

Chapter 2

Some General Issues of Propagation of Magnetoelastic Waves in Electroconductive Isotropic and Anisotropic Media

This chapter is devoted to the study of propagation of magnetoelastic waves in isotropic and anisotropic electro-conductive media. The basic linearized equations, boundary conditions and relations of perfectly conducting media, describing magnetoelastic wave processes, are derived. Magnetoelastic plane waves in infinite media are investigated. The characteristic equation of bulk magnetoelastic waves is derived. The classification of fast and slow, as well as of quasi-longitudinal and quasi-transverse, magnetoelastic waves is given. It is shown in which cases, depending on the orientation of the external magnetic field, the medium is under plane-strain conditions. The condition of complete hyperbolicity, which ensures the possibility of propagation of magnetoelastic waves in any direction, is established. Investigation of the roots of the characteristic equation reveals the nature of propagation of fast and slow waves, depending on the physical and mechanical properties of the medium and the magnitude of the external magnetic field intensity. The changes in wave's phase velocities depending on the direction of propagation characteristics of the elastic medium and magnetic field.

2.1 Basics of the Linear Theory of Magnetoelasticity of Electroconductive Media

In this section, on the basis of the coupled nonlinear equations and the relations of magnetoelasticity of electrically conductive media given in Chap. 1 (Sect. 1.1), the basic linear equations and relations of perfectly conducting anisotropic elastic media describing the wave processes in these media in the presence of external magnetic fields, are obtained by linearization.

On the basis of the results of the Chap. 1 (Sect. 1.1), hereinafter we consider media with an infinite conductivity (i.e. perfectly conducting media). In this case, from Ohm's law written in its own coordinate system K^* [the first equation

in (1.1.5)], it follows that inside an ideally conducting medium the following condition is fulfilled:

$$\mathbf{E}^* = 0, \quad (2.1.1)$$

as the current density $\mathbf{j}^*$ must be finite. From (2.1.1) and (1.1.8) we have:

$$\mathbf{E} = -\frac{1}{c}(\mathbf{V} \times \mathbf{B}). \quad (2.1.2)$$

Hence it follows that the electric field intensity $\mathbf{E}$ in a perfectly conducting medium is determined by the field of magnetic induction $\mathbf{B}$ and the velocity of the medium particles $\mathbf{V}$. On the other hand, in perfectly conducting media under the influence of quasi-stationary electromagnetic fields, the displacement current density is so small in comparison with the conduction current density [1] that with high accuracy the following equation can be used instead of the second equation of (1.1.1):

$$\text{rot } \mathbf{H} = \frac{4\pi}{c} \mathbf{j}. \quad (2.1.3)$$

From (2.1.2) and (2.1.3) it follows that the interaction of fields of magnetic induction $\mathbf{B}$ and elastic displacement $\mathbf{u}$ plays an important role in magnetoelasticity of a conducting body. Distribution of these values determines all other parameters characterizing the magnetoelastic problem.

In this chapter, when considering the problems of magnetoelastic waves in conducting media, we will use the linearized equations of the theory of elasticity and electrodynamics. Linearization of the basic relations of magnetoelasticity is carried out under the assumption that along with the smallness of elastic disturbance, electromagnetic disturbance is also small.

As usual, we consider two states magnetoelastic system: disturbed and undisturbed. The values related to the first state will be marked with index "0" from above or below (depending on convenience). The values related to the second state will be represented as a sum of the undisturbed values and the corresponding disturbances. The latter values will be considered to be small and denoted by the corresponding small letters.

So, let us consider an elastic conducting body in a stationary external magnetic field. Assume that the magnetostatic problem for the initial (undisturbed) state is solved, i.e. the magnetic induction vectors for the outer $\mathbf{B}_0^{(e)}$ and inner $\mathbf{B}_0$ regions are known. These characteristics of the electromagnetic field satisfy the following equations and boundary conditions:

$$\begin{aligned} \text{rot } \mathbf{H}_0 = 0, \quad \text{div } \mathbf{B}_0 = 0, \quad \mathbf{E}_0 = 0, \\ \text{rot } \mathbf{H}_0^{(e)} = 0, \quad \text{div } \mathbf{B}_0^{(e)} = 0, \quad \mathbf{E}_0^{(e)} = 0, \end{aligned} \quad (2.1.4)$$

$$\begin{aligned} \left[\mathbf{B}_0^{(e)} - \mathbf{B}_0 \right] \cdot \mathbf{n}_0 &= 0, \\ \left[\mathbf{H}_0^{(e)} - \mathbf{H}_0 \right] \times \mathbf{n}_0 &= 0, \end{aligned} \quad (2.1.5)$$

where $\mathbf{H}_0^{(e)} = \mathbf{B}_0^{(e)}$, $\mathbf{H}_0 = \hat{\mu}^{-1} \mathbf{B}_0$ are the vectors of intensity of the initial magnetic field in vacuum and in medium, respectively; $\mathbf{n}_0$ is the unit outward normal vector of the surface of discontinuity (the boundary of the medium) in undisturbed state; $\hat{\mu}$ is the tensor of magnetic permeability of the medium.

Using the usual notation without “ $\sim$ ” sign [based on Formula (1.1.33)], representing the disturbance components of the elastic and electromagnetic fields in the form [2–5]

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h}, \quad \mathbf{E} = \mathbf{e}, \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \quad \mathbf{D} = \mathbf{d}, \quad \mathbf{j}, \mathbf{u}, \quad \rho_e, \quad (2.1.6)$$

and considering that the characteristics of the induced electromagnetic field and the elastic perturbations are small, we linearize the original equations and relations (1.1.26)–(1.1.36) taking into account (2.1.2) and (2.1.3), neglecting the squares and products of all the values, arising from the disturbance. For the sake of simplicity let restrict us with the investigation of the following cases: (a) the mechanical stresses of unperturbed state are equal to zero; (b) external forces of non-magnetic origin and disturbances of the points of bodies surface are independent of perturbations. As a result, we obtain the following linear equations and the magnetoelasticity relations for perfectly conducting anisotropic elastic media in the initial region Ω_0 in Cartesian coordinates x_1, x_2, x_3 of the observer’s frame of reference K .

The equations and relations of magnetoelasticity for a perfectly conducting body in the inner region:

$$\begin{aligned} \operatorname{div} \hat{\sigma} + \frac{1}{c} \mathbf{j} \times \mathbf{B}_0 &= \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2}, \quad \mathbf{B} = \hat{\mu} \mathbf{H}, \\ \operatorname{rot} \mathbf{h} &= \frac{4\pi}{c} \mathbf{j}, \quad \operatorname{div} \mathbf{b} = 0, \quad \mathbf{b} = \hat{\mu} \mathbf{h}, \quad \mathbf{B}_0 = \hat{\mu} \mathbf{H}_0, \\ \operatorname{rot} \mathbf{e} &= -\frac{1}{c} \frac{\partial \mathbf{b}}{\partial t}, \quad \mathbf{e} = -\frac{1}{c} \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{B}_0, \\ \mathbf{j} &= \frac{c}{4\pi} \operatorname{rot} \mathbf{h}, \quad \rho_e = \frac{1}{4\pi} \operatorname{div} \mathbf{d}, \\ \sigma_{ik} &= C_{iklm} \gamma_{lm}, \quad \gamma_{lm} = \frac{1}{2} \left(\frac{\partial u_l}{\partial x_m} + \frac{\partial u_m}{\partial x_l} \right), \\ \mathbf{d} &= \hat{\varepsilon}_e \mathbf{e} + \frac{1}{c} \left[\hat{\varepsilon}_e \left(\frac{\partial \mathbf{u}}{\partial t} \times \hat{\mu} \mathbf{H}_0 \right) - \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{H}_0 \right], \end{aligned} \quad (2.1.7)$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the elastic displacement vector; γ_{mn} are the components of the Green deformation tensor; σ_{ij} are the components of the Cauchy stress tensor; $\mathbf{e}, \mathbf{h}$ and $\mathbf{d}, \mathbf{b}$ are the vectors of intensities and inductions of the disturbed

electromagnetic field; $\mathbf{j}$ is the conduction current density vector; $\mathbf{B}_0$ —the displacement vector of the external magnetic field; ρ_e —the density of the induced electric charge; $\hat{\epsilon}_e$ —the permittivity tensor; $\hat{\mu}$ —the magnetic permeability tensor; ρ_0 —the density of the medium in the undeformed state; c —the electrodynamic constant; C_{ijmn} —the components of the tensor of elastic constants, which have the following symmetry [6–9]:

$$C_{ijkl} = C_{jikl} = C_{ijnk} = C_{nkij} \quad (2.1.8)$$

In the case of an isotropic medium, the components of the elastic constants tensor are given by:

$$C_{iklm} = \lambda \delta_{ik} \delta_{lm} + G(\delta_{il} \delta_{km} + \delta_{im} \delta_{kl}) \quad (2.1.9)$$

Maxwell's equations in the outer region for vacuum:

$$\begin{aligned} \operatorname{rot} \mathbf{e}^{(e)} &= -\frac{1}{c} \frac{\partial \mathbf{h}^{(e)}}{\partial t}, \quad \operatorname{div} \mathbf{e}^{(e)} = 0, \\ \operatorname{rot} \mathbf{h}^{(e)} &= \frac{1}{c} \frac{\partial \mathbf{e}^{(e)}}{\partial t}, \quad \operatorname{div} \mathbf{h}^{(e)} = 0, \end{aligned} \quad (2.1.10)$$

from which the following wave equations are obtained:

$$\begin{aligned} \Delta \mathbf{h}^{(e)} - \frac{1}{c^2} \frac{\partial^2 \mathbf{h}^{(e)}}{\partial t^2} &= 0, \\ \Delta \mathbf{e}^{(e)} - \frac{1}{c^2} \frac{\partial^2 \mathbf{e}^{(e)}}{\partial t^2} &= 0. \end{aligned} \quad (2.1.11)$$

Boundary conditions on the surface of the elastic body S_0 :

- (a) The conditions characterizing the electrodynamic part of the problem take the form (it is assumed that there is no surface current in the unperturbed state):

$$\mathbf{n}_0 \times (\mathbf{e} - \mathbf{e}^{(e)}) = \frac{1}{c} \mathbf{n}_0 \times \left[\frac{\partial^2 \mathbf{u}}{\partial t^2} \times (\mathbf{B}_0 - \mathbf{B}_0^{(e)}) \right], \quad (2.1.12)$$

$$\mathbf{n}_0 \cdot (\mathbf{b} - \mathbf{b}^{(e)}) = 0,$$

$$4\pi \mathbf{j}_s^* = -c \mathbf{n}_0 \times (\mathbf{h} - \mathbf{h}^{(e)}), \quad (2.1.13)$$

$$4\pi \rho_s^* = -\mathbf{n}_0 \cdot (\mathbf{d} - \mathbf{d}^{(e)}), \quad (2.1.14)$$

where relations (2.1.13) serve to find the surface current $\mathbf{j}_s^*$ and charge ρ_s^* , appearing on the surface of the body as it moves in a magnetic field;

(b) The conditions characterizing the mechanical part of the problem:

when the surface forces $\mathbf{f}_{0s}$ are given on the surface and have the form:

$$\mathbf{n}_0 \cdot (\hat{\sigma} + \hat{t} - \hat{t}^{(e)}) = 0, \quad (2.1.15)$$

where $\hat{t}$ and $\hat{t}^{(e)}$ are the Maxwell tensor perturbations in medium and in vacuum, respectively, the components of which are determined by the formulas:

$$t_{ik} = \frac{1}{4\pi} (H_i^0 b_k + B_k^0 h_i) - \frac{1}{4\pi} \delta_{ik} \mathbf{H}_0 \cdot \mathbf{h}, \quad (2.1.16)$$

$$t_{ik}^{(e)} = \frac{1}{4\pi} (H_i^{(e)0} b_k^{(0)} + B_k^{(e)0} h_i^{(e)}) - \frac{1}{4\pi} \delta_{ik} \mathbf{H}_0^{(e)} \cdot \mathbf{h}^{(e)}. \quad (2.1.17)$$

At the end of this section, we note the following. Excluding the vector $\mathbf{e}$ from the Eq. (2.1.7) and integrating it with respect to time, we find:

$$\mathbf{b}(x_i, t) = \text{rot}[\mathbf{u}(x_i, t) \times \mathbf{B}_0] + q(x_i) \quad (2.1.18)$$

where $q(x_i)$ is an arbitrary time-independent function which should be determined from the initial conditions. Usually for a perfectly conducting body it is assumed that $\mathbf{q} \equiv 0$ and Eq. (2.1.18) is used in the form of:

$$\mathbf{b} = \text{rot}(\mathbf{u} \times \mathbf{B}_0) \quad (2.1.19)$$

or according to (2.1.7)

$$\mathbf{h} = \hat{\mu}^{-1} \text{rot}(\mathbf{u} \times \mathbf{B}_0) = \hat{\mu}^{-1} \text{rot}(\mathbf{u} \times \mu^{-1} \mathbf{B}_0), \quad (2.1.20)$$

where $\hat{\mu}^{-1}$ is the inverse tensor of $\hat{\mu}$. It should be noted that such an assumption ($\mathbf{q} \equiv 0$) is usually true when investigating stationary processes. But in the general case, when we study the problem of propagation of initial perturbations, it does not allow to independently set the initial conditions for the elastic displacement and the generated magnetic field. In this case, the initial conditions must be compatible with each other by an equation at the initial instant of time.

2.2 Equations of Magnetoelastic Waves in Displacements. One-Dimensional Quasi-longitudinal and Quasi-transversal Magnetoelastic Waves

In this section we derive the equations of motion in displacements which describe the magnetoelastic wave process in perfectly conducting anisotropic media in the presence of external homogeneous magnetic fields. The problem of structure of one-dimensional magnetoelastic waves and the waves separation into quasi-longitudinal and quasi-transverse is investigated [10–14].

2.2.1 The Equations of Magnetoelastic Waves in Displacements

The equations obtained in the preceding section will be reduced to a form we will use further.

By determining the vector of the electric current density $\mathbf{j}$ from the fourth equation of system (2.1.7) and using relation (2.1.20), we obtain the following expression for the ponderomotive force $\mathbf{F}$:

$$\mathbf{F} = \frac{1}{c} (\mathbf{j} \times \mathbf{B}_0) = \frac{1}{4\pi} [\text{rot } \hat{\mu}^{-1} \text{rot}(\mathbf{u} \times \mathbf{B}_0)] \times \mathbf{B}_0. \quad (2.2.1)$$

Hence, in the case of a homogeneous external magnetic field, we obtain the following expressions for the projection of the ponderomotive force in the Cartesian coordinate system $Ox_1x_2x_3$:

$$F_i = \tilde{C}_{ijmn} \frac{\partial^2 u_n}{\partial x_j \partial x_m}, \quad (2.2.2)$$

with the following notation:

$$\tilde{C}_{ijmn} = \eta_0 B_\alpha^0 B_\beta^0 \mu_{\gamma\delta}^* (\delta_{i\gamma} \delta_{\alpha j} - \delta_{ij} \delta_{\alpha\gamma}) (\delta_{n\delta} \delta_{\beta m} - \delta_{nm} \delta_{\delta\beta}), \quad (2.2.3)$$

$$\begin{aligned} \mathbf{B}_0 &= \{B_i^0\}, \\ \mathbf{H}_0 &= \{H_i^0\}, \\ \eta_0 &= \frac{1}{4\pi} |\hat{\mu}|, \quad (i, j, m, n, \alpha, \beta, \gamma, \delta = 1, 2, 3) \end{aligned} \quad (2.2.4)$$

where $|\hat{\mu}|$ is the determinant of tensor $\hat{\mu}$; $\mu_{\gamma\delta}^*$ —the components of the tensor μ^* joined to tensor $\hat{\mu}$; δ_{ij} —the Kronecker symbols.

Using the convolution of tensor product and the reverse tensor criterion [15], it is not difficult to prove that the values $\tilde{C}_{ijmn}$ constitute a fourth-order tensor and satisfy the symmetry condition of the following type:

$$\tilde{C}_{ijmn} = \tilde{C}_{nmji} \quad (2.2.5)$$

Using the symmetry property of tensor μ^* , from (2.2.3) we obtain the expressions for the coefficients of Eq. (2.2.2):

$$\begin{aligned} \tilde{C}_{1111} &= \eta_0 [B_2^0 (B_2^0 \mu_{22}^* + 2B_3^0 \mu_{23}^*) + B_3^0 B_3^0 \mu_{33}^*], \\ \tilde{C}_{1123} &= -\eta_0 [B_2^0 (B_2^0 \mu_{23}^* + B_3^0 \mu_{33}^*)], \\ \tilde{C}_{1112} &= -\eta_0 [B_1^0 (B_2^0 \mu_{22}^* + B_3^0 \mu_{23}^*)], \\ \tilde{C}_{1131} &= -\eta_0 [B_3^0 (B_2^0 \mu_{12}^* + B_3^0 \mu_{13}^*)], & \tilde{C}_{1223} &= \eta_0 [B_2^0 B_2^0 \mu_{13}^*], \\ \tilde{C}_{1113} &= -\eta_0 [B_1^0 (B_2^0 \mu_{23}^* + B_3^0 \mu_{33}^*)], & \tilde{C}_{1213} &= \eta_0 [B_1^0 B_2^0 \mu_{13}^*], \\ \tilde{C}_{1312} &= \eta_0 [B_1^0 B_3^0 \mu_{12}^*], & \tilde{C}_{1221} &= \eta_0 [B_2^0 B_2^0 \mu_{11}^*], \\ \tilde{C}_{1131} &= -\eta_0 [B_3^0 (B_2^0 \mu_{12}^* + B_3^0 \mu_{13}^*)], & \tilde{C}_{1231} &= \eta_0 [B_3^0 B_2^0 \mu_{11}^*], \\ \tilde{C}_{1121} &= -\eta_0 [B_2^0 (B_2^0 \mu_{12}^* + B_3^0 \mu_{13}^*)], & \tilde{C}_{1331} &= \eta_0 [B_3^0 B_3^0 \mu_{11}^*], \\ \tilde{C}_{1122} &= \eta_0 [B_1^0 B_2^0 \mu_{12}^* + B_3^0 (B_1^0 \mu_{13}^* + B_2^0 \mu_{23}^* + B_3^0 \mu_{33}^*)], \\ \tilde{C}_{1132} &= -\eta_0 [B_3^0 (B_2^0 \mu_{22}^* + B_3^0 \mu_{23}^*)], & \tilde{C}_{1212} &= \eta_0 [B_1^0 B_2^0 \mu_{12}^*], \end{aligned} \quad (2.2.6)$$

The remaining coefficients can be obtained from the above by cyclic permutation of the indices 1, 2, 3 and by using the symmetry relation (2.2.5).

Using Hooke's law and geometric relations included in (2.1.7), the expression of the ponderomotive force (2.2.2), from the first equation of system (2.1.7) in the absence of usual volume forces, we obtain the following equations of magnetoelastic waves in displacements:

$$A_{ikln} \frac{\partial^2 u_n}{\partial x_k \partial x_l} = \frac{\partial^2 u_i}{\partial t^2} \quad (2.2.7)$$

where the coefficients A_{ikln} are determined by the expressions:

$$A_{ikln} = \rho_0^{-1} (C_{ikln} + \tilde{C}_{ikln}) \quad (2.2.8)$$

and satisfy, according to (2.2.5) and (2.1.8), the following symmetry conditions:

$$A_{ikln} = A_{nlki} \quad (2.2.9)$$

The remaining characteristics of the magnetoelastic problem are determined by the vectors of elastic displacement $\mathbf{u}$ and the induction of the external magnetic field $\mathbf{B}_0$ according to the formulas above.

Thus, the equations of motion of an ideally conducting elastic body in the presence of an external homogeneous magnetic field (2.2.7) are linear homogeneous second-order differential equations with constant coefficients which depend on the physical and mechanical properties of the medium and the intensity of the given external magnetic field. These equations are independent from the equations of the electromagnetic field, but the connection between elastic and electromagnetic fields in general is realized through the boundary conditions [10–12].

2.2.2 One-Dimensional Quasi-longitudinal and Quasi-transverse Waves

Let us consider the solutions of Eq. (2.2.7) of the form:

$$u_i = U_i \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t), \quad U_i = \text{const}, \quad (i = 1, 2, 3), \quad (2.2.10)$$

which describe the propagation of plane monochromatic magnetoelastic waves with the wave vector $\mathbf{k}$, frequency ω , amplitude $\mathbf{U}$ and phase $\varphi = \mathbf{k} \cdot \mathbf{r} - \omega t$.

The equation of equiphase surface

$$\varphi = \mathbf{k} \cdot \mathbf{r} - \omega t = \text{const} \quad (2.2.11)$$

at any given moment is a plane (a phase plane), the velocity of which (the phase velocity of the wave) is determined by the expression [1, 16–18]:

$$\begin{aligned} \mathbf{V} &= -|\text{grad } \varphi|^{-1} \mathbf{n} \frac{\partial \varphi}{\partial t}, \\ \mathbf{n} &= |\text{grad } \varphi|^{-1} \text{grad } \varphi. \end{aligned} \quad (2.2.12)$$

According to (2.2.11) $\text{grad } \varphi = \mathbf{k}$, $\mathbf{n} = \mathbf{k}/k$ (k is the wavenumber), $\varphi_t = -\omega$.

Substituting these values into (2.2.12), we obtain the expression for the phase velocity:

$$\mathbf{V} = \mathbf{n}\omega/k = V\mathbf{n} \quad (2.2.13)$$

Substituting the solution (2.2.10) into Eq. (2.2.7), we obtain:

$$(A_{ijn}k_jk_l - \omega^2\delta_{in})U_n = 0 \quad (2.2.14)$$

or

$$(\hat{A} - V^2 \hat{I})\mathbf{U} = 0, \quad (2.2.15)$$

where $\hat{I}$ is a unit tensor, $\hat{A}$ —a second-order tensor, defined as follows:

$$A_{in} = A_{ijln} n_j n_l \quad (2.2.16)$$

It follows from (2.2.15) that the displacement vector $\mathbf{U}$ of a monochromatic plane wave is an eigenvector, and the square of the phase velocity of the wave V^2 is an eigenvalue of the tensor. Consequently, V^2 is the root of the characteristic equation:

$$|\hat{A} - V^2 \hat{I}| = 0 \quad (2.2.17)$$

of the equation system (2.2.15).

It is seen from (2.2.6), (2.2.8) and (2.2.16) that the components of the tensor $\hat{A}$ depend on the direction $\mathbf{n}$ of the wave normal, on the physical-mechanical properties of the medium and on the components of the external magnetic field.

Consequently, the problem of propagation of magnetoelastic waves consists in finding the displacement $\mathbf{U}$ and the phase velocity $\mathbf{V}$ of magnetoelastic plane waves with an arbitrarily direction of the wave normal $\mathbf{n}$ according to the given elastic properties of the medium and to the given magnetic field.

Let us show that for any direction the tensor $\hat{A}$ is symmetric.

Indeed, according to (2.2.16), we have:

$$\begin{aligned} A_{in} = A_{ijln} n_j n_l &= A_{i11n} n_1^2 + A_{i22n} n_2^2 + A_{i33n} n_3^2 \\ &+ (A_{i12n} + A_{i21n}) n_1 n_2 + (A_{i13n} + A_{i31n}) n_1 n_3 \\ &+ (A_{i32n} + A_{i23n}) n_2 n_3, \end{aligned} \quad (2.2.18)$$

which, in view of (2.2.9), can be written as:

$$\begin{aligned} A_{in} = A_{ijln} n_j n_l &= A_{n11i} n_1^2 + A_{n22i} n_2^2 + A_{n33i} n_3^2 \\ &+ (A_{n12i} + A_{n21i}) n_1 n_2 + (A_{n13i} + A_{n31i}) n_1 n_3 \\ &+ (A_{n32i} + A_{n23i}) n_2 n_3. \end{aligned} \quad (2.2.19)$$

Now, calculating A_{ni} from (2.2.16) and comparing the resulting expression with the expression of A_{in} in (2.2.19), we see that they coincide. Thus, the tensor $\hat{A}$ is symmetric and, consequently, has three real eigenvalues V_i^2 [19].

In the case of an arbitrary anisotropic medium in the absence of a magnetic field, the tensor $\hat{A}$ is proved to be positive definite for any direction of the wave normal $\mathbf{n}$ [17]. For physical reasons it is natural to assume that, even in the presence of a magnetic field, the tensor $\hat{A}$ has this property. In what follows, this property is

proved in the case of the plane problem of an orthotropic medium, when the external magnetic field is directed along one of the principal directions of elasticity of the medium. Below the proof of this property in the case of an isotropic medium is given. In the general case, this property can be proved using the property of positive definiteness of the magnetoelastic energy, as it is done in the case of the purely elastic problem [17].

For an isotropic medium, according to (1.1.6), we have: $\mu_{ik} = \mu\delta_{ik}$, $\mu_{ik}^* = \delta_{ik}$. Using these relations, we obtain from (2.2.6):

$$\begin{aligned}\tilde{C}_{1111} &= \eta_0(B_{02}^2 + B_{03}^2), & \tilde{C}_{1122} &= \eta_0 B_{03}^2, \\ \tilde{C}_{1221} &= \eta_0 B_{02}^2, & \tilde{C}_{1331} &= \eta_0 B_{03}^2, \\ \tilde{C}_{1112} &= -\eta_0 B_{01} B_{02}, & \tilde{C}_{1123} &= -\eta_0 B_{03} B_{02}, \\ \tilde{C}_{1132} &= -\eta_0 B_{03} B_{02}, & \tilde{C}_{1112} &= -\eta_0 B_{01} B_{02}, \\ \tilde{C}_{1113} &= \tilde{C}_{1121} = \tilde{C}_{1131} = \tilde{C}_{1212} = \tilde{C}_{1213} = \tilde{C}_{1223} = \tilde{C}_{1312} = 0.\end{aligned}\quad (2.2.20)$$

Directing the $0x_1$ axis along the direction of the wave vector $\mathbf{n}$ and substituting (2.2.20), (2.1.9) into (2.2.8), we find A_{iklm} , and, according to (2.2.16), the components of the $\hat{A}$ tensor.

As a result, the tensor $\hat{A}$ for an isotropic medium is represented in the following form:

$$\hat{A} = \begin{bmatrix} d_0 + c_0 + B_{*2}^2 + B_{*3}^2 & -B_{*1}B_{*2} & -B_{*1}B_{*3} \\ -B_{*1}B_{*2} & d_0 + B_{*1}^2 & 0 \\ -B_{*1}B_{*3} & 0 & d_0 + B_{*1}^2 \end{bmatrix} \quad (2.2.21)$$

with the notation:

$$\begin{aligned}d_0 &= \frac{G}{\rho_0} > 0, & c_0 &= \frac{(\lambda + G)}{\rho_0} > 0, \\ \mu &> 0, & B_{*i} &= \frac{1}{4\pi} \frac{B_{0i}}{\rho_0} > 0, \quad (i = 1, 2, 3).\end{aligned}\quad (2.2.22)$$

It is clear from (2.2.22) that the principal minors of the tensor are positive:

$$\begin{aligned}A_1 &= d_0 + c_0 + B_{*2}^2 + B_{*3}^2 > 0, \\ A_2 &= d_0(d_0 + c_0 + B_{*1}^2) + (c_0 + B_{*2}^2)B_{*1}^2 > 0, \\ A_3 &= (d_0 + B_{*1}^2)[d_0(d_0 + c_0 + B_{*1}^2) + c_0 B_{*1}^2] > 0.\end{aligned}\quad (2.2.23)$$

Consequently, for isotropic media in the presence of a magnetic field, the tensor $\hat{A}$ is positive definite in any direction of the wave normal $\mathbf{n}$. Hence it follows that in this case the characteristic equation has three positive real roots V_i^2 . This means that plane monochromatic magnetoelastic waves can propagate at any angle to the direction of the external magnetic field $\mathbf{B}_0$.

In the general case, the real symmetric tensor $\hat{A}$ will have three distinct eigenvalues V_1^2 , V_2^2 , V_3^2 , to each of which there corresponds an eigenvector which determines the direction of the displacement vector $\mathbf{U}$ in the wave. Consequently, in both isotropic and anisotropic media, in general, for any given direction of the wave normal $\mathbf{n}$, three magnetoelastic waves with different phase velocities can propagate. These waves, having a common wave normal, are called isonormal waves. The displacement vectors of isonormal waves, as the eigenvectors of the real symmetric tensor $\hat{A}$, are mutually perpendicular. In those particular cases, when the tensor $\hat{A}$ becomes uniaxial, the two eigenvalues of this tensor coincide, hence two of the three isonormal waves have the same phase velocities, and their displacement vectors may have any direction in a plane perpendicular to the displacement vector of the third wave.

As an example, consider the case of an isotropic medium [12]. On the basis of (2.2.21), the characteristic equation determining the phase velocities of propagation of magnetoelastic waves can be represented in the form:

$$(V^2 - d_0 - B_{*1}^2) \left[(V^2 - d_0)^2 + c_0 B_{*1}^2 - (V^2 - d_0)(c_0 + B_*^2) \right] = 0. \quad (2.2.24)$$

Hence, to determine the phase velocities of isonormal magnetoelastic waves with a given wave normal $\mathbf{n}$, we obtain the expressions:

$$\begin{aligned} V_{1,2}^2 &= 0, 5 \left[c_0 + 2d_0 + \chi_* \pm \sqrt{(c_0 + \chi_*)^2 - 4c_0 \chi_* \cos^2(\mathbf{B}_0, \mathbf{n})} \right], \\ V_3^2 &= d_0 + \chi_* \cos^2(\mathbf{B}_0, \mathbf{n}), \end{aligned} \quad (2.2.25)$$

where

$$\chi_* = \frac{B_0^2}{4\pi\rho_0} \mu^{-1}. \quad (2.2.26)$$

Let us consider the specific cases.

- (a) The wave normal $\mathbf{n}$ is parallel to the magnetic field $\mathbf{B}_0$. In this case we find from (2.2.21) and (2.2.25) that:

$$\hat{A} = \begin{bmatrix} d_0 + c_0 & 0 & 0 \\ 0 & d_0 + B_*^2 & 0 \\ 0 & 0 & d_0 + B_*^2 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} V_1^2 = d_0 + c_0 = a_0 \\ V_2^2 = d_0 + \chi_* \\ V_3^2 = d_0 + \chi_* \end{bmatrix}. \quad (2.2.27)$$

It follows that the tensor $\hat{A}$ is uniaxial. In this case one of the eigenvectors of the tensor $\hat{A}$ will have the same direction as $\mathbf{n} = \mathbf{i}_1$ (the corresponding wave will be longitudinal). Two other eigenvectors of the tensor $\hat{A}$ may be chosen arbitrarily in a

plane perpendicular to $\mathbf{n}$ (the corresponding waves will be transverse and will have the same phase velocity). It is seen from (2.2.27) that the longitudinal wave is purely elastic, and the transverse waves are magnetoelastic.

- (b) The wave normal $\mathbf{n}$ is perpendicular to the direction of the vector $\mathbf{B}_0$. In this case we have according to (2.2.21) and (2.2.27):

$$\hat{A} = \begin{bmatrix} d_0 + c_0 + \chi_* & 0 & 0 \\ 0 & d_0 & 0 \\ 0 & 0 & d_0 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} V_1^2 = d_0 + c_0 + \chi_* \\ V_2^2 = d_0 \\ V_3^2 = d_0 \end{bmatrix}. \quad (2.2.28)$$

It follows from (2.2.28) that there propagates one longitudinal wave perpendicular to the magnetic field $\mathbf{B}_0$, and its velocity depends on the magnetic field, and two transverse waves propagate with coinciding phase velocities independent from the magnitude of the magnetic field intensity.

Thus, the presence of a magnetic field in the case of an isotropic medium leads to the tensor $\hat{A}$ to become biaxial for any direction of the wave normal $\mathbf{n}$, with the exception of those directions that are parallel or perpendicular to the direction of the magnetic field.

Important properties of propagation of magnetoelastic waves in isotropic media follow directly from this conclusion.

- (1) None of the three isonormal magnetoelastic waves (except when the vector $\mathbf{B}_0$ is parallel or perpendicular to the vector $\mathbf{n}$) is neither purely longitudinal nor purely transverse. However, in any case, one of the three isonormal waves will have a displacement vector $\mathbf{U}_1$, which forms the smallest angle with the wave normal $\mathbf{n}$. Such a wave is called quasi-longitudinal, and the other two are called quasi-transverse.
- (2) Usually the phase velocities of all three isonormal waves are different for any direction of the wave normal $\mathbf{n}$. The wave with a phase velocity V_1 (for $V_1 > V_2$, $V_1 > V_3$) is called fast, the waves with phase velocities V_2 and V_3 are called slow.
- (3) Linearly polarized magnetoelastic waves can propagate in any direction of the wave normal $\mathbf{n}$. Elliptically polarized magnetoelastic waves, in contrast to purely elastic waves, can propagate only in directions that are parallel or perpendicular to the direction of the magnetic field.
- (4) In the presence of a magnetic field, the phase velocities of the transverse waves coincide, and besides, when the vectors $\mathbf{B}_0$ and $\mathbf{n}$ are parallel, the velocities of these waves increase with an increase of the magnetic field intensity, and when these vectors are perpendicular, the velocities do not depend on the magnetic field intensity.

The character of wave propagation (both elastic and magnetoelastic) is very complicated in the case of anisotropic media. With some reservations, the above

properties also occur in this case. The influence of medium anisotropy on propagation of magnetoelastic waves will be discussed later. Here we note the following. If the direction of the wave normal $\mathbf{n}$ in an anisotropic medium is specific in the absence of a magnetic field, then by choosing the direction of the magnetic field it can be made ordinary (specific directions of $\mathbf{n}$ are those along which purely longitudinal waves can propagate).

In conclusion, let us consider one qualitative aspect of the influence of the magnetic field on the character of propagation of magnetoelastic waves. It is known [17] that in an isotropic medium in the absence of a magnetic field the frequency of a wave is proportional to the absolute value of the wave vector $\mathbf{k}$ and therefore the group velocity of the wave $\mathbf{V}_g$ (the velocity of propagation of perturbations) coincides with the phase velocity. In the case of magnetoelastic waves, this property is violated for any direction of the wave normal $\mathbf{n}$ (except for specific directions). Indeed, the group velocity of a wave of any type is defined as follows [17]:

$$\mathbf{V}_g = \frac{\partial \omega}{\partial \mathbf{k}} = \frac{\partial \omega}{\partial k_j} \mathbf{i}_j. \quad (2.2.29)$$

From (2.2.14) we have:

$$\omega^2 = A_{ijlm} k_j k_l U_i U_m / U^2. \quad (2.2.30)$$

Taking into account (2.2.30), we find from (2.2.29):

$$V_j^g = \frac{\partial \omega}{\partial k_j} = \frac{1}{V} A_{ijlm} \frac{U_i U_m}{U^2} n_l \quad (2.2.31)$$

or

$$\mathbf{U} = V^{-1} \hat{A}^* \cdot \vec{n}, \quad A_{il}^* = A_{ijlm} U_j U_m U^{-2} \quad (2.2.32)$$

Hence it follows that the group and phase velocities are related by

$$\mathbf{U} \cdot \mathbf{n} = V. \quad (2.2.33)$$

The obtained relations show that the group and phase velocities of magnetoelastic waves (except for specific directions) differ both in direction and in magnitude, and, as can be seen from (2.2.33), the magnitude of the group velocity can not be less than the phase velocity. They coincide: when purely elastic waves are considered to be propagating in an isotropic medium; when the wave normal is specific in an anisotropic medium without any magnetic field; in the presence of a magnetic field in an isotropic medium as well as in an anisotropic medium.

2.3 Propagation Character of Two-Dimensional Magnetoelastic Waves

In this section, two-dimensional equations of magnetoelastic waves are derived from the general equations of magnetoelastic waves for conductive anisotropic media, given in the second section of this chapter, in the case of planar deformation, when an anisotropic medium in the absence of a magnetic field in the planar problem is characterized by four elastic constants, and the induction vector of the external magnetic field is parallel to the main directions of elasticity of the medium. The question of complete hyperbolicity of the system of equations of the plane problem is discussed.

2.3.1 Equations of Two-Dimensional Magnetoelastic Waves

Let us assume that an elastic body is in a state of plane deformation, i.e. the displacement vector $\mathbf{u}$ does not depend on one of the coordinates, for example x_3 , and there is no elastic displacement in the direction of this coordinate:

$$\begin{aligned} u_1 &= u_1(x_1, x_2, t), \\ u_2 &= u_2(x_1, x_2, t), \\ u_3 &\equiv 0. \end{aligned} \quad (2.3.1)$$

In this case, the remaining unknown quantities, as follows from the foregoing relations, will also depend only on the variables x_1, x_2, t .

Let us find the necessary conditions under which Eq. (2.2.7) admit solutions of the form (2.3.1) [11].

Let the deformation plane $0x_1x_2$ be a plane of elastic symmetry. In this case, the tensor $\hat{\mu}$ and the matrix of elastic coefficients $[C_{ik}]$ in the ordinary two-index notations have the forms [9, 17]:

$$\begin{aligned} \hat{\mu} &= \begin{bmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{21} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{bmatrix}, \\ [C_{ik}] &= \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{16} \end{bmatrix}. \end{aligned} \quad (2.3.2)$$

In view of the positive definiteness of the tensor $\hat{\mu}$ [9], its components satisfy the following conditions:

$$\begin{aligned} \mu_{11} > 0, \quad \mu_{22} > 0, \quad \mu_{33} > 0, \\ |\hat{\mu}| = \mu_{33}(\mu_{11}\mu_{22} - \mu_{12}\mu_{21}) > 0. \end{aligned} \quad (2.3.3)$$

According to (2.3.1)–(2.3.3) and (2.2.2)–(2.2.4), we obtain from (2.2.7) two equations that determine the components u_1, u_2 of the displacement vector $\mathbf{u}$ and the following additional condition (the third equation of the system (2.2.7):

$$B_{03} \left[B_{01} \frac{\partial(\operatorname{div} \mathbf{u})}{\partial x_1} + B_{02} \frac{\partial(\operatorname{div} \mathbf{u})}{\partial x_2} \right] = 0. \quad (2.3.4)$$

Consequently, the condition (2.3.4) is necessary for the existence of the solution of the system of Eq. (2.2.7) in the form (2.3.1), when the deformation plane $0x_1x_2$ is a plane of elastic symmetry.

It follows from condition (2.3.4) that it will be satisfied in the following three cases [11, 12]:

- (1) The specified magnetic field $\mathbf{B}_0$ is perpendicular to the deformation plane $0x_1x_2$, i.e. $\mathbf{B}_0 = \{0, 0, B_{03}\}$. This is the case of a transverse magnetic field.
- (2) The specified magnetic field $\mathbf{B}_0$ is parallel to the deformation plane $0x_1x_2$, i.e. $\mathbf{B}_0 = \{B_{01}, B_{02}, 0\}$. This is the case of a longitudinal magnetic field.
- (3) The medium is incompressible: $\operatorname{div} \mathbf{u} = 0$. The magnetic field has an arbitrary direction with respect to the deformation plane $0x_1x_2$.

It should be noted that in the case of presence of a plane of elastic symmetry, conditions (1) and (2) can also be formulated in the same way for the intensity vector $\mathbf{H}_0$ of the external magnetic field. Indeed, from the equality $\mathbf{B}_0 = \hat{\mu}\mathbf{H}_0$, by virtue of (2.3.2), the following relations are obtained:

$$\begin{aligned} B_{01} &= \mu_{11}H_{01} + \mu_{12}H_{02}, \\ B_{02} &= \mu_{12}H_{01} + \mu_{22}H_{02}, \\ B_{03} &= \mu_{33}H_{03}. \end{aligned} \quad (2.3.5)$$

From this and (2.3.3) it follows that if $\mathbf{H}_0 = \{H_{01}, H_{02}, 0\}$ then the vector $\mathbf{B}_0$ has the form $\mathbf{B}_0 = \{B_{01}, B_{02}, 0\}$; and if $\mathbf{H}_0 = \{0, 0, H_{03}\}$, then $\mathbf{B}_0 = \{0, 0, B_{03}\}$.

In the following we will investigate only cases (1) and (2), considering such media, whose equations of motion in the absence of a magnetic field under plain-strain conditions contain four elastic constants and are written in the form [20–23, 32]:

$$\begin{aligned} a_0 u_{xx} + c_0 v_{xy} + d_0 u_{yy} &= u_{tt}, \\ d_0 v_{xx} + c_0 u_{xy} + b_0 v_{yy} &= v_{tt}, \end{aligned} \quad (2.3.6)$$

where

$$\begin{aligned} a_0 &= C_{11}/\rho_0, \\ b_0 &= C_{22}/\rho_0, \\ d_0 &= C_{66}/\rho_0, \\ c_0 &= (C_{12} + C_{66})/\rho_0, \end{aligned} \quad (2.3.7)$$

$$\begin{aligned} x_1 &\equiv x, & x_2 &\equiv y, & x_3 &\equiv z, \\ u_1 &\equiv u, & u_2 &\equiv v, & u_3 &\equiv w, \\ \frac{\partial u}{\partial x_1} &\equiv u_x, & \frac{\partial u}{\partial x_2} &\equiv u_y, & \frac{\partial u}{\partial x_3} &\equiv u_z, \\ \frac{\partial^2 u}{\partial x_1^2} &\equiv u_{xx}, & \frac{\partial^2 u}{\partial x_1 \partial x_2} &\equiv u_{xy}, & \frac{\partial^2 u}{\partial x_2^2} &\equiv u_{yy} \end{aligned} \quad (2.3.8)$$

In particular, the following media are among the above media [7, 22, 23]:

- (a) Orthotropic media and rhombic crystals. The equations of motion of these media have the form (2.3.6), when the coordinate axes are directed along the principal directions of the elasticity of the medium.
- (b) Transversely isotropic media and hexagonal crystals. In this case, Eq. (2.3.6) are true when the axis Ox is directed along the principal axis of the sixth-order crystal. When the axis Oz is analogously oriented, we obtain the equations of an isotropic medium.
- (c) Cubic crystals and anisotropic media with cubic symmetry. The orientation of the axes is the same as in the case of orthotropic media (case a). For these media the following relation holds:

$$a_0 = b_0. \quad (2.3.9)$$

- (d) Isotropic media. For such media:

$$\begin{aligned} a_0 &= b_0, \\ c_0 &= a_0 - d_0. \end{aligned} \quad (2.3.10)$$

In all the above cases, the tensor $\hat{\mu}_m$ is diagonal [9]:

$$\hat{\mu} = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}, \quad (2.3.11)$$

and in case (b) and in cases (c) and (d), respectively, we have:

$$\mu_1 = \mu_2, \quad (2.3.12)$$

$$\mu_1 = \mu_2 = \mu_3 = \mu. \quad (2.3.13)$$

Assuming that the specified magnetic field $\mathbf{H}_0$ has one of the following directions:

$$\begin{aligned} (1) \quad \mathbf{H}_0 &= \{H_{01}, 0, 0\}, \\ (2) \quad \mathbf{H}_0 &= \{0, H_{02}, 0\}, \\ (3) \quad \mathbf{H}_0 &= \{0, 0, H_{03}\}, \end{aligned} \quad (2.3.14)$$

the equations of motion (2.2.7) for the considered media under plain-strain conditions in the presence of a magnetic field can be represented in the form [11, 12]:

$$\begin{aligned} a_m u_{xx} + c_m v_{xy} + d_m u_{yy} &= u_{tt}, \\ e_m v_{xx} + c_m u_{xy} + b_m v_{yy} &= v_{tt}, \end{aligned} \quad (2.3.15)$$

where the cases $m = 1$, $m = 2$, $m = 3$ correspond to the directions of the magnetic field (2.3.14), and the case $m = 0$ corresponds to the absence of a magnetic field.

The coefficients a_m , b_m , d_m , e_m , c_m are determined as follows:

$$\begin{aligned} a_1 &= a_0, \quad b_1 = b_0 + \chi_1, \quad d_1 = d_0, \\ e_1 &= d_0 + \delta_1 \chi_1, \quad c_1 = c_0, \\ a_2 &= a_0 + \chi_2, \quad b_2 = b_0, \quad d_2 = d_0 + \delta_2 \chi_2, \\ e_2 &= d_0, \quad c_2 = c_0, \\ a_3 &= a_0 + \chi_3, \quad b_3 = b_0 + \chi_3, \quad d_3 = d_0, \\ e_3 &= d_0, \quad c_3 = c_0 + \chi_3, \\ \chi_i &= \mu_i \frac{H_{0i}^2}{4\pi\rho_0}, \quad \delta_1 = \frac{\mu_1}{\mu_2}, \quad \delta_2 = \frac{1}{\delta_1}, \quad (i = 1, 2, 3). \end{aligned} \quad (2.3.16)$$

2.3.2 The Condition of Complete Hyperbolicity. The Area of Values of the Coefficients of the Equations of Magnetoelastic Waves

For physical reasons, let us consider those restrictions that must be imposed on the coefficients of the system of Eq. (2.3.15) [11, 12].

First, let us consider the area of allowable values of the parameters a_0 , b_0 , d_0 , c_0 , i.e. the coefficients of the equations system (2.3.6). This region is

found from the positivity condition of the elastic energy, which, for plane deformation, can be written in the form [22–24]:

$$a_0 > 0, \quad b_0 > 0, \quad d_0 > 0, \quad (2.3.17)$$

$$E_0 \equiv a_0 b_0 - (c_0 - d_0)^2 > 0. \quad (2.3.18)$$

In view of (2.3.17), inequality (2.3.18) can be represented as follows:

$$-C_D^{(0)} < c_0 < C_E^{(0)}, \quad (2.3.19)$$

where

$$\begin{aligned} C_D^{(0)} &= \sqrt{a_0 b_0} - d_0, \\ C_E^{(0)} &= \sqrt{a_0 b_0} + d_0. \end{aligned} \quad (2.3.20)$$

Using the relations (2.3.17), the condition of complete hyperbolicity for the system of Eq. (2.3.6) can be written in the form [24]:

$$-C_E^{(0)} < c_0 < C_E^{(0)}. \quad (2.3.21)$$

Comparing inequalities (2.3.19) and (2.3.21), we note that under the conditions (2.3.17), (2.3.18) the system (2.3.6) is completely hyperbolic [20].

According to (2.3.3) and (2.3.15), the parameters δ_i and χ_i satisfy the inequalities:

$$\delta_i > 0, \quad \chi_i > 0. \quad (2.3.22)$$

From this and (2.3.16) it follows that the parameters a_m , b_m , d_m , e_m are positive:

$$a_m > 0, \quad b_m > 0, \quad d_m > 0, \quad e_m > 0. \quad (2.3.23)$$

Let us show that the system (2.3.15) is completely hyperbolic under the conditions (2.3.17), (2.3.18) and (2.3.22). Indeed, the condition of complete hyperbolicity for system (2.3.15) can be represented in the form [11]:

$$a_m e_m p^4 + l_m p^2 + b_m d_m > 0, \quad (2.3.24)$$

which must be satisfied for any value of the parameter p (here and in consequent two paragraphs there is no summation with respect to repeated indices). It follows from (2.3.23) and (2.3.24) that the necessary and sufficient conditions for the fulfillment of inequality (2.3.24) are the following inequalities:

$$D_m \leq 0 \quad \text{or} \quad D_m > 0, \quad l_m > 0. \quad (2.3.25)$$

where

$$\begin{aligned} l_m &= a_m b_m + d_m e_m - c_m^2, \\ D_m &= l_m^2 - 4a_m b_m d_m e_m. \end{aligned} \quad (2.3.26)$$

From (2.3.26) we have:

$$\begin{aligned} l_m &> 0 & \text{for} & \quad |c_m| < c_l^{(m)}; \\ D_m &\leq 0 & \text{for} & \quad C_D^{(m)} \leq |c_m| \leq C_E^{(m)}, \\ D_m &> 0 & \text{for} & \quad C_D^{(m)} > |c_m| \text{ and } |c_m| > C_E^{(m)}, \end{aligned} \quad (2.3.27)$$

where

$$\begin{aligned} C_l^{(m)} &= \sqrt{a_m b_m + d_m e_m}, \\ C_D^{(m)} &= \left| \sqrt{a_m b_m} - \sqrt{d_m e_m} \right|, \\ C_E^{(m)} &= \left| \sqrt{a_m b_m} + \sqrt{d_m e_m} \right|, \end{aligned} \quad (2.3.28)$$

$$C_D^{(m)} < C_l^{(m)} \leq C_E^{(m)}. \quad (2.3.29)$$

On the basis of (2.3.27)–(2.3.29), (2.3.23) and (2.3.25) we obtain the domain of admissible values of the parameters a_m , b_m , d_m , e_m , c_m for which the system of Eq. (2.3.15) is completely hyperbolic:

$$a_m > 0, \quad b_m > 0, \quad d_m > 0, \quad e_m > 0, \quad (2.3.30)$$

$$-C_E^{(m)} < c_m < C_E^{(m)}. \quad (2.3.31)$$

According to (2.3.16), condition (2.3.31) can be written in the form:

$$|c_0| < C_E^{(m)} = \sqrt{a_m b_m} + \sqrt{d_m e_m} \quad (m = 1, 2), \quad (2.3.32)$$

$$|c_0 + \chi_3| < C_E^{(3)} = \sqrt{a_3 b_3} + d_3 \quad (m = 3). \quad (2.3.33)$$

When $m = 1, 2$, by virtue of (2.3.28), we have: $0 < |C_D^{(0)}| < C_E^{(0)} < C_E^{(m)}$. From this and from comparison of the inequalities (2.3.31) and (2.3.21) it follows that under the conditions (2.3.17), (2.3.18) and (2.3.22) the inequalities (2.3.30), (2.3.31) are fulfilled. When $m = 3$ from (2.3.3) we have:

$$-\left(\sqrt{a_3 b_3} + d_0 + \chi_3\right) < c_0 < \sqrt{a_3 b_3} + d_0 - \chi_3. \quad (2.3.34)$$

According to (2.3.16), (2.3.17) and (2.3.22) it is easy to see that the left-hand side of (2.3.34) satisfies the condition:

$$-\left(\sqrt{a_3 b_3} + d_0 + \chi_3\right) < -\left|C_D^{(0)}\right| = -\left|\sqrt{a_0 b_0} - d_0\right|. \quad (2.3.35)$$

Let us show that the right-hand side of (2.3.34) satisfies the condition

$$C_E^{(0)} < \sqrt{a_3 b_3} + d_0 - \chi_3 \quad (2.3.36)$$

Inequality (2.3.36) is reduced to the form which shows the validity of the condition (2.3.36). It follows from (2.3.34) to (2.3.36) that the inequality (2.3.33) is satisfied under the condition (2.3.19).

Consequently, the system of equations of magnetoelastic waves (2.3.15) for the values of the coefficients a_m, b_m, c_m, d_m, e_m , taken from the region (2.3.17), (2.3.18) and (2.3.22), is completely hyperbolic. This means that under the above conditions, magnetoelastic waves can propagate in any direction.

In view of the fact that the elastic constants of real known media of the considered class satisfy the conditions [22–24]:

$$a_0 > d_0, \quad b_0 > d_0, \quad d_0 > 0, \quad c_0 > 0, \quad E_0 > 0, \quad (2.3.37)$$

or, which is the same,

$$A_0 = a_0 - d_0 > 0, \quad B_0 = b_0 - d_0 > 0, \quad d_0 > 0, \quad 0 < c_0 < C_E^{(0)}. \quad (2.3.38)$$

in [22–24], the behavior of elastic waves is investigated under these assumptions. In this section, we also proceed from the inequalities (2.3.37) or (2.3.38) and (2.3.22). In addition, to simplify further calculations, we take

$$\delta_i = 1 \quad (i = 1, 2). \quad (2.3.39)$$

For isotropic media and anisotropic media with cubic symmetry, (2.3.39) is satisfied exactly. For the remaining conductive anisotropic media of the considered class, in view of $\mu_i \approx 1$, this equality is satisfied approximately. In view of the above, the relations (2.3.16) will be used in the form:

$$\begin{aligned}
a_1 &= a_0, & b_1 &= b_0 + \chi, & d_1 &= d_0, \\
e_1 &= d_0 + \chi, & c_1 &= c_0, \\
a_2 &= a_0 + \chi, & b_2 &= b_0, & d_2 &= d_0 + \chi, \\
e_2 &= d_0, & c_2 &= c_0, \\
a_3 &= a_0 + \chi, & b_3 &= b_0 + \chi, \\
d_3 &= d_0, & e_3 &= d_0, & c_3 &= c_0 + \chi,
\end{aligned} \tag{2.3.40}$$

where

$$\chi = \frac{H_0^2}{4\pi\rho_0}. \tag{2.3.41}$$

The parameter $\sqrt{\chi}$ that characterizes the intensity of the external magnetic field has the dimension of the velocity and, similarly to magnetohydrodynamics [25, 26], is often called the Alfvén velocity.

Thus, in what follows, the system of equations for magnetoelastic waves (2.3.15) will be considered under the conditions (2.3.37) and (2.3.40) imposed on the coefficients of this system. Under these conditions the system will be completely hyperbolic, which ensures the possibility of propagation of magnetoelastic waves in any direction.

2.4 Investigation of the Roots of Characteristic Equation of Two-Dimensional Magnetoelastic Waves and Drawing of the Acceptable Area on the Basis of the Roots

In this section, functional-invariant solutions of two-dimensional equations of magnetoelastic waves for electrically conductive anisotropic media, derived in the previous section, are constructed. The corresponding characteristic equation of magnetoelastic waves is obtained. The behavior of the roots of this equation is investigated in detail, depending on the values of the elastic constants and the magnitude of the external magnetic field. The roots of the characteristic equation, depending on the above parameters, are shown to be as follows: either each root has its own unique interval of realness, or one of them is real in two disjoint intervals, and the other is real in one interval. On the basis of the results obtained here, the behavior of plane magnetoelastic waves is subsequently investigated.

2.4.1 Characteristic Equation of Two-Dimensional Magnetoelastic Waves

According to the previous paragraph, the plane magnetoelastic waves in anisotropic perfectly conducting elastic media in the plane problem in the presence of a constant external homogeneous magnetic field are described by the equations of motion in displacements (2.3.15). The coefficients of the Eq. (2.3.15) are determined by expressions (2.3.40) and (2.3.7) and satisfy the conditions (2.3.37). In (2.3.15), $u = f(x, y, t)$, $v = g(x, y, t)$ are the components of the elastic displacement vector $\mathbf{u} = (u, v, 0)$ in a Cartesian coordinate system $Oxyz$ whose axes coincide with the principal directions of the elasticity of the medium.

According to the Smirnov-Sobolev method of complex solutions [24, 27], the functionally invariant solutions of the system of Eq. (2.3.15) let's present in the form:

$$u = f(\Omega), \quad v = g(\Omega) \quad (2.4.1)$$

where Ω is a function defined implicitly by a linear equation with respect to x, y, t :

$$\delta \equiv l(\Omega)t + m(\Omega)x + n(\Omega)y - k(\Omega) = 0 \quad (2.4.2)$$

Here, f and g are continuous twice differentiable functions if the quantities $l(\Omega), \dots, k(\Omega)$ are real. And if some of these quantities are complex, then f and g are analytic functions of Ω .

Defining the derivatives of the functions (2.4.1) using the known formulas for differentiating complex and implicit functions [24, 28] and substituting their values into the system (2.3.15), we obtain the conditions:

$$\begin{aligned} (a_m m^2 + d_m n^2 - l^2) f'(\Omega) + c_m m n g'(\Omega) &= 0 \\ c_m m n f'(\Omega) + (e_m m^2 + b_m n^2 - l^2) g'(\Omega) &= 0 \end{aligned} \quad (2.4.3)$$

which establish the relationship between the unknown functions (2.4.1).

The system (2.4.3) has a nonzero solution if the following condition holds:

$$\Delta \equiv (a_m m^2 + d_m n^2 - l^2)(e_m m^2 + b_m n^2 - l^2) - c_m^2 m^2 n^2 = 0 \quad (2.4.4)$$

Equation (2.4.4) establishes the relationship between the functions $l(\Omega)$, $m(\Omega)$, $n(\Omega)$, and, according to (2.4.3), between the derivatives $f'(\Omega)$ and $g'(\Omega)$.

Thus, the class of functions (2.4.1) expresses the solution of system (2.3.15) if the argument Ω is defined by Eq. (2.4.2) with coefficients subject to Eq. (2.4.4), and the functions themselves are subject to condition (2.4.3).

It is possible to separate the solutions that are plane waves from the class of functionally invariant solutions. To do this we take in Eq. (2.4.2)

$$\begin{aligned} l(\Omega) &\equiv 1, & m(\Omega) &\equiv -\theta = \text{const}, \\ k(\Omega) &\equiv \Omega, & n(\Omega) &\equiv \lambda = \text{const}. \end{aligned}$$

Then we get:

$$u_k = f_k(\Omega_k), \quad v_k = g_k(\Omega_k) \quad (2.4.5)$$

$$\Omega_k = t - \theta x \pm \lambda_k y \quad (k = 1, 2), \quad (2.4.6)$$

where λ_k are the roots of Eq. (2.4.4), which in this case takes the form:

$$b_m d_m \lambda^4 - p_m(\theta) \lambda^2 + r_m(\theta) = 0 \quad (2.4.7)$$

We represent the solutions of (2.4.7) in the form:

$$\lambda = \pm \lambda_k(\theta) = \pm \sqrt{\left(p_m(\theta) + (-1)^k \sqrt{q_m(\theta)}\right) / 2b_m d_m} \quad (k = 1, 2). \quad (2.4.8)$$

We use here the notation:

$$\begin{aligned} p_m &= b_m + d_m - l_m \theta^2, \\ r_m &= (a_m \theta^2 - 1)(e_m \theta^2 - 1), \\ q_m &= p_m^2 - 4b_m d_m r_m, \end{aligned} \quad (2.4.9)$$

$$D_m = l_m^2 - 4a_m b_m d_m e_m, \quad l_m = a_m b_m + d_m e_m - c_m^2, \quad (2.4.10)$$

$$M_m = (b_m + d_m)N_m - (a_m d_m - b_m e_m)B_m, \quad (2.4.11)$$

$$A_m = a_m - e_m, \quad B_m = b_m - d_m, \quad N_m = A_m B_m - c_m^2. \quad (2.4.12)$$

It follows from (2.4.5) and (2.4.6) that for real values of θ the functions (2.4.5) express homogeneous and inhomogeneous plane waves.

To construct the solutions, both the first and the second Eq. (2.4.3) can be used. However, for some values of θ the coefficients of one of the equations can simultaneously vanish and the solution compiled on the basis of this equation becomes meaningless for such values. In the case of purely elastic waves, in order to eliminate this deficiency, it is proposed in [28] first to formulate a generalized condition from conditions of the type (2.4.3), summing the latter. The solution based on the following generalized condition

$$\begin{aligned} &(a_m \theta_k^2 + d_m \lambda_k^2 - c_m \theta_k \lambda_k - 1) f'_k(\Omega_k) \\ &+ (e_m \theta_k^2 + b_m \lambda_k^2 - c_m \theta_k \lambda_k - 1) g'_k(\Omega_k) = 0, \end{aligned} \quad (2.4.13)$$

according to (2.4.5), can be represented as:

$$\begin{aligned} u_k &= (e_m \theta^2 + b_m \lambda_k^2 - c_m \theta \lambda_k - 1) W_k(\Omega_k), \\ v_k &= -(a_m \theta^2 + d_m \lambda_k^2 - c_m \theta \lambda_k - 1) W_k(\Omega_k) \quad (k = 1, 2). \end{aligned} \quad (2.4.14)$$

Here W_k are the branches of an arbitrary continuous twice differentiable function W if the coefficients of the variables x, y, t in (2.4.6) are real. If some of these coefficients in some domain of variables x, y, t are complex quantities, then W is an analytic function of the complex variable Ω .

In this section, in what follows we will consider only homogeneous waves.

The choice of the sign “+” or “−” in front of λ_k in (2.4.6) shows that the plane waves (2.4.14) propagate in opposite directions “forward” and “backward”. In accordance with this it is clear that, by choosing the coordinate system (relative to the principal directions of the elasticity of the medium and the direction of the magnetic field), four waves corresponding to the hyperbolic system (2.3.15) are divided into two pairs propagating in opposite directions. In view of this, in what follows, we will consider waves with the plus sign in front of λ_k in (2.4.8).

Equations of the fronts of plane waves (2.4.14) have the form:

$$\Omega_k = t - \theta x + \lambda_k y = C = \text{const} \quad (2.4.15)$$

The phase velocities and directions of propagation of these waves are determined by the following formulas:

$$\begin{aligned} \mathbf{V}_k &= -\frac{\partial \phi / \partial t}{|\text{grad } \phi_k|} \vec{n}_k, \quad \mathbf{n}_k = \frac{\text{grad } \phi_k}{|\text{grad } \phi_k|}, \quad \phi_k = C - t + \theta x - \lambda_k y, \\ \mathbf{V}_k &= \frac{\theta}{\theta^2 + \lambda_k^2} \mathbf{i} - \frac{\lambda_k}{\theta^2 + \lambda_k^2} \mathbf{j}, \quad \mathbf{n}_k = \frac{\theta}{\sqrt{\theta^2 + \lambda_k^2}} \mathbf{i} - \frac{\lambda_k}{\sqrt{\theta^2 + \lambda_k^2}} \mathbf{j}, \\ V_k &= \frac{1}{\sqrt{\theta^2 + \lambda_k^2}}, \quad \text{tg } \alpha_k = \frac{\theta}{\lambda_k(\theta)} \end{aligned} \quad (2.4.16)$$

where $\mathbf{n}_k$ are the unit normal vectors to the wave fronts, $\mathbf{V}_k$ are the phase velocities of propagation, $\alpha_k = \angle(-\mathbf{j}, \mathbf{n}_k)$, C is an arbitrary constant (Fig. 2.1).

It follows from (2.4.16) that to each value of the parameter θ from the interval $(-\infty; \infty)$, for which the quantities $\lambda_k(\theta)$ are real, there correspond plane homogeneous waves whose phase velocities and propagation directions are determined by Formula (2.4.16). In view of the symmetry, we will consider the interval $0 \leq \theta < \infty$.

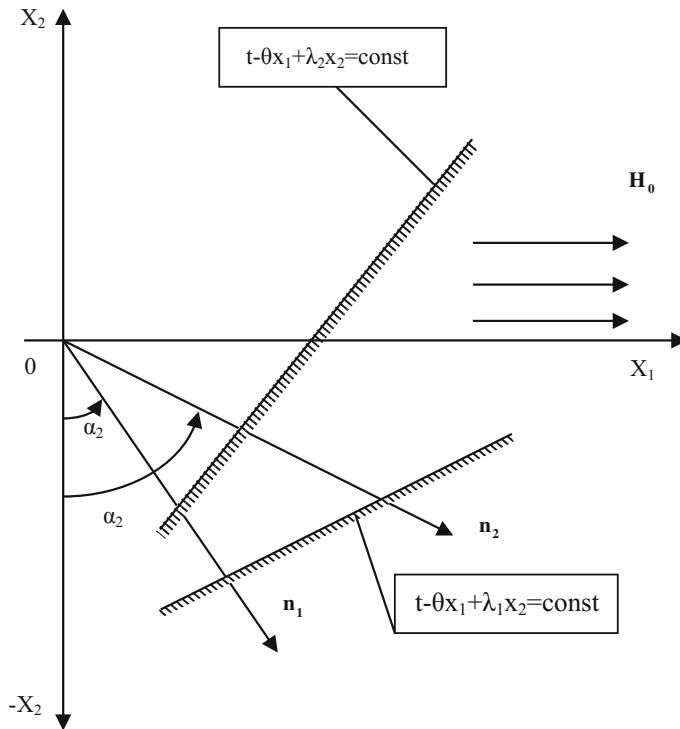


Fig. 2.1 Investigation of the roots of characteristic equation

2.4.2 Investigation of the Roots of the Characteristic Equation

As is known, when investigating wave processes in anisotropic media [23, 24, 27, 29], the problem of separating waves into quasi-longitudinal and quasi-transverse, or fast and slow, depending on the parameter θ for any of the above admissible values of the parameters a_0, b_0, c_0, d_0 (conditions 2.3.37), is very important. This problem is connected with the study of the properties of the roots $\lambda_k(\theta)$ of the corresponding characteristic equation.

An investigation of this problem in the case of propagation of plane magnetoelastic waves in both isotropic and anisotropic media is given in [12, 30, 31]. It is related to the study of the properties of the roots $\lambda_k(\theta)$ of the characteristic Eq. (2.4.7) of the equations system (2.3.15) for any values of the parameters a_0, b_0, c_0, d_0 and χ , satisfying condition (2.3.37). In particular, it is important to determine the realness domain of functions $\lambda_k(\theta)$ depending on the values of the indicated parameters. It should be noted that the properties of the roots of the characteristic equation are also used in solving other problems related to the propagation of waves.

In what follows, by virtue of symmetry, the behavior of the roots $\lambda_k(\theta)$ is considered only in the interval $0 \leq \theta < \infty$. As already noted, we will investigate only the waves corresponding to the plus sign in front of λ_k in (2.4.8). The choice of the sign “plus” or “minus” in front of λ_k in the solution shows that the plane waves propagate in elastic medium in opposite directions with coincident phase velocities.

On the basis of (2.4.8) and (2.4.12), it is easy to see that the function $\lambda_2(\theta)$ takes on real values when one of the following conditions is fulfilled:

$$\begin{aligned} (a) \quad & q_m > 0, \quad r_m > 0, \quad p_m > 0; \\ (b) \quad & q_m = 0, \quad p_m \geq 0; \\ (c) \quad & q_m > 0, \quad r_m < 0, \end{aligned} \tag{2.4.17}$$

and the function $\lambda_1(\theta)$ takes on real values if conditions (a) or (b) of (2.4.17) are satisfied.

Consequently, it is necessary to investigate the behavior of the functions $q_m(\theta)$, $r_m(\theta)$ and $p_m(\theta)$ in the interval $0 \leq \theta < \infty$.

It follows from (2.4.9) that, depending on the sign of the quantity l_m , the function $p_m(\theta)$ varies as follows:

$$\begin{aligned} (1) \quad & p_m(\theta) > 0, \quad \text{for } l_m \leq 0, \quad \theta \in [0, +\infty); \\ (2) \quad & p_m(\theta) > 0, \quad \text{for } l_m > 0, \quad \theta \in [0, \theta_p^{(m)}); \\ (3) \quad & p_m(\theta) \leq 0, \quad \text{for } l_m > 0, \quad \theta \in [\theta_p^{(m)}, +\infty), \end{aligned} \tag{2.4.18}$$

where

$$\theta_p^{(m)} = \sqrt{(b_m + d_m)/l_m} \tag{2.4.19}$$

We now turn to the study of the function $r_m(\theta)$. Note that the zeros of the function $r_m(\theta)$, which are provided by the index r according to (2.4.9) are determined from equation

$$r_m(\theta) = (a_m \theta^2 - 1)(e_m \theta^2 - 1) = 0$$

and are expressed as follows:

$$\begin{aligned} \theta &= \pm \theta_{rk}^{(m)}; \\ \theta_{r1}^{(m)} &= a_m^{-1/2}, \\ \theta_{r2}^{(m)} &= e_m^{-1/2}, \quad (k = 1, 2) \end{aligned} \tag{2.4.20}$$

It follows from (2.4.20) that the location of the roots $\theta_{rk}^{(m)}$ ($k = 1, 2$) on the positive semiaxis θ depends on the sign of the quantity A_m and has the form:

$$\begin{aligned}
\theta_{r1}^{(m)} &< \theta_{r2}^{(m)}, \quad \text{for } A_m > 0 \quad (m = 0, 1, 2, 3); \\
\theta_{r1}^{(1)} &= \theta_{r2}^{(1)}, \quad \text{for } A_1 = 0; \\
\theta_{r1}^{(1)} &> \theta_{r2}^{(1)}, \quad \text{for } A_1 < 0.
\end{aligned} \tag{2.4.21}$$

Wherein

$$\begin{aligned}
A_m &> 0, \quad \text{for } 0 \leq \chi < +\infty \quad (m = 1, 2); \\
A_3 &> 0, \quad \text{for } 0 \leq \chi < \chi_A; \\
A_3 &\leq 0, \quad \text{for } \chi_A < \chi < +\infty,
\end{aligned} \tag{2.4.22}$$

where

$$\chi_A = A_0 = a_0 - d_0 > 0. \tag{2.4.23}$$

Based on the foregoing, the behavior of the function $r_m(\theta)$ can be described as follows:

(1) for $A_m > 0$ ($m = 0, 1, 2, 3$):

$$\begin{aligned}
r_m &> 0, \quad \text{for } \theta \in [0, \theta_{r1}^{(m)}) \cup (\theta_{r2}^{(m)}, \infty); \\
r_m &\leq 0, \quad \text{for } \theta \in [\theta_{r1}^{(m)}, \theta_{r2}^{(m)}];
\end{aligned}$$

(2) for $A_1 = 0$:

$$\begin{aligned}
r_1 &> 0 \quad \text{for } \theta \in [0, +\infty), \quad \theta \neq \theta_{r1}^{(1)} = \theta_{r2}^{(1)}; \\
r_1 &= 0 \quad \text{for } \theta = \theta_{r1}^{(1)} = \theta_{r2}^{(1)} = a_0^{-1/2},
\end{aligned} \tag{2.4.24}$$

(3) for $A_1 < 0$:

$$\begin{aligned}
r_1 &> 0, \quad \text{for } \theta \in [0, \theta_{r1}^{(1)}) \cup (\theta_{r2}^{(1)}, \infty); \\
r_1 &\leq 0, \quad \text{for } \theta \in [\theta_{r1}^{(1)}, \theta_{r2}^{(1)}].
\end{aligned}$$

Now we will clarify the conditions under which the zeros of the functions $r_m(\theta)$ are roots for functions $\lambda_k(\theta)$. Substituting (2.4.20) into (2.4.8), we obtain:

$$\begin{aligned}\lambda_k(\theta_{r1}^{(m)}) &= \sqrt{\frac{-S_m + (-1)^k |S_m|}{2a_m b_m d_m}}, \\ \lambda_k(\theta_{r2}^{(m)}) &= \sqrt{\frac{-K_m + (-1)^k |K_m|}{2a_m b_m e_m}},\end{aligned}\tag{2.4.25}$$

where S_m and K_m denote:

$$\begin{aligned}S_m &= A_m d_m + c_m^2, \\ K_m &= A_m b_m - c_m^2 \quad (m = 0, 1, 2, 3).\end{aligned}\tag{2.4.26}$$

In the absence of a magnetic field ($m = 0$), in view of (2.3.40) and (2.4.26), we have:

$$\begin{aligned}A_0 > 0; S_0 > 0; \quad K_0 > 0, \quad \text{for} \quad 0 < c_0 < c_k^{(0)}; \\ K_0 \leq 0, \quad \text{for} \quad c_k^{(0)} \equiv \sqrt{b_0 A_0} \leq c_0 < C_E^{(0)};\end{aligned}\tag{2.4.27}$$

From (2.4.26), by virtue of (2.3.40), it follows directly that

$$\begin{aligned}S_m &> 0, \quad \text{for} \quad A_m \geq 0 \\ K_1 &< 0, \quad \text{for} \quad A_1 \leq 0.\end{aligned}$$

On the other hand, from (2.4.26) we have:

$$S_1 = S_0 - d_0 \chi.\tag{2.4.28}$$

It is clear from (2.4.28) that:

$$\begin{aligned}S_1 &> 0, \quad \text{for} \quad \chi \in [0, \chi_S); \\ S_1 &\leq 0, \quad \text{for} \quad \chi \in [\chi_S, \infty)\end{aligned}\tag{2.4.29}$$

where

$$\chi_S = S_0/d_0 = (A_0 d_0 + c_0^2)\tag{2.4.30}$$

From (2.4.26) we have:

$$\begin{aligned}K_1 &= K_0 + (A_0 - b_0)\chi - \chi^2, \\ K_2 &= K_0 + b_0 \chi, \\ K_3 &= K_0 + (A_0 - b_0 + 2c_0)\chi.\end{aligned}\tag{2.4.31}$$

On the basis of (2.4.26)–(2.4.30), (2.3.37) and (2.3.41) we come to the conclusion that the quantities K_m and S_m , depending on the values of the parameters a_0 , b_0 , d_0 , c_0 and χ can have the following signs:

(1) for $A_m > 0$ ($m = 0, 1, 2, 3$) :

$$(a) \quad S_m > 0, K_m \geq 0,$$

$$(b) \quad S_m > 0, K_m < 0;$$

(2) for $A_1 = 0$:

$$S_1 > 0, K_1 < 0; \quad (2.4.32)$$

(3) for $A_1 < 0$:

$$(a) \quad S_1 > 0, K_1 < 0,$$

$$(b) \quad S_1 \leq 0, K_1 < 0.$$

Taking into account (2.4.32), we obtain from (2.4.25) the values of the functions $\lambda_k(\theta)$ at the points $\theta = \theta_{rk}^{(m)}$ where $r_m(\theta)$ vanishes. The result of the calculations is given in Table 2.1, where the following notations are done:

$$\begin{aligned} \lambda_0^{(m)} &= \sqrt{\frac{|S_m|}{a_m b_m d_m}}, \\ \lambda_*^{(m)} &= \sqrt{\frac{|K_m|}{b_m d_m e_m}}. \end{aligned} \quad (2.4.33)$$

Table 2.1 The result of calculations and sign of the roots

m	A_m	S_m	K_m	$\lambda_1(\theta_{r1}^{(m)})$	$\lambda_1(\theta_{r2}^{(m)})$	$\lambda_2(\theta_{r1}^{(m)})$	$\lambda_2(\theta_{r2}^{(m)})$
0, 1, 2, 3	>0	>0	>0	=0	$i\lambda_0^{(m)}$	$\lambda_0^{(m)}$	=0
		>0	=0	=0	=0	$\lambda_0^{(m)}$	=0
		>0	<0	=0	=0	$\lambda_0^{(m)}$	$\lambda_*^{(m)}$
1	=0	>0	<0	=0	=0	$\lambda_0^{(1)} = \lambda_*^{(1)}$	$\lambda_0^{(1)} = \lambda_*^{(1)}$
	<0	>0	<0	=0	=0	$\lambda_0^{(1)}$	$\lambda_*^{(1)}$
		=0	<0	=0	=0	=0	$\lambda_*^{(1)}$
		<0	<0	$i\lambda_0^{(1)}$	=0	=0	$\lambda_*^{(1)}$

On the basis of Table 2.1, the following conclusion can be drawn:

(1) If the parameters a_0 , b_0 , d_0 , c_0 and χ satisfy the inequalities:

$$\begin{aligned} \text{(a)} \quad S_m &> 0, \quad K_m > 0 \\ \text{(b)} \quad S_m &< 0, \quad K_m < 0, \end{aligned} \tag{2.4.34}$$

then one of the roots of the equation $r_m(\theta) = 0$ is a zero of the function $\lambda_1(\theta)$, and the other root is a zero of the function $\lambda_2(\theta)$; in case (a) $\theta_{r1}^{(m)}$ is the root of the function $\lambda_1(\theta)$, $\theta_{r2}^{(m)}$ is the root of the function $\lambda_2(\theta)$; and in case (b) it is vice versa.

(2) If these parameters satisfy the inequalities

$$S_m > 0, \quad K_m < 0, \tag{2.4.35}$$

then both roots of the equation $r_m(\theta) = 0$ are zeros of the function $\lambda_1(\theta)$, and the function $\lambda_2(\theta)$ does not vanish anywhere.

(3) Cases

$$S_m > 0, \quad K_m = 0 \quad \text{or} \quad S_m = 0, \quad K_m < 0 \tag{2.4.36}$$

are transitive between cases (1) and (2), where if $S_m > 0$, $K_m = 0$, then both functions $\lambda_k(\theta)$ vanish at the point $\theta_{r2}^{(m)}$ and only $\lambda_1(\theta)$ vanishes at $\theta_{r1}^{(m)}$; and if $S_m = 0$, $K_m < 0$ ($m = 1$), then both functions vanish at $\theta_{r1}^{(m)}$ and only $\lambda_1(\theta)$ vanishes at $\theta_{r2}^{(m)}$.

Comparing the values of $\theta_{r1}^{(m)}$, $\theta_{r2}^{(m)}$ and $\theta_p^{(m)}$, we find their location depending on the values of the parameters a_0 , b_0 , d_0 , c_0 and χ (for real $\theta_p^{(m)}$):

(1) for $A_m > 0$ ($m = 0, 1, 2, 3$):

- (a) if $K_m \geq 0$, $S_m > 0$, then $\theta_{r1}^{(m)} < \theta_p^{(m)} \leq \theta_{r2}^{(m)}$,
- (b) if $K_m < 0$, $S_m > 0$, then $\theta_{r1}^{(m)} < \theta_{r2}^{(m)} < \theta_p^{(m)}$,

(2) for $A_1 = 0$:

$$\theta_{r1}^{(1)} = \theta_{r2}^{(1)} < \theta_p^{(1)}, \tag{2.4.37}$$

(3) for $A_1 < 0$:

(a) if $K_1 < 0$, $S_1 > 0$, then $\theta_{r2}^{(1)} < \theta_{r1}^{(1)} < \theta_p^{(1)}$,

(b) if $K_1 < 0$, $S_1 \leq 0$, then $\theta_{r2}^{(1)} < \theta_p^{(1)} \leq \theta_{r1}^{(1)}$

where $\theta_p^{(m)}$ are the roots of the function $p_m(\theta)$.

Let us now turn to the investigation of the function $q_m(\theta)$ under the internal radical in (2.4.8). The zeros of $q_m(\theta)$ according to (2.4.10) are defined by expressions

$$\theta = \pm \theta_{qk}^{(m)}, \quad (2.4.38)$$

$$\theta_{qk}^{(m)} = \sqrt{D_m^{-1}(M_m + (-1)^k \sqrt{\Delta_m})} \quad (k = 1, 2),$$

$$\Delta_m = M_m^2 - B_m^2 D_m = -4b_m d_m c_m^2 N_m. \quad (2.4.39)$$

It follows from (2.4.39) that when the following conditions are satisfied:

- (1) $N_m = A_m B_m - c_m^2 > 0$: all four roots (2.4.38) are pairwise complex conjugate;
- (2) $N_m \leq 0$, $D_m > 0$, $M_m < 0$: all roots (2.4.38) are imaginary;
- (3) (a) $N_m \leq 0$, $D_m > 0$, $M_m > 0$ or (b) $N_m = D_m = M_m = 0$: all roots (2.4.38) are real;
- (4) $N_m < 0$, $D_m < 0$: the roots $\pm \theta_{q1}^{(m)}$ are real, and the roots $\pm \theta_{q2}^{(m)}$ are imaginary.

It follows from the foregoing that the function $q_m(\theta)$ under the conditions (2.4.37) varies as follows:

- (I) for (1) or (2) or (b) from (3): $q_m(\theta) > 0$, if $\theta \in [0, \infty)$;
- (II) for (a) from (3):

$$\begin{aligned} q_m > 0 & \quad \text{if } \theta \in [0, \theta_{q1}^{(m)}) \cup (\theta_{q2}^{(m)}, \infty); \\ q_m \leq 0 & \quad \text{if } \theta \in [\theta_{q1}^{(m)}, \theta_{q2}^{(m)}]; \end{aligned} \quad (2.4.40)$$

(III) for (4):

$$\begin{aligned} q_m > 0 & \quad \text{if } \theta \in [0, \theta_{q1}^{(m)}); \\ q_m \leq 0 & \quad \text{if } \theta \in [\theta_{q1}^{(m)}, \infty). \end{aligned}$$

Finally, consider the location of the roots of function $r_m(\theta)$, $q_m(\theta)$ and $p_m(\theta)$. From (2.4.20) and (2.4.38) we obtain

$$\begin{aligned}
\left(\theta_{r1}^{(m)}\right)^{-2} - \left(\theta_{qk}^{(m)}\right)^{-2} &= B_m^{-2} \left[\sqrt{d_m |N_m|} - (-1)^k c_m \sqrt{b_m} \right]^2 \geq 0, \\
\left(\theta_{r2}^{(m)}\right)^{-2} - \left(\theta_{qk}^{(m)}\right)^{-2} &= B_m^{-2} \left[\sqrt{b_m |N_m|} - (-1)^k c_m \sqrt{d_m} \right]^2 \geq 0.
\end{aligned} \tag{2.4.41}$$

Equating the expressions (2.4.41) to zero, we obtain $S_m = 0$, $K_m = 0$. On the other hand, it follows from (2.4.41) that the real values $\theta_{qk}^{(m)}$ are to the right of the points $\theta_{rk}^{(m)}$ on the positive semiaxis. Further, when $K_m > 0$, according to (2.4.37), the value $\theta_p^{(m)}$ lies between the points $\theta_{r1}^{(m)}$ and $\theta_{r2}^{(m)}$. Hence it follows that for $K_m > 0$ the following arrangement holds: $\theta_{r1}^{(m)} < \theta_p^{(m)} < \theta_{r2}^{(m)} < \theta_{q1}^{(m)} < \theta_{q2}^{(m)}$. If $K_m < 0$, then according to (2.4.37) $\theta_p^{(m)}$ is to the right of the point $\theta_{rk}^{(m)}$ on the axis θ . In this case $r_m(\theta_p^{(m)}) > 0$. Consequently, the point $\theta_p^{(m)}$ is located in the region where $q_m(\theta) < 0$. By virtue of (2.4.40) this means that in the case of two real roots $\theta_{q1}^{(m)}$ and $\theta_{q2}^{(m)}$, the point $\theta_p^{(m)}$ lies between them, and in the case of an imaginary root $\theta_{q2}^{(m)}$, the condition $\theta_{q1}^{(m)} < \theta_p^{(m)}$ holds.

On the basis of the foregoing and the relation (2.4.37) and (2.4.40) we have the following arrangement of points $\theta_{rk}^{(m)}$, $\theta_{qk}^{(m)}$, $\theta_p^{(m)}$:

(1) for $A_m > 0$ ($m = 0, 1, 2, 3$):

- (a) $\theta_{r1}^{(m)} < \theta_p^{(m)} \leq \theta_{r2}^{(m)} \leq \theta_{q1}^{(m)} < \theta_{q2}^{(m)}$, if $K_m \geq 0$, $S_m > 0$,
- (b) $\theta_{r1}^{(m)} < \theta_{r2}^{(m)} < \theta_{q1}^{(m)} < \theta_p^{(m)} < \theta_{q2}^{(m)}$, if $K_m < 0$, $S_m > 0$,

(2) for $A_1 = 0$:

$$\theta_{r1}^{(1)} = \theta_{r2}^{(1)} < \theta_{q1}^{(1)} < \theta_p^{(1)} < \theta_{q2}^{(1)}; \tag{2.4.42}$$

(3) for $A_1 < 0$:

- (a) $\theta_{r2}^{(1)} < \theta_{r1}^{(1)} < \theta_{q1}^{(1)} < \theta_p^{(1)} < \theta_{q2}^{(1)}$, if $K_1 < 0$, $S_1 > 0$,
- (b) $\theta_{r2}^{(1)} < \theta_p^{(1)} \leq \theta_{r1}^{(1)} \leq \theta_{q1}^{(1)} < \theta_{q2}^{(1)}$, if $K_1 < 0$, $S_1 \leq 0$.

To determine the boundary of realness intervals of the functions $\lambda_k(\theta)$, it is also necessary to establish what values the roots $\theta_{qk}^{(m)}$ take on under the conditions $K_m \geq 0$ or $K_m < 0$. On the basis of conditions (2.3.37) and (2.4.39), it can be shown that for $K_m \geq 0$ the roots $\theta_{qk}^{(m)}$ can be either real or complex. In the case of the fulfillment of the condition $K_m < 0$, only the following conditions hold:

$$(a) D_m > 0, \quad M_m > 0; \quad (b) D_m < 0, \quad (2.4.43)$$

from which on the basis of (2.4.39) it follows that for $K_m < 0$ either $\theta_{q1}^{(m)}$ and $\theta_{q2}^{(m)}$ are real [case (a) of (2.4.43)], either $\theta_{r1}^{(m)}$ is real, and $\theta_{r2}^{(m)}$ is imaginary [case (b) from (2.4.43)].

Thus, the roots $\pm\theta_{q1}^{(m)}$ of the equation $q_m(\theta) = 0$ under the condition $K_m < 0$ are necessarily real and determine the boundaries of the realness domains of the roots of the characteristic equation $\lambda_k(\theta)$ when the condition $S_m > 0$ is satisfied. Using (2.4.18), (2.4.24), (2.4.40) and (2.4.42), on the basis of (2.4.17), we find the realness domains of the roots of the characteristic equation $\lambda_k(\theta)$ depending on the values of the parameters a_0 , b_0 , d_0 , c_0 and χ . The results are given in Table 2.2.

The functions $\lambda_k(\theta)$ take on imaginary or complex values at those points of the interval $0 \leq \theta < +\infty$ which do not belong to the realness domain of these functions. The inhomogeneous plane waves correspond to these imaginary or complex values.

Thus, depending on the values of the parameters a_0 , b_0 , d_0 , c_0 and χ when considering the behavior of the roots $\lambda_k(\theta)$ of the characteristic equation, we have two main cases [30, 31].

- (1) These parameters satisfy the conditions (2.4.34). In this case, the functions $\lambda_k(\theta)$ are real in the intervals $[0, \theta_{ri}^{(m)}]$ of the positive semi-axis θ ($\theta_{ri}^{(m)}$ are the roots of the equation $r_m(\theta) = 0$) and vanish at points $\theta_{ri}^{(m)}$. The roots $\lambda_k(\theta)$ possessing such properties, we attribute to the first type.

Table 2.2 Realness intervals

m	Cases			Realness intervals	
	A_m	$K_m S_m$		$\lambda_1(\theta)$	$\lambda_2(\theta)$
0, 1, 2, 3	>0	>0	>0	$0 \leq \theta \leq \theta_{r1}^{(m)}$	$0 \leq \theta \leq \theta_{r2}^{(m)}$
		=0	>0	$0 \leq \theta \leq \theta_{r1}^{(m)};$ $\theta = \theta_{r2}^{(m)}$	$0 \leq \theta \leq \theta_{r2}^{(m)}$ $\theta_{r2}^{(m)} = \theta_{q1}^{(m)} = \theta$
		<0	>0	$0 \leq \theta \leq \theta_{r1}^{(m)}$ $\theta_{r2}^{(m)} \leq \theta \leq \theta_{q1}^{(m)}$	$0 \leq \theta \leq \theta_{q1}^{(m)}$
1	=0	<0	>0	$0 \leq \theta \leq \theta_{q1}^{(1)}$	$0 \leq \theta \leq \theta_{q1}^{(1)}$
	<0	<0	>0	$0 \leq \theta \leq \theta_{r2}^{(1)}$ $\theta_{r1}^{(1)} \leq \theta \leq \theta_{q1}^{(1)}$	$0 \leq \theta \leq \theta_{q1}^{(1)}$
		<0	=0	$0 \leq \theta \leq \theta_{r2}^{(1)};$ $\theta = \theta_{r1}^{(1)}$	$0 \leq \theta \leq \theta_{r1}^{(1)}$
		<0	<0	$0 \leq \theta \leq \theta_{r2}^{(1)}$	$0 \leq \theta \leq \theta_{r1}^{(1)}$

- (2) These parameters satisfy the conditions (2.4.35). In this case, the function $\lambda_1(\theta)$ takes on real values in two intervals: $[0, \theta_{r1}^{(m)}]$ and $[\theta_{r2}^{(m)}, \theta_{q1}^{(m)}]$, if $A_m > 0$; $[0, \theta_{r2}^{(m)}]$ and $[\theta_{r2}^{(m)}, \theta_{q1}^{(m)}]$ if $A_1 < 0$. The function $\lambda_2(\theta)$ takes on real values in the interval $[0, \theta_{q1}^{(m)}]$. The roots $\lambda_k(\theta)$, possessing such properties, we attribute to the second type.
- (3) If either of the quantities K_m or S_m vanishes, then we are dealing with the case, transitive between the above cases.

The so-called conical refraction ($A_1 = 0$) is a special case [25]. In this case the functions $\lambda_k(\theta)$ are real in the interval $[0, \theta_{q1}^{(m)}]$, and the function $\lambda_1(\theta)$ vanishes at the double point $\theta_{r1}^{(1)} = \theta_{r2}^{(1)}$ of this interval, and the function $\lambda_2(\theta)$ does not vanish anywhere.

2.5 Classification of Plane Magnetoelastic Waves. Behavior of Their Velocities

Here, using the results of the previous section, the classification of magnetoelastic waves is categorized into fast and slow, or quasi-longitudinal and quasi-transverse, depending on the physical and mechanical properties of the medium and on the magnitude and orientation of the external magnetic field. The changes in phase velocities are studied depending on the indicated parameters. In particular, intervals of monotonicity and extreme values of phase velocities are determined [12, 14, 31–33].

Magnetoelastic wave processes in anisotropic elastic media are complex and diverse, depend on the anisotropy class, the relationships between the elastic constants, the magnitude and direction of the external magnetic field, the directions of wave propagation, and so on. In contradistinction to isotropic media, where purely longitudinal and purely transverse waves propagate, in anisotropic media, as well as in isotropic and anisotropic media under a magnetic field, quasi-longitudinal and quasi-transverse, or fast and slow waves propagate. The wave process in isotropic media has been described by functionally invariant or complex solutions of wave equations. The wave processes in anisotropic media were investigated in [24, 27, 28] by the noted method. In [12], the wave processes in isotropic media under a magnetic field were studied by the method of complex solutions. The studies [14, 31, 32] are devoted to the behavior of magnetoelastic waves in perfectly conducting anisotropic media.

Further, by virtue of the symmetry of the cases $m = 2$ and $m = 3$ of the direction of the external magnetic field, we will consider the cases $m = 1$, $m = 3$.

2.5.1 Classification of Plane Magnetoelastic Waves. Fast and Slow Magnetoelastic Waves

Let us consider the behavior of the functions $\alpha_k(\theta)$ and $V_k(\theta)$ from (2.4.16). Let us find the derivatives of these functions in the realness intervals of the roots $\lambda_k(\theta)$.

According to (2.4.16), we have:

$$\begin{aligned}
 \varphi_k &= b_m + d_m + (-1)^k \frac{B_m^2 - M_m \theta^2}{\sqrt{q_m(\theta)}}, \\
 \varphi'_k &= \frac{(-1)^{k+1} 8b_m d_m c_m^2 \theta^3 N_m}{\sqrt{q_m(\theta)}}, \\
 \Psi_k &= \frac{2b_m d_m - l_m + (-1)^k (D_m \theta^2 - M_m)}{\sqrt{q_m(\theta)}}, \\
 \Psi'_k &= \frac{(-1)^k 8b_m d_m c_m^2 \theta N_m}{\sqrt{q_m^3(\theta)}}, \\
 (tg \alpha_k) &= \frac{\varphi_k(\theta)}{2b_m d_m \lambda_k^3(\theta)}, \\
 V'_k &= -\frac{\theta V_k^3(\theta)}{2b_m d_m} \Psi_k(\theta).
 \end{aligned} \tag{2.5.1}$$

Let us consider the possible cases.

Case 1 Let $K_m > 0$. Then $A_m > 0, S_m > 0$ and N_m can take a value of any sign. According to Table 2, the functions $\lambda_1(\theta)$ and $\lambda_2(\theta)$ are real in the following intervals and vanish at the ends of these segments:

$$\begin{aligned}
 0 \leq \theta \leq \theta_{r1}^{(m)}, \\
 0 \leq \theta \leq \theta_{r2}^{(m)}, \quad (\theta_{r1}^{(m)} < \theta_{r2}^{(m)}).
 \end{aligned} \tag{2.5.2}$$

Let $N_m > 0$. It follows from (2.5.1)₄ that $\phi'_1 > 0, \phi'_2 < 0$, that is the function ϕ_1 increases and ϕ_2 decreases. Further, according to (2.3.40) and (2.4.12) we have:

$$\begin{aligned}
 B_1 &= B_0 + \chi > 0, \\
 B_3 &= B_0 + \chi > 0
 \end{aligned} \tag{2.5.3}$$

Taking into account $A_m > 0, S_m > 0$ and (2.5.3) from (2.5.1)₃ we obtain:

$$\begin{aligned}
 \varphi_1(0) &= 2d_m > 0, \\
 \varphi_2(\theta_{r2}^{(m)}) &= 2b_m d_m A_m K_m^{-1} > 0.
 \end{aligned}$$

It follows that in the segments of (2.5.2) the right-hand sides of (2.5.1) take on positive values and therefore $\operatorname{tg} \alpha_k(\theta)$ increases from zero to infinity. Therefore, the angles $\alpha_k(\theta)$ in (2.5.2) increase from 0 to $\pi/2$:

$$\begin{aligned} 0 &\leq \alpha_1 \leq \pi/2, \\ 0 &\leq \alpha_2 \leq \pi/2 \end{aligned} \quad (2.5.4)$$

Let $N_m = 0$. According to (2.5.1) we have $\varphi'_k(\theta) \equiv 0$, $\varphi_k(\theta) = b_m + d_m = \operatorname{const} > 0$, $(\operatorname{tg} \alpha_k)'_{\theta} > 0$. Therefore, in the segments (2.5.2) the angles $\alpha_k(\theta)$, monotonically increasing, run through the values of the interval (2.5.4).

Let us assume that the condition $N_m < 0$ is fulfilled. According to (2.5.1), φ_2 increases and φ_1 decreases. In addition, by virtue of $A_m > 0$, $S_m > 0$ and (2.5.2), we have:

$$\begin{aligned} \varphi_1(\theta_{r1}^{(m)}) &= 2b_m d_m A_m S_m^{-1} > 0, \\ \varphi_2(0) &= 2b_m > 0. \end{aligned}$$

From this and (2.5.1) it follows that the angles $\alpha_k(\theta)$ in the corresponding segments (2.5.2) increase from 0 to $\pi/2$.

Thus, under the condition $K_m > 0$ to the values from the intervals (2.5.2) there correspond waves (2.4.14) with directions of propagation (2.5.4).

It follows from (2.4.16) and (2.4.7) that the relation between the phase velocities is:

$$V_1(\alpha) > V_2(\alpha) \quad \text{for} \quad \alpha \in [0; \pi/2]. \quad (2.5.5)$$

Thus, in the case of $K_m < 0$ the waves (2.4.14) for $k = 1$ are fast, and for $k = 2$ they are slow, and:

$$\begin{aligned} V_f &= V_1(\theta) \quad \text{for} \quad \theta \in [0; \theta_{r1}^{(m)}]; \\ V_s &= V_2(\theta) \quad \text{for} \quad \theta \in [0; \theta_{r2}^{(m)}]. \end{aligned} \quad (2.5.6)$$

Case 2 Let $K_m < 0$. Suppose that the condition $A_m > 0$ is also fulfilled. Then $S_m > 0$. Under these conditions, i.e. when

$$A_m > 0, \quad S_m > 0, \quad K_m < 0 \quad (2.5.7)$$

as it follows from Table 2.2, the roots $\lambda_k(\theta)$ belong to the second type, and $\lambda_1(\theta)$ takes on real values in two segments:

$$\begin{aligned} 0 \leq \theta \leq \theta_{r1}^{(m)}, \\ \theta_{r2}^{(m)} \leq \theta \leq \theta_{q1}^{(m)} \quad \left(\theta_{r1}^{(m)} < \theta_{r2}^{(m)} \right) \end{aligned} \quad (2.5.8)$$

vanishing at the end of the first and at the beginning of the second segment; the function $\lambda_2(\theta)$ is real only in $0 \leq \theta \leq \theta_{q1}^{(m)}$, where it is nonzero. In addition, since $q_m(\theta_{q1}^{(m)}) = 0$, the condition $\lambda_1(\theta_{q1}^{(m)}) = \lambda_2(\theta_{q1}^{(m)})$ is true. Representing N_m from (2.4.12) in the form $N_m = K_m - d_m A_m$, we note that $N_m < 0$ for (2.5.7). As the conditions $A_m > 0$, $S_m > 0$, $N_m < 0$ hold, on the basis of the results of Case 1 we conclude that when θ takes on values from the first interval (2.5.8), the angle $\alpha_1(\theta)$ increases from 0 to $\pi/2$, and when θ runs through the interval $0 \leq \theta \leq \theta_{q1}^{(m)}$, the angle $\alpha_2(\theta)$ increases and varies from 0 to the value $\alpha_2(\theta_{q1}^{(m)}) < \pi/2$. When $N_m < 0$ the function $\phi_1(\theta)$ decreases, and since for $K_m < 0$ the condition $\phi_1(\theta_{r2}^{(m)}) = -2b_m d_m |K_m|^{-1} < 0$ holds, the right-hand side of (2.5.5) for $k = 1$ is negative in the second interval (2.5.8), and therefore the angle $\alpha_1(\theta)$ in this interval decreases from $\pi/2$ to the value $\alpha_1(\theta_{q1}^{(m)}) \equiv \alpha_2(\theta_{q1}^{(m)})$.

Thus, for $A_m > 0, K_m < 0$ to the values θ from $0 \leq \theta \leq \theta_{r1}^{(m)}$ there correspond waves propagating along the directions $0 \leq \alpha_1(\theta) \leq \pi/2$. To the values θ from $0 \leq \theta \leq \theta_{q1}^{(m)}$ there correspond waves propagating along the directions $0 \leq \alpha_2(\theta) \leq \alpha_2(\theta_{q1}^{(m)}) \equiv \alpha_1(\theta_{q1}^{(m)})$. Finally, to the values θ from the interval $\theta_{r2}^{(m)} \leq \theta \leq \theta_{q1}^{(m)}$ there correspond waves propagating along the directions $\pi/2 \geq \alpha_1(\theta) \geq \alpha_1(\theta_{q1}^{(m)}) \equiv \alpha_2(\theta_{q1}^{(m)})$. Now can be verified that

$$V_f(\alpha) > V_s(\alpha) \quad \text{for } \alpha \in [0; \pi/2], \quad (2.5.9)$$

$$\begin{aligned} V_f(\alpha) &= V_1(\alpha_1(\theta)) \quad \text{for } \theta \in [0; \theta_{r1}^{(m)}], \\ V_s(\alpha) &= \begin{cases} V_2(\alpha_2(\theta)) & \text{for } \theta \in [0; \theta_{q1}^{(m)}] \\ V_1(\alpha_1(\theta)) & \text{for } \theta \in [\theta_{r2}^{(m)}; \theta_{q1}^{(m)}]. \end{cases} \end{aligned} \quad (2.5.10)$$

Therefore, in the case of $A_m > 0, K_m < 0$, the waves (2.4.14) for $k = 1$ in the interval $0 \leq \theta \leq \theta_{r1}^{(m)}$ are fast, and for $k = 2$ in the interval $0 \leq \theta \leq \theta_{q1}^{(m)}$ they are slow. The velocities of fast and slow waves are determined by (2.4.16) and (2.5.10), and their directions—by (2.4.16), and:

$$\alpha = \alpha_1(\theta) \quad \text{for} \quad \theta \in \left[0; \theta_{r1}^{(m)}\right], \quad (\text{for fast waves}), \quad (2.5.11)$$

$$\alpha = \alpha_2(\theta) \quad \text{for} \quad \theta \in \left[0; \theta_{q1}^{(m)}\right]; \quad (\text{for slow waves}). \quad (2.5.12)$$

$$\alpha = \alpha_1(\theta) \quad \text{for} \quad \theta \in \left[\theta_{r2}^{(m)}; \theta_{q1}^{(m)}\right].$$

Now suppose that the condition $A_m = 0$ is satisfied for $K_m < 0$, which is possible only for $m = 1$. In this case, according to Table 2.2 the functions λ_k are real in the interval $0 \leq \theta \leq \theta_{q1}^{(1)}$, the function $\lambda_1(\theta)$ vanishes at the double point $\theta = \theta_{r1}^{(1)} = \theta_{r2}^{(1)} = 1/\sqrt{a_0}$ and the function $\lambda_2(\theta)$ does not vanish anywhere. This is the case of conical refraction [25]. The separation of waves into fast and slow is carried out as in the case of (2.5.7). But, in contrast to that case, here we have $\theta_{r1}^{(1)} = \theta_{r2}^{(1)}$, as a result of which in the direction of the magnetic field $\mathbf{H}_0$ the magnetoelastic waves propagate with the same phase velocity $V_f(\pi/2) = V_s(\pi/2) = \sqrt{a_1} = \sqrt{a_0}$:

Finally, consider the case $K_1 < 0, A_1 < 0$. Let us first assume that the condition $S_1 > 0$ is also satisfied. Then, according to Table 2.2, the function $\lambda_2(\theta)$ is real in the interval $0 \leq \theta \leq \theta_{q1}^{(m)}$, and the function $\lambda_1(\theta)$ is real in the intervals $0 \leq \theta \leq \theta_{r2}^{(1)}$ and $\theta_{r1}^{(1)} \leq \theta \leq \theta_{q1}^{(1)}$, where

$$\begin{aligned} \lambda_1\left(\theta_{rk}^{(1)}\right) &= 0, \\ \lambda_1\left(\theta_{q1}^{(1)}\right) &= \lambda_2\left(\theta_{q1}^{(1)}\right) > 0. \end{aligned}$$

It follows that this case is analogous to the case $K_1 < 0, A_1 < 0$. Analogy here means that the waves are separated into fast and slow waves in the same way. The corresponding results can be obtained from (2.5.9) to (2.5.12) by interchanging $\theta_{r1}^{(1)}$ and $\theta_{r2}^{(1)}$.

Further, on the basis of Table 2.2, it is easy to see that the case $K_1 < 0, A_1 < 0, S_1 < 0$ is analogous to the case $K_1 > 0, A_1 > 0, S_1 > 0$. But here we must make the above substitution in (2.5.5) and (2.5.6). $K_m = 0$ or $S_m = 0$ are transitional between the above cases.

Thus, the separation of the magnetoelastic waves (2.4.14) into fast and slow depends on the type of the roots $\lambda_k(\theta)$, and when the roots $\lambda_k(\theta)$ are of the first type, the wave separation is performed by the Formulas (2.5.5) and (2.5.6), and if

the roots $\lambda_k(\theta)$ are of the second type, then the wave separation is performed by the Formulas (2.5.9) and (2.5.10).

We note the following changes in the sign of the quantity A_1 depending on the magnitude of the magnetic field $\mathbf{H}_0$. For $0 < \chi < \chi_A^{(1)}$, ($A_1 \geq 0$) in the direction of the axis Ox (parallel to $\mathbf{H}_0$), the slow waves propagate at a velocity $V_s = \sqrt{e_1(\chi)}$. Further, with increasing of the magnetic field intensity $\chi \geq \chi_A^{(1)}$, ($A_1 \leq 0$), the slow waves propagate at a constant velocity $V_s(\chi) = \sqrt{a_0} = \text{const}$. The picture of the fast waves propagation is opposite: for $\chi \leq \chi_A^{(1)}$ the fast waves propagate at a constant velocity $V_f(\chi) = \sqrt{a_0} = \text{const}$, and for $\chi \geq \chi_A^{(1)}$ —at a velocity $V_f(\chi) = \sqrt{e_1} \geq \sqrt{a_0}$ that increases from $\sqrt{a_0}$ to $+\infty$. Hence it follows that the fast waves are longitudinal for $\chi < \chi_A^{(1)}$, while the slow waves are transverse. For $\chi > \chi_A^{(1)}$ we have the opposite picture. It also follows that for $\chi < \chi_A^{(1)}$ the longitudinal waves propagate faster than the transverse waves.

In the case $\chi > \chi_A^{(1)}$ the transverse waves propagate faster than the longitudinal waves, which is usually impossible with the propagation of purely elastic waves. In the case $\chi = \chi_A^{(1)}$ the longitudinal and transverse waves propagate at the same velocity, and this separation is formal here—there is one wave, and its displacement vector $\vec{u}$ can have any direction with respect to $\mathbf{H}_0$ and any value (conical refraction case).

2.5.2 Behavior of the Phase Velocity of Plane Magnetoelastic Waves

Let us now turn to the investigation of the extremum of phase velocities inside the realness segments of the roots $\lambda_k(\theta)$.

The critical points of the extreme of the velocities $V_k(\theta)$ according to (2.5.6) are determined from the equation:

$$v_m(\theta) \equiv N_0 D_m \theta^4 - 2N_0 M_m \theta^2 + N_0 B_m^2 - N_m c_m^2 = 0. \quad (2.5.13)$$

Let us determine the conditions for the solutions of Eq. (2.5.13) to be the extremal points for the functions $\lambda_k(\theta)$.

Case 1. The effect of a longitudinal magnetic field Let us first consider the case of a longitudinal magnetic field ($m = 1$). According to (2.5.1), (2.4.25), (2.4.32) and (2.4.42) we have:

$$\begin{aligned}
\Psi_1(0) &= 2d_0B_1^{-1}(B_0B_1 - c_1^2), \quad \Psi_2(0) = -2b_1B_1^{-1}(A_0B_1 - c_1^2) \\
\Psi_1(\theta_{r2}^{(1)}) &= -2b_1d_1|K_1|^{-1}(B_0A_1 - c_1^2), \quad \text{for } K_1 < 0, A_1 > 0 \\
\Psi_1(\theta_{q1}^{(1)}) &= +\infty, \quad \Psi_2(\theta_{q1}^{(1)}) = -\infty, \quad \text{for } K_1 < 0 \\
\Psi_1(\theta_{r1}^{(1)}) &= 2b_1d_1c_1^4 > 0, \quad \text{for } A_1 = 0 \quad (\theta_{r1}^{(1)} = \theta_{r2}^{(1)}) \\
\Psi_1(\theta_{r2}^{(1)}) &= 2b_1d_1|K_1|^{-1}(B_0|A_1| + c_1^2) > 0, \quad \text{for } A_1 < 0 \\
\Psi_1(\theta_{r1}^{(1)}) &= 2b_1d_1S_1^{-1}(A_0|A_1| + c_1^2) > 0, \quad \text{for } A_1 < 0, S_1 > 0. \\
\Psi_2(\theta_{r1}^{(1)}) &= -2b_1d_1|S_1|^{-1}(A_0|A_1| + c_1^2) < 0, \quad \text{for } S_1 < 0
\end{aligned} \tag{2.5.14}$$

Using the usual methods of mathematical analysis, on the basis of the Formulas (3.4.16), (2.5.1)₁–(2.5.1)₅ we investigate the maximum and minimum of the velocities $V_k(\theta)$. Here are the final results:

- (1) If $N_1 > 0$ and $A_0A_1 - c_1^2 \leq 0$ then the velocity $V_1(\theta)$ in the interval $(0, \theta_{r1}^{(1)})$ decreases from $V_1(0) = \sqrt{b_1}$ to $V_1(\theta_{r1}^{(1)}) = \sqrt{a_0}$. If $N_1 > 0$ and $B_0B_1 - c_1^2 \leq 0$ then the velocity $V_1(\theta)$ in the interval $(0, \theta_{r1}^{(1)})$ increases from $V_1(0) = \sqrt{b_1}$ to $V_1(\theta_{r1}^{(1)}) = \sqrt{a_0}$.
- (2) If $N_1 > 0$ and $B_0A_1 - c_1^2 \leq 0$, then the velocity $V_2(\theta)$ in the interval $(0, \theta_{r2}^{(1)})$ increases from $V_2(0) = \sqrt{d_0}$ to $V_2(\theta_{r2}^{(1)}) = \sqrt{e_1}$; and if $N_1 > 0$, $B_0A_1 - c_1^2 > 0$, then $V_2(\theta)$ in the indicated interval reaches the maximum value, and the maximum point is the zero of Eq. (2.5.13).
- (3) If $N_1 = 0$, then the velocity $V_2(\theta)$ in the interval $(0, \theta_{r2}^{(1)})$ increases monotonically from $V_2(0) = \sqrt{d_0}$ to $V_2(\theta_{r2}^{(1)}) = \sqrt{e_1}$.
- (4) If $N_1 = 0$ and (a) $b_1 - a_1 > 0$ or (b) $b_1 - a_1 < 0$, then the velocity $V_1(\theta)$ in the interval $(0, \theta_{r1}^{(1)})$ varies monotonically and in case (a) it decreases from $V_1(0) = \sqrt{b_1}$ to $V_1(\theta_{r1}^{(1)}) = \sqrt{a_0}$, and in case (b) it increases from $V_1(0) = \sqrt{b_1}$ to $V_1(\theta_{r1}^{(1)}) = \sqrt{a_0}$; and if $b_1 - a_1 = 0$, then $V_1(\theta)$ in the considered interval retains a constant value equal to $V_1(\theta) = \sqrt{a_0}$.
- (5) If $N_1 < 0$, $K_1 \geq 0$ and (a) $A_0A_1 - c_1^2 \geq 0$ or (b) $B_0B_1 - c_1^2 \geq 0$ then the velocity $V_1(\theta)$ in the interval $(0, \theta_{r1}^{(1)})$ in case (b) decreases from $V_1(0) = \sqrt{b_1}$ to $V_1(\theta_{r1}^{(1)}) = \sqrt{a_0}$, and in case (a) increases from $V_1(0) = \sqrt{b_1}$ to $V_1(\theta_{r1}^{(1)}) = \sqrt{a_0}$.

- (6) If $N_1 < 0$, $K_1 \geq 0$ and $A_0A_1 - c_1^2 < 0, B_0B_1 - c_1^2 < 0$, then the velocity $V_1(\theta)$ in the interval $(0, \theta_{r1}^{(1)})$ has a maximum point, at which $v_1(\theta) = 0$.
- (7) If $N_1 < 0$, $K_1 \geq 0$ and $A_0B_1 - c_1^2 \geq 0$, then the velocity $V_2(\theta)$ in the interval $0 \leq \theta \leq \theta_{r2}^{(1)}$ increases monotonically from $V_2(0) = \sqrt{a_0}$ to $V_2(\theta_{r2}^{(1)}) = \sqrt{e_1}$; if $A_0B_1 - c_1^2 < 0$, then the velocity $V_2(\theta)$ in the indicated interval has a minimum point, at which $v_1(\theta) = 0$.
- (8) If $K_1 < 0$, $A_1 \geq 0$ (then also $N_1 < 0$), then the velocity $V_1(\theta)$ in the interval $(0, \theta_{r1}^{(1)})$ behaves as in the case $N_1 < 0$, $K_1 > 0$, and in the interval $(0, \theta_{r1}^{(1)})$ decreases, and $V_1(\theta_{r2}^{(1)}) = \sqrt{e_1}$, $V_1(\theta_{q1}^{(1)}) = \sqrt{(2b_1d_1)^{-1} [b_1 + d_1 + \theta_{q1}^{(1)2}(2b_1d_1 - l_1)]}$.
- (9) If $K_1 < 0$, $A_1 \geq 0$ and $A_0B_1 - c_1^2 \geq 0$, then the velocity $V_2(\theta)$ in the interval $(0, \theta_{q1}^{(1)})$ increases from $V_2(0) = \sqrt{a_0}$ to $V_2(\theta_{q1}^{(1)}) = V_1(\theta_{q1}^{(1)})$; and if $A_0B_1 - c_1^2 < 0$, then the velocity $V_2(\theta)$ in the indicated interval has a minimum point, which is the solution of the Eq. (2.5.13).
- (10) If $K_1 < 0$, $A_1 < 0$, $S_1 > 0$ and $B_0B_1 - c_1^2 \geq 0$, then the velocity $V_1(\theta)$ in the interval $(0, \theta_{r1}^{(1)})$ decreases from $V_1(0) = \sqrt{b_1}$ to $V_1(\theta_{r2}^{(1)}) = \sqrt{e_1}$; if $B_0B_1 - c_1^2 < 0$, then the velocity $V_1(\theta)$ in the indicated interval has a maximum point, which is the solution of the Eq. (2.5.13).
- (11) If $K_1 < 0$, $A_1 < 0$, $S_1 > 0$ then the velocity $V_1(\theta)$ and $(\theta_{r1}^{(1)}, \theta_{q1}^{(1)})$ decreases, and:

$$V_1(\theta_{r1}^{(1)}) = \sqrt{a_0}, \quad V_1(\theta_{q1}^{(1)}) = \sqrt{(2b_1d_1)^{-1} [b_1 + d_1 + \theta_{q1}^{(1)2}(2b_1d_1 - l_1)]};$$

and the velocity $V_2(\theta)$ in the interval $(0, \theta_{q1}^{(1)})$ varies, as in the case $K_1 < 0$, $A_1 > 0$.

- (12) If $S_1 < 0$ (then also $N_1 < 0$, $A_1 < 0$, $K_1 < 0$), then the velocity $V_1(\theta)$ in the interval $(0, \theta_{r2}^{(1)})$ varies, as in case (10), and the velocity $V_2(\theta)$ varies in the interval $(0, \theta_{r1}^{(1)})$ the same way as in the interval $(0, \theta_{r2}^{(1)})$ of the case $K_1 > 0$, $N_1 < 0$.

Case 2. The effect of a transverse magnetic field Let us now consider the case of a transverse magnetic field ($m = 3$). In this case, the critical points are determined by the expressions:

$$\theta_1^{(3)} = (B_3 - c_3) \left[a_3 b_3 - (c_3 + d_3)^2 \right]^{-1},$$

$$\theta_2^{(3)} = (B_3 + c_3) \left[a_3 b_3 - (c_3 - d_3)^2 \right]^{-1},$$

where the point $\theta_1^{(3)}$ is critical for the function $V_1(\theta)$, and the point $\theta_2^{(3)}$ is critical for $V_2(\theta)$, and instead of the Formula (2.5.14) we have the following formulas:

$$\begin{aligned} \Psi_1(0) &= 2d_3 B_3^{-1} (B_3 + c_3) (B_0 - c_0), \quad \Psi_2(0) = -2b_3 B_3^{-1} N_3, \\ \Psi_1(\theta_{r2}^{(3)}) &= -2b_3 d_3 |K_3|^{-1} N_3 \quad \text{for } K_3 < 0, \\ \Psi_2(\theta_{r2}^{(3)}) &= -2b_3 d_3 |K_3|^{-1} N_3 \quad \text{for } K_3 < 0, \\ \Psi_1(\theta_{q1}^{(3)}) &= +\infty, \quad \Psi_2(\theta_{q1}^{(3)}) = -\infty, \\ \Psi_1(\theta_{r1}^{(3)}) &= -2b_3 d_3 S_3^{-1} (A_3 + c_3) (A_0 - c_0), \\ \Psi_2(\theta_{r2}^{(3)}) &= 2b_3 d_3 K_3^{-1} N_3 \quad \text{for } K_3 > 0. \end{aligned} \tag{2.5.15}$$

According to (2.4.16), (2.5.1) and (2.5.15) we arrive at the conclusion:

- (1) If $N_3 > 0$ and (a) $A_0 - c_0 \leq 0$ or (b) $B_0 - c_0 \leq 0$, then the velocity $V_1(\theta)$ in the interval $(0, \theta_{r1}^{(3)})$ in case (a) decreases from $\sqrt{b_3}$ to $\sqrt{a_3}$, and in case (b) it increases from $\sqrt{b_3}$ to $\sqrt{a_3}$; and if $A_0 - c_0 > 0$, $B_0 - c_0 > 0$, then the velocity $V_1(\theta)$ at the point $\theta_1^{(3)} \in (0, \theta_{r1}^{(3)})$ has a minimum:

$$\begin{aligned} V_1(\theta_1^{(3)}) &= \sqrt{\frac{a_3 b_3 - (c_3 + d_3)^2}{A_0 + B_0 - 2c_0}}, \\ \alpha_1(\theta_1^{(3)}) &= \text{arctg} \left(\frac{B_0 - c_0}{A_0 - c_0} \right)^{\frac{1}{2}} \end{aligned} \tag{2.5.16}$$

- (2) If $N_3 > 0$, then $V_2(\theta)$ at the point $\theta_2^{(3)} \in (0, \theta_{r2}^{(3)})$ has a maximum:

$$\begin{aligned} V_2(\theta_2^{(3)}) &= \sqrt{\frac{a_3 b_3 - (c_3 - d_3)^2}{A_0 + B_0 - 2c_0}}, \\ \alpha_2(\theta_2^{(3)}) &= \text{arctg} \left(\frac{B_3 + c_3}{A_3 + c_3} \right)^{\frac{1}{2}} \end{aligned} \tag{2.5.17}$$

- (3) If $N_3 = 0$ and (a) $b_0 < a_0$ or (b) $b_0 > a_0$, then the velocity $V_1(\theta)$ in the interval $(0, \theta_{r1}^{(3)})$ in case (a) decreases, and in case (b) increases, and $V_1(0) = \sqrt{b_3}$,

- $V_1(\theta_{r1}^{(3)}) = \sqrt{a_3}$; and if $b_0 = a_0$, then $V_1(\theta)$ is constant and is equal to $\sqrt{a_3}$: $V_1(\theta) \equiv \sqrt{a_3}$.
- (4) If $N_3 = 0$, then $V_2(\theta)$ in the interval $(0, \theta_{r2}^{(3)})$ is constant and equals to $\sqrt{d_0}$.
- (5) If $N_3 < 0$, $K_1 \geq 0$ and (a) $B_0 - c_0 \geq 0$ or (b) $A_0 - c_0 \geq 0$, then the velocity $V_1(\theta)$ in the interval $(0, \theta_{r1}^{(3)})$ in case (a) decreases from $\sqrt{b_3}$ to $\sqrt{a_3}$, and in case (b) it increases from $\sqrt{b_3}$ to $\sqrt{a_3}$; and if $A_0 - c_0 < 0$, $B_0 - c_0 < 0$, then the velocity $V_1(\theta)$ at the point $\theta_1^{(3)} \in 0 \leq \theta \leq \theta_{r1}^{(3)}$ has a maximum, and $V_1(\theta_1^{(3)})$, $\alpha_1(\theta_1^{(3)})$ are determined by the Formulas (2.5.16).
- (6) If $N_3 < 0$, $K_3 \geq 0$ then $V_2(\theta)$ at the point $\theta_2^{(3)} \in 0 \leq \theta \leq \theta_{r2}^{(3)}$ always has a minimum, and the Formulas (2.5.17) hold.
- (7) If $K_3 < 0$, (and also $N_3 < 0$), then the functions $V_1(\theta)$ in the interval $(0, \theta_{r1}^{(3)})$ is described as in the case $N_3 < 0$, $K_3 > 0$, and in the interval $(\theta_{r2}^{(3)}, \theta_{q1}^{(3)})$ it decreases from $\sqrt{d_0}$ to $V_1(\theta_1^{(3)})$; $V_2(\theta)$ at the point $\theta_2^{(3)} \in 0 \leq \theta \leq \theta_{q1}^{(3)}$ has a minimum, and $V_2(0) = \sqrt{d_0}$, $V_2(\theta_{q1}^{(1)}) = V_1(\theta_{q1}^{(1)})$.

This analysis in the Fig. 2.2 is drawn. From this the results for an isotropic medium can be obtained [12].

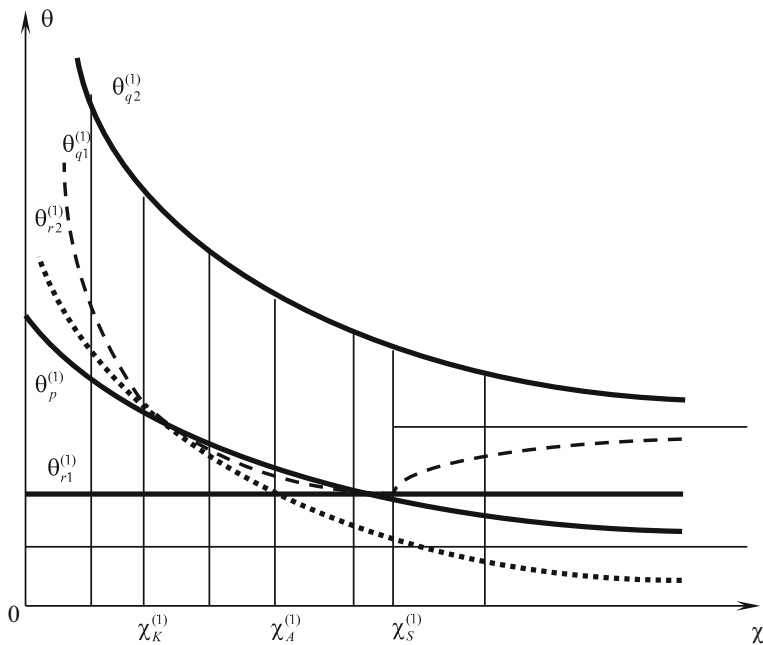


Fig. 2.2 The effect of a transverse magnetic field

2.5.3 The Case of an Isotropic Medium

In the case of a longitudinal magnetic field ($m = 1$), according to (2.3.10), from (2.4.23), (2.4.36), (2.4.28), (2.4.32), (2.4.9)–(2.4.12) and (2.3.40) we have:

$$\begin{aligned} A_1 &= A_0 - \chi, \\ B_1 &= B_0 + \chi, \\ N_1 &= -\chi^2, \\ K_1 &= -\chi^2 - d_0\chi + d_0A_0. \end{aligned} \quad (2.5.18)$$

It follows that for any isotropic medium, in the presence of a magnetic field of arbitrary intensity, the following relationships are valid:

$$B_1 > 0, \quad I_1 > 0, \quad N_1 > 0, \quad D_1 > 0 \quad \text{for} \quad 0 < \chi < +\infty \quad (2.5.19)$$

$$\begin{aligned} A_1 > 0, K_1 > 0, S_1 > 0 & \quad \text{for} \quad 0 < \chi < \chi_k^{(1)} \\ A_1 \geq 0, K_1 < 0, S_1 > 0, & \quad \text{for} \quad \chi_k^{(1)} < \chi \leq \chi_A^{(1)} \\ A_1 < 0, K_1 < 0, S_1 \geq 0 & \quad \text{for} \quad \chi_A^{(1)} < \chi \leq \chi_S^{(1)} \\ A_1 < 0, K_1 < 0, S_1 < 0 & \quad \text{for} \quad \chi_S^{(1)} < \chi < \infty \end{aligned} \quad (2.5.20)$$

In the case of isotropic media, all the singular points $\theta_{rk}^{(1)}$, $\theta_{qk}^{(1)}$ and $\theta_p^{(1)}$ ($k = 1, 2$) turn out to be real, and [12]:

$$\begin{aligned} \theta_{r1}^{(1)} &= a_0^{-\frac{1}{2}}, \\ \theta_{r2}^{(1)} &= e_1^{-\frac{1}{2}} = (d_0 + \chi)^{-\frac{1}{2}}, \\ \theta_p^{(1)} &= \left[\frac{a_0 + d_0 + \chi}{(a_0 + d_0)\chi + 2a_0d_0} \right]^{\frac{1}{2}}, \\ \theta_{qk}^{(1)} &= (\chi A_0)^{-1/2} \left[\sqrt{b_1} + (-1)^k \sqrt{d_0} \right]. \end{aligned} \quad (2.5.21)$$

By virtue of (2.5.18), (2.5.19) and (2.4.24), the collocation of the points (2.5.21) is the following [12]:

$$\begin{aligned} (1) \quad & 0 < \theta_{r1}^{(1)} < \theta_p^{(1)} \leq \theta_{r2}^{(1)} \leq \theta_{q1}^{(1)} < \theta_{q2}^{(1)} \quad \text{for} \quad 0 < \chi \leq \chi_k^{(1)}, \\ (2) \quad & 0 < \theta_{r1}^{(1)} \leq \theta_{r2}^{(1)} < \theta_{q1}^{(1)} < \theta_p^{(1)} < \theta_{q2}^{(1)} \quad \text{for} \quad \chi_k^{(1)} < \chi \leq \chi_A^{(1)}, \\ (3) \quad & 0 < \theta_{r2}^{(1)} < \theta_{r1}^{(1)} \leq \theta_{q1}^{(1)} \leq \theta_p^{(1)} < \theta_{q2}^{(1)} \quad \text{for} \quad \chi_A^{(1)} < \chi \leq \chi_S^{(1)}, \\ (4) \quad & 0 < \theta_{r2}^{(1)} < \theta_p^{(1)} < \theta_{r1}^{(1)} < \theta_{q1}^{(1)} < \theta_{q2}^{(1)} \quad \text{for} \quad \chi_S^{(1)} < \chi < +\infty, \end{aligned} \quad (2.5.22)$$

where

$$\begin{aligned}\chi_k^{(1)} &= 0,5 \left(\sqrt{d_0(4a_0 - 3d_0)} - d_0 \right), \\ \chi_A^{(1)} &= A_0, \\ \chi_S^{(1)} &= \frac{A_0 a_0}{d_0}.\end{aligned}\tag{2.5.23}$$

The change of the collocation of these singular points depending on the intensity of the external magnetic field is shown in Fig. 2.3.

In this case, the separation of magnetoelastic waves into fast and slow waves is carried out in the following way according to the general case.

In cases (1) and (4) from (2.5.22) the wave with the velocity $V_1(\theta)$ is fast, and the one with the velocity $V_2(\theta)$ is slow; in cases (2) and (3) the wave with the velocity $V_1(\theta)$ for $0 \leq \theta \leq \theta_{rk}^{(1)}$ [$k = 1$ —in case (2), $k = 2$ —in case (3)]—is fast, the wave with the velocity $V_2(\theta)$ for $0 \leq \theta \leq \theta_{q1}^{(1)}$ and $V_1(\theta)$ for $\theta_{rk}^{(1)} \leq \theta \leq \theta_{q1}^{(1)}$ [$k = 2$ —in case (2), $k = 1$ —in case (3)]—is slow.

In the case of isotropic media, the following estimates are used to investigate the extreme of the functions $V_k(\theta)$ [12]:

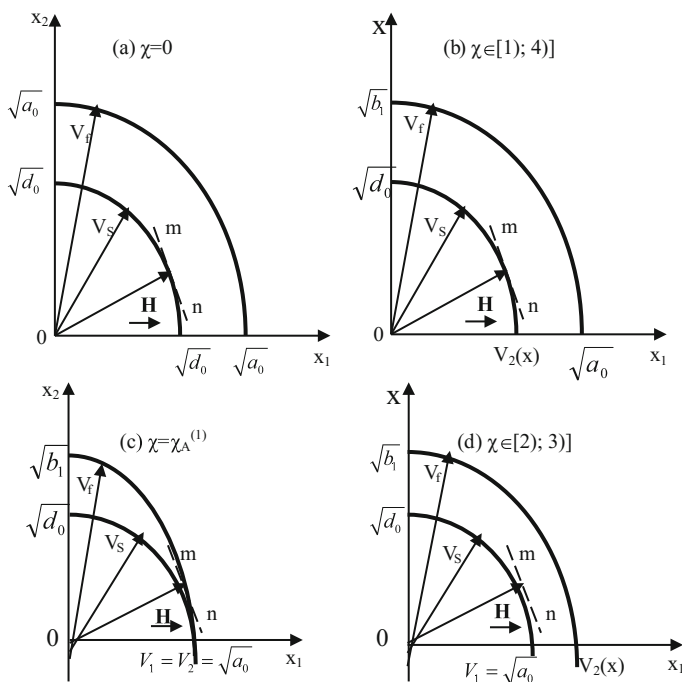


Fig. 2.3 The case of isotropic media

$$\begin{aligned}
N_1(\chi) &= -\chi^2 < 0, \quad B_0B_1 - c_1^2 = A_0\chi > 0, \quad A_0A_1 - c_1^2 = -A_0\chi < 0, \\
B_0A_1 - c_1^2 &= -A_0\chi < 0, \quad A_0B_1 - c_1^2 = A_0\chi > 0, \quad \Psi_1\left(\theta_{q1}^{(1)}\right) = +\infty, \\
\Psi_2\left(\theta_{q1}^{(1)}\right) &= -\infty, \quad V_1\left(\theta_{q1}^{(1)}\right) = V_2\left(\theta_{q1}^{(1)}\right) = \sqrt{4}b_1d_1.
\end{aligned} \tag{2.5.24}$$

Hence, on the basis of the results of 2.5.2 of this section, we arrive at the following conclusion [12].

- (1) If $0 < \chi \leq \chi_k^{(1)}$, then the velocity $V_1(\theta)$ decreases monotonically from $V_1(0) = \sqrt{b_1}$ to $V_1\left(\theta_{r1}^{(1)}\right) = \sqrt{a_0}$, and the velocity $V_2(\theta)$ increases monotonically from $V_2(0) = \sqrt{d_0}$ to $V_2\left(\theta_{r2}^{(1)}\right) = \sqrt{e_1}$, and $V_S(\alpha) = V_2(\theta)$, $V_f(\alpha) = V_1(\theta)$.
- (2) If $\chi_k^{(1)} < \chi \leq \chi_A^{(1)}$, then the velocity $V_1(\theta)$ in the interval $(0, \theta_{r1}^{(1)})$ varies as in case (1), and in the interval $(\theta_{r2}^{(1)}, \theta_{rq}^{(1)})$ it decreases monotonically from $V_1\left(\theta_{r2}^{(1)}\right) = \sqrt{e_1}$ to $V_1\left(\theta_{q1}^{(1)}\right) = \sqrt{4}b_1d_1$, the velocity $V_2(\theta)$ increases monotonically from $V_2(0) = \sqrt{d_0}$ to $V_2\left(\theta_{q1}^{(1)}\right) = V_1\left(\theta_{q1}^{(1)}\right) = \sqrt{4}b_1d_1$, and the speed of the fast wave $V_f(\alpha)$ decreases from $V_f(0) = \sqrt{b_1}$ to $V_f(\pi/2) = \sqrt{a_0}$, while the velocity of the slow wave increases from $V_S(0) = \sqrt{d_0}$ to $V_f(\pi/2) = \sqrt{e_1}$.
- (3) If $\chi = \chi_A^{(1)}$, then a refraction of waves take place, i.e. magnetoelastic waves propagate with the same velocities in the direction of the magnetic field ($\alpha = \pi/2$).
- (4) If $\chi_A^{(1)} < \chi < \chi_S^{(1)}$, then the velocity $V_1(\theta)$ in the interval $(0, \theta_{r2}^{(1)})$ decreases monotonically from $V_1(0) = \sqrt{b_1}$ to $V_1\left(\theta_{r2}^{(1)}\right) = \sqrt{e_1}$, and in the interval $(\theta_{r1}^{(1)}, \theta_{q1}^{(1)})$ —from $\sqrt{a_1}$ to $\sqrt{4}b_1d_1$; the velocity $V_2(\theta)$ increases monotonically from $V_2(0) = \sqrt{d_0}$ to $V_2\left(\theta_{q1}^{(1)}\right) = \sqrt{4}b_1d_1$, and the velocity of the fast wave $V_f(\alpha)$ decreases monotonically from $V_f(0) = \sqrt{b_1}$ to $V_f(\pi/2) = \sqrt{e_1}$, and the velocity of the slow wave $V_S(\alpha)$ increases monotonically from $V_S(0) = \sqrt{d_0}$ to $V_S(\pi/2) = \sqrt{a_0}$.
- (5) If $\chi \geq \chi_S^{(1)}$, then the velocity $V_1(\theta)$ decreases monotonically from $V_1(0) = \sqrt{b_1}$ to $V_1\left(\theta_{r2}^{(1)}\right) = \sqrt{e_1}$, and the velocity $V_2(\theta)$ increases monotonically from $V_2(0) = \sqrt{d_0}$ to $V_2\left(\theta_{r1}^{(1)}\right) = \sqrt{a_0}$.

Figure 2.3 [12] shows the curves of phase velocity changes (phase polars) of fast and slow magnetoelastic waves in an isotropic medium for various characteristic values of the magnetic field. Because of the symmetry, only one quadrant is shown

(the line mn perpendicular to the radius-vector of the phase velocity gives the position of the plane discontinuity in a unit of time after passing through the origin).

In conclusion, to quantify the effect of the magnetic field, let us consider a numerical example. For the calculation we take: $C_{11} = 0.414 \times 10^{11}$ dyne/cm², $C_{66} = 0.264 \times 10^{11}$ dyne/cm², $C_{12} = 0.33 \times 10^{11}$ dyne/cm², $\rho_0 = 0.87$ g/cm³ (a perfectly conductive medium of the potassium type). Then, according to (2.4.23) and (2.4.30), we find: $\chi_A^{(1)} = 1.73 \times 10^{10}$ cm/s², $\chi_S^{(1)} = 17.1 \times 10^{10}$ cm/s², to which there correspond magnetic fields with the intensities $H_A = 4.34 \times 10^5$ Oe, $H_S = 1.36 \times 10^5$ Oe. For these values of the magnetic field we have the following value for the phase velocity $V_2(\chi)$ of transverse waves propagating along the magnetic field: $V_2(\chi_A^{(1)}) = 2.18 \times 10^5$ cm/s (the refraction case), $V_2(\chi_S^{(1)}) = 4.49 \times 10^5$ cm/s (fast wave velocity). For comparison, let us give the value V_2 in the absence of a magnetic field: $V_2(0) = 1.74 \times 10^5$ cm/s.

From this it is clear that to the critical values $\chi_A^{(1)}$ and $\chi_S^{(1)}$ of the parameter χ there correspond sufficiently strong magnetic fields, for which the phase velocity of the wave, as expected, increases substantially (at $\chi = \chi_A^{(1)}$ —1.6 times, at $\chi = \chi_S^{(1)}$ —2.6 times) [12].

References

1. Sedov, L.I.: The mechanics of a continuous medium, vol.1,2, 492 p. Nauka, Moscow (1976)
2. Ambartsumyan, S.A., Baghdasaryan, G.Y.: Electrically conductive plates and shells in a magnetic field. M: Publishing House "Physics and Mathematics", 288p (1996)
3. Baghdasaryan, G.Y.: Vibrations and stability of magnetoelastic systems, 440p. Publishing House. YSU, Yerevan (1999)
4. Guz, A.N., Mahort, F.G. Mechanics of connected fields in construction elements, vol. 3. Acousto-Electro-Magneto-Elasticity. Naukova dumka, Kiev, 288p (1988)
5. Novozhilov, V.V.: Fundamentals of the nonlinear theory of elasticity, 212p. Gostekhizdat, Moscow (1948)
6. Das, N.C., Bhattacharya, S.K.: Love waves in elastic media in presence of magnetic field. Geophys. Res. Bull. **16**(2), 105–110 (1978)
7. Lehnitsky, S.G.: Theory of Elasticity of an Anisotropic Body. Nauka, Moscow, 416 pp (1977)
8. Novozhilov, V.V.: Theory of Elasticity, 370p. Sudpromgiz, Leningrad (1958)
9. Sirotnin, Y.I., Shaskolskaya, M.P.: Fundamentals of Crystallophysics. Nauka, Moscow, 639p (1979)
10. Baghdasaryan, G.Y., Danoyan, Z.N.: Propagation of monochromatic magnetoelastic waves in ideally conducting media. In: Investigations on the Mechanics of a Solid Deformed Body. Ed. AN Arm.SSR, no. 2, pp. 42–50 (1983)
11. Baghdasaryan, G.Y., Danoyan, Z.N.: Equations of motion in displacements of ideally conducting elastic anisotropic media in the presence of a magnetic field. Mech., Interuniversity. Sat. Sci. Works, Yerevan **3**, 32–42 (1984)
12. Danoyan, Z.N.: To the plane problem of propagation of magnetoelastic waves in ideally conducting isotropic media. Izv. AN Arm.SSR, Mechanics, T. **25**(5):27–46 (1974)

13. Danoyan, Z.N.: On the propagation of magnetoelastic waves in ideally conducting isotropic media with cubic symmetry. In: *Studies in Mech. Tv. Def. The bodies*. Yerevan: Ed. AN Arm. SSR, pp. 104–109 (1981)
14. Danoyan, Z.N.: Propagation of plane magnetoelastic waves in anisotropic ideally conducting media. *Izv. NAS of Armenia, Mech.* **56**(3), 37–48 (2003)
15. Kochin, N.E.: *Vector calculus and the origin of the tensor calculus*. Izd. AN SSSR, Moscow, 426 p (1961)
16. Dielesan, E., Royer, D.: *Elastic waves in solids*. Nauka, Moscow, 424p (1982)
17. Fedorov, F.I.: *Theory of elastic waves in crystals*. Nauka, Moscow, 386 pp (1965)
18. Grinchenko, V.T., Meleshko, V.V.: *Harmonic oscillations and waves in elastic media*. Naukova Dumka, Kiev, 283 p (1981)
19. Baghdasaryan, G.Y., Danoyan, Z.N.: Surface magnetoelastic Rayleigh waves. *Mechanics, Rep. Interuniversity. Sat*, Yerevan, №2, pp. 31–37 (1982)
20. Beetroot, V.A.: Elastic vibrations of an anisotropic body. *Uch. App. Leningrad State University*, issue 17, pp. 28–71 (1949)
21. Beetroot, V.A.: To the solution of dynamic problems of the plane theory of elasticity for an anisotropic body. *PMM* **25**(5), 885–896 (1961)
22. Budaev, V.S.: Elastic waves in crystals and anisotropic media. *PMTF* (6), 143–153 (1974)
23. Osipov, I.O.: Reflection and refraction of plane elastic waves on the boundary of two anisotropic media. *Izv. ANSSSR, ser geofiz* (5), 649–665 (1961)
24. Sveklo, B.A.: On the solution of dynamic problems of plane theory of elasticity for anisotropic body. *AMM.* **25**(5), 885–896 (1961)
25. Kulikovskiy, A.G., Lyubimov, G.A.: *Magnetic hydrodynamics*. Physmatgiz, Moscow, 246p (1962)
26. Selezov, I.T., Selezova, L.V. *Waves in magnetohydroelastic media*. Nauk Dumka, Kiev, 164p (1975)
27. Osipov, I.O. To the method of functionally invariant solutions for problems of the dynamic theory of elasticity of anisotropic media. *Izv. AN SSSR, sero geophys.* (3), 391–396 (1963)
28. Osipov, I.O.: To the method of complex solutions of dynamical problems of the plane theory of elasticity of anisotropic media. *Izv. RAS MTT.* (4), 102–112 (1999)
29. Viktorov, I.A.: *Sound surface waves in solids*. Nauka, Moscow, 286 p (1981)
30. Danoyan, Z.N.: To the method of functionally invariant solutions for the problem of magnetoelasticity of ideally conducting anisotropic media. *Mechanics, Uche. Notes by YSU*, №1, pp. 52–61 (1984)
31. Danoyan, Z.N.: Investigation of the roots of the characteristic equation of plane magnetoelastic waves for ideally conducting anisotropic media. *Math. Methods Physicomechanical Fields* **46**(3), 116–120 (2003)
32. Danoyan, Z.N.: Plane magnetoelastic waves in an anisotropic perfectly conductive medium. *Math. Methods Physical-mechanical Fields* **46**(3), 121–135 (2003)
33. Viktorov, I.A.: *Physical basis for the application of ultrasonic waves of Rayleigh and Lamb in engineering*. Nauka, Moscow, p. 168 (1966)

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Baghdasaryan, G.; Danoyan, Z.

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