

Chapter 2

Methodology

2.1 Framework of the Stability Analysis

In Fig. 2.1, the numerical methods used in this thesis are outlined. Starting from the Navier-Stokes equations, the base flow is solved with boundary layer assumption. When perturbations are introduced into the flow, the stability equations can be obtained by subtracting the N-S equations of the laminar base flow. By parabolizing the stability equations, we make use of the high efficiency of the parabolized stability equations (PSE). The objective function and constraint are then made by introducing the adjoint equations. As a result, the optimal disturbances can be recovered. On the other hand, after linearizing the stability equations (LST), the equations can be solved locally as an eigenvalue problem or a singular value problem.

The study follows the procedure:

1. Solve the laminar base flow;
2. Perform the local modal and non-modal stability analysis;
3. Iterate the (linear) PSE and its adjoint equation to recover the optimal perturbations;
4. Calculate the nonlinear development and interactions of perturbations with the initial profiles from step 2 and/or 3;
5. Secondary instability analysis.

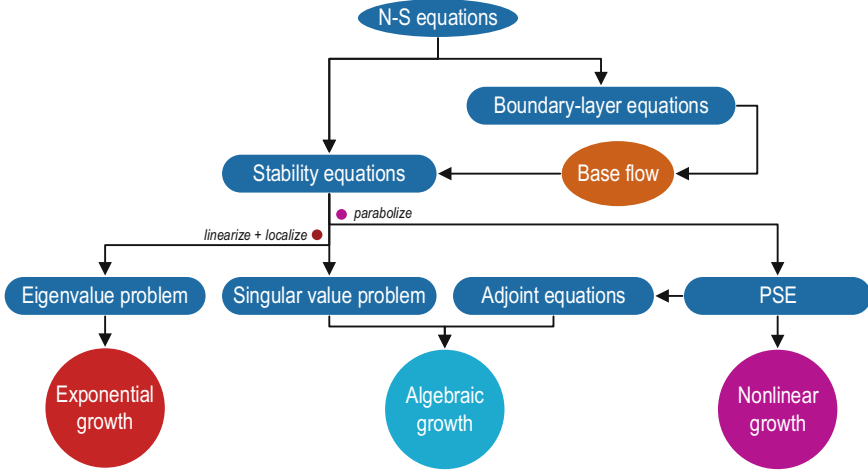


Fig. 2.1 Framework of the stability analysis employed in this book

2.2 Governing Equations and the Base Flow

2.2.1 Governing Equations

We start by writing the dimensional N-S equations in vector form:

$$\left. \begin{aligned} \frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot (\rho^* \mathbf{V}^*) &= 0 \\ \rho^* \left(\frac{\partial \mathbf{V}^*}{\partial t^*} + (\mathbf{V}^* \cdot \nabla^*) \mathbf{V}^* \right) &= -\nabla^* p^* + \nabla^* (\lambda^* (\nabla^* \cdot \mathbf{V}^*)) \\ &\quad + \nabla^* \cdot (\mu^* (\nabla^* \mathbf{V}^* + \nabla^* \mathbf{V}^{*T})) \\ \rho^* C_p^* \left(\frac{\partial T^*}{\partial t^*} + (\mathbf{V}^* \cdot \nabla^*) T^* \right) &= \nabla^* \cdot (\kappa^* \nabla^* T^*) \\ &\quad + \frac{\partial p^*}{\partial t^*} + (\mathbf{V}^* \cdot \nabla^*) p^* + \Phi^* \end{aligned} \right\}. \quad (2.1)$$

The dissipation function in (2.1) is

$$\Phi^* = \lambda^* (\nabla^* \cdot \mathbf{V}^*)^2 + \frac{\mu^*}{2} (\nabla^* \mathbf{V}^* + \nabla^* \mathbf{V}^{*T})^2, \quad (2.2)$$

$\mathbf{V}^* = (u^*, v^*, w^*)^T$ is the flow velocity. The coordinates employed in this thesis in shown in Fig. 1.5. x , y and z denote streamwise, normal-to-wall and spanwise directions respectively. The curvature of the wall is present in the streamwise direction.

The N-S equations are then scaled to remove the dimensions. The velocity is divided by U_∞^* ; thermodynamical quantities ρ^* , T^* , μ^* , λ^* and κ^* by their freestream values; pressure p^* by $\rho_\infty^* U_\infty^{*2}$; length and curvature by boundary layer thickness scale $\delta_0^* = \sqrt{\nu_\infty^* x_0^* / U_\infty^*}$ at x_0^* and time t^* by δ_0^* / U_∞^* , i.e.,

$$\left. \begin{aligned} u &= \frac{u^*}{U_\infty^*}, \quad v = \frac{v^*}{U_\infty^*}, \quad w = \frac{w^*}{U_\infty^*} \\ \rho &= \frac{\rho^*}{\rho_\infty^*}, \quad T = \frac{T^*}{T_\infty^*}, \quad \mu = \frac{\mu^*}{\mu_\infty^*}, \quad \lambda = \frac{\lambda^*}{\lambda_\infty^*}, \quad \kappa = \frac{\kappa^*}{\kappa_\infty^*} \\ x &= \frac{x^*}{\delta_0^*}, \quad y = \frac{y^*}{\delta_0^*}, \quad z = \frac{z^*}{\delta_0^*}, \quad k = k^* \delta_0^* \\ p &= \frac{p^*}{\rho_\infty^* (U_\infty^*)^2}, \quad t = \frac{t^* U_\infty^*}{\delta_0^*} \end{aligned} \right\} \quad (2.3)$$

As a result, the dimensionless form of the N-S equations are characterized with:

$$Re_0 = \frac{\rho_\infty^* U_\infty^* \delta_0^*}{\mu_\infty^*}, \quad Ma = \frac{U_\infty^*}{\sqrt{\gamma R_{\text{gas}}^* T_\infty^*}}, \quad Pr = \frac{\mu_\infty^* C_p^*}{\kappa_\infty^*}. \quad (2.4)$$

The N-S equations consists of 5 equations and 11 unknowns: ρ , u , v , w , T , p , μ , λ , κ , R_{gas} and C_p . Six complementary equations are prescribed for the closure of the system:

Equation of state (EoS) for perfect gas

$$\left. \begin{aligned} C_p^* &= \text{CONST} \\ R_{\text{gas}}^* &= \text{CONST} \\ p^* &= \rho^* R_{\text{gas}}^* T^* \Leftrightarrow p = \frac{\rho T}{\gamma Ma^2} \end{aligned} \right\} \quad (2.5)$$

Constant Pr

$$Pr = \frac{C_p^* \mu^*}{\kappa^*} = \text{CONST} \Leftrightarrow \frac{\mu}{\mu_\infty} = \frac{\kappa}{\kappa_\infty} \quad (2.6)$$

Sutherland's law for viscosity

$$\left. \begin{aligned} \mu^* &= \mu_s^* \frac{T_s^* T_s^* + S^*}{T_s^* T_s^* + S^*} \Leftrightarrow \mu = \mu_s \frac{T T_s + S}{T_s T_s + S} \\ T_s^* &= 273 K \\ \mu_s^* &= 1.71 \times 10^{-5} \text{ kg/(m} \cdot \text{s)} \\ S^* &= 110.4 K \end{aligned} \right\} \quad (2.7)$$

Stoke's hypothesis

$$\lambda^* + 2/3\mu^* = 0 \Leftrightarrow \lambda = -2/3\mu. \quad (2.8)$$

2.2.2 The Base Flow

For steady two-dimensional flows, the governing boundary layer equations can be derived from (2.1)[1]

$$\left. \begin{aligned} \frac{\partial(\rho^* u^*)}{\partial x^*} + \frac{\partial(\rho^* v^*)}{\partial y^*} &= 0 \\ \rho^* \left(u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \right) &= -\frac{dp^*}{dx^*} + \frac{\partial}{\partial y^*} \left(\mu^* \frac{\partial u^*}{\partial y^*} \right) \\ \rho^* \left(u^* \frac{\partial H^*}{\partial x^*} + v^* \frac{\partial H^*}{\partial y^*} \right) &= \frac{\partial}{\partial y^*} \left(\frac{\mu^*}{Pr} \frac{\partial H^*}{\partial y^*} \right) + \frac{\partial}{\partial y^*} \left[\left(1 - \frac{1}{Pr} \right) \mu^* u^* \frac{\partial u^*}{\partial y^*} \right] \end{aligned} \right\} \quad (2.9)$$

where H^* is the total enthalpy:

$$H^* = h^* + \frac{1}{2}u^*u^* = C_p^*T^* + \frac{1}{2}u^*u^* = \frac{\gamma}{\gamma-1}R_g^*T^* + \frac{1}{2}u^*u^* \quad (2.10)$$

Introduce the Levy-Lees transformation:

$$\left. \begin{aligned} d\xi &= \rho_e^* \mu_e^* u_e^* dx^* \\ d\eta &= \frac{\rho_e^* u_e^*}{\sqrt{2\xi}} dy^* \end{aligned} \right\} \quad (2.11)$$

Therefore, for perfect gas

$$\frac{\partial \eta}{\partial y^*} = \frac{\rho_e^* u_e^*}{\sqrt{2\xi}} = \frac{\rho_e^*}{\rho_e^*} \frac{\rho_e^* u_e^*}{\sqrt{2\xi}} \propto \rho = \frac{1}{T} \quad (2.12)$$

Define the stream function ψ

$$\rho^* u^* = \frac{\partial \psi^*}{\partial y^*}, \quad \rho^* v^* = -\frac{\partial \psi^*}{\partial x^*} \quad (2.13)$$

A dimensionless stream function f is related to ψ^* as:

$$\psi^*(x, y) = \sqrt{2\xi} f(\xi, \eta) \quad (2.14)$$

The dimensionless function g is defined as

$$g(\xi, \eta) = \frac{H^*}{H_\infty^*} \quad (2.15)$$

Substituting (2.11), (2.13), (2.14) and (2.15) into the boundary layer Eq. (2.9), yields

$$\left. \begin{aligned} 2\xi \left(f' \frac{\partial f'}{\partial \xi} - \frac{\partial f}{\partial \xi} f'' \right) &= \beta_p [(k+1)(g - f'^2)] + (Cf'')' + ff'' \\ 2\xi \left(f' \frac{\partial g}{\partial \xi} - \frac{\partial f}{\partial \xi} g' \right) &= (a_1 g')' + (a_2 f' f'')' + fg' \end{aligned} \right\} \quad (2.16)$$

where

$$\left. \begin{aligned} \beta_p &= \frac{2\xi}{u_e^*} \frac{du_e^*}{d\xi} \\ C &= \frac{\rho^* \mu^*}{\rho_e^* \mu_e^*} = \frac{\mu}{T} \\ k_M &= \frac{\gamma - 1}{2} Ma^2 \\ a_1 &= \frac{C}{Pr} \\ a_2 &= C \left(1 - \frac{1}{Pr} \right) \frac{2k_M}{1 + k_M} \end{aligned} \right\} \quad (2.17)$$

The flow temperature can be recovered from

$$\frac{T^*}{T_e^*} = (k_M + 1)g - k_M f'^2. \quad (2.18)$$

The boundary conditions are specified

$$\left. \begin{aligned} u^* &= v^* = 0 \\ \frac{\partial H^*}{\partial y^*} &= 0 \text{ (adiabatic)} \\ H^* &= H_{\text{wall}}^* \text{ (isothermal)} \end{aligned} \right\} \text{ at } y^* = 0 \quad (2.19)$$

$$\left. \begin{aligned} u^* &= U_\infty^* \\ H^* &= H_\infty^* \end{aligned} \right\} \text{ at } y^* = \infty \quad (2.20)$$

Accordingly, (2.16) is subject to

$$\left. \begin{aligned} f &= f' = 0, \quad g' = 0 \text{ (or } g = g_{\text{wall}}) \text{ at } \eta = 0 \\ f' &= 1, \quad g = 1 \text{ at } \eta = \infty \end{aligned} \right\} \quad (2.21)$$

The resulting boundary value problem of (2.16) provides the similarity solution for compressible boundary layers which serves as the base flow. One may notice the

streamwise curvature is omitted in the base flow. This is a high-order influence on Görtler instability [2].

2.3 Stability Equations and Numerical Methods

The instantaneous flow field $\mathbf{q} = (\rho, u, v, w, T)$ consists of the laminar base flow $\mathbf{q}_0$ plus the perturbation $\tilde{\mathbf{q}}$:

$$\mathbf{q}(x, y, z, t) = \mathbf{q}_0(x, y) + \tilde{\mathbf{q}}(x, y, z, t) \quad (2.22)$$

The perturbed flow $\mathbf{q}$ and the base flow $\mathbf{q}_0$ both satisfy the N-S equations. We substitute (2.22) into the N-S Eq. (2.1) and subtract equation of the base flow. The stability equation is then derived:

$$\begin{aligned} \mathbf{\Gamma} \frac{\partial \tilde{\mathbf{q}}}{\partial t} + \mathbf{A} \frac{\partial \tilde{\mathbf{q}}}{\partial x} + \mathbf{B} \frac{\partial \tilde{\mathbf{q}}}{\partial y} + \mathbf{C} \frac{\partial \tilde{\mathbf{q}}}{\partial z} + \mathbf{D} \tilde{\mathbf{q}} = & \mathbf{V}_{xx} \frac{\partial^2 \tilde{\mathbf{q}}}{\partial x^2} + \mathbf{V}_{yy} \frac{\partial^2 \tilde{\mathbf{q}}}{\partial y^2} \\ & + \mathbf{V}_{zz} \frac{\partial^2 \tilde{\mathbf{q}}}{\partial z^2} + \mathbf{V}_{xy} \frac{\partial^2 \tilde{\mathbf{q}}}{\partial x \partial y} + \mathbf{V}_{yz} \frac{\partial^2 \tilde{\mathbf{q}}}{\partial y \partial z} + \mathbf{V}_{zx} \frac{\partial^2 \tilde{\mathbf{q}}}{\partial z \partial x} + \tilde{\mathbf{N}} \end{aligned} \quad (2.23)$$

where the 5×5 matrices $\mathbf{\Gamma}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{V}_{xx}$, $\mathbf{V}_{yy}$, $\mathbf{V}_{zz}$, $\mathbf{V}_{xy}$, $\mathbf{V}_{xz}$, $\mathbf{V}_{yz}$ are functions of the base flow, curvature and the dimensionless parameters Re , Ma , Pr . Detailed expressions can be found in one of the author's journal articles [3]. The vector $\tilde{\mathbf{N}}$ indicates nonlinear terms.

2.3.1 Modal Stability: The Eigenvalue Problem

Assume wavelike solutions of the form:

$$\tilde{\mathbf{q}}(x, y, z, t) = \hat{\mathbf{q}}(y) \exp(i(\alpha x + \beta z - \omega t)) + c.c. \quad (2.24)$$

Substitute into (2.23) and ignore the nonlinear terms. In this work, β and ω are prescribed (spatial problem) and α is to be solved.

$$\mathbf{A} \hat{\mathbf{q}} + \mathbf{B} \frac{\partial \hat{\mathbf{q}}}{\partial y} + \mathbf{C} \frac{\partial^2 \hat{\mathbf{q}}}{\partial y^2} = \alpha \left(\mathbf{M} \hat{\mathbf{q}} + \mathbf{N} \frac{\partial \hat{\mathbf{q}}}{\partial y} \right) + \alpha^2 \mathbf{P} \hat{\mathbf{q}} \quad (2.25)$$

where

$$\left. \begin{aligned} \mathbf{A} &= -i\omega \mathbf{\Gamma} + i\beta \mathbf{C} + \mathbf{D} + \beta^2 \mathbf{V}_{zz} \\ \mathbf{B} &= \mathbf{B} - i\beta \mathbf{V}_{yz} \\ \mathbf{C} &= -\mathbf{V}_{yy} \\ \mathbf{M} &= -i\mathbf{A} - \beta \mathbf{V}_{xz} \\ \mathbf{N} &= i\mathbf{V}_{xy} \\ \mathbf{P} &= -\mathbf{V}_{xz} \end{aligned} \right\} \quad (2.26)$$

(2.25) is the eigenvalue equation to be solved. We make use of the 4th-order central differential scheme:

$$\left. \begin{aligned} \frac{\partial \hat{q}_j}{\partial y} &= \frac{\hat{q}_{j-2} - 8\hat{q}_{j-1} + 8\hat{q}_{j+1} - \hat{q}_{j+2}}{12\Delta y} \\ \frac{\partial^2 \hat{q}_j}{\partial y^2} &= \frac{-\hat{q}_{j-2} + 16\hat{q}_{j-1} - 30\hat{q}_j + 16\hat{q}_{j+1} - \hat{q}_{j+2}}{12(\Delta y)^2} \end{aligned} \right\} \quad (2.27)$$

We defined the discretized vector $\hat{\mathbf{Q}} = (\hat{q}_1, \hat{q}_2, \dots, \hat{q}_j, \dots, \hat{q}_N)$. The finite-difference procedure therefore can be written in matrix form:

$$\left. \begin{aligned} \mathbf{F}_y \hat{\mathbf{Q}} &= \left(\frac{\partial \hat{q}_1}{\partial y}, \frac{\partial \hat{q}_2}{\partial y}, \dots, \frac{\partial \hat{q}_j}{\partial y}, \dots, \frac{\partial \hat{q}_N}{\partial y} \right) \\ \mathbf{F}_{yy} \hat{\mathbf{Q}} &= \left(\frac{\partial^2 \hat{q}_1}{\partial y^2}, \frac{\partial^2 \hat{q}_2}{\partial y^2}, \dots, \frac{\partial^2 \hat{q}_j}{\partial y^2}, \dots, \frac{\partial^2 \hat{q}_N}{\partial y^2} \right) \end{aligned} \right\} \quad (2.28)$$

(2.25) becomes

$$(\mathbf{A}' + \mathbf{B}' \mathbf{F}_y + \mathbf{C}' \mathbf{F}_{yy}) \hat{\mathbf{Q}} = \alpha (\mathbf{M}' + \mathbf{N}' \mathbf{F}_y) \hat{\mathbf{Q}} + \alpha^2 \mathbf{P}' \hat{\mathbf{Q}} \quad (2.29)$$

where matrices of the size $5N \times 5N$ $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{C}'$, $\mathbf{M}'$, $\mathbf{N}'$, $\mathbf{P}'$ are the discretizations of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{M}$, $\mathbf{N}$, $\mathbf{P}$. (2.29) is a nonlinear eigenvalue equation and can be linearized through:

$$\left(\begin{matrix} \mathbf{O} \\ \mathbf{A}' + \mathbf{B}' \mathbf{F}_y + \mathbf{C}' \mathbf{F}_{yy} - \mathbf{M}' - \mathbf{N}' \mathbf{F}_y \end{matrix} \right) \begin{pmatrix} \hat{\mathbf{Q}} \\ \alpha \hat{\mathbf{Q}} \end{pmatrix} = \alpha \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{P}' \end{pmatrix} \begin{pmatrix} \hat{\mathbf{Q}} \\ \alpha \hat{\mathbf{Q}} \end{pmatrix} \quad (2.30)$$

The boundary conditions for (2.30) is

$$\begin{cases} \hat{u} = \hat{v} = \hat{w} = \hat{T} = 0 & \text{at } y = 0 \\ \hat{u} = \hat{v} = \hat{w} = \hat{T} = 0 & \text{at } y = \infty \end{cases} \quad (2.31)$$

Numerically solving (2.30), one obtains the eigenvector $\hat{\mathbf{Q}}$ (the perturbation) and the eigenvalue α , where $-\alpha_i$ is the growth rate and α_r is the streamwise wavenumber.

2.3.2 Algebraic Stability: The Singular Value Problem

In algebraic stability analysis, no assumption is made on the perturbation along x :

$$\tilde{q}(x, y, z, t) = \hat{q}(x, y) \exp(i(\beta z - \omega t)) + c.c. \quad (2.32)$$

Substitute into (2.23), ignore the nonlinear terms, in an operator form, we have

$$\frac{d\hat{q}}{dx} = \mathcal{L}\hat{q} \quad (2.33)$$

The solution is

$$\hat{q}(x) = \hat{q}(0)e^{x\mathcal{L}} \quad (2.34)$$

Define the inner product of any two vectors:

$$\langle \hat{p}, \hat{q} \rangle = \int_0^\infty \hat{p}^H \hat{q} dy \quad (2.35)$$

An energy norm is defined on the inner product space:

$$\|\hat{q}\|_E = \langle \hat{q}, \mathbf{M}\hat{q} \rangle \quad (2.36)$$

where

$$\mathbf{M} = \text{diag} \left(\frac{\hat{T}}{\bar{\rho}\gamma Ma^2}, \bar{\rho}, \bar{\rho}, \bar{\rho}, \frac{\bar{\rho}}{\gamma(\gamma-1)\hat{T}Ma^2} \right) \quad (2.37)$$

is a diagonal matrix (positive definite), take Cholesky decomposition:

$$\mathbf{M} = \mathbf{F}^H \mathbf{F} \quad (2.38)$$

The energy norm (2.36) [4] measures the perturbation energy at a specified stream-wise location and, can be related to the 2-norm through

$$\begin{aligned} \|\hat{q}\|_E &= \langle \hat{q}, \mathbf{M}\hat{q} \rangle = \int_0^\infty \hat{q}^H \mathbf{M}\hat{q} dy \\ &= \int_0^\infty \hat{q}^H \mathbf{F}^H \mathbf{F} \hat{q} dy = \langle \mathbf{F}\hat{q}, \mathbf{F}\hat{q} \rangle = \|\mathbf{F}\hat{q}\|_2^2 \end{aligned} \quad (2.39)$$

(2.36) can be further extended to the energy norm of matrices:

$$\|\mathbf{A}\|_E = \max \frac{\|\mathbf{A}\hat{q}\|_E}{\|\hat{q}\|_E} = \max \frac{\|\mathbf{F}\mathbf{A}\mathbf{F}^{-1}\mathbf{F}\hat{q}\|_2^2}{\|\mathbf{F}\hat{q}\|_2^2} = \|\mathbf{F}\mathbf{A}\mathbf{F}^{-1}\|_2^2 \quad (2.40)$$

The eigenvalue and eigenvector solved in Sect. 2.3.1 give the solution for $x \rightarrow \infty$ which is not necessarily consistent with the flow physics. Take Couette flow as an example, its growth rate is negative (stable) at any given Re . However, flow instability is still observed in experiments. The reason is that the matrix of the eigenvalue problem is usually non-regular. Thus its eigenvectors are not orthogonal to each other. The perturbation may become unstable after linear combinations:

$$\check{\mathbf{q}}(x) = \sum_{k=1}^K \phi_k(x) \hat{\mathbf{q}}_k \quad (2.41)$$

where $\phi_k(x) = \phi_k(0)e^{i\alpha_k x}$. In a finite streamwise domain from 0 to x , we measure the maximum possible amplification factor of the perturbation energy G .

$$\begin{aligned} G(\beta, \omega, Re, x) &= \max \frac{\|\check{\mathbf{q}}(x)\|_E}{\|\check{\mathbf{q}}(0)\|_E} = \max \frac{\|\check{\mathbf{q}}(0)e^{x\mathcal{L}}\|_E}{\|\check{\mathbf{q}}(0)\|_E} \\ &= \|e^{x\mathcal{L}}\|_E = \|\mathbf{F}\mathbf{A}\mathbf{F}^{-1}\|_2^2 \end{aligned} \quad (2.42)$$

where

$$\mathbf{A} = \text{diag}(e^{i\alpha_1 x}, e^{i\alpha_2 x}, \dots, e^{i\alpha_K x}) \quad (2.43)$$

The matrix $\mathbf{F}$ satisfies $\mathbf{F}^H \mathbf{F} = \mathbf{A}$. The matrix $\mathbf{A}$ is given by the elements $a_{ij} = \langle \hat{\mathbf{q}}_i, \hat{\mathbf{q}}_j \rangle$. The 2-norm of $\mathbf{F}\mathbf{A}\mathbf{F}^{-1}$ therefore can be determined from its maximum singular value σ_1 .

2.3.3 PSE

The limitations for local eigenvalue or singular problem are:

1. The stability equations are solved locally, thus, the base flow must be quasi-parallel. For example, Görtler instability can not be solved locally except when the streamwise coordinate is large where the flow is quasi-parallel (Sect. 1).
2. Nonlinear terms are ignored. Therefore, the amplitude of the perturbation must be small enough.

The parabolized stability equations (PSE) overcome the above and is more efficient than eigenvalue problem. Take Fourier expansions for the perturbation $\tilde{\mathbf{q}}$ and nonlinear terms $\tilde{\mathbf{N}}$:

$$\left. \begin{aligned} \tilde{\mathbf{q}} &= \sum_{m=-M}^M \sum_{n=-N}^N \hat{\mathbf{q}}_{mn}(x, y) \exp \left(i \int \alpha_{mn} dx + in\beta z - im\omega t \right) \\ \tilde{N} &= - \sum_{m=-M}^M \sum_{n=-N}^N \hat{N}_{mn}(x, y) \exp \left(i \int \alpha_{mn} dx + in\beta z - im\omega t \right) \end{aligned} \right\} \quad (2.44)$$

Substitute into (2.23), we have

$$\begin{aligned} \mathbf{A}\hat{\mathbf{q}}_{mn} + \mathbf{B}\frac{\partial \hat{\mathbf{q}}_{mn}}{\partial y} + \mathbf{C}\frac{\partial^2 \hat{\mathbf{q}}_{mn}}{\partial y^2} + \mathbf{D}\frac{\partial \hat{\mathbf{q}}_{mn}}{\partial x} \\ + \mathbf{E}\frac{\partial^2 \hat{\mathbf{q}}_{mn}}{\partial x \partial y} + \mathbf{F}\frac{\partial^2 \hat{\mathbf{q}}_{mn}}{\partial x^2} + \hat{N}_{mn} \exp \left(-i \int \alpha_{mn} dx \right) = 0 \end{aligned} \quad (2.45)$$

where

$$\left. \begin{aligned} \mathbf{A} &= -im\omega \mathbf{\Gamma} + in\beta \mathbf{C} + \mathbf{D} + \beta^2 n^2 \mathbf{V}_{zz} + i\alpha_{mn} \mathbf{A} \\ &\quad + (\alpha_{mn}^2 - i\alpha_{mn,x}) \mathbf{V}_{xx} + n\beta \alpha_{mn} \mathbf{V}_{xz} \\ \mathbf{B} &= \mathbf{B} - in\beta \mathbf{V}_{yz} - i\alpha_{mn} \mathbf{V}_{xy} \\ \mathbf{C} &= -\mathbf{V}_{yy} \\ \mathbf{D} &= \mathbf{A} - in\beta \mathbf{V}_{xz} - 2i\alpha_{mn} \mathbf{V}_{xx} \\ \mathbf{E} &= -\mathbf{V}_{xy} \\ \mathbf{F} &= -\mathbf{V}_{xx} \end{aligned} \right\} \quad (2.46)$$

An axillary equation for α_{mn} (2.47) is required to make the shape function $\hat{\mathbf{q}}_{mn}$ evolves slowly in the streamwise direction. i.e., $\frac{\partial \hat{\mathbf{q}}_{mn}}{\partial x} \sim \mathcal{O}(1/Re)$.

$$\alpha^{\text{new}} = \alpha^{\text{old}} - i \frac{\int_0^\infty \rho (\hat{u}^\dagger \partial \hat{u} / \partial x + \hat{v}^\dagger \partial \hat{v} / \partial x + \hat{w}^\dagger \partial \hat{w} / \partial x) dy}{\int_0^\infty \rho (\hat{u}^\dagger \hat{u} + \hat{v}^\dagger \hat{v} + \hat{w}^\dagger \hat{w}) dy} \quad (2.47)$$

Through the magnitude analysis of (2.45), we have $\mathbf{A} \sim \mathbf{B} \sim \mathbf{D} \sim \mathcal{O}(1)$, $\mathbf{C} \sim \mathbf{E} \sim \mathbf{F} \sim \mathcal{O}(1/Re)$, $\partial \hat{\mathbf{q}}_{mn} / \partial y \sim \mathcal{O}(1)$, $\partial \hat{\mathbf{q}}_{mn} / \partial x \sim \mathcal{O}(1/Re)$. Ignore terms of order $\mathcal{O}(1/Re^2)$ and above, the equation is parabolized:

$$\mathbf{A}\hat{\mathbf{q}}_{mn} + \mathbf{B}\frac{\partial \hat{\mathbf{q}}_{mn}}{\partial y} + \mathbf{C}\frac{\partial^2 \hat{\mathbf{q}}_{mn}}{\partial y^2} + \mathbf{D}\frac{\partial \hat{\mathbf{q}}_{mn}}{\partial x} + \hat{N}_{mn} \exp \left(-i \int \alpha_{mn} dx \right) = 0 \quad (2.48)$$

The fourth order central difference (2.27) and implicit Euler scheme is applied in the y and x directions respectively. The discretized equations become

$$(\mathbf{A}' + \mathbf{B}' F_y + \mathbf{C}' F_{yy}) \hat{\mathbf{Q}}_{mn} + \mathbf{D}' \frac{\partial \hat{\mathbf{Q}}_{mn}}{\partial x} + \hat{N}'_{mn} \exp \left(-i \int \alpha_{mn} dx \right) = 0 \quad (2.49)$$

The nonlinear terms in (2.49) are calculated iteratively with local physical quantities. By marching downstream, the spatial development of the perturbation is solved. Discussions on the residual ellipticity of PSE can be found in [5, 6] and validation is provided by [7–9].

It is important to note, the streamwise wavenumber of Görtler mode and Klebanoff mode $\alpha_r \equiv 0$. Their shape functions evolve slowly in the streamwise physically. As a result, the governing equations are parabolic in nature. No auxiliary condition is applied. (2.48) is simplified as

$$\mathbf{A}\hat{\mathbf{q}}_{mn} + \mathbf{B}\frac{\partial\hat{\mathbf{q}}_{mn}}{\partial y} + \mathbf{C}\frac{\partial^2\hat{\mathbf{q}}_{mn}}{\partial y^2} + \mathbf{D}\frac{\partial\hat{\mathbf{q}}_{mn}}{\partial x} + \hat{\mathbf{N}}_{mn} = 0 \quad (2.50)$$

The detailed expressions of matrices **A**, **B**, **C**, **D** are provided in the Appendix.

2.3.4 Secondary Instability Equations

In the methodology of the linear secondary instability (Herbert 1988; Schmid and Henningson 2001), the stability analysis is performed typically in a y-z cross-section (so-called Bi-global). The disturbances, therefore, are assumed to be inhomogeneous in the wall-normal and spanwise direction but periodic in time and streamwise direction, i.e.,

$$\left. \begin{aligned} \tilde{\mathbf{q}}_{0s}(y, z) &= \sum_{n_1=-\infty}^{\infty} \hat{\mathbf{q}}_{0n_1}(y) e^{in_1\beta z} \\ \tilde{\mathbf{q}}_s(x, y, z, t) &= e^{\gamma_s t} e^{i\omega_s t + i\alpha_s x} \sum_{n_2=-\infty}^{\infty} \hat{\mathbf{q}}_{n_2}(y) e^{in_2\beta z} \\ 0 &\leq \gamma_i \leq \frac{\beta}{2} \end{aligned} \right\} \quad (2.51)$$

where $0 \leq \gamma_i/\beta \leq 0.5$ is the Floquet parameter and decides the type of the secondary instability. $\gamma_i/\beta = 0$ produces the fundamental, $\gamma_i/\beta = 0.5$ the subharmonic and the other values result in detuned types. Substitute (2.51) into the N-S Eq. (2.1), after a similar procedure as in the primary instability (Sect. 2.3.1) the secondary instability equations can be obtained. The temporal problem is solved with α_s prescribed and ω_s is the eigenvalue to be determined.

2.4 Adjoint Equations and Optimal Perturbations

2.4.1 Adjoint Equations

The stability equations described in Sects. 2.3.1 and 2.3.3 can be written in the form of operators after linearization:

$$\mathcal{L}\hat{\mathbf{q}} = 0 \quad (2.52)$$

With the inner product defined in Sect. 2.3.2, we define the adjoint operator $\mathcal{L}^*$ which satisfies

$$\langle \hat{\mathbf{p}}, \mathcal{L}\hat{\mathbf{q}} \rangle = \langle \mathcal{L}^*\hat{\mathbf{p}}, \hat{\mathbf{q}} \rangle + \text{B.C.} \quad (2.53)$$

where $\hat{\mathbf{p}}$ is the adjoint vector to $\hat{\mathbf{q}}$, B.C. denotes the boundary terms after integration. For the equations of PSE,

$$\mathcal{L} = \mathbf{A} + \mathbf{B}\frac{\partial}{\partial y} + \mathbf{C}\frac{\partial^2}{\partial y^2} + \mathbf{D}\frac{\partial}{\partial x} \quad (2.54)$$

Integrating (2.53) by parts yields

$$\mathcal{L}^* = \mathbf{A}^* + \mathbf{B}^*\frac{\partial}{\partial y} + \mathbf{C}^*\frac{\partial^2}{\partial y^2} + \mathbf{D}^*\frac{\partial}{\partial x} \quad (2.55)$$

where

$$\left. \begin{aligned} \mathbf{A}^* &= \mathbf{A}^H - \frac{\partial \mathbf{B}^H}{\partial y} + \frac{\partial^2 \mathbf{C}^H}{\partial y^2} - \frac{\partial \mathbf{D}^H}{\partial x} \\ \mathbf{B}^* &= -\mathbf{B}^H + 2\frac{\partial \mathbf{C}^H}{\partial y} \\ \mathbf{C}^* &= \mathbf{C}^H \\ \mathbf{D}^* &= -\mathbf{D}^H \end{aligned} \right\} \quad (2.56)$$

To remove the boundary terms in (2.53), the boundary condition for $\hat{\mathbf{p}} = (\hat{\rho}^\dagger, \hat{u}^\dagger, \hat{v}^\dagger, \hat{w}^\dagger, \hat{T}^\dagger)$ is specified:

$$\begin{cases} \hat{u}^\dagger = \hat{v}^\dagger = \hat{w}^\dagger = \hat{T}^\dagger = 0 & y = 0 \\ \hat{u}^\dagger = \hat{v}^\dagger = \hat{w}^\dagger = \hat{T}^\dagger = 0 & y = \infty \end{cases} \quad (2.57)$$

The adjoint equations are therefore written:

$$\mathcal{L}^*\hat{\mathbf{p}} = 0 \quad (2.58)$$

The numerical procedure is similar to solving PSE, except for the marching is backward from downstream to upstream.

2.4.2 Optimal Perturbations

The optimal perturbation can be recovered with Lagrange-multiplier. The objective function is the perturbation energy at $x = x_1$ divided by the value at the inlet $x = x_0$:

$$J(\hat{\mathbf{q}}) = \frac{\|\hat{\mathbf{q}}_1\|_E}{\|\hat{\mathbf{q}}_0\|_E} \quad (2.59)$$

We define the functional

$$\mathcal{F}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = J(\hat{\mathbf{q}}) - \langle \hat{\mathbf{p}}, \mathcal{L}\hat{\mathbf{q}} \rangle \quad (2.60)$$

To seek the maximum value of the objective $J(\hat{\mathbf{q}})$, we seek the stagnation point of the functional (2.60). Take variation of (2.60):

$$\delta \mathcal{F} = \langle \nabla_{\hat{\mathbf{p}}} \mathcal{F}, \delta \hat{\mathbf{p}} \rangle + \langle \nabla_{\hat{\mathbf{q}}} \mathcal{F}, \delta \hat{\mathbf{q}} \rangle \quad (2.61)$$

In order the variation is 0, the two parts in (2.61) must both equal to 0. The first part:

$$\langle \nabla_{\hat{\mathbf{p}}} \mathcal{F}, \delta \hat{\mathbf{p}} \rangle = 0 \Leftrightarrow \mathcal{L}\hat{\mathbf{q}} = 0 \quad (2.62)$$

is satisfied by solving the stability equations. With the definition of the adjoint operator (2.53), the second part is

$$\begin{aligned} \langle \nabla_{\hat{\mathbf{q}}} \mathcal{F}, \delta \hat{\mathbf{q}} \rangle = 0 \Leftrightarrow & -\langle \mathcal{L}^* \hat{\mathbf{p}}, \delta \hat{\mathbf{q}} \rangle \\ & + \langle \mathbf{D}^H \hat{\mathbf{p}}_0, \delta \hat{\mathbf{q}}_0 \rangle - \langle \mathbf{D}^H \hat{\mathbf{p}}_1, \delta \hat{\mathbf{q}}_1 \rangle \\ & - \left\langle \frac{\|\hat{\mathbf{q}}_1\|_E}{\|\hat{\mathbf{q}}_0\|_E^2} \mathbf{M} \hat{\mathbf{q}}_0, \delta \hat{\mathbf{q}}_0 \right\rangle \\ & + \left\langle \frac{1}{\|\hat{\mathbf{q}}_0\|_E} \mathbf{M} \hat{\mathbf{q}}_1, \delta \hat{\mathbf{q}}_1 \right\rangle \end{aligned} \quad (2.63)$$

where $\langle \mathcal{L}^* \hat{\mathbf{p}}, \delta \hat{\mathbf{q}} \rangle = 0$ is equivalent to solving the adjoint equations. At $x = x_0$ and $x = x_1$:

$$\left. \begin{aligned} \mathbf{D}^H \hat{\mathbf{p}}_0 - \frac{\|\hat{\mathbf{q}}_1\|_E}{\|\hat{\mathbf{q}}_0\|_E^2} \mathbf{M} \hat{\mathbf{q}}_0 &= 0 \\ -\mathbf{D}^H \hat{\mathbf{p}}_1 + \frac{1}{\|\hat{\mathbf{q}}_0\|_E} \mathbf{M} \hat{\mathbf{q}}_1 &= 0 \end{aligned} \right\} \quad (2.64)$$

Therefore, we have the initial condition for the direct and adjoint equations:

$$\begin{cases} \hat{\mathbf{q}}_0 = c_0 \mathbf{M}^{-1} \mathbf{D}^H \hat{\mathbf{p}}_0 \\ \hat{\mathbf{p}}_1 = c_1 (\mathbf{D}^H)^{-1} \mathbf{M} \hat{\mathbf{q}}_1 \end{cases} \quad (2.65)$$

where c_0 and c_1 are constants. For linear problems, they have no influence on the results. The optimal perturbation is obtained by solving iteratively (2.52), (2.58) and (2.65) :

1. Determine the location of the inlet x_0 and the outlet of the optimal perturbation x_1 ;
 2. Solve the eigenvalue problem to obtain the initial perturbation $\hat{\mathbf{q}}_0$;
 3. March the direct Eq. (2.52) from x_0 to x_1 ;
 4. From (2.65), obtain the initial condition $\hat{\mathbf{p}}_1$ for the adjoint equation at x_1 ;
 5. March the adjoint Eq. (2.58) from x_1 to x_0 ;
 6. From (2.65), obtain the initial condition $\hat{\mathbf{q}}_0$ for the direct equation at x_0 .
- repeat step 3–6, until the objective function (2.59) reaches a convergence. Usually it takes only 3–4 iterations.

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