{1}{2}(\sigma_1 - \sigma_3), \tau_{12} = \frac{1}{2}(\sigma_1 - \sigma_2), \tau_{23} = \frac{1}{2}(\sigma_2 - \sigma_3) \tag{2.15}$$

The maximum shear stress acts on the plane bisecting the angle between the largest and smallest principal stresses and is equal to half of the difference between these principal stresses

$$\tau_{\max} = \tau_{13} = \frac{1}{2}(\sigma_1 - \sigma_3) \tag{2.16}$$

The corresponding normal stresses σ_{13}, σ_{12} and σ_{23} acting on the sections where τ_{13}, τ_{12} and τ_{23} are acting, respectively, are

$$\sigma_{13} = \frac{1}{2}(\sigma_1 + \sigma_3), \sigma_{12} = \frac{1}{2}(\sigma_1 + \sigma_2), \sigma_{23} = \frac{1}{2}(\sigma_2 + \sigma_3) \tag{2.17}$$

It is seen from Eq. 2.10 that the maximum principal shear stress τ_{13} equals to the sum of the other two (τ_{12} + τ_{23}), i.e.,

$$\tau_{13} = \tau_{12} + \tau_{23} \tag{2.18}$$

2.4.3 Octahedral Shear Stress

If the normal of the oblique plane makes equal angles with all the principal axes, and

$$l = m = n = \pm \frac{1}{\sqrt{3}} \tag{2.19}$$

These planes are called the octahedral plane and the shear stresses acting on it are called the octahedral shear stresses. The normal stress, called the octahedral normal stress σ_8 (or σ_{oct}), acting on this plane equals the mean stress

$$\sigma_8 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \sigma_m \tag{2.20}$$

A tetrahedron similar to this one can be constructed in each of the four quadrants above the *x–y* plane and in each of the four quadrants below the *x–y* plane. On the oblique face of each of these eight tetrahedra the condition $l^2 = m^2 = n^2 = 1/3$ will apply. The difference between the tetrahedra will be in the signs attached to *l, m* and *n*. The eight tetrahedra together form an octahedra, and on each of the eight planes form the faces of this octahedron.

The octahedral normal stress is given by Eq. (2.20) and the octahedral shear stress τ_8 (sometimes denoted as τ_{oct}) acting on the octahedral plane is

$$\begin{aligned} \tau_8 &= \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \\ &= \frac{1}{\sqrt{3}}\left[(\sigma_1 - \sigma_m)^2 + (\sigma_2 - \sigma_m)^2 + (\sigma_3 - \sigma_m)^2\right]^{1/2} \end{aligned} \tag{2.21}$$

The direction cosines *l, m* and *n* of principal plane, principal shear stress plane and the octahedral plane, as well as the principal shear stresses and corresponding normal stresses are listed in Table 2.1.

Table 2.1 Direction cosines of the principal planes and the principal shear stress planes etc

	Principal plane			Principal shear stress plane			Octa. plane
$l =$	± 1	0	0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{3}}$
$m =$	0	± 1	0	$\pm\frac{1}{\sqrt{2}}$	0	$\pm\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{3}}$
$n =$	0	0	± 1	0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{3}}$
$\sigma =$	σ_1	σ_2	σ_3	$\sigma_{12} = \frac{\sigma_1+\sigma_2}{2}$	$\sigma_{13} = \frac{\sigma_1+\sigma_3}{2}$	$\sigma_{23} = \frac{\sigma_2+\sigma_3}{2}$	σ_8
$\tau =$	0	0	0	$\tau_{12} = \frac{\sigma_1-\sigma_2}{2}$	$\tau_{13} = \frac{\sigma_1-\sigma_3}{2}$	$\tau_{23} = \frac{\sigma_2-\sigma_3}{2}$	τ_8

2.5 Hexahedron, Octahedron, Dodecahedron

According to the stress state, various polyhedral elements can be illustrated as shown in Figs. 2.4, 2.5, 2.6 and 2.7.

2.5.1 Principal Stress Element (σ_1, σ_2, σ_3)

Principal stress element is a cubic element, the three principal stresses σ_1, σ_2, σ_3 act on this element. The principal stress element and three principal stresses (σ_1, σ_2, σ_3) are shown in Fig. 2.4.

2.5.2 Isoclinal Octahedron Element (τ_8, σ_8)

Isoclinal octahedral element subjected to the octahedral normal stresses σ_8 and octahedral shear stresses τ_8, as shown in Fig. 2.5, is a regular octahedron.

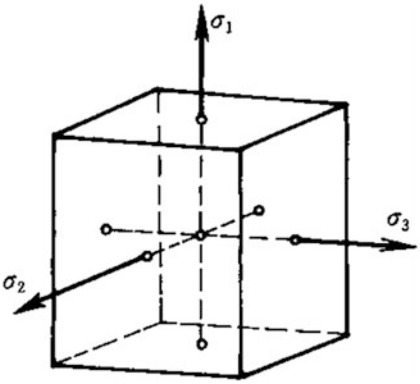


Fig. 2.4 Principal stress element

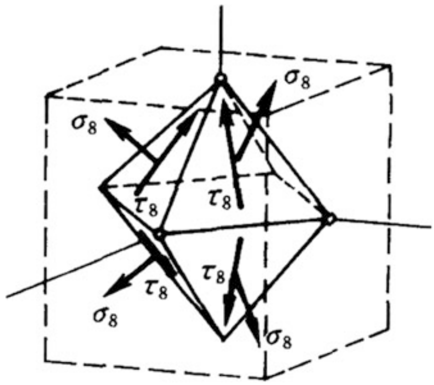
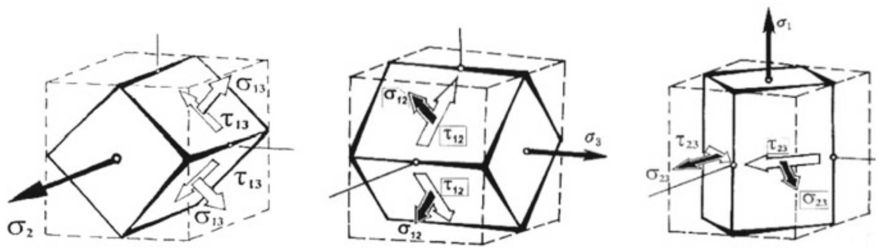


Fig. 2.5 σ_8 and τ_8 element

2.5.3 Single-Shear Element (τ_{13}–σ_{13}, τ_{12}–σ_{12}; τ_{23}–σ_{23})

The maximum shear stress element (τ_{13}, σ_{13}, σ_2) is a quadrangular prism element, which the maximum shear stress τ_{13}, corresponding normal stress σ_{13}, as well as the intermediate principal stress σ_2 act on. This kind of element, as shown in Fig. 2.6a, may be referred to as the single-shear element because only one shear stress and corresponding normal stress act on the element.

Quadrangular prism element (τ_{12}, σ_{12}, σ_3) is shown in Fig. 2.6b, acted by the intermediate principal shear stress element (when $\tau_{12} \geq \tau_{23}$), the intermediate principal shear stress τ_{12} and the corresponding normal stress σ_{12}, as well as the minimum principal stress σ_3. Quadrangular prism element (τ_{23}, σ_{23}, σ_1) is shown in Fig. 2.6c, acted by the minimum principal shear stress element (when $\tau_{12} \leq \tau_{23}$), the minimum principal shear stress τ_{23} and the corresponding normal stress σ_{23}, as well as the maximum principal stress σ_1.



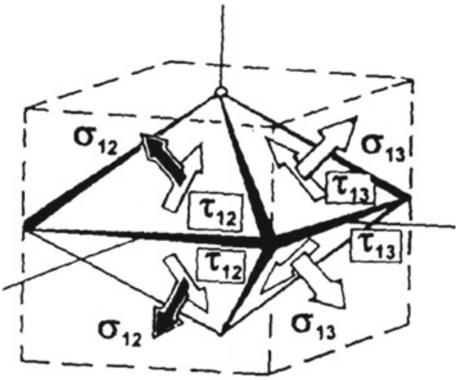
(a) ($\tau_{13}, \sigma_{13}, \sigma_2$) (b) ($\tau_{12}, \sigma_{12}, \sigma_3$) (c) ($\tau_{23}, \sigma_{23}, \sigma_1$)

Fig. 2.6 Single-shear elements

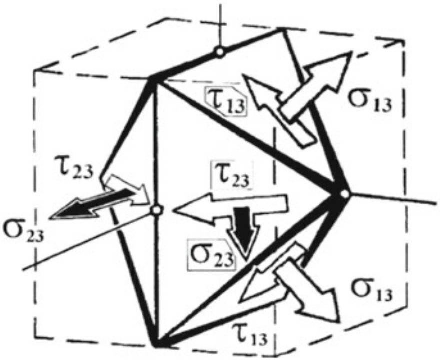
2.5.4 Twin-Shear Element (τ_{13}–σ_{13}, τ_{12}–σ_{12}); (τ_{13}–σ_{13}, τ_{23}–σ_{23})

Figure 2.7a shows an orthogonal octahedron (τ_{13}, τ_{12}; σ_{13}, σ_{12}), in which the principal shear stresses τ_{13}, τ_{12} and the corresponding normal stresses σ_{13}, σ_{12} act on this element. This new element was proposed by Yu (Yu 1988, 1989). It can be referred to as the twin-shear element.

The principal shear stresses τ_{13}, τ_{23} and the corresponding normal stresses σ_{13}, σ_{23} act on an orthogonal octahedron element (τ_{13}, τ_{23}; σ_{13}, σ_{23}), as shown in Fig. 2.7b. This element can also be referred to as the twin-shear element. They are available to use for the mechanical model of strength theory.



(a) (τ_{13}, τ_{12}; σ_{13}, σ_{12})



(b) (τ_{13}, τ_{23}; σ_{13}, σ_{23})

Fig. 2.7 Twin-shear elements

How to get the twin-shear element from principal stress element to single-shear element and then from single-shear element to twin-shear element? The process is illustrated in Fig. 2.8. It can be easy obtained that the twin-shear element is a spatial equipartition, as shown in Fig. 2.1.

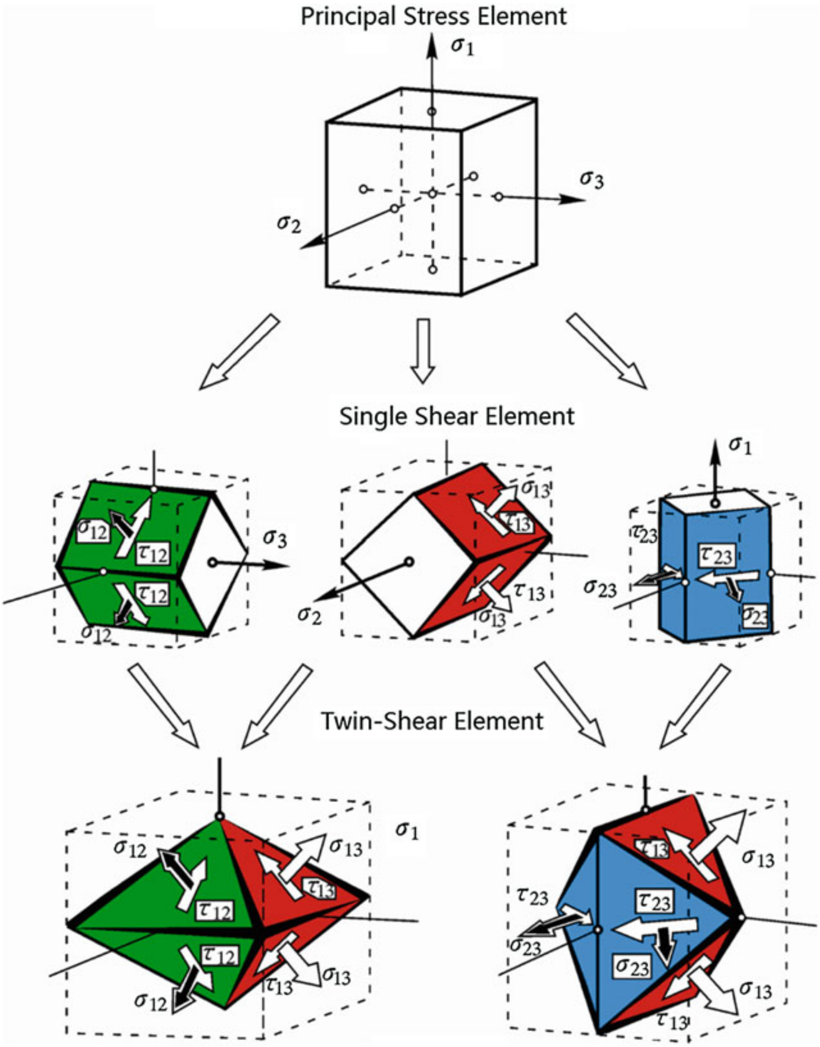


Fig. 2.8 From principal stress element to twin-shear element

2.5.5 Three-Shear Element (τ_{13}, τ_{12}, τ_{23}; σ_{13}, σ_{12}, σ_{23})

The three principal shear stresses τ_{13}, τ_{12}, τ_{23} and the corresponding normal stresses σ_{13}, σ_{12}, σ_{23} acting on a element is shown in Fig. 2.9. The first presentation of the dodecahedron element may be by Walczak at Krakov, Poland in 1951. This element of dodecahedron can be referred to as the three-shear element. It is interesting that the three principal shear stresses τ_{13}, τ_{12}, τ_{23} only have two independent variations because the maximum principal shear stress equals the sum of the other two, i.e. $\tau_{13} = \tau_{12} + \tau_{23}$. The formation of the three-shear element is illustrated in Fig. 2.9.

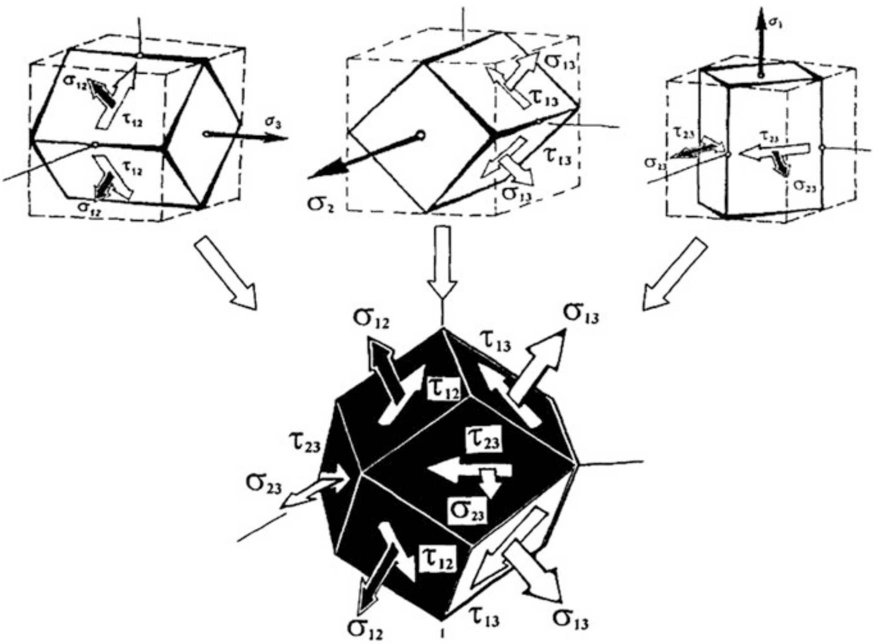


Fig. 2.9 The formation of the three-shear element

2.5.6 Twin-Shear Stress State and Twin-Shear Element

The stress state at a point can be determined by the combination of the three principal stresses (σ_1, σ_2, σ_3). It is expressed by $f(\sigma_1, \sigma_2, \sigma_3)$. The principal stress state $f(\sigma_1, \sigma_2, \sigma_3)$ can be converted to principal shear stress state $f(\tau_{13}, \tau_{12}, \tau_{23})$. However, only two principal shear stresses of the three are dependent variables because the maximum principal shear stress τ_{13} equals the sum of the other two shear stresses. This relationship can be expressed as follows:

$$\tau_{13} \equiv \tau_{12} + \tau_{23} \tag{2.22}$$

The concept of twin-shear can be illustrated by Mohr's stress circle, as shown in Fig. 2.10.

As is described, there are only two independent variables in the all three shear stresses. We can understand the relationship among three shear stresses in figure the maximum is twice the diameter of the sum of other two. Like the twin-shear element, there are two types of twin-shear stress circle, as shown in Fig. 2.11a, b.

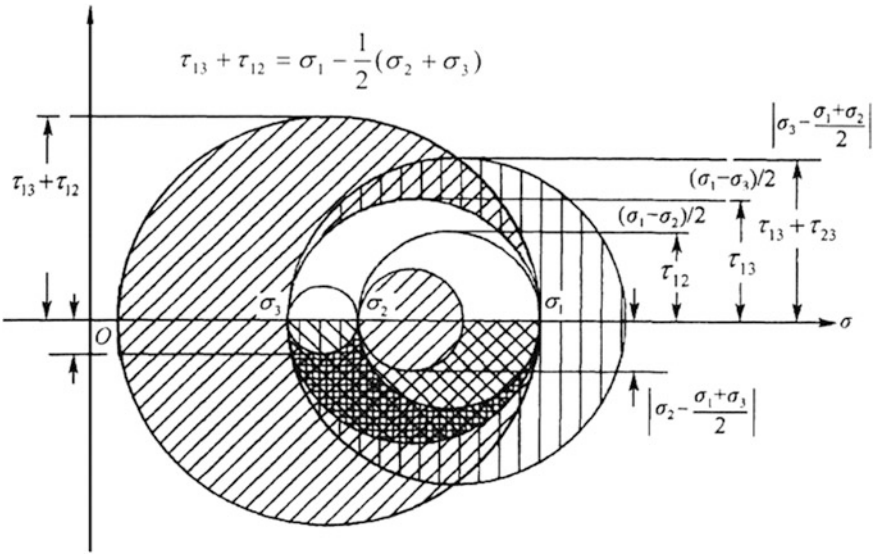


Fig. 2.10 The concept of twin-shear can be illustrated by Mohr's stress circle

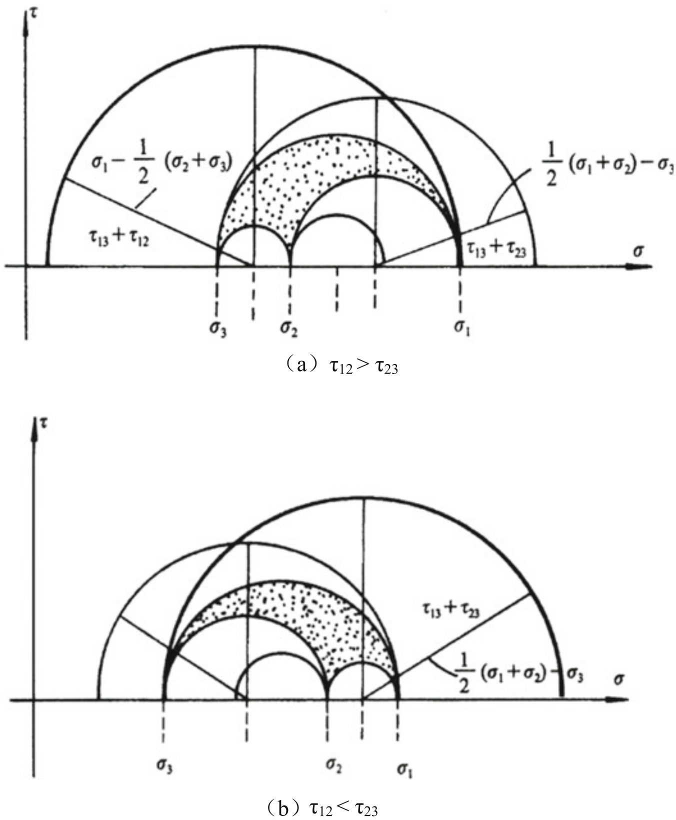


Fig. 2.11 Two types of twin-shear stress circle

2.6 Stress Space

The stress point P $(\sigma_1, \sigma_2, \sigma_3)$ in stress space can be expressed by other forms, such as $P(x, y, z)$, $P(r, \theta, \xi)$, or $P(J_2, \theta, \xi)$. The geometrical representation of these transfers can be seen in Figs. 2.12 and 2.13.

For the straight line OZ passing through the origin and making the same angle with each of the coordinate axes, the equation is

$$\sigma_1 = \sigma_2 = \sigma_3 \tag{2.23}$$

The equation of the π_0-plane is

$$\sigma_1 + \sigma_2 + \sigma_3 = 0 \tag{2.24}$$

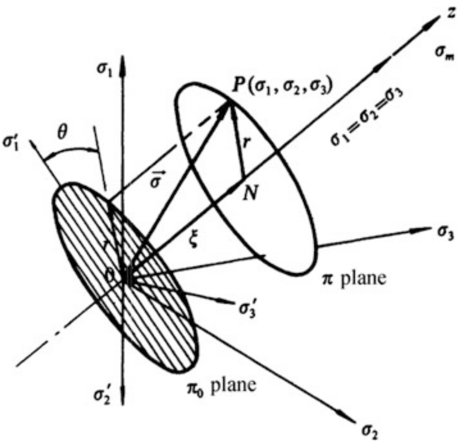


Fig. 2.12 Cylindrical coordinates

The stress tensor σ_{ij} can be divided into the spherical stress tensor and deviatoric stress tensor. The stress vector σ can also be divided into two parts: the hydrostatic stress vector σ_m and the mean shear stress vector τ_m.

$$\sigma = \sigma_\mathrm{m} + \tau_\mathrm{m} \tag{2.25}$$

Their magnitudes are given by

$$\xi = \frac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3) \tag{2.26}$$

$$r = \sqrt{\frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} = \sqrt{3}\tau_8 = \sqrt{2J_2} = 2\tau_m \tag{2.27}$$

in which σ_8 is the octahedral normal stress and τ_8 is the octahedral shear stresses.

$$\tau_\mathrm{m} = \sqrt{\frac{\tau_{13}^2 + \tau_{12}^2 + \tau_{23}^2}{3}} = \sqrt{\frac{1}{12}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} \tag{2.28}$$

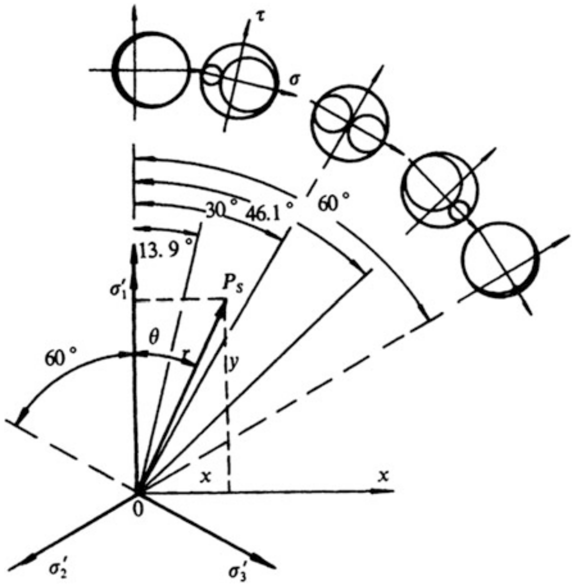


Fig. 2.13 Stress state in the π-plane

The π-plane is parallel to the π_0-plane and is given by

$$\sigma_1 + \sigma_2 + \sigma_3 = C \tag{2.29}$$

in which C is a constant. The spherical stress tensor σ_m is the same for all points in the π-plane of stress space and

$$\sigma_m = C/3 \tag{2.30}$$

The projections of the three principal stress axes in stress space σ_1, σ_2, σ_3 are $\sigma_1', \sigma_2', \sigma_3'$. The relationship between them is

$$\sigma_1' = \sigma_1 \cos\beta = \sqrt{\frac{2}{3}}\sigma_1, \sigma_2' = \sigma_2 \cos\beta = \sqrt{\frac{2}{3}}\sigma_2, \sigma_3' = \sigma_3 \cos\beta = \sqrt{\frac{2}{3}}\sigma_3 \tag{2.31}$$

in which β is the angle between $O'A, O'B, O'C$ and the three coordinates as shown in Fig. 2.14.

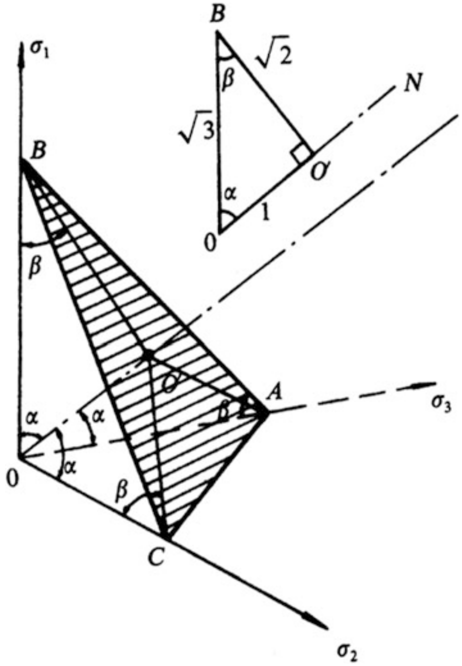


Fig. 2.14 Deviatoric plane

2.6.1 Relation Between (σ_1, σ_2, σ_3) and (X, Y, Z)

The relations between the coordinates of the deviatoric plane and the principal stresses are:

$$x = \frac{1}{\sqrt{2}}(\sigma_3 - \sigma_2), y = \frac{1}{\sqrt{6}}(2\sigma_1 - \sigma_2 - \sigma_3), z = \frac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3) \tag{2.32}$$

$$\begin{aligned} \sigma_1 &= \frac{1}{3}\left(\sqrt{6}y + \sqrt{3}z\right), \sigma_2 = \frac{1}{6}\left(2\sqrt{3}z - \sqrt{6}y - 3\sqrt{2}x\right), \\ \sigma_3 &= \frac{1}{6}\left(3\sqrt{2}x - \sqrt{6}y + 2\sqrt{3}z\right) \end{aligned} \tag{2.33}$$

2.6.2 Relation (σ_1, σ_2, σ_3) and (ξ, r, θ) or (J_2, τ_m, θ)

The relations between the cylindrical coordinates (ξ, r, θ) and the principal stresses $(\sigma_1, \sigma_2, \sigma_3)$ are

$$\xi = |ON| = \frac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{I_1}{3} = \sqrt{3}\sigma_m \tag{2.34}$$

$$\begin{aligned} r = |NP| &= \frac{1}{\sqrt{3}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{\frac{1}{2}} \\ &= \left(S_1^2 + S_2^2 + S_3^2\right)^{\frac{1}{2}} = \sqrt{2J_2} = \sqrt{3}\tau_8 = 2\tau_m \end{aligned} \tag{2.35}$$

$$\theta = \tan^{-1}\left(\frac{x}{y}\right) \tag{2.36}$$

From Eqs. (2.32) and (2.35) we can obtain

$$\cos\theta = \frac{y}{r} = \frac{\sqrt{6}S_1}{\sqrt{2J_2}} = \frac{\sqrt{3}}{2}\frac{S_1}{\sqrt{J_2}} = \frac{2\sigma_1 - \sigma_2 - \sigma_3}{2\sqrt{3}\sqrt{J_2}} \tag{2.37}$$

The second and third invariants of the deviatoric stress tensor are

$$J_2 = -(S_1S_2 + S_2S_3 + S_3S_1) \tag{2.38}$$

$$J_3 = S_1S_2S_3 \tag{2.39}$$

Three principal deviatoric stresses can be deduced

$$S_1 = \frac{2}{\sqrt{3}}\sqrt{J_2}\cos\theta, S_2 = \frac{2}{\sqrt{3}}\sqrt{J_2}\cos\left(\frac{2\pi}{3} - \theta\right), S_3 = \frac{2}{\sqrt{3}}\sqrt{J_2}\cos\left(\frac{2\pi}{3} + \theta\right) \tag{2.40}$$

These relations are suitable to the conditions $\sigma_1 \geq \sigma_2 \geq \sigma_3$ and $0 \leq \theta \leq \pi/3$. The limit loci in the π-plane has threefold symmetry, so if the limit loci in the range of 60° are given, then the limit loci in π-plane can be obtained.

The three principal stresses can be expressed as follows:

$$\begin{aligned} \sigma_1 &= \frac{1}{\sqrt{3}}\xi + \sqrt{\frac{2}{3}}r\cos\theta \\ \sigma_2 &= \frac{1}{\sqrt{3}}\xi + \sqrt{\frac{2}{3}}r\cos(\theta - 2\pi/3) \quad 0 \leq \theta \leq \frac{\pi}{3} \\ \sigma_3 &= \frac{1}{\sqrt{3}}\xi + \sqrt{\frac{2}{3}}r\cos(\theta + 2\pi/3) \end{aligned} \tag{2.41}$$

The principal stresses can also be expressed in terms of the first invariant I_1 of the stress tensor and the second invariant of the deviatoric stress J_2 as follows:

$$\begin{aligned}\sigma_1 &= \frac{I_1}{3} + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos\theta \\ \sigma_2 &= \frac{I_1}{3} + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos(\theta - \frac{2\pi}{3}) \\ \sigma_3 &= \frac{I_1}{3} + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos(\theta + \frac{2\pi}{3})\end{aligned} \tag{2.42}$$

The principal shear stresses can also be obtained

$$\begin{aligned}\tau_{13} &= \sqrt{J_2}\sin\left(\theta + \frac{\pi}{3}\right) = \sqrt{2}\tau_{\mathrm{m}}\sin\left(\theta + \frac{\pi}{3}\right) \\ \tau_{12} &= \sqrt{J_2}\sin\left(\frac{\pi}{3} - \theta\right) \\ \tau_{23} &= \sqrt{J_2}\sin(\theta)\end{aligned} \tag{2.43}$$

2.7 Stress State Parameters

The stress state at a point (element) is determined by the combination of the three principal stresses (σ_1, σ_2, σ_3). Based on the characteristics of the stress state and by introducing a certain parameter, it can be divided into several types. In 1926, Lode introduced a stress parameter μ_σ as follows:

$$\mu_\sigma = (2\sigma_2 - \sigma_1 - \sigma_3)/(\sigma_1 - \sigma_3) \tag{2.44}$$

which is referred to as the Lode stress parameter. The Lode parameter can be expressed in terms of principal shear stress as follows

$$\mu_\sigma = \frac{2\sigma_2 - \sigma_1 - \sigma_3}{\sigma_1 - \sigma_3} = \frac{\tau_{23} - \tau_{12}}{\tau_{13}} \tag{2.45}$$

Subsequently, Yu M-H introduced the “twin shear stress” concept into the analysis of the stress state and offered two twin–shear stress parameters (Yu M-H 1991, 1992):

$$\mu_\tau = \frac{\tau_{12}}{\tau_{13}} = \frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3} = \frac{S_1 - S_2}{S_1 - S_3} \tag{2.46}$$

$$\mu'_\tau = \frac{\tau_{23}}{\tau_{13}} = \frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3} = \frac{S_2 - S_3}{S_1 - S_3} \tag{2.47}$$

$$\mu_\tau + \mu'_\tau = 1, \quad 0 \le \mu_\tau \le 1, 0 \le \mu'_\tau \le 1 \tag{2.48}$$

The twin-shear stress parameters are simpler and have an explicit physical meaning. They can reflect the state of the intermediate principal stress and can represent the status of stress state.

The twin-shear stress parameters have nothing to do with the hydrostatic stress. They instead represent the status of the deviatoric stress state and the stress angle on the deviatoric plane in stress space, as shown in Fig. 2.13. Five different stress states are shown in Fig. 2.13. They are $\theta = 0^\circ (\mu_\tau = 1)$, $\theta = 13.9^\circ$ $(\mu_\tau = 3/4$, $\mu'_\tau = 1/4)$, $\theta = 30°$ $(\mu_\tau = \mu'_\tau = 0.5)$, $\theta = 46.1^\circ$ $(\mu_\tau = 1/4$, $\mu'_\tau = 3/4)$ and $\theta = 60^\circ$ $(\mu_\tau = 0$, $\mu'_\tau = 1)$. According to the meaning of the twin–shear stress parameters, we know that:

If $\mu_\tau = 1$ ($\mu'_\tau = 0$, (stress angle equals $\theta = 0°$), the stress states include three following cases:

1. $\sigma_1 > 0, \quad \sigma_2 = \sigma_3 = 0$, uniaxial tension stress state;
2. $\sigma_1 = 0, \quad \sigma_2 = \sigma_3 < 0$, equal biaxial compression stress state;
3. $\sigma_1 > 0, \quad \sigma_2 = \sigma_3 < 0$, uniaxial tension, equal biaxial compression stress state.

If $\mu_\tau = \mu'_\tau = 0.5$ (stress angle equals $\theta = 30°$), the corresponding stress states are as follows:

1. $\sigma_2 = \frac{1}{2}(\sigma_1 + \sigma_3) = 0$, pure shear stress state;
2. $\sigma_2 = (\sigma_1 + \sigma_3)/2 > 0$, biaxial tension and uniaxial compression stress state;
3. $\sigma_2 = (\sigma_1 + \sigma_3)/2 < 0$, uniaxial tension and biaxial compression stress state.

If $\mu_\tau = 0, \mu'_\tau = 1$, (stress angle equals $\theta = 60°$), then the corresponding stress states are as follows:

1. $\sigma_1 = \sigma_2 = 0, \ \sigma_3 < 0$, uniaxial compression stress state;
2. $\sigma_1 = \sigma_2 > 0, \ \sigma_3 = 0$, equal biaxial tension stress state;
3. $\sigma_1 = \sigma_2 > 0, \ \sigma_3 < 0$, equal biaxial tension and uniaxial compression stress state.

According to the twin-shear stress parameters and the magnitude of the two smaller principal shear stresses, the stress state can be divided into three kinds of conditions as follows:

1. Extended tension stress state, that is, $\tau_{12} > \tau_{23}$, $0 \le \mu'_\tau < 0.5 < \mu_\tau \le 1$. The stress state (uniaxial tension and biaxial compression) can be expressed by deviatoric stress, and the absolute magnitude of the tensile stress is a maximum, so it can be called the extended tension stress state. When the intermediate principal stress σ_2 equals the minimum principal stress σ_3, then $\mu_\tau = 1$ ($\mu'_\tau = 0$). If $\sigma_2 = \sigma_3 = 0$, the extended tension stress state becomes the uniaxial tension stress state.

2. Extended shear stress state, that is, $\tau_{12} = \tau_{23}$, $\sigma_2 = (\sigma_1 + \sigma_3/2)$. The two smaller stress circulars are equal, the second deviatoric stress $S_2 = 0$ and the magnitude of the other two deviatoric stresses are identical, but one is tensile and the other is compressive. The two twin-shear stress parameters are identical, that is, $\mu_\tau = \mu'_\tau = 0.5$. If $\sigma_2 = (\sigma_1 + \sigma_3)/2 = 0$, the extended shear stress state becomes the pure shear stress state.
3. Extended compression stress state, that is, $\tau_{12} < \tau_{23}$, $0 \le \mu_\tau < 0.5 < \mu'_\tau \le 1$. If $\sigma_1 = \sigma_2 = 0$, $\sigma_3 < 0$, this stress state becomes the uniaxial compression stress state.

The twin-shear parameters simplify the Lode parameter and have a clear physical meaning. Their relations are:

$$\mu_\tau = \frac{1 - \mu_\sigma}{2} = 1 - \mu'_\tau \tag{2.49}$$

$$\mu'_\tau = \frac{1 + \mu_\sigma}{2} = 1 - \mu_\tau \tag{2.50}$$

Some types of stress states and stress state parameters including the Lode parameter and the twin-shear stress parameters are summarized in Table 2.2.

Table 2.2 Principal stresses, shear stresses and stress state parameters

Stress state		Principal stress	Principal shear stress	Stress angle	Parameter of stress state		
					μ_τ	μ'_τ	μ_σ
Extended tension	Pure tension, equal biaxial compression	$\sigma_2 = \sigma_3$	$\tau_{23} = 0$	0°	1	0	−1
	$\tau_{23} = \frac{\tau_{12}}{3}$, $\tau_{13} = 4\tau_{23}$	$\sigma_2 < \frac{\sigma_1 + \sigma_3}{2}$	$\tau_{12} > \tau_{23}$	13.9°	$\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$
Pure shear		$\sigma_2 = \frac{\sigma_1 + \sigma_3}{2}$	$\tau_{12} = \tau_{23}$	30°	0.5	0.5	0
Extended compression	$\tau_{12} = \frac{\tau_{23}}{3}$, $\tau_{13} = 4\tau_{12}$	$\sigma_2 > \frac{\sigma_1 + \sigma_3}{2}$	$\tau_{12} < \tau_{23}$	46.1°	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{2}$
	Pure compression equal biaxial compression	$\sigma_2 = \sigma_1$	$\tau_{12} = 0$ $\tau_{23} = \tau_{13}$	60°	0	1	1

2.8 Summary

Elements and stress states are described briefly in this chapter. Stress state theory is studied in many courses, such as mechanics of materials, elasticity, plasticity, mechanics of solids, rock mechanics, soil mechanics. Only the basic formulas are given here.

The twin-shear stresses, the twin-shear element and the twin-shear stress parameter are new concepts. These new concepts will be used in following chapters.

2.9 Readings

[Readings 2-1] There are three pictures about the statue of Venus collected at Italian Naples Country Archaeology Museum, as shown in Fig. 2.15. They are taken from the statue of Goddess of Venus by Yu Shu-qi from three directions. These images are different, but they are all the identical Venus at Naples. The concept of stress state is somewhat similar in this point. This statue had approximately 2200 history, and discovered at Naples, Italy.

Fig. 2.15 The Love Goddess of Venus collected at Italian Naples Country Archaeology Museum (Yu Shu-qi taking from the three directions for Venus)

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