Chapter 2
Fourier Analysis and Fourier Transform

2.1 Overview

The origins of Fourier analysis in science can be found in Ptolemy's decomposing celestial orbits into cycles and epicycles and Pythagoras' decomposing music into consonances. Its modern history began with the eighteenth century work of Bernoulli, Euler, and Gauss on what later came to be known as Fourier series. J. Fourier in 1822 [Theorie analytique de la Chaleur] was the first to claim that arbitrary periodic functions could be expanded in a trigonometric (later called a Fourier) series, a claim that was eventually shown to be incorrect, although not too far from the truth. It is an amusing historical sidelight that this work won a prize from the French Academy, in spite of serious concerns expressed by the judges (Laplace, Lagrange, and Legendre) regarding Fourier's lack of rigor. Fourier was apparently a better engineer than mathematician. Dirichlet later made rigorous the basic results for Fourier series and gave precise conditions under which they applied. The rigorous theoretical development of general Fourier transforms did not follow until about one hundred years later with the development of the Lebesgue integral.

Fourier analysis is a prototype of beautiful mathematics with many-faceted applications not only in mathematics, but also in science and engineering. Since the work on heat flow of Jean Baptiste Joseph Fourier (March 21, 1768 to May 16, 1830) in the treatise entitled Theorie Analytique de la Chaleur, Fourier series and Fourier transforms have gone from triumph to triumph, permeating mathematics such as partial differential equations, harmonic analysis, representation theory, number theory and geometry. Their societal impact can best be seen from spectroscopy to the effect that atoms, molecules and hence matters can be identified by means of the frequency spectrum of the light that they emit. Equipped with the fast Fourier transform in computations and fulled by recent technological innovations in digital signals and images, Fourier analysis has stood out as one of the very top achievements of mankind comparable with the Calculus of Sir Isaac Newton.

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The Fourier transform is of fundamental importance to image processing. It allows us to perform tasks which would be impossible to perform any other way; its efficiency allows us to perform other tasks more quickly. The Fourier Transform provides, among other things, a powerful alternative to linear spatial filtering; it is more efficient to use the Fourier transform than a spatial filter for a large filter. The Fourier Transform also allows us to isolate and process particular image frequencies, and so perform low-pass and high-pass filtering with a great degree of precision.

2.2 Fourier Series

The concept of frequency and the decomposition of waveforms into elementary "harmonic" functions first arose in the context of music and sound.

2.2.1 Periodic Functions

A periodic function is a function that repeats its value in regular intervals or periods. A function f is said to be periodic with period T $(T \neq 0)$ if $f(x+T) = f(x)$ for all values of x in the domain. The most important examples are the trigonometric functions (i.e. sine or cosine), which repeat values over the intervals of 2π.

The sine function $f(x) = sin(x)$ has the value 0 at the origin and performs exactly one full cycle between the origin and the point $x = 2\pi$. Hence $f(x) = sin(x)$ is a periodic function with period 2π, i.e.

$$sin(x) = sin(x + 2\pi) = sin(x + 4\pi) = \cdots = sin(x + 2n\pi), \tag{2.2.1}$$

for all $n \in \mathbb{Z}$. The same is true for cosine function except its value is 1 at the origin i.e. $cos(0) = 1$, see Fig. 2.1.

2.2.2 Frequency and Amplitude

The number of oscillations of $sin(x)$ over the distance $T = 2\pi$ is one and thus the value of the angular frequency $\omega = 2\pi/T = 1$. If $f(x) = sin(3x)$, we obtain a compressed sine wave that oscillates three times faster than the original function $sin(x)$. The function $sin(3x)$ performs five full cycles over a distance of 2π and thus has the angular frequency $\omega = 3$ and a period $T = 2\pi/3$, see Fig. 2.2.

In general, the period T relates the angular frequency ω as

$$T = \frac{2\pi}{\omega} \tag{2.2.2}$$

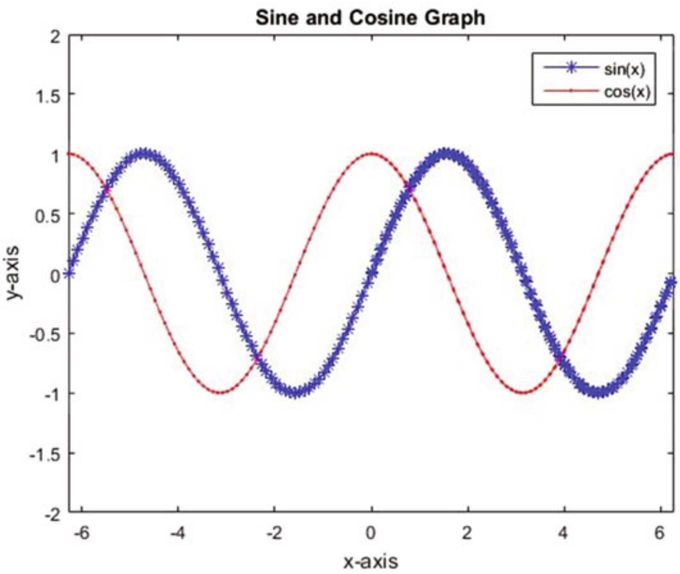

Fig. 2.1 Sine and Cosine Graph

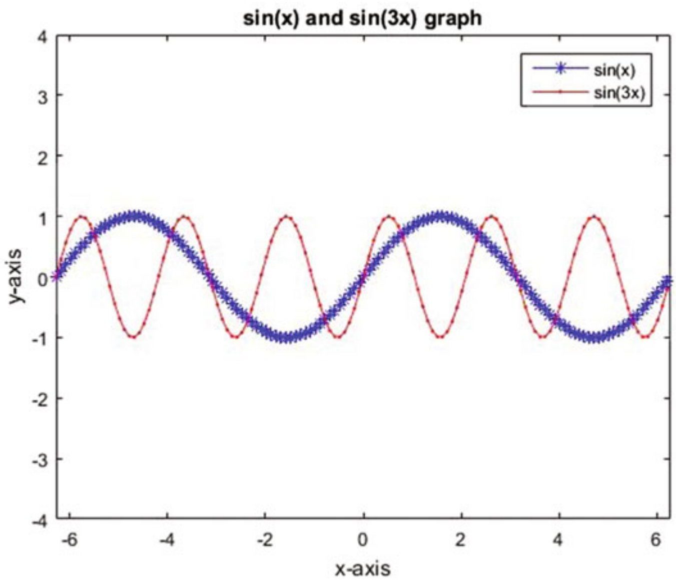

Fig. 2.2 $sin(x)$ and $sin(3x)$ Graph

for $\omega > 0$. The relationship between the angular frequency ω and the common frequency f is given by

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \text{or} \quad \omega = 2\pi f, \tag{2.2.3}$$

where f is measured in cycles per length or time unit. A sine or cosine function oscillates between the peak values $+1$ and -1 and its amplitude is 1. Multiplying by a constant $a \in \mathbb{R}$ changes the peak values of the function to $+a$ and $-a$ and its amplitude to a. In general, the expression

$$a \cdot sin(\omega x) \quad \text{and} \quad a \cdot cos(\omega x) \tag{2.2.4}$$

denotes a sine or cosine function with amplitude a and angular velocity ω, evaluated at position (or point in time) x.

2.2.3 Phase

Phase is the position of a point in time (an instant) on a waveform cycle. A complete cycle is defined as the interval required for the waveform to return to its arbitrary initial value. In sinusoidal functions or in waves "phase" has two different, but closely related, meanings. One is the initial angle of a sinusoidal function at its origin and is sometimes called phase offset or phase difference. Another usage is the fraction of the wave cycle that has elapsed relative to the origin.

Shifting a sine function along the x-axis by distance φ,

$$sin(x) \to sin(x - \varphi), \tag{2.2.5}$$

changes the phase of the sine wave and φ denotes the phase angle of the resulting function, see Fig. 2.3. Thus, we have

$$sin(nx) = cos(\omega x - \pi/2). \tag{2.2.6}$$

i.e. cosine and sine functions are orthogonal in a sense and we can use this fact to create new sinusoidal function with arbitrary frequency, phase and amplitude. In particular, adding a cosine and a sine function with the identical frequency ω and arbitrary amplitude A and B, respectively create another sinusoids

$$A \cdot cos(\omega x) + B \cdot sin(\omega x) = C \cdot cos(\omega x - \varphi). \tag{2.2.7}$$

where $c = \sqrt{A^2 + B^2}$ and $\varphi = tan^{-1}(B/A)$.

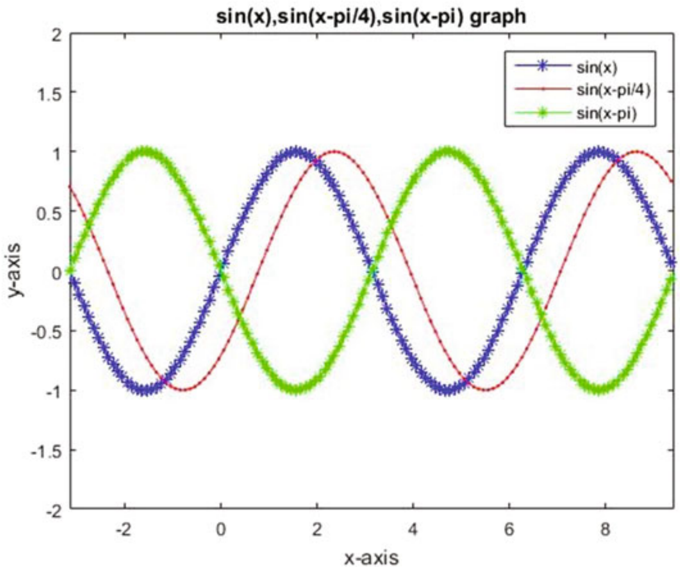

Fig. 2.3 $sin(x), sin(x - \pi/4)$ and $sin(x - \pi)$ Graph

2.2.4 Fourier Series of Periodic Functions

As we seen earlier, sinusoidal function of arbitrary frequency, amplitude and phase can be described as the sum of suitably weighted cosine and sine functions. Is it possible to write non-sinusoidal functions to sum of cosine and sine functions? It was Fourier [Jean Baptiste Joseph de Fourier (1768–1830)] who first extended this idea to arbitrary functions and showed that (almost) any periodic function $f(x)$ with a fundamental frequency ω_0 can be described as a infinite sum of harmonic sinusoids i.e.

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}[A_n cos(\omega_0 n x) + B_n sin(\omega_0 n x)]. \tag{2.2.8}$$

This is called Fourier series and A_0, A_n and B_n are called Fourier coefficients of the function $f(x)$, where

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \tag{2.2.9}$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(\omega_0 n x)dx, \tag{2.2.10}$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(\omega_0 n x)dx. \tag{2.2.11}$$

A Fourier series is an expression of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines. Fourier series make use of the orthogonality relationships of sine and cosine functions, since these functions form a complete orthogonal system over $[-\pi, \pi]$ or any interval of length 2π. The computation and study of Fourier series is known as Harmonic analysis and is extremely useful as a way to break up an arbitrary periodic function into a set of simple terms that can be plugged in, solved individually and then recombined to obtain the solution of the original problem of an approximation to it to whatever accuracy is desired or practical.

More general form of Fourier series is

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}[A_n cos(nx) + B_n sin(nx)], \tag{2.2.12}$$

where

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \tag{2.2.13}$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx, \tag{2.2.14}$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx. \tag{2.2.15}$$

The miracle of Fourier series is that as long as $f(x)$ is continuous (or even piecewise-continuous, with some caveats discussed in the Stewart text), such a decomposition is always possible.

2.2.5 Complex Form of Fourier Series

The Fourier series representation for a periodic function f, can be expressed more simply using complex exponentials. Moreover, because of the unique properties of the exponential function, Fourier series are often easier to manipulate in complex form. The transition from the real form to the complex form of a Fourier series is made using the following identities, called Euler identities,

$$e^{i\theta} = cos(\theta) + isin(\theta) \quad \text{and} \quad e^{-i\theta} = cos(\theta) - isin(\theta). \tag{2.2.16}$$

By adding these identities, and then dividing by 2, or subtracting them, and then dividing by $2i$, we have

$$cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{2.2.17}$$

The complex Fourier series is obtained from (2.2.12) by writing $cos(nx)$ and $sin(nx)$ in their complex exponential form and rearranging as follows:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + B_n \left(\frac{e^{inx} - e^{-inx}}{2i} \right) \right]$$
$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[\frac{A_n - iB_n}{2} \right] e^{inx} + \sum_{m=-\infty}^{-1} \left[\frac{A_{-m} + iB_{-m}}{2} \right] e^{imx}$$

where we substituted $m = n$ in the last term on the last line. Equation clearly suggests the much simpler complex form of the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}, \tag{2.2.18}$$

with the coefficients given by

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx. \tag{2.2.19}$$

Note that the Fourier coefficients C_n are complex valued. It is seen that for a real-valued function $f(x)$, the following holds for the complex coefficients C_n

$$\overline{C_{-n}} = C_n, \tag{2.2.20}$$

where $\overline{C_n}$ denotes the complex conjugate of C_n.

2.3 Fourier Transform

In the previous section we have seen how to expand a periodic function as a trigonometric series. This can be thought of as a decomposition of a periodic function in terms of elementary modes, each of which has a definite frequency allowed by the periodicity. This concept can be generalized to functions periodic on any interval.

If the function has period L, then the frequencies must be integer multiples of the fundamental frequency $k = 2\pi/L$. The Fourier series of functions of arbitrary periodicity is

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n cos(2\pi nx/L) + B_n sin(2\pi nx/L)], \tag{2.3.1}$$

where

$$A_0 = \frac{1}{L}\int_{-L/2}^{L/2} f(x)dx, \tag{2.3.2}$$

$$A_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)cos(2\pi nx/L)dx, \tag{2.3.3}$$

$$B_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)sin(2\pi nx/L)dx, \tag{2.3.4}$$

or in the exponential notation,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi nx/L}, \tag{2.3.5}$$

where

$$C_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-i2\pi nx/L}dx. \tag{2.3.6}$$

Fourier series was a powerful one and forms the backbone of the Fourier transform. The Fourier transform can be viewed as an extension of the above Fourier series to non-periodic functions and allows us to deal with non-periodic functions. A non-periodic function can be thought of as a periodic function in the limit $L \to \infty$. Clearly, the larger L is, the less frequently the function repeats, until in the limit $L \to \infty$ the function does not repeat at all. In the limit $L \to \infty$ the allowed frequencies become a continuum and the Fourier sum goes over to a Fourier integral.

Consider a function $f(x)$ defined on the real line. If $f(x)$ were periodic with period L, then $f(x)$ can be expand by Eq. (2.3.5) as Fourier series converging to it almost everywhere within each period $[-L/2, L/2]$. Even if $f(x)$ is not periodic, we can still define a function

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi nx/L}, \tag{2.3.7}$$

with the same C_n as above. Consider the limit in which L become very large.

Define

$$k_n = \frac{2n\pi}{L},$$

then

$$f_L(x) = \sum_{n=-\infty}^{\infty} C_n e^{ik_n x}. \tag{2.3.8}$$

It is clear that for very large L the sum contains a very large number of waves with wave-vector k_n and that each successive wave differs from the last by a tiny change in wave-vector (or wavelength),

$$\triangle k = k_{n+1} - k_n = \frac{2\pi}{L}.$$

In the limit $L \to \infty$ the allowed k becomes a continuous variable, the discrete coefficients, C_n, become a continuous function of k, denoted by $C(k)$ and the summation can be replaced by an integral and we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(k) e^{ikx} dx, \tag{2.3.9}$$

$$C(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \tag{2.3.10}$$

The functions f and C are called a Fourier transform pair, C is called the Fourier transform of f and f is called the inverse Fourier transform of C.

This prompts us to define the 1D-*Fourier transform* of the function $f(x)$ as

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx, \tag{2.3.11}$$

provided that the integral exists. Not every function $f(x)$ has a Fourier transform. A sufficient condition is that it be square-integrable; that is, so that the following integral converges:

$$||f||^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx. \tag{2.3.12}$$

If in addition of being square-integrable, the function is continuous, then one also has the inversion formula

$$f(x) = (\check{\hat{f}}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dx. \tag{2.3.13}$$

2.3.1 2D-Fourier Transform

Two-dimensional (2D) *Fourier transform* of the function $f(x, y)$ is defined as

$$\hat{f}(k,l) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{-i(kx+ly)}dxdy, \tag{2.3.14}$$

provided that the integral exists and a the two-dimensional (2D) *inverse Fourier transform* is defined by

$$f(x, y) = (\check{\hat{f}}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(k,l)e^{i(kx+ly)}dkdl. \tag{2.3.15}$$

2.3.2 Properties of Fourier Transform

Let $\hat{f}(k)$ and $\hat{g}(k)$ are Fourier transform of functions $f(t)$ and $g(t)$, respectively. Then we have the following:

1. Linearity

$$\widehat{af + bg}(k) = a\widehat{f}(k) + b\widehat{g}(k),$$

here $a,\ b$ are constants, i.e. if we add two functions then the Fourier transform of the resulting function is simply the sum of the individual Fourier transforms and if we multiply a function by any constant then we must multiply the Fourier transform by the same constant.

2. Shifting There are two basic shift properties of the Fourier transform:

i. Time Shifting

$$(\widehat{f(t \pm t_0)})(k) = \widehat{f}(k)e^{\pm ikt_0}.$$

ii. Frequency Shifting

$$(\widehat{f(t)e^{\pm ik_0x}})(k) = \widehat{f}(k \pm k_0).$$

Here t_0 and k_0 are constants. i.e. Translating a function in one domain corresponds to a multiplication by a complex exponential function in the other domain.

3. Scaling

$$\widehat{f(ax)}(k) = \frac{1}{a}\widehat{f}(\frac{k}{a}),$$

here a is constant. When a signal is expanded in time, it is compressed in frequency, and vice versa i.e. we cannot be simultaneously short in time and short in frequency.

4. Differentiation

i. Time differentiation property

$$\widehat{f(t)}(k) = ik\widehat{f}(k).$$

Differentiating a function is said to amplify the higher frequency components because of the additional multiplying factor k.

ii. Frequency differentiation property

$$\widehat{tf(t)}(k) = i\frac{d\widehat{f}(k)}{dk}.$$

5. Conjugate Symmetry The Fourier transform is conjugate symmetric for time functions that are real-valued,

$$\widehat{f}(-k) = \overline{\widehat{f}(k)}.$$

From this it follows that the real part and the magnitude of the Fourier transform of real valued time functions are even functions of frequency and that the imaginary part and phase are odd functions of frequency. By property of conjugate symmetry, in displaying or specifying the Fourier transform of a real-valued time function it is necessary to display the transform only for positive values of k.

6. Reversal

$$\widehat{f(-x)}(k) = \widehat{f}(-k), \quad \text{for} \quad x,\ k \in \mathbb{R}.$$

7. Duality This property relates to the fact that the analysis equation and synthesis equation look almost identical except for a factor of $\frac{1}{2\pi}$ and the difference of a minus sign in the exponential in the integral.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikt}dk \iff \hat{f}(k) = \int_{-\infty}^{\infty} f(t)e^{-ikt}dt$$

8. Convolution The convolution theorem states that convolution in time domain corresponds to multiplication in frequency domain and vice versa:

$$(\widehat{f * g)(t)}(k) = \widehat{f}(k)\widehat{g}(k).$$

9. Parseval's Relation

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk.$$

2.4 Discrete Fourier Transform

We assume that vectors in $\mathbb{C}^N$, i.e., sequences of N complex numbers, are indexed from 0 to $N-1$ instead of $\{1, 2, 3, \ldots, N\}$. we regard x as a function defined on the finite set

$$\mathbb{Z}_N = \{0, 1, 2, \ldots, N-1\}, \tag{2.4.1}$$

and we identify x with column vector

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_{N-1} \end{bmatrix}.$$

This allows us to write the product of $N \times N$ matrix A by x as Ax. To save space, we usually writ such a x horizontally instead of vertically $x = (x_0, x_1, \ldots, x_{N-1})$. In order to be consistent with the notation for functions used later in the infinite dimensional context, we write $l^2(\mathbb{Z}_N)$ in place of $\mathbb{C}^N$. So, formally,

$$l^2(\mathbb{Z}_N) = \{x = (x_0, x_1, \ldots, x_{N-1}) : \quad x_j \in \mathbb{C}, \quad 0 \le j \le N-1\}. \tag{2.4.2}$$

With the usual component-wise addition and scalar multiplication, $l^2(\mathbb{Z}_N)$ is an N-dimensional vector space over $\mathbb{C}$. One basis for $l^2(\mathbb{Z}_N)$ is an standard or Euclidean basis $E = \{e_0, e_1, \ldots, e_{N-1}\}$, where

$$e_j(n) = \begin{cases} 1, & \text{if } n = j \\ 0, & \text{otherwise.} \end{cases} \tag{2.4.3}$$

In this notation, the complex inner product on $l^2(\mathbb{Z}_N)$ is

$$\langle x, y \rangle = \sum_{k=0}^{N-1} x_k \overline{y}_k, \tag{2.4.4}$$

with the associated norm

$$||x|| = \left(\sum_{k=0}^{N-1} |x_k|^2 \right)^{1/2}, \tag{2.4.5}$$

called the l^2-norm.

2.4.1 1D-Discrete Fourier Transform

Definition 2.1 Define $E_0, E_1, \ldots, E_{N-1} \in l^2(\mathbb{Z}_N)$ by

$$E_m(n) = \frac{1}{\sqrt{N}} e^{2\pi imn/N}, \quad \text{for} \quad 0 \le m, \quad n \le N-1. \tag{2.4.6}$$

Clearly, the set $\{E_0, E_1, \ldots, E_{N-1}\}$ is an orthonormal basis for $l^2(\mathbb{Z}_N)$. We have

$$x = \sum_{m=0}^{N-1} \langle x, E_m \rangle E_m, \tag{2.4.7}$$

$$\langle x, y \rangle = \sum_{m=0}^{N-1} \langle x, E_m \rangle \overline{\langle y, E_m \rangle}, \tag{2.4.8}$$

$$||x||^2 = \sum_{m=0}^{N-1} |\langle x, E_m \rangle|^2. \tag{2.4.9}$$

By definition of inner product

$$\langle x, E_m \rangle = \sum_{n=0}^{N-1} x_n \overline{\frac{1}{\sqrt{N}} e^{2\pi imn/N}} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} x_n e^{-2\pi imn/N}. \tag{2.4.10}$$

Definition 2.2 Suppose $x = (x_0, x_1, \ldots, x_{N-1}) \in l^2(\mathbb{Z}_N)$. For $m = 0, 1, 2, \ldots, N-1$, define

$$\widehat{x}_m = \sum_{n=0}^{N-1} x_n e^{-2\pi imn/N}. \tag{2.4.11}$$

Then $\widehat{x} = (\widehat{x}_0, \widehat{x}_1, \ldots, \widehat{x}_{N-1}) \in l^2(\mathbb{Z}_N)$. The map $\widehat{\ }: l^2(\mathbb{Z}_N) \to l^2(\mathbb{Z}_N)$, which takes x to $\widehat{x}$, is called the 1D-*discrete Fourier transform* (DFT).

It can easily see that $\widehat{x}_m, \ m \in \mathbb{Z}$ is periodic with period N:

$$\widehat{x}_{m+N} = \sum_{m=0}^{N-1} x_n e^{-2\pi i(m+N)n/N} = \sum_{m=0}^{N-1} x_n e^{-2\pi imn/N} e^{-2\pi iNn/N} = \sum_{m=0}^{N-1} x_n e^{-2\pi imn/N} = \widehat{x}_m,$$

since $e^{-2\pi iNn/N} = e^{-2\pi in} = 1$, for every $n \in \mathbb{Z}$. Comparing the Eqs. (2.4.10) and (2.4.11), we have

$$\widehat{x}_m = \sqrt{N} \langle x, E_m \rangle. \tag{2.4.12}$$

Remark 2.1 Equation (2.4.12) actually defines the DFT coefficients x_k for any index k and resulting x_k are periodic with period N in the index. We will thus sometimes refers to the x_k on the other ranges of the index, for example $-N/2 < k \leq N/2$ when N is even. Actually, even if N is odd, the range $-N/2 < k \leq N/2$ works because k is required to be an integer.

Equation (2.4.12) leads to the following reformulation of formulae (2.4.7), (2.4.8) and (2.4.9). Let $\widehat{x} = (\widehat{x}_0, \widehat{x}_1, \ldots, \widehat{x}_{N-1})$ and $\widehat{y} = (\widehat{y}_0, \widehat{y}_1, \ldots, \widehat{y}_{N-1}) \in l^2(\mathbb{Z}_N)$. Then

(i) Fourier Inversion Formula:

$$x_n = \frac{1}{N}\sum_{m=0}^{N-1} \widehat{x}_m e^{-2\pi i m n/N}, \qquad \text{for} \quad n = 0, 1, 2, \ldots, N-1. \tag{2.4.13}$$

(ii) Parseval's Relation:

$$\langle x, y\rangle = \frac{1}{N}\sum_{m=0}^{N-1} \widehat{x}_m \overline{\widehat{y}_m} = \frac{1}{N}\langle \widehat{x}_m, \widehat{y}_m\rangle. \tag{2.4.14}$$

(ii) Plancherel Theorem:

$$||x||^2 = \frac{1}{N}\sum_{m=0}^{N-1} |\widehat{x}_m|^2 = \frac{1}{N}||\widehat{x}_m||^2. \tag{2.4.15}$$

The DFT can be represented by matrix, since Eq. (2.4.11) shows that the map taking x to $\widehat{x}$ is a linear transformation. Define

$$w_N = e^{-2\pi i/N}.$$

Then we have

$$e^{-2\pi i m n/N} = w_N^{mn} \quad \text{and} \quad e^{2\pi i m n/N} = w_N^{-mn},$$

and

$$\widehat{x}_m = \sum_{n=0}^{N-1} x_n w_N^{mn}. \tag{2.4.16}$$

Definition 2.3 Let W_N be the matrix $[w_{mn}]_{0\leq m,n\leq N-1}$, such that $w_{mn} = w_N^{mn}$. Hence

$$W_N = \begin{bmatrix} 1 & 1 & 1 & \cdot\cdot & 1 \\ 1 & w_N & w_N^2 & \cdot\cdot & w_N^{N-1} \\ 1 & w_N^2 & w_N^4 & \cdot\cdot & w_N^{2(N-1)} \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ 1 & w_N^{(N-1)} & w_N^{2(N-1)} & \cdot\cdot & w_N^{(N-1)(N-1)} \end{bmatrix}.$$

Regarding $x, \widehat{x} \in l^2(\mathbb{Z}_N)$ as column vectors (as Eq. 2.4.11), the m^{th} component of $W_N x$ is

$$\sum_{n=0}^{N-1} w_{mn} x_n = \sum_{n=0}^{N-1} x_n w_N^{mn} = \widehat{x}_m \quad 0 \le m \le N-1.$$

Hence

$$\widehat{x} = W_N x. \tag{2.4.17}$$

Now, we only compute a simple example in order to demonstrate the definitions. We could use Eq. (2.4.12) for this, but it is easier to use Eq. (2.4.17). The values of matrices W_2 and W_4 are as follows:

$$W_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}.$$

Example 2.1 Let $x = (1, 0, -3, 4) \in l^2(\mathbb{Z}_4)$. Find $\widehat{x}$.

Solution

$$\widehat{x} = W_4 x = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4+4i \\ -6 \\ 4-4i \end{bmatrix}.$$

The matrix W_N has a lot of structure. This structure can even be exploited to develop an algorithm called the fast Fourier transform that provides a very efficient method for computing DFT's without actually doing the full matrix multiplication.

Definition 2.4 (*Convolution*) For $x, y \in l^2(\mathbb{Z}_N)$, the *convolution* $x * y \in l^2(\mathbb{Z}_N)$ is the vector with components

$$(x * y)(m) = \sum_{m=0}^{N-1} x(m-n) y(n), \tag{2.4.18}$$

for all m.

Suppose $x, y \in l^2(\mathbb{Z}_N)$. Then for each m,

$$\widehat{(x * y)}_m = \widehat{x}_m \widehat{y}_m. \tag{2.4.19}$$

2.4.2 Inverse 1D-Discrete Fourier Transform

To interpret the Fourier inversion formula (2.4.13), we make the following definition:

Definition 2.5 Let $m = 0, 1, 2, \ldots, N-1$. Define $F_m \in l^2(\mathbb{Z}_N)$ by

$$F_m(n) = \frac{1}{N}e^{2\pi imn/N}, \quad \text{for} \quad 0 \le n \le N-1. \tag{2.4.20}$$

Then $F = \{F_0, F_1, \ldots, F_{N-1}\}$ is called the Fourier basis for $l^2(\mathbb{Z}_N)$.

Form Eq. (2.4.6) we have

$$F_m = \frac{1}{\sqrt{N}}E_m. \tag{2.4.21}$$

Since E_m is orthonormal basis for $l^2(\mathbb{Z}_N)$, F is an orthogonal basis for $l^2(\mathbb{Z}_N)$. With this notation, Eq. (2.4.13) becomes

$$x = \sum_{m=0}^{N-1} \widehat{x}_m F_m. \tag{2.4.22}$$

The Fourier inversion formula (2.4.13) shows that the linear transformation $\widehat{\ }: l^2(\mathbb{Z}_N) \to l^2(\mathbb{Z}_N)$ is a one-one map. Therefore $\widehat{\ }$ is invertible. Hence, Eq. (2.4.13) gives us a formula for the inverse of discrete Fourier transform and it is denoted by $\check{\ }$.

Definition 2.6 Let $y = (y_0, y_1, \ldots, y_{N-1}) \in l^2(\mathbb{Z}_N)$. Define

$$\check{y}_n = \frac{1}{N}\sum_{m=0}^{N-1} y_n e^{2\pi imn/N} \quad \text{For} \quad n = 0, 1, 2, \ldots, N-1. \tag{2.4.23}$$

Then $\check{y} = (\check{y}_0, \check{y}_1, \ldots, \check{y}_{N-1}) \in l^2(\mathbb{Z}_N)$. The map $\check{\ }: l^2(\mathbb{Z}_N) \to l^2(\mathbb{Z}_N)$, which takes y to $\check{y}$, is called the 1D-*inverse discrete Fourier transform* (IDFT).

We can easily see that $\check{y}_m, m \in \mathbb{Z}$ is also periodic function with period N. Fourier inversion formula states that for $x \in l^2(\mathbb{Z}_N)$,

$$(\check{\widehat{x}}_n) = x_n \ \text{ or } \ \widehat{(\check{x}_n)} = x_n, \quad \text{for} \ \ n = 0, 1, 2, \ldots, N-1.$$

Since the DFT is an invertible linear transformation, the matrix W_N is invertible and we must have $x = W_N^{-1}\widehat{x}$. Substituting $\widehat{x} = y$ and equivalently $x = \check{y}$ in equations, we have

$$\check{y} = W_N^{-1}y. \tag{2.4.24}$$

In the notation of formula (2.4.16), formula (2.4.21) becomes

$$\check{y} = \sum_{n=0}^{N-1} y_n \frac{1}{N} w_N^{-mn} = \sum_{n=0}^{N-1} y_n \frac{1}{N} \overline{w_N^{mn}}.$$

This shows that the (n, m) entry of W_N^{-1} is $\frac{\overline{w_N^{mn}}}{N}$, which is $1/N$ times of the complex conjugate of the (n, m) entry of W_N. If we denote by $\overline{W_N}$ the matrix whose entries are the complex conjugates of the entries of W_N, we have

$$W_N^{-1} = \frac{1}{N}\overline{W_N}.$$

We have

$$W_2^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } W_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}.$$

Example 2.2 Let $y = (2, 4 + 4i, -6, 4 - 4i) \in l^2(\mathbb{Z}_4)$. Find $\check{y}$.

Solution

$$W_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}\begin{bmatrix} 2 \\ 4+4i \\ -6 \\ 4-4i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 4 \end{bmatrix}.$$

2.4.3 2D-Discrete Fourier Transform and 2D-Inverse Discrete Fourier Transform

The definition of the two-dimensional (2D) discrete Fourier transform is very similar to that for one dimension. The forward and inverse transforms for an $M \times N$ matrix, where for notational convenience we assume that the m indices are from 0 to $M - 1$ and the n indices are from 0 to $N - 1$ are:

$$\widehat{x}_{(r,s)} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} x_{(m.n)} e^{-2\pi i \left(\frac{mr}{M} + \frac{ns}{N}\right)}, \tag{2.4.25}$$

and

$$\check{x}_{(r,s)} = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} x_{(m,n)} e^{2\pi i \left(\frac{mr}{M} + \frac{ns}{N}\right)}. \tag{2.4.26}$$

2.4.4 Properties of 2D-Discrete Fourier Transform

All the properties of the one-dimensional DFT transfer into two dimensions. But there are some further properties not previously mentioned, which are of particular use for image processing.

1. Similarity First notice that the forward and inverse transforms are very similar, with the exception of the scale factor $1/MN$ in the inverse transform, and the negative sign in the exponent of the forward transform. This means that the same algorithm, only very slightly adjusted, can be used for both the forward an inverse transforms.

2. The DFT as a spatial filter Note that the values

$$e^{\pm 2\pi i(\frac{mr}{M}+\frac{ns}{N})} \tag{2.4.27}$$

are independent of the values $\widehat{x}$ or $\check{x}$. This means that they can be calculated in advance, and only then put into the formulas above. It also means that every value of $\widehat{x}_{(r,s)}$ is obtained by multiplying every value of $\check{x}_{(r,s)}$ by a fixed value, and adding up all the results. But this is precisely what a linear spatial filter does: it multiplies all elements under a mask with fixed values, and adds them all up. Thus we can consider the DFT as a linear spatial filter which is as big as the image. To deal with the problem of edges, we assume that the image is tiled in all directions, so that the mask always has image values to use.

3. Separability Notice that the discrete Fourier transform filter elements can be expressed as products:

$$e^{2\pi i(\frac{mr}{M}+\frac{ns}{N})} = e^{2\pi i(\frac{mr}{M})}e^{2\pi i(\frac{ns}{N})}. \tag{2.4.28}$$

The first product value

$$e^{2\pi i(\frac{mr}{M})} \tag{2.4.29}$$

depends only on m and r, and is independent of n and s. Conversely, the second product value

$$e^{2\pi i(\frac{ns}{N})} \tag{2.4.30}$$

depends only on n and s, and is independent of m and r. This means that we can break down our formulas above to simpler formulas that work on single rows or columns:

$$\widehat{x}_m = \sum_{r=0}^{M-1} x_r e^{-2\pi imr/M}, \tag{2.4.31}$$

$$x_r = (\check{\widehat{x}_m})_r = \frac{1}{M}\sum_{m=0}^{M-1} \widehat{x}_m e^{2\pi imr/M}. \tag{2.4.32}$$

If we replace m and r with n and s we obtain the corresponding formulas for the DFT of matrix columns. These formulas define the one-dimensional DFT of a vector, or simply the DFT.

The 2D DFT can be calculated by using this property of separability; to obtain the 2D DFT of a matrix, we first calculate the DFT of all the rows, and then calculate the DFT of all the columns of the result. Since a product is independent of the order, we can equally well calculate a 2D DFT by calculating the DFT of all the columns first, then calculating the DFT of all the rows of the result.

4. Linearity An important property of the DFT is its linearity. the DFT of a sum is equal to the sum of the individual DFT's, and the same goes for scalar multiplication:

$$\widehat{x+y} = \widehat{x} + \widehat{y}$$

and

$$\widehat{kx} = k\widehat{x},$$

where k is a scalar, and x and y are matrices.

This property is of great use in dealing with image degradation such as noise which can be modelled as a sum:

$$d = f + \eta,$$

where f is the original image, η is the noise, and d is the degraded image. Since

$$\widehat{d} = \widehat{f} + \widehat{\eta}$$

we may be able to remove or reduce η by modifying the transform. As we shall see, some noise appears on the DFT in a way which makes it particularly easy to remove.

5. Convolution Theorem This result provides one of the most powerful advantages of using the DFT. Suppose we wish to convolve an image M with a spatial filter S. Our method has been place S over each pixel of M in turn, calculate the product of all corresponding gray values of M and elements of S, and add the results. The result is called the *digital convolution* of M and S, is denoted by

$$M * S.$$

This method of convolution can be very slow, especially if S is large.

The DC coefficient The value $\widehat{x}_{(0,0)}$ of the DFT is called the *DC coefficient*. If we put $r = s = 0$ in the Eq. (3.2.25), we have

$$\widehat{x}_{(0,0)} = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x_{(m,n)} e^{-2\pi i(\frac{m0}{M}+\frac{n0}{N})} = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x_{(m,n)}.$$

That is, this term is equal to the sum of all terms in the original matrix.

6. Shifting For purposes of display, it is convenient to have the DC coefficient in the centre of the matrix. This will happen if all elements $x_{(m,n)}$ in the matrix are multiplied by $(-1)^{m+n}$ before the transform.

7. Conjugate Symmetry An analysis of the Discrete Fourier transform definition leads to a symmetry property, i.e., if we make the substitutions $m = -m$ and $n = -n$ in Eq. (2.4.25), then

$$\widehat{x}_{(m,n)} = \overline{\widehat{x}_{(-m+pM,-n+qN)}},$$

for any integers p and q. This means that half of the transform is a mirror image of the conjugate of the other half. We can think of the top and bottom halves, or the left and right halves, being mirror images of the conjugates of each other.

8. Displaying Transforms Having obtained the Fourier transform $\widehat{x}$ of an image x, we would like to see what it looks like. As the elements $\widehat{x}$ are complex numbers, we can't view them directly, but we can view their magnitude $|\widehat{x}|$, since these will be numbers of type double, generally with large range. The display of the magnitude of a Discrete Fourier transform is called the *spectrum* of the transform.

2.5 Fast Fourier Transform

The fast Fourier transform (FFT) is a class of algorithms for computing the DFT and IDFT efficiently and it is one of the most influential algorithms of the twentieth century. Cooley and Tukey [6] in 1965 published the modern version of the FFT algorithms, but most of the ideas appeared earlier. Given the matrix W_N, it is not surprising that shortcuts can be found for computing the matrix-vector product $W_N x$. All modern software uses the FFT algorithm for computing discrete Fourier transforms and the details are usually transparent to the user. There are many of points of view on the FFT as a computer scientist would classify the FFT as a classic divide and conquer algorithm, a mathematician might view the FFT as a natural consequence of the structure of certain finite groups and other practitioners might simply view it as a efficient technique for organizing the sums involved in Eq. (2.4.11).

From Eq. (2.4.17), we have $\hat{x} = W_N x$, where W_N is the matrix in Definition 2.3. Clearly, direct computation of $\hat{x}$ takes N^2 complex multiplications. More precisely, we could also count the number of additions. Though, we get a good idea of the speed of computation by just considering the number of complex multiplications required because multiplication is much slower on a computer than addition. Here, complex multiplication, we mean the multiplication of two complex numbers. This would appear to require four real multiplications, but by a trick, it requires only three real multiplications. In signal and image processing, the vectors under consideration can be very large. Computation of the DFTs of these vectors in real time by direct means may be beyond the capacity of ones computational hardware. So a fast algorithm is needed known as FFT. Our intention is merely to demonstrate a very brief introduction to the idea behind one form of the FFT algorithm.

Now, We begin with the simplest version of the FFT, in which the length N of the vector is assumed to be even.

Theorem 2.1 *Let $M \in \mathbb{N}$, with $N = 2M$ and let $x \in l^2(\mathbb{Z}_N)$. Define $u, v \in l^2(\mathbb{Z}_M)$ by*

$$u_k = x_{2k} \quad for \quad k = 0, 1, 2, \ldots, M-1, \ \ or \ \ u_k = \left(x_0,\ x_2,\ x_4,\ \ldots x_{N-4},\ x_{N-2}\right),$$

and

$$v_k = x_{2k+1} \quad for \quad k = 0, 1, 2, \ldots, M-1 \ \ or \ \ v_k = \left(x_1,\ x_3,\ x_5,\ \ldots x_{N-3},\ x_{N-1}\right).$$

Let $\hat{x}$ denote the DFT of x defined on N points, that is, $\hat{x} = W_N x$. Let $\hat{u}$ and $\hat{v}$ denote the DFTs of u and v respectively,defined on $M = N/2$ points, that is, $\hat{u} = W_M u$ and $\hat{v} = W_M v$. Then for $m = 0, 1, 2, \ldots, M-1$,

$$\hat{x}(m) = \hat{u}(m) + e^{-2\pi i m/N} \hat{v}(m). \tag{2.5.1}$$

Also, for $m = M, M+1, M+2, \ldots, N-1$, let $l = m - M$. Note that the corresponding values of l are $l = 0, 1, 2, \ldots, M-1$. Then

$$\hat{x}(m) = \hat{x}(l+M) = \hat{u}(l) - e^{-2\pi i l/N} \hat{v}(l). \tag{2.5.2}$$

Proof By definition, for $m = 0, 1, 2, \ldots, N-1$,

$$\widehat{x_m} = \sum_{n=0}^{N-1} x_n e^{-2\pi i m n/N},$$

The sum over $n = 0, 1, 2, \ldots, N-1$ can be broken up into the sum over the even values $n = 2k, k = 0, 1, 2, \ldots, M-1$, plus the sum over the odd values $n = 2k+1, k = 0, 1, 2, \ldots, M-1$:

$$\begin{aligned}
\widehat{x_m} &= \sum_{k=0}^{M-1} x_{2k} e^{-2\pi i 2km/N} + \sum_{k=0}^{M-1} x_{2k+1} e^{-2\pi i (2k+1)m/N}, \\
&= \sum_{k=0}^{M-1} u_k e^{-2\pi i k m/(N/2)} + e^{-2\pi i m/N} \sum_{k=0}^{M-1} v_k e^{-2\pi i k m/(N/2)}, \\
&= \sum_{k=0}^{M-1} u_k e^{-2\pi i k m/M} + e^{-2\pi i m/N} \sum_{k=0}^{M-1} v_k e^{-2\pi i k m/M}.
\end{aligned}$$

In the case $m = 0, 1, 2, \ldots, M-1$, the last expression is $\hat{u}(m) + e^{-2\pi i m/N}\hat{v}(m)$, so we have Eq. (2.5.1). Now suppose $m = M, M+1, M+2, \ldots, N-1$. By writing $m = l + M$ as in the statement of the theorem and substituting this for m above, we get

$$\begin{aligned}\widehat{x}_m &= \sum_{k=0}^{M-1} u_k e^{-2\pi i k(l+m)/M} + e^{-2\pi i(l+m)/N} \sum_{k=0}^{M-1} v_k e^{-2\pi i k(l+m)/M}, \\ &= \sum_{k=0}^{M-1} u_k e^{-2\pi i k l/M} - e^{-2\pi i l/N} \sum_{k=0}^{M-1} v_k e^{-2\pi i k l/M},\end{aligned}$$

since the exponentials $e^{-2\pi i k l/M}$ are periodic with period M, and $e^{-2\pi i M/N} = e^{-\pi i} = -1$ for $N = 2M$. Hence Eq. (2.5.2) proves. □

Notice that the same values are used in Eqs. (2.5.1) and (2.5.2) and to apply Eqs. (2.5.1) and (2.5.2), we first compute $\hat{u}$ and $\hat{v}$. Each can be computed directly with M^2 complex multiplications since each of these is a vector of length $M = N/2$.Then compute the products $e^{-2\pi i m/N}\hat{v}(m)$ for $m = 0, 1, 2, \ldots, M-1$, this requires an additional M multiplications. Rest is done using only additions and subtractions of these quantities, which we do not count. Hence, the total number of complex multiplications required to compute $\hat{x}$ by Eqs. (2.5.1) and (2.5.2) is at most

$$2M^2 + M = 2\left(\frac{N}{2}\right)^2 + \frac{N}{2} = \frac{1}{2}(N^2 + N).$$

For N large, this is essentially $N^2/2$, whereas the number of complex multiplications required to compute $\hat{x}$ directly is N^2. Thus, Theorem 2.1 already cuts the computation time nearly in half.

If N is divisible by 4 instead of just 2, we can proceed further. Similarly, if N is divisible by 8, we can carry this one step further, and so on. Since u and v have even order, we can then apply the same method to reduce the time required to compute them. A more general way to describe this is to define $\#_N$, for any positive integer N, to be the least number of complex multiplications required to compute the DFT of a vector of length N. If $N = 2M$, then Eqs. (2.5.1) and (2.5.2) reduce the computation of $\hat{x}$ to the computation of two DFTs of size M, plus M additional complex multiplications. Hence

$$\#_N \leq 2\#_M + M. \tag{2.5.3}$$

The most favorable case is when N is a power of 2. He we have the following:

Theorem 2.2 *Let $N = 2^n$ for some $n \in \mathbb{N}$. Then*

$$\#_N \leq \frac{1}{2} N log_2 N.$$

Proof For $n = 1$, a vector of length 2^1 is of the form $x = (a, b)$. Then $\hat{x} = (a + b, a - b)$. Notice that this computation does not require any complex multiplications, so $\#_2 = 0 < 1 = (2\log_2 2)/2$. The result holds for $n = 1$. By induction, suppose it holds for $n = k - 1$. Then for $n = k$, we have by Eq. (2.5.3) and

$$\#_{2^k} \leq 2\#_{2^{k-1}} + 2^{k-1} \leq 2\frac{1}{2}2^{k-1}(k-1) + 2^{k-1} = k2^{k-1} = \frac{1}{2}k2^k = \frac{1}{2}N\log_2 N.$$

Hence result holds for $n = k$. Thus the result true for all n. □

For a vector of size $262,144 = 2^{18}$, the FFT reduces the number of complex multiplications needed to compute the DFT from 6.87×10^{10} to 2, 359, 296, thus making the computation more than 29, 000 times faster. Hence, if it takes 8 hours to do this via DFT directly, then it would take about 1 second to do it via the FFT. As N increases, this ratio becomes more extreme to the point that some computations that can be done by the FFT in a reasonable length of time could not be done directly in an entire lifetime. The FFT is usually implemented without explicit recursion. The FFT is not limited to N that are the powers of 2. What if N is not even? If N is prime, the method of the FFT does not apply. However, an efficient FFT algorithm can be derived most easily when N is "highly Composite" i.e. factors completely into small integers. In general, if N is composite, say $N = pq$, a generalization of Theorem 2.1 can be applied.

Theorem 2.3 *Let $p, q \in \mathbb{N}$, and $N = pq$. Let $x \in l^2(\mathbb{Z}_N)$. Define $w_0, w_1, \ldots, w_{p-1} \in l^2(\mathbb{Z}_q)$ by*

$$w_l(k) = x_{kp+l} \quad for \quad k = 0, 1, 2, \ldots, q-1.$$

For $b = 0, 1, 2, \ldots, q-1$, define $v_b \in l^2(\mathbb{Z}_p)$ by

$$v_b(l) = e^{-2\pi i b l/N} \hat{w}_l(b) \quad for \quad l = 0, 1, 2, \ldots, p-1.$$

Then for $a = 0, 1, 2, \ldots, p-1$ and $b = 0, 1, 2, \ldots, q-1$,

$$\hat{x}(aq + b) = \hat{v}_b(a). \tag{2.5.4}$$

Note that by the division algorithm, every $m = 0, 1, 2, \ldots, N-1$ is of the form $aq + b$ for some $a \in \{0, 1, 2, \ldots, p-1\}$ and $b \in \{0, 1, 2, \ldots, q-1\}$, so Eq. (2.5.4) gives the full DFT of x.

Proof We can write each $n = 0, 1, \ldots, N-1$ uniquely in the form $kp + l$ for some $k \in \{0, 1, 2, \ldots, q-1\}$ and $l \in \{0, 1, 2, \ldots, p-1\}$. Hence

$$\widehat{x}_{aq+b} = \sum_{n=0}^{N-1} x_n e^{-2\pi i(aq+b)n/N} = \sum_{l=0}^{p-1}\sum_{k=0}^{q-1} x_{kp+l} e^{-2\pi i(aq+b)(kp+l)/(pq)}.$$

Note that

$$e^{-2\pi i(aq+b)(kp+l)/(pq)} = e^{-2\pi iak} e^{-2\pi ial/p} e^{-2\pi ibk/q} e^{-2\pi ibl/(pq)}.$$

Since $e^{-2\pi iak} = 1$ and $pq = N$, using the definition of $w_l(k)$ we obtain

$$\widehat{x}_{aq+b} = \sum_{l=0}^{p-1} e^{-2\pi ial/p} e^{-2\pi ibl/N} \sum_{k=0}^{q-1} e^{-2\pi ibk/q}$$

$$= \sum_{l=0}^{p-1} e^{-2\pi ial/p} e^{-2\pi ibl/N} \hat{w}_l(b)$$

$$= \sum_{l=0}^{p-1} e^{-2\pi ial/p} v_b(l) = \widehat{v}_b(a).$$

□

This proof shows the basic principle behind the FFT. In computing $\widehat{x}_{aq+b}$, the same quantities $v_b(l), 0 \le l \le p-1$, arise for each value of a. The FFT algorithm recognizes this and computes these values only once. We first compute the vectors $\widehat{w}_l$, for $l = 0, 1, ..., p-1$. Each of these is a vector of length q, so computing each $\widehat{w}_l$ requires $\#_q$ complex multiplications. So this step requires a total of $p\#q$ complex multiplications. The next step is to multiply each $\widehat{w}_l(b)$ by e $e^{-2\pi ibl/N}$ to obtain the vectors $v_b(l)$. This requires a total of pq complex multiplications, one for each of the q values of b and p values of l. Finally we compute the vectors $\hat{v}_b$ for $b = 0, 1, ..., q-1$. Each v_b is a vector of length p, so each of the q vectors $\hat{v}_b$ requires $\#_p$ complex multiplications, for a total of $q^{\#_p}$ multiplications. Adding up, we have an estimate for the number of multiplications required to compute a DFT of size $N = pq$, namely

$$\#_{pq} \le p^{\#_q} + q^{\#_p} + pq. \tag{2.5.5}$$

This estimate can be used inductively to make various estimates on the time required to compute the FFT. The advantage of using the FFT is greater the more composite N is. There are many variations on the FFT algorithm, sometimes leading to slight advantages over the basic one given here. But the main point is that the DFT of a vector of length $N = 2^n$ can be computed with at most $n2^{n-1} = (N/2)\log_2 N$ complex multiplications as opposed to $N^2 = 2^{2n}$ if done directly.

Since $\check{x}_n = \frac{1}{N}\hat{x}_{N-n}$ and $x * y = (\hat{x}\hat{y})\check{}$, the FFT algorithm can be used to compute the IDFT and convolutions quickly also. In computing IDFT, at most $(N/2)\log_2 N$ steps requires if N is a power of 2. We do not count division by N because integer division is relatively fast. If $x, y \in l^2(\mathbb{Z}_N)$, for N a power of 2, it takes at most

$N\log_2 N$ multiplications to compute $\hat{x}$ and $\hat{y}$, and N multiplications to compute $\hat{x}\hat{y}$, and at most $(N/2)\log_2 N$ multiplications to take the IDFT of $\hat{x}\hat{y}$. Thus overall it takes no more than $N + (3N/2)\log_2 N$ multiplications to compute $z * w$.

2.6 The Discrete Cosine Transform

The Fourier transform and the DFT are designed for processing complex valued signals, and they always produce a complex-valued spectrum even in the case where the original signal was strictly real-valued. The reason is that neither the real nor the imaginary part of the Fourier spectrum alone is sufficient to represent (i.e., reconstruct) the signal completely. In other words, the corresponding cosine (for the real part) or sine functions (for the imaginary part) alone do not constitute a complete set of basis functions.

Further, we know that a real-valued signal has a symmetric Fourier spectrum, so only one half of the spectral coefficients need to be computed without losing any signal information.

There are several spectral transformations that have properties similar to the DFT but do not work with complex function values. The discrete cosine transform (DCT) is well known example that is particularly interesting in our context because it is frequently used for image and video compression. The DCT uses only cosine functions of various wave numbers as basis functions and operators on real-valued signals and spectral coefficients. Similarly, there is also a discrete sine transform (DST) based on a system of sine functions.

2.6.1 1D-Discrete Cosine Transform

In the one-dimensional case, the *discrete cosine transform* (DCT) for a signal $g(n)$ of length N is defined as

$$G(m) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} g(n) \cdot c_m cos\left(\pi \frac{m(2n+1)}{2N}\right), \tag{2.6.1}$$

for $0 \leq m < N$, and the *inverse discrete cosine transform* (IDCT) is

$$g(n) = \sqrt{\frac{2}{N}} \sum_{m=0}^{N-1} G(m) \cdot c_m cos\left(\pi \frac{m(2n+1)}{2N}\right), \tag{2.6.2}$$

for $0 \leq n < N$, respectively, with

$$c_m = \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } m = 0 \\ 1, & \text{otherwise.} \end{cases} \tag{2.6.3}$$

Note that the index variables (n, m) are used differently in the forward transform and the inverse transform, so the two transform are, in contrast to the DFT, not symmetric. One may ask why it is possible that the DCT can work without any sine functions, while they are essential in the DFT. The trick is to divide all frequencies in half such that they are spaced more densely and thus the frequency resolution in the spectrum is doubled. Comparing the cosine parts of the DFT basis functions (Eq. 2.4.11) and those of the DCT (Eq. 2.6.1), we have

DFT: $C_n^N(m) = cos\left(2\pi \frac{mn}{N}\right)$
DCT: $D_n^N(m) = cos\left(\pi \frac{m(2n+1)}{2N}\right) = cos\left(\pi \frac{2m(n+0.5)}{2N}\right),$

one can only see that the period of the DCT basis functions $(2N/m)$ is double the period of DFT functions (M/m) and DCT functions are also phase-shifted by 0.5 units. of course, much more efficient (fast) algorithms exist. Moreover, the DCT can also be computed in $O(M \log_2 M)$ time using FFT. The DCT is often used for image compression, in particular JPEG compression, where the size of the transformed sub images is fixed at 8×8 and the processing can highly be optimized.

2.6.2 2D-Discrete Cosine Transform

The two-dimensional form of the DCT follows immediately from the one-dimensional definition, resulting in 2D forward transform

$$G(m_1, m_2) = \sqrt{\frac{2}{N_1 N_2}} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} g(n_1, n_2) \cdot c_{m_1} cos\left(\pi \frac{m_1(2n_1+1)}{2N_1}\right) \cdot c_{m_2} cos\left(\pi \frac{m_2(2n_2+1)}{2N_2}\right), \tag{2.6.4}$$

for $0 \le m_1 < N_1, \quad 0 \le m_2 < N_2$, and the 2D-inverse DCT is

$$g(n_1, n_2) = \sqrt{\frac{2}{N_1 N_2}} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} G(m_1, m_2) \cdot c_{m_1} cos\left(\pi \frac{m_1(2n_1+1)}{2N_1}\right) \cdot c_{m_2} cos\left(\pi \frac{m_2(2n_2+1)}{2N_2}\right), \tag{2.6.5}$$

for $0 \le m_1 < N_1, \quad 0 \le m_2 < N_2$.

2.7 Heisenberg Uncertainty Principle

The Heisenberg uncertainty principle was originally stated in physics, and claims that it is impossible to know both the position and momentum of a particle simultaneously. However, it has an analog basis in signal processing. In terms of signals, the Heisenberg uncertainty principle is given by the rule that it is impossible to know both the frequency and time at which they occur. The time and frequency domains are complimentary. If one is local, the other is global. Formally, the uncertainty principle is expressed as

$$(\triangle t)^2(\triangle\omega)^2 \geq \frac{1}{4}. \tag{2.7.1}$$

In the case of an impulse signal, which assumes a constant value for a brief period of time, the frequency spectrum is finite; whereas in the case of a step signal which extends over infinite time, its frequency spectrum is a single vertical line. This fact shows that we can always localize a signal in time or in frequency but not both simultaneously. If a signal has a short duration, its band of frequency is wide and vice-versa.

2.8 Windowed Fourier Transform or Short-Time Fourier Transform

The short-time Fourier transform (STFT) is a modified version of Fourier transform. The Fourier transform separates the input signal into a sum of sinusoids of different frequencies and also identities their respective amplitudes. Thus, the Fourier transform gives the frequency-amplitude representation of an input signal. The Fourier transform is not an effective tool to analyses non-stationary signals. In STFT, the non-stationary signal is divided into small portions, which are assumed to be stationary. This is done using a window function of a chosen width, which is shifted and multiplied with the signal to obtain small stationary signals.

2.8.1 1D and 2D Short-Time Fourier Transform

The short-time Fourier transform maps a signal into two-dimensional function of time and frequency. The STFT of a one-dimensional signal $f(t)$ is represented by $X(\tau, \omega)$ where

$$X(\tau,\omega) = \int_{-\infty}^{\infty} f(t)g^*(t-\tau)e^{-i\omega t}dt, \tag{2.8.1}$$

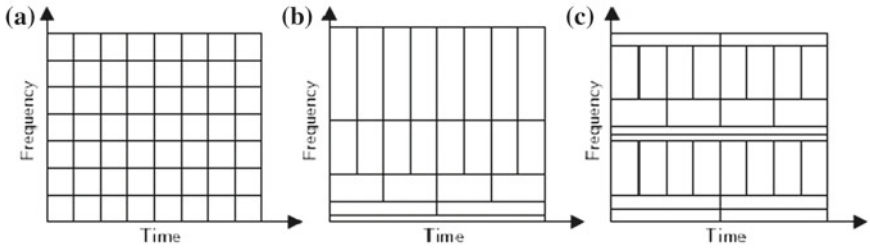

Fig. 2.4 The time-frequency tiling of STFT

Here, g^* denotes the conjugate of g, $f(t)$ represents the input signal, $g_{\tau,\omega}(t) = g(t-\tau)e^{i\omega t}$ is a temporal window with finite support and $X(\tau, \omega)$ is the time frequency atom. Also, the non-stationary signal $f(t)$ is assumed to be approximately stationary in the span of the temporal window $g_{\tau,\omega}(t)$.

In the case of a 2D signal $f(x, y)$, the space-frequency atom or STFT is given by

$$X(\tau_1, \tau_2, \omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)g^*(x-\tau_1, x-\tau_2)e^{-i(\omega_1 x+\omega_2 y)}dxdy, \quad (2.8.2)$$

where τ_1, τ_2 represents the spatial position of the two-dimensional window $g_{\tau_1,\tau_2,\omega_1,\omega_2}(x, y)$ and ω_1, ω_2 represents the spatial frequency parameters. The performance of STFT for specific application depends on the choice of the window. Different types of windows that can be used in STFT are Hamming, Hanning, Gaussian and Kaiser windows. The time-frequency tiling of STFT is given in Fig. 2.4.

2.8.2 Drawback of Short-Time Fourier Transform

The main drawback of STFT is that once a particular size time window is chosen, the window remains the same for all frequencies. To analyze the signal effectively, a more flexible approach is needed where the window size can vary in order to determine more accurately either the time or frequency information of the signal. This problem is known as resolution problem.

2.9 Other Spectral Transforms

Apparently, the Fourier transform is not the only way to represent a given signal in frequency space; in fact, numerous similar transforms exist. Some of these, such as the discrete cosine transform, also use sinusoidal basis functions, while others, such as the Hadamard transform (also known as the Walsh transform), build on binary 0/1-functions. All of these transforms are of global nature; i.e., the value of any spectral coefficient is equally influenced by all signal values, independent of the spatial position in the signal. Thus a peak in the spectrum could be caused by a high-amplitude event of local extent as well as by a widespread, continuous wave of low amplitude. Global transforms are therefore of limited use for the purpose of detecting or analyzing local events because they are incapable of capturing the spatial position and extent of events in a signal. A solution to this problem is to use a set of local, spatially limited basis functions (wavelets) in place of the global, spatially fixed basis functions. The corresponding wavelet transform, of which several versions have been proposed, allows the simultaneous localization of repetitive signal components in both signal space and frequency space.

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