PREFACE

This manual provides solutions to the problems that have answers in the back of our book, *ADVANCED ENGINEERING MATHEMATICS*. In many cases, the solutions are not as detailed as the examples in the book; they are intended to provide the primary steps in each solution so the student is able to quickly review how a problem is solved. The discussion of a subtle point, should one exist in a particular problem, is left as a task for the instructor. In general, some knowledge of a problem may be needed to fully understand all of the steps presented. This manual is not intended to be a self-paced workbook for the student; the instructor is critically needed to provide explanations and discussions of significant points in many of the problems.

The degree of difficulty and length of solution for each problem varies considerably. Some are relatively easy and others quite difficult. This allows for flexibility in assignments or in practice sessions. Typically, the easier problems are the first problems for a particular section.

The problems have been carefully solved with the hope that errors have not been introduced. Even though care is taken and problems and equations are reviewed, errors still creep in. We would appreciate knowing about any errors that you may find. They can be eliminated in future printings and/or included on an appropriate web page.

We'd also like to thank Professor Matthew Boelkins of Grand Valley State University for his contributions to the solutions of the problems in Chapters 4, 5, and 6.

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1. Ordinary Differential Equations

Section 1.2

2. $uu'' = 1 + x$ is nonlinear, 1st order.

4. $u'' - 2u' + u = \cos x$ is linear, nonhomogeneous, 2nd order.

6. $u'' - u = 0$ is linear, homogeneous, 2nd order.

8. $(u^2)' + u = 0$ is linear, 1st order, homogeneous (divide by *u*).

10. $u' = \sin x + e^x$ implies $u(x) = \cos x + e^x + C.$

12. $u'' = 2x$ implies $u'(x) = x^2 + C,$ which implies $u(x) = \dfrac{x^3}{3} + Cx + D.$

13. $u''' = x^2$ implies $u'' = \dfrac{x^3}{3} + C$, which implies $u' = \dfrac{x^4}{12} + Cx + D,$ which implies
$$u(x) = \frac{x^5}{60} + C\frac{x^2}{2} + Dx + E.$$

14. $u^{iv} = x - 2$ implies $u''' = \dfrac{x^2}{2} - 2x + C_1 \quad \Rightarrow u'' = \dfrac{x^3}{6} - x^2 + C_1 x + C_2$
$$\Rightarrow u' = \frac{x^4}{24} - \frac{x^3}{3} + C_1\frac{x^2}{2} + C_2 x + C_3 \quad \Rightarrow u(x) = \frac{x^5}{120} - \frac{x^4}{12} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4$$

22. $s'' = -20.$ $s' = -20t + c_1.$ $s = -10t^2 + c_1 t + c_2.$ $s'(0) = 100.$ $\therefore c_1 = 100.$
$s(0) = 0.$ $\therefore c_2 = 0.$ $\therefore s(t) = -10t^2 + 100t.$ $s' = -20t + 100 = 0.$ $\therefore t = 5$ at $s_{\max}.$
$$\therefore s_{\max} = -10 \times 5^2 + 100 \times 5 = \underline{250 \text{ m}}$$

Section 1.3.1

2. $\dfrac{u'}{u^2} = 10$ implies $d\left(-\dfrac{1}{u}\right) = d(10x).$ So, $-\dfrac{1}{u} = 10x + C$ and $\underline{u(x) = -\dfrac{1}{10x + C}}$

4. We have $\dfrac{u'}{u} = \sin x$ so $d(\ln u) = d(-\cos x).$
$$\therefore \ln x = -\cos x + \ln C \text{ or } \underline{u(x) = Ce^{-\cos x}}$$

6. $xu' = 1 - u^2$ so $\dfrac{u'}{1-u^2} = \dfrac{1}{x}$. Thus, $\dfrac{1}{2}\left(\dfrac{1}{1-u} + \dfrac{1}{1+u}\right)u' = d(\ln x)$

or $\dfrac{1}{2}d\left[-\ln(1-u) + \ln(1+u)\right] = d(\ln x)$ or $\dfrac{1}{2}d\left[\ln\dfrac{1+u}{1-u}\right] = d(\ln x)$

Thus, $\dfrac{1}{2}\ln\dfrac{1+u}{1-u} = \ln x + \ln C$. $\therefore \ln\left(\dfrac{1+u}{1-u}\right)^{1/2} = \ln Cx$

Hence, $\dfrac{1+u}{1-u} = (Cx)^2 = Kx^2$. $\therefore \underline{u(x) = \dfrac{Kx^2 - 1}{Kx^2 + 1}}$

8. $5xdu + x^2udx = 0$ so $\dfrac{5}{u}du + xdx = 0$. $\therefore 5\ln u + \dfrac{x^2}{2} = C$.

Thus, $\ln u = C - \dfrac{x^2}{10}$. Finally, $\underline{u(x) = Ke^{-x^2/10}}$

10. $x^2u' = xu + u^2$ so $u' = \dfrac{u}{x} + \dfrac{u^2}{x^2}$. Set $u = vx$ so $u' = v + xv'$.

Then $v + xv' = v + v^2$ or $\dfrac{v'}{v^2} = \dfrac{1}{x}$. Then, $d\left(-\dfrac{1}{v}\right) = d(\ln Cx)$.

Hence, $-\dfrac{1}{v} = \ln Cx$ so $v = -\dfrac{1}{\ln Cx}$. Finally, $\underline{u(x) = -\dfrac{x}{\ln Cx}}$

12. $3u + (u+x)u' = 0$ becomes $3\dfrac{u}{x} + \left(\dfrac{u}{x} + 1\right)u' = 0$. Let $u = vx$ so $u' = v + xv'$.

Then $3v + (v+1)(v + xv') = 0$. This leads to $xv'(1+v) = -4v - v^2$ or $\dfrac{v'(1+v)}{4v+v^2} = -\dfrac{1}{x}$

or $v'\left(\dfrac{1/4}{v} + \dfrac{3/4}{v+4}\right) = -\dfrac{1}{x}$. So $\dfrac{1}{4}\ln v - \dfrac{3}{4}\ln(v+4) = -\ln(cx)$.

$\therefore \dfrac{v}{(v+4)^3} = \dfrac{K}{x^4}$ or $\underline{\dfrac{u}{(u+4x)^3} = \dfrac{K}{x^6}}$

14. Let $x + 2u = y$ so $y' = 1 + 2u'$. Then $(x + 2u + 1)u' = x + 2u + 4$ becomes

$(y+1)\dfrac{y'-1}{2} = y + 4$ or $\left(1 - \dfrac{2}{y+3}\right)y' = 3$. $\therefore y - 2\ln(y+3) = 3x + c$.

Finally, $\underline{x + 2u - 2\ln(3 + x + 2u) = 3x + c}$

16. From $\dfrac{u'}{u+1} = \cot x$, we obtain $\ln(u+1) = \ln(c\sin x)$, so the general solution is

$u(x) = -1 + c\sin x$. But, $u(2) = 0 = -1 + c\sin 2$. $\therefore c = 1/\sin 2$. $\therefore \underline{u(x) = -1 + \sin x/\sin 2}$

18. Set $v = u - x$ in $xu' = (u-x)^3 + u$ to obtain $x(v'-1) = v^3 + v - x$

or $\dfrac{v'}{v} - \dfrac{v}{v^2+1}v' = \dfrac{1}{x}$. $\therefore \dfrac{v^2}{v^2+1} = cx^2$ so $u(x) = x\left(1 + \dfrac{1}{\sqrt{K-x^2}}\right)$

$u(1) = 2 = 1 + \dfrac{1}{\sqrt{K-1}}$. $\therefore K = 2$ and $\underline{u(x) = x\left[1 + (2-x^2)^{-1/2}\right]}$

Section 1.3.3

4. For $u' + 2u = 2x$, $F(x) = e^{\int 2dx} = e^{2x}$. $\therefore \dfrac{d}{dx}(e^{2x}u) = 2xe^{2x}$ and

$ue^{2x} = \int 2xe^{2x}dx + c = xe^{2x} - \dfrac{1}{2}e^{2x} + c$. $\therefore \underline{u(x) = x - \dfrac{1}{2} + ce^{-2x}}$

6. For $u' - 2u = e^x$, $F(x) = e^{\int -2dx} = e^{-2x}$. $\dfrac{d}{dx}(e^{-2x}u) = e^{-2x}e^x = e^{-x}$

$\therefore e^{-2x}u(x) = -e^{-x} + c$. $\therefore \underline{u(x) = -e^x + ce^{2x}}$

10. For $u' - \dfrac{1}{x^2}u = \dfrac{2}{x^2}\sin\dfrac{1}{x}$, $F(x) = e^{-\int \frac{1}{x^2}dx} = e^{1/x}$. $\dfrac{d}{dx}(e^{1/x}u) = \dfrac{2}{x^2}\sin\dfrac{1}{x}e^{1/x}$

$\therefore e^{1/x}u(x) = \int_1^x \dfrac{2}{t^2}e^{1/t}\sin\dfrac{1}{t}dt + C$. $\therefore \underline{u(x) = e^{-1/x}\int_1^x \dfrac{2}{t^2}e^{1/t}\sin\dfrac{1}{t}dt + Ce^{-1/x}}$

12. For $u' + xu = e^{-x^2}$, $F(x) = e^{\int xdx} = e^{x^2/2}$. $\dfrac{d}{dx}(e^{x^2/2}u) = e^{-x^2/2}$ so

$e^{x^2/2}u(x) = \int_1^x e^{-t^2/2}dt + c$. $\therefore u(x) = e^{-x^2/2}\int_1^x e^{-t^2/2}dt + ce^{-x^2/2}$

$u(1) = 0 = ce^{-1/2}$. $\therefore c = 0$. $\therefore \underline{u(x) = e^{-x^2/2}\int_1^x e^{-t^2/2}dt}$

Section 1.4

2. $Li' + Ri = v$. $10^{-3}i' + 20i = 0.2e^{2t}$ or $\dfrac{di}{dt} + 2\times10^4 i = 200e^{2t}$

$F(t) = e^{\int 2\times10^4 dt} = e^{2\times10^4 t}$. $\therefore i(t) = e^{-2\times10^4 t}\left(\int 200e^{2t}e^{2\times10^4 t}dt + C\right)$

$= e^{-2\times10^4 t}\left(\dfrac{200}{20002}e^{20002t} + C\right) = 0.01e^{2t} + Ce^{-2\times10^4 t}$. $i(0) = 0 = 0.01 + C$

$\therefore C = -0.01$. $\therefore \underline{i(t) = 0.01\left(e^{2t} - e^{-2\times10^4 t}\right)}$

4. $12 = 200q' + q/10^{-6}$ or $\dfrac{dq}{dt} + 5000q = \dfrac{6}{100}$. $F(t) = e^{5000t}$

$\therefore q(t) = e^{-5000t}\left(\int 0.06e^{5000t}dt + C\right) = 12\times10^{-6} + Ce^{-5000t}$. $q(0) = 0$. $\therefore C = -12\times10^{-6}$

$\therefore q(t) = 12\times10^{-6}(1 - e^{-5000t})$. as $t \to \infty$, $q \to 12\times0^{-6}$. Let $q = 6\times10^{-6}$

Then $\dfrac{1}{2} = 1 - e^{-5000t}$. $\therefore \underline{t = 1.386\times10^{-4}\text{ s}}$

6. $\dfrac{dC}{dt} + \dfrac{10/60}{1500}C = \dfrac{10/60}{1500}C_1$. To find C_1: CO_2 entering $= 0.0016\times18\times300\times0.04$

$= 0.3456$ m^3/min. Air entering = 10 m^3/min. $\therefore C_1 = \dfrac{0.3456}{10} = a$

$\dfrac{dC}{dt} = 1.111\times10^{-4}(a - C)$ or $\dfrac{dC}{a-C} = 1.111\times10^{-4}dt$. $\therefore \ln(C-a) = -1.111\times10^{-4}t + \ln K$

or $C(t) = Ke^{-1.111\times10^{-4}t} + a$. At $t = 0$, $C(0) = 0.004$. $\therefore K = 0.03056$

$\therefore \underline{C(t) = 0.03456 - 0.03056e^{-1.111\times10^{-4}t}}$

9. $\Sigma F_y = ma_y$. Sum forces in vertical direction: $-D + Mg = M\dfrac{dV}{dt}$. $\therefore$ Eq. is OK.

a) $100\dfrac{dV}{dt} = 100\times9.81 - 0.01V = -0.01(V - 9.81\times10^4)$. $\therefore \dfrac{dV}{V - 9.81\times10^4} = -10^{-4}dt$

$\therefore \ln(V - 9.81\times10^4) = -10^{-4}t + \ln K$ or $V(t) = 9.81\times10^4(1 - e^{-10^{-4}t})$.

Let $V = 50$: $50 = 9.81\times10^4(1 - e^{-10^{-4}t})$. $\therefore \underline{t = 5.098\text{ s}}$

b) $100\dfrac{dV}{dt} = 981 - 0.004V^2$ or $\dfrac{dV}{V^2 - 2.453\times10^5} = -4\times10^{-5}dt$ or

$\left(\dfrac{1}{V - 495.3} - \dfrac{1}{V + 495.3}\right)\dfrac{dV}{990.6} = -4\times10^{-5}dt$. $\therefore \dfrac{V - 495.3}{V + 495.3} = Ce^{-0.0396t}$.

$V(0) = 0$. $\therefore C = -1$. $\therefore V(t) = \dfrac{495.3(1 - e^{-.0396t})}{1 + e^{-.0396t}}$. If $V = 50$ $\underline{t = 5.116\text{ s}}$

11. With insulated sides (no heat transfer) the heat flux is constant at each *x*-location.

$\therefore 10 = -kA\dfrac{\partial T}{\partial x} = -100\dfrac{1200}{10^6}\dfrac{dT}{dx}$. $\therefore \dfrac{dT}{dx} = -83.33$ and $T(x) = -83.33x + C$

$T(2) = 50 = -83.33\times2 + C$. $\therefore C = 217$ and $\underline{T(x) = -83.33x + 217}$

13. $\dfrac{de}{dt} = -\alpha e$. $\therefore \ln e = -\alpha t + \ln C$. $\therefore e(t) = Ce^{-\alpha t}$. $e(0) = e^0 = C$. $\therefore e(t) = e_0e^{-\alpha t}$

$e(20) = \dfrac{e_0}{2} = e_0e^{-20\alpha}$. $\therefore \alpha = 0.0347$. $0.05e_0 = e_0e^{-.0347t}$. $\therefore \underline{t = 86.3\text{ min}}$

Section 1.5.1

2. A discontinuity at 0, not a jump because $\lim_{x\to 0^-}(\ln x) = -\infty$.

4. Not a jump discontinuity because $\lim_{x\to 0}\left(\frac{1}{x^2}\right) = \infty$.

5. No discontinuity at $x = 0$.

6. Not a jump discontinuity because $\lim_{\substack{x\to 0\\ x>0}}\left(\frac{\sin x}{x}\right) = 1 = \lim_{\substack{x\to 0\\ x<0}}\left(\frac{\sin x}{x}\right)$.

8. Not sectionally continuous because $\lim_{x\to 0}(\ln x) = -\infty$.

10. Not sectionally continuous because the singularity at $x = 0$ is not a jump discontinuity.

12. Not sectionally continuous because the singularity at $x = 0$ is not a jump discontinuity.

Section 1.5.3

2. $p_0(x) = \dfrac{1}{x}$. $\therefore W(x) = Ke^{-\int dx/x} = Ke^{-\ln x} = Ke^{\ln 1/x} = \underline{K/x}$

4. $p_0(x) = 0$. $\therefore W(x) = \underline{K}$

10. $$\begin{vmatrix} \alpha u_1 + \beta u_2 & \gamma u_1 + \delta u_2 \\ \alpha u_1' + \beta u_2' & \gamma u_1' + \delta u_2' \end{vmatrix} = \alpha\gamma u_1 u_1' - \alpha\delta u_1 u_2' + \beta\gamma u_2 u_1' + \beta\delta u_2 u_2' - \alpha\gamma u_1 u_1' - \alpha\delta u_2 u_1'$$

$$-\beta\gamma u_1 u_2' - \beta\delta u_2 u_2' = \alpha\delta(u_1 u_2' - u_2 u_1') + \beta\gamma(u_2 u_1' - u_1 u_2') = (\alpha\delta - \beta\gamma)\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = (\alpha\delta - \beta\gamma)W$$

$$\therefore \underline{\alpha\delta - \beta\gamma \neq 0}$$

Section 1.6

2. $m^2 - 9 = (m-3)(m+3) = 0$. $\therefore m = 3,\ -3$. $\therefore \underline{u(x) = c_1 e^{3x} + c_2 e^{-3x}}$

4. $4m^2 + 1 = (2m - i)(2m + i) = 0$. $\therefore m = i/2,\ -i/2$. $\therefore \underline{u(x) = c_1 e^{ix/2} + c_2 e^{-ix/2}}$

6. $m^2+4m+4=(m+2)^2=0.\quad \therefore m=-2,\ -2.\qquad \therefore \underline{u(x)=c_1e^{-2x}+c_2xe^{-2x}}$

8. $m^2+4m-4=0.\quad \therefore m=-2\pm 2\sqrt{2}.\qquad \therefore \underline{u(x)=c_1e^{-(2-2\sqrt{2})x}+c_2e^{-(2+2\sqrt{2})x}}$

10. $m^2-4m+8=0.\quad \therefore m=2\pm 2i.\qquad \therefore \underline{u(x)=c_1e^{(2+2i)x}+c_2e^{(2-2i)x}}$

12. $2m^2+6m+5=0.\quad \therefore m=-\dfrac{3}{2}\pm i.\qquad \therefore \underline{u(x)=c_1e^{-(\frac{3}{2}-i)x}+c_2e^{-(\frac{3}{2}+i)x}}$

14. From No.2, $a=0$ so $\underline{u(x)=c_1e^{3x}+c_2e^{-3x}}$

16. From No.4, $a=0$ so $\underline{u(x)=c_1\cos x/2+c_2\sin x/2}$

18. From No.8, $\underline{u(x)=e^{-2x}\left(c_1e^{2\sqrt{2}x}+c_2e^{-2\sqrt{2}x}\right)}$

20. From No.10, $\underline{u(x)=e^{2x}\left(c_1\cos 2x+c_2\sin 2x\right)}$

22. $m^2+5m+3=0,\quad m=\dfrac{5}{2}\pm\dfrac{\sqrt{13}}{2}.\qquad \therefore \underline{u(x)=e^{-5x/2}\left(c_1e^{\frac{\sqrt{13}}{2}x}+c_2e^{-\frac{\sqrt{13}}{2}x}\right)}$

24. $m^2-9=0,\quad a=0,\ b=-9.\qquad \therefore \underline{u(x)=c_1\sinh\left(3x+c_2\right)}$

26. $m^2+\dfrac{1}{4}=0,\quad a=0,\ b=\dfrac{1}{4}.\qquad \therefore \underline{u(x)=c_1\cos\left(x/2+c_2\right)}$

28. $m^2+4m-4=0,\quad a=4,\ b=-4.\qquad \therefore \underline{u(x)=c_1e^{-2x}\sinh\left(2\sqrt{2}x+c_2\right)}$

30. $m^2-4m+8=0,\quad a=-4,\ b=8.\qquad \therefore \underline{u(x)=c_1e^{2x}\cos\left(2x+c_2\right)}$

32. $m^2+5m+3=0,\quad a=5,\ b=3.\qquad \therefore \underline{u(x)=c_1e^{-2.5x}\cos\left(\sqrt{13}x/2+c_2\right)}$

34. $m^2+5m+6=0,\quad a=5,\ b=6.\qquad \therefore u(x)=c_1e^{-2x}+c_2e^{-3x}.\quad u(0)=2=c_1+c_2$
$u'(0)=0=-2c_1-3c_2.\qquad \therefore c_1=6,\ c_2=-4$ and $\underline{u(x)=6e^{-2x}-4e^{-3x}}$

36. $m^2-4,\quad a=0,\quad b=-4.\quad \therefore u(x)=c_1e^{2x}+c_2e^{-2x}.\quad u(0)=2=c_1+c_2$
$u'(0)=1=2c_1-2c_2.\quad \therefore c_1=5/4,\quad c_2=3/4$ and $\underline{u(x)=\dfrac{5}{4}e^{2x}+\dfrac{3}{4}e^{-2x}}$

38. See No.34: $u(x)=c_1e^{-5x/2}\sinh(\frac{x}{2}+c_2)$. $u(0)=2=c_1\sinh c_2$.

$u'(0)=0=-\frac{5}{2}\sinh c_2+c_1\cosh c_2$. $\therefore 10=c_1\cosh c_2$ and $\tanh c_2=\frac{1}{5}$.

$\therefore c_1=9.8,\quad c_2=0.2027$ and $\underline{u(x)=9.8\sinh(\tfrac{x}{2}+0.2027)}$

40. See No.33: $\underline{u(x)=\frac{1}{3}\sin 3x}$

42. See No.36: $\underline{u(x)=\frac{5}{4}e^{2x}+\frac{3}{4}e^{-2x}}$

44. $m^3-1=(m-1)(m^2+m+1)$. $\therefore m=1,\quad \frac{-1+i\sqrt{3}}{2},\quad \frac{-1-i\sqrt{3}}{2}$

Hence, $\underline{u(x)=c_1e^x+e^{-x/2}(c_2\cos\sqrt{3}x/2+c_3\sin\sqrt{3}x/2)}$

46. $m^4-m^2=0$. $\therefore m=0,\ 0,\ 1,\ -1$. $\therefore \underline{u(x)=c_1+c_2x+c_3e^x+c_4e^{-x}}$

48. $m^4-m^3=0$. $\therefore m=0,\ 0,\ 0,\ 1$. $\therefore \underline{u(x)=c_1+c_2x+c_3x^2+c_4e^x}$

Section 1.7.1

2. $\sum F=ma$. $-Ks=m\frac{d^2s}{dt^2}$. $10\frac{d^2s}{dt^2}+10s=0$. $m^2=-1$. $\therefore m_1=i,\quad m_2=-i$

$\therefore s(t)=A\cos t+B\sin t$. $s(0)=10=A,\quad s'(0)=0=B$. $\underline{s(t)=10\cos t}$

4. $\sum F=ma$. $-Ky=m\ddot{y}$. $0.03\ddot{y}+0.5y=0$. $m^2=-16.67$. $\therefore m=\pm 4.08i$

$\therefore y(t)=A\cos 4.08t+B\sin 4.08t$. $4.08t=2\pi$. $\therefore \underline{t_{cycle}=1.54\text{ s}}$

6. $2\ddot{y}+50y=0$. $m^2=-25$. $m=\pm 5i$. $\therefore y(t)=A\cos 5t+B\sin 5t$

$y(0)=2=A,\quad y'(0)=-10=5B$. $\therefore \underline{y(t)=2(\cos 5t-\sin 5t)}$

Section 1.7.2

2. $4\ddot{y}+40\dot{y}+64y=0.\quad m^2+10m+16=0.\quad m_1=-8,\quad m_2=-2.\quad \therefore y(t)=Ae^{-8t}+Be^{-2t}$

$$y(0)=0=A+B,\quad \dot{y}(0)=50=-8A-2B.\quad \therefore A=-B=\frac{25}{3}.\quad \therefore \underline{y(t)=\frac{25}{3}(e^{-8t}-e^{-2t})}$$

7. $y(t)=1.336(e^{-4.258t}=e^{-11.742t}).\quad \therefore \dot{y}(t)=-5.689e^{-4.258t}+15.69e^{-11.742t}$

Set $\dot{y}=0$ and solve for t. This can be done by trial and error. Try t = 0.1: $\dot{y}=1.13$.
Try t = 0.12: $\dot{y}=0.42$. Try t = 0.13: $\dot{y}=0.14$. Try t = 0.14: $\dot{y}=-0.10$. $\therefore \underline{t=0.136\text{ s}}$
It would be nice to have an expression to find t at $y_{\max}$. See No.8.

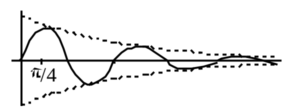
9. Substitute t = 0.136 s back into the expression for $y(t)$:

$$y_{\max}=1.336(e^{-4.258\times 0.136}-e^{-11.742\times 0.136})=\underline{0.478\text{ m}}$$

12. $y(t)=2e^{-t}\sin t.\quad \therefore C=2M,\quad \dfrac{\sqrt{4KM-4M^2}}{2M}=1.\quad \therefore K=2M.\quad D=\dfrac{2\pi\times M}{\sqrt{4(2M)M-4M^2}}=2\pi$

$$\therefore \frac{C}{C_C}=\frac{D}{\sqrt{D^2+4\pi^2}}=0.707.\quad \therefore C=70.7\%\text{ of }C_C$$

$y_{\max}$ occurs at $\dot{y}=0,\quad \therefore \tan t_m=1.\quad \therefore \underline{t_m=\pi/4\text{ s}}$



Section 1.7.3

1. a) $Li''+Ri'+i/C=0.\quad 10^{-3}i''+Ri'+i/2\times 10^{-5}=0.\quad m=\dfrac{-R\pm\sqrt{R^2-200}}{0.002}$

If $R^2<200$ then oscillatory. $\underline{R_{\text{crit}}=14.14\text{ ohms}}$

b) For a parallel circuit:

$$2\times 10^{-5}v''+v'/R+v/10^{-3}=0.\quad m=\frac{R^{-2}\pm\sqrt{R^{-2}-8\times 10^{-2}}}{4\times 10^{-5}}.\quad R^{-2}<0.08.\quad \therefore \underline{R_{crit}=3.54}$$

3. $10^{-4}i''+20i'+\dfrac{2}{10^{-6}}i=0.\quad \therefore m=\dfrac{-20\pm\sqrt{400-800}}{2\times 10^{-4}}=-10^5\pm 10^5 j$ (here $j=\sqrt{-1}$)

$$\therefore i(t)=e^{-10^5 t}(A\cos 10^5 t+B\sin 10^5 t).\quad i(0)=10=A,\quad i'(0)=0=-10^5A+10^5B.\quad \therefore B=10$$

$$\therefore \underline{i(t)=10(\cos 10^5 t+\sin 10^5 t)e^{-10^5 t}}$$

Section 1.8

2. Let $u_p(x) = Ax + B$. Then $0 + A + 2(Ax + B) = 2x$. $\therefore A = 1, \quad B = -\frac{1}{2}$. $\quad \therefore \underline{u_p(x) = x - \frac{1}{2}}$

4. Let $u_p(x) = Axe^{-x}$ since e^x is a solution to the homogeneous eqn. Then

$$u_p' = Ae^x + Axe^x, \quad u_p'' = 2Ae^x + Axe^x. \quad \therefore 2Ae^x + Axe^x - Axe^x - Axe^x = e^x$$

$$\therefore 2A = 1 \text{ and } \underline{u_p(x) = \frac{1}{2}xe^x}$$

6. Let $u_p(x) = Ax\sin 3x$. $\quad u_p' = A\sin 3x + 3Ax\cos 3x$. $\quad u_p'' = 6A\cos 3x - 9Ax\sin 3x$

$$\therefore 6A\cos 3x - 9Ax\sin 3x + 9Ax\sin 3x = \cos 3x. \quad A = \frac{1}{6} \text{ and } \underline{u_p(x) = \frac{1}{6}x\sin 3x}$$

8. Since $\sin 3x$ is a solution to the homogeneous eqn, we assume

$$u_p(x) = Ax^2 + Bx + C + Dx\cos 3x$$

Then substitute into the given eqn:

$$2A - 6D\sin 3x - 9Dx\cos 3x + 9Ax^2 + 9Bx + 9C + 9Dx\cos 3x = x^2 + \sin 3x$$

$$\therefore 9A = 1, \quad 9B = 0, \quad 9C + 2A = 0, \quad -6D = 1. \quad \therefore \underline{u_p(x) = \frac{1}{81}(9x^2 - 2) - \frac{1}{6}x\cos 3x}$$

10. $m^2 + 4m + 4 = 0$. $\quad m_1 = m_2 = -2$. $\quad \therefore u_h(x) = c_1e^{-2x} + c_2xe^{-2x}$. $\quad$ Let $u_p(x) = Ax^2 + Bx + C$

$$2A + 8Ax + 4B + 4Ax^2 + 4Bx + 4C = x^2 + x + 4. \quad \therefore 4A = 1, \quad 8A + 4B = 1, \quad 2A + 4B + 4C = 4$$

$$\therefore A = \frac{1}{4}, \quad B = -\frac{1}{4}, \quad C = \frac{9}{8}. \quad \therefore \underline{u(x) = c_1e^{-2x} + c_2xe^{-2x} + \frac{1}{4}x^2 - \frac{1}{4}x + \frac{9}{8}}$$

12. $m^2 = -4$. $\quad m = \pm 2i$. $\quad \therefore u_h(x) = c_1\cos 2x + c_2\sin 2x$. $\quad$ Let $u_p(x) = Ax\cos 2x$

$$-4A\sin 2x - 4Ax\cos 2x + 4Ax\cos 2x = \sin 2x. \quad \therefore -4A = 1 \text{ and } A = -\frac{1}{4}$$

$$\therefore \underline{u(x) = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}x\sin 2x}$$

14. $m^2+5m+6=0.\quad m_1=-2,\quad m_2=-3.\quad \therefore u_h(x)=c_1e^{-2x}+c_2xe^{-3x}$

Let $u_p(x)=A\sin 2x+B\cos 2x$. There results:

$-4A\sin 2x-4B\cos 2x+10A\cos 2x-10B\sin 2x+6A\sin 2x+6B\cos 2x=3\sin 2x$

$$\left.\begin{aligned}\cos 2x:&\quad -4B+10A+6B=0\\ \sin 2x:&\quad -4A-10B+6A=3\end{aligned}\right\}\qquad \therefore A=\frac{3}{52},\quad B=-\frac{15}{52}$$

$$\therefore \underline{u(x)=c_1e^{-2x}+c_2e^{-3x}+\frac{3}{52}(\sin 2x-5\cos 2x)}$$

16. $m=\pm 2i.\quad \therefore u_h(x)=c_1\cos 2x+c_2\sin 2x.$ Let $u_p(x)=A\sin x.\quad -A\sin x+4A\sin x=2\sin x$

$\therefore A=\dfrac{2}{3}$ and u(x)=$c_1\cos 2x+c_2\sin 2x+\dfrac{2}{3}\sin x.\quad u(0)=1=c_1.\quad u'(0)=0=2c_2+\dfrac{2}{3}$

$\therefore c_2=-\dfrac{1}{3}$ and $\underline{u(x)=\cos 2x-\dfrac{1}{3}\sin 2x+\dfrac{2}{3}\sin x}$

18. $m=\pm 2i.\quad \therefore u_h(x)=c_1\cos 2x+c_2\sin 2x.$ Let $u_p(x)=Ax\cos 2x$

$-4A\sin 2x-4Ax\cos 2x+4Ax\cos 2x=2\sin 2x$

$\therefore A=-1/2.\quad u(x)=c_1c\cos 2x+c_2\sin 2x-\dfrac{1}{2}x\cos 2x.\quad u(0)=0=c_1.\quad u'(0)=0=2c_2-\dfrac{1}{2}$

$\therefore c_2=\dfrac{1}{4}$ and $\underline{u(x)=\dfrac{1}{4}\sin 2x-\dfrac{1}{2}x\cos 2x}$

20. $m=\pm 4.\quad \therefore u_h(x)=c_1e^{4x}+c_2e^{-4x}.$ Let $u_p(x)=Axe^{4x}.\quad 8Ae^{4x}+16Axe^{4x}-16Axe^{4x}=2e^{4x}$

$\therefore 8A=2$ and $A=\dfrac{1}{4}.\quad \therefore u(x)=c_1e^{4x}+c_2e^{-4x}+\dfrac{1}{4}e^{4x}.\quad u(0)=0=c_1+c_2.\quad u'(0)=0=4c_1-4c_2+\dfrac{1}{4}$

Section 1.9.1 and 1.9.2

2. $y(t)=c_1+c_2e^{-0.2t}-49.1t.\quad y(0)=0=c_1+c_2.\quad \dot{y}(0)=100=-0.2c_2-49.1.\quad \therefore c_1=746,\quad c_2-746$

$\therefore y(t)=746(1-e^{-0.2t})-49.1t.$ To find $y_{\max}$: $\dot{y}(t)=0=0.2\times 746e^{-0.2t}-49.1.\quad \therefore t_m=5.557$ s

$\therefore y(t_m)=746(1-e^{-0.2\times 5.557})-49.1\times 5.557=\underline{228\text{ m}}$

4. The solution is (see Nos. 2 & 3) $y(t)=c_1+c_2e^{-0.049t}+200t$ where $C/M=0.5/(100/9.81)$ and $Mg/C=100/0.5.\quad y(0)=0=c_1+c_2.\quad \dot{y}(0)=0=-0.049c_2+200.\quad \therefore c_2=4080,\quad c_1=-4080$

$\therefore y(t)=4080(e^{-0.049t}-1)+200t$ so that $v=\dot{y}=-200e^{-0.049t}+200.$ As $t\to\infty,\quad v\to 200$

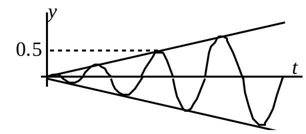
Let $v=0.99\times 200=198=-200e^{-0.049t}+200.\quad \therefore \dfrac{2}{200}=e^{-0.049t}.\qquad \underline{t=94\text{ s}}$

6. $2\ddot{y}+32y=0.1\sin 4t.\quad 2m^2+32=0.\quad \therefore m=\pm 4i.\quad \therefore y_h(t)=c_1\cos 4t+c_2\sin 4t.$

Let $y_p(t)=At\cos 4t.\quad 16A\sin 4t-32At\cos 4t+32At\cos 4t=0.1\sin 4t.\quad \therefore -16A=0.1$

$A=-1/160.\quad \therefore y(t)=c_1\cos 4t+c_2\sin 4t-\dfrac{t}{160}\cos 4t.\quad y(0)=0=c_1.\quad \dot{y}(0)=0=4c_2-\dfrac{1}{160}$

$\therefore y(t)=1/160(\sin 4t-4t\cos 4t)$



Ignore the sine term (it's at most 1) and let $\cos 4t=-1$.

Then $0.5=4t/640$ and $\underline{t=80\text{ s}}$

8. $m^2+9=0.\quad \therefore m=\pm 3i$ and $y_h(t)=c_1\cos 3t+c_2\sin 3t.$ Let $y_p(t)=At\sin 3t.$ Then

$6A\cos 3t-9At\sin 3t+9At\sin 3t=8\cos 3t.\quad \therefore A=4/3.\quad y(t)=c_1\cos 3t+c_2\sin 3t+\dfrac{4}{3}t\sin 3t$

$y(0)=0=c_1,\quad \dot{y}(0)=0=3c_2.\qquad \therefore \underline{y(t)=\dfrac{4}{3}t\sin 3t}$

10. $m^2+16=0.\quad \therefore m=\pm 4i$ and $y_h(t)=c_1\cos 4t+c_2\sin 4t.$ Let $y_p(t)=A\sin t.$ Then

$-A+10A=2.\quad \therefore A=\dfrac{2}{15}.\quad \therefore y(t)=c_1\cos 4t+c_2\sin 4t+\dfrac{2}{15}\sin t$

$y(0)=0=c_1,\quad \dot{y}(0)=10=4c_2+\dfrac{2}{15}.\qquad \therefore \underline{y(t)=\dfrac{37}{15}\sin 4t+\dfrac{2}{15}\sin t}$

12. $m^2+1=0.\quad \therefore m=\pm i$ and $y_h(t)=c_1\cos t+c_2\sin t.$ Let $y_p(t)=Ae^{-t}.$ Then

$Ae^{-t}+Ae^{-t}=2e^{-t}.\quad \therefore A=1.\quad \therefore y(t)=c_1\cos t+c_2\sin t+e^{-t}$

$y(0)=0=c_1+1,\quad \dot{y}(0)=2=c_2-1.\qquad \therefore \underline{y(t)=3\sin t-\cos t+e^{-t}}$

14. $Li''+Ri'+i/C=v'.\quad i''+10\,000i=12\,000\sin 2t.\quad m^2+10\,000=0.\quad \therefore m=\pm 100i$

$\therefore i_h(t)=c_1\cos 100t+c_2\sin 100t.\quad$ Let $i_p(t)=At\cos 100t.\quad -200A\sin 100t-10\,000At\cos 100t$

$+10\,000At\cos 100t=12\,000\sin 100t.\quad \therefore A=-60.\quad \therefore i(t)=c_1\cos 10t+c_2\sin 10t-60t\cos 10t$

$i(0)=0=c_1,\quad i'(0)=0=100c_2-60.\qquad \therefore \underline{i(t)=0.6\sin 100t-60t\cos 100t}$

16. $\dfrac{20}{9.81}\ddot{y}+98y=2\cos 7t.\quad \dfrac{20}{9.81}m^2+98=0.\quad \therefore m=\pm 6.933i$ and $y_h(t)=c_1\cos\omega_0 t+c_2\sin\omega_0 t$

Let $y_p(t)=A\cos 7t.$ Then $-99.898A+98A=2.\quad \therefore A=-1.054$

$y(t)=c_1\cos 6.933t+c_2\sin 6.933t-1.054\cos 7t.\quad y(0)=0=c_1=1.054,\quad \dot{y}(0)=0=0.6933c_2$

$\therefore y(t)=1.054(\cos 6.933t-\cos 7t).$ Using Eq.1.9.18, $\varepsilon=\dfrac{7-6.933}{2}=0.0335.\quad \omega\cong 7$

Therefore, using Eq. 1.9.19, $y(t)=2.1\sin 0.0335t\sin 7t.$ The beat frequency is $0.0335/2\pi$ which is <u>0.00533 hertz</u>. The maximum amplitude occurs when $\sin 0.0335t$ and $\sin 7t$ are 1. $\therefore \underline{y_{\max}=2.10\text{ m}}$

Section 1.9.3

4. Let $y_p(t) = A\sin 3t + B\cos 3t$. Then $-9AS - 9BC + 6AC - 6BS + AS + BC = C$

$$\left.\begin{array}{r}-9A - 6B + A = 0\\ -9B + 6A + B = 1\end{array}\right\} \quad \therefore A = \frac{3}{50}, \quad B = -\frac{2}{25} \quad \text{and} \quad \underline{y_p(t) = \frac{1}{50}(3\sin 3t - 4\cos 3t)} \quad \text{where} \quad \begin{array}{l}S = \sin 3t\\ C = \cos 3t\end{array}$$

6. Let $y_p(t) = A\sin 2t + B\cos 2t$. Then $-4AS - 4BC + 0.2AC - 0.2BS + 2AS + 2BC = 2S$

$$\left.\begin{array}{ll}S: & -4A - 0.2B + 2A = 2\\ C: & -4B + 0.2A + 2B = 1\end{array}\right\} \quad \therefore B = -0.099, \quad A = -0.99. \qquad \underline{y_p(t) = -0.99\sin 2t - 0.099\cos 2t}$$

8. Let $y_p(t) = A_1 \sin t + B_1 \cos t + A_2 \sin 2t + B_2 \cos 2t$.

$$-A_1S_1 - B_1C_1 - 4A_2S_2 - 4B_2C_2 + A_1C_1 - B_1S_1 + 2A_2C_2 - 2B_2S_2 + 2A_1S_1 + 2B_1C_1$$
$$+2A_2S_2 + 2B_2C_2 = C_1 - S_2.$$

$$\left.\begin{array}{ll}\left.\begin{array}{ll}S_1: & -A_1 - B_1 + 2A_1 = 0\\ C_1: & -B_1 + A_1 + 2B_1 = 1\end{array}\right\} & \begin{array}{l}A_1 = 1/2\\ B_1 = 1/2\end{array}\\ \left.\begin{array}{ll}S_2: & -4A_2 - 2B_2 + 2A_2 = -1\\ C_2: & -4B_2 + 2A_2 + 2B_2 = 0\end{array}\right\} & \begin{array}{l}A_2 = 1/4\\ B_2 = 1/4\end{array}\end{array}\right\} \quad \therefore \underline{y_p(t) = \frac{1}{2}(\sin t + \cos t) + \frac{1}{4}(\sin 2t + \cos 2t)}$$

10. Let $y_p(t) = A_1 \sin t + B_1 \cos t + A_2 \sin 2t + B_2 \cos 2t$. $-A_1S_1 - B_1C_1 - 4A_2S_2 - 4B_2C_2$

$$+7A_1C_1 - 7B_1S_1 + 14A_2C_2 - 14B_2S_2 + 10A_1S_1 + 10B_1C_1 + 10A_2S_2 + 10B_2C_2 = 2S_1 - C_2$$

$$\left.\begin{array}{ll}S_1: & -A_1 - 7B_1 + 10A_1 = 2\\ C_1: & -B_1 + 7A_1 + 10B_1 = 0\end{array}\right\} \quad \begin{array}{l}A_1 = 0.1385\\ B_1 = -0.1077\end{array}$$

$$\left.\begin{array}{ll}S_2: & -4A_2 - 14B_2 + 10A_2 = 0\\ C_2: & -4B_2 + 14A_2 + 10B_2 = -1\end{array}\right\} \quad \begin{array}{l}A_2 = 0.0875\\ B_2 = 0.0375\end{array} \qquad m^2 + 7m + 10 = 0. \quad \therefore m = -5, \quad -2$$

$$\therefore \underline{y(t) = c_1e^{-5t} + c_2e^{-2t} + 0.1385\sin t - 0.1077\cos t + 0.0875\sin 2t + 0.0375\cos 2t}$$

12. Let $y_p(t) = A\sin 2t + B\cos 2t$. Then $-4AS - 4BC + 0.2AC - 0.2BS + 2AS + 2BC = C$

$$\left.\begin{array}{ll}S: & -4A - 0.2B + 2A = 0\\ C: & -4B + 0.2A + 2B = 1\end{array}\right\} \quad \therefore \begin{array}{l}B = -0.495\\ A = 0.0495\end{array} \qquad m^2 + 0.1m + 2 = 0. \quad m = 0.05 \pm 1.413i$$

$$\therefore \underline{y(t) = e^{-0.05t}(c_1 \cos 1.413t + c_2 \sin 1.413t) + 0.0495\sin 2t - 0.495\cos 2t}$$

14. Let $y_p(t) = A\sin t + B\cos t$. Then $-AS - BC + 2AC - 2BS + AS + BC = 2S$

$$\left.\begin{array}{ll}S: & -A - 2B + A = 2\\ C: & -B + 2A + B = 0\end{array}\right\} \quad B = -1, \quad A = 0. \qquad m^2 + 2m + 1 = 0. \quad m = 1, \quad -1$$

$$\therefore y(t) = (c_1 + c_2t)e^{-t} - \cos t$$

$$\left.\begin{array}{l}y(0) = 0 = c_1 - 1\\ \dot{y}(0) = 0 = c_2 - c_1\end{array}\right\} \quad c_1 = 1, \quad c_2 = 1. \qquad \therefore \underline{y(t) = (1+t)e^{-t} - \cos t}$$

16 . Let $y_p(t) = A\cos t + B\sin t$. Then $-AC - BS + 0.1BC - 0.1AS + 2AC + 2BS = 20.2C$

$$\left.\begin{array}{ll} S: & -B - 0.1A + 2B = 0 \\ C: & -A + 0.1B + 6A = 20.2 \end{array}\right\} \therefore \begin{array}{l} B = 2 \\ A = 20 \end{array} \qquad m^2 + 0.1m + 2 = 0. \quad m = -0.05 \pm 1.413i$$

$$\therefore y(t) = e^{-0.05t}(c_1 \cos 1.413t + c_2 \sin 1.413t) + 20\cos 2t + 2\sin 2t$$

$$\left.\begin{array}{l} y(0) = 0 = c_1 + 20 \\ \dot{y}(0) = 10 = 1.413c_2 - 0.05c_1 + 4 \end{array}\right\} \begin{array}{l} c_1 = -20 \\ c_2 = 4 \end{array}$$

$$\therefore \underline{y(t) = e^{-0.05t}(4\sin 1.413t - 20\cos 1.413t) + 20\cos t + 2\sin t}$$

18. Let $y_p(t) = A\sin 4t + B4\cos t$. $\therefore -16AS - 16BC + .08AC - .08BS + 16AS + 16BC = 2S$

$$\left.\begin{array}{ll} S: & -16A - .08B + 16A = 10 \\ C: & -16B + .08A + 16B = 0 \end{array}\right\} \quad B = 25, \quad A = 0. \qquad m^2 + .02m + 16 = 0. \quad m = 0.01 \pm 4i$$

$$\therefore y(t) = (c_1 \cos 4t + c_2 \sin 4t)e^{-.01t} + 25\cos 4t$$

$$\left.\begin{array}{l} y(0) = 0 = c_1 + 25 \\ \dot{y}(0) = 0 = 4c_2 - .01c_1 - 5/4 \end{array}\right\} \begin{array}{l} c_1 = -25 \\ c_2 = -.0625 \end{array} \qquad \therefore \underline{y(t) = 25\cos 4t - e^{-.01t}(25\cos 4t + .0615\sin 4t)}$$

20. Using Eq. 1.9.24: $\Delta_{\max} = \dfrac{2 \times 20 \times 3}{5\sqrt{4 \times 3^2 \times 2^2 - 5^2}} = \underline{2.2 \text{ m}}$

Eq. 1.9.22: $\omega^2 = \omega_0^2 - \dfrac{C^2}{2M^2} = 2^2 - \dfrac{5^2}{2 \times 3^2} = 2.61.$ $\quad \therefore \underline{\omega = 1.62 \text{ rad/s}}$

22. $10^{-4}q'' + 30q' + q/10^{-6} = 12.$ $\quad m^2 + 3 \times 10^5 m + 10^{10} = 0.$ $\quad \therefore m - 2.62 \times 10^5, \quad -3.82 \times 10^4$

By inspection, $q_p(t) = 12 \times 10^{-6}$. $\quad \therefore q(t) = c_1 e^{-2.62 \times 10^5 t} + c_2 e^{-3.82 \times 10^4 t} + 12 \times 10^{-6}$

$$\left.\begin{array}{l} q(0) = 0 = c_1 + c_2 + 12 \times 10^{-6} \\ q'(0) = 0 = -2.62 \times 10^5 c_1 - 3.82 \times 10^4 c_2 \end{array}\right\} \therefore c_1 = 2.05 \times 10^{-6}, \quad c_2 = 14.05 \times 10^{-6}$$

$\therefore \underline{q(t) = \left(2.05e^{-2.62 \times 10^5 t} + 14.05e^{-3.82 \times 10^4 t} + 12\right) \times 10^{-6}}$ and $\underline{i(t) = 0.537\left(e^{-3.28 \times 10^4 t} - e^{-2.62 \times 10^5 t}\right)}$

24. See Ex.1.7.2. $Cv'' + v'/R + v/L = 0.$ $\quad v(0) = q(0)/C = 10^{-4}/10^{-6} = 100$

and $v'(0) = i(0)/C = 0.$ $\quad 10^{-6}v'' + v'/80 + v/10^{-4} = 0.$ $\quad m^2 + 12\,500m + 10^{10} = 0$

$m = -6250 \pm 99\,800i.$ $\quad \therefore v(t) = e^{-6250t}(c_1 \cos 99\,800t + c_2 \sin 99\,800t).$ $\quad v(0) = 100 = c_1$

$v'(0) = 0.$ $\quad \therefore c_2 = 6.26.$ $\quad i_R(t) = \dfrac{v(t)}{R} e^{-6250t}(100\cos 99\,800t + 6.26\sin 99\,800t)/80$

$t = 10^4, \quad i_R = e^{-0.625}(100\cos 9.98 + 6.26\sin 9.98)/80 = \underline{-0.591 \text{ amp}}$

26. $40i_1 + 10^{-4} i_3' = 12 \quad \boxed{1}, \qquad 40i_1 + q/10^{-6} + 20i_2 = 12 \quad \boxed{2}, \qquad q' = i_2 \quad \boxed{3}, \quad i_1 = i_2 + i_3 \quad \boxed{4}$

Substitute $\boxed{2}$ and $\boxed{3}$ into $\boxed{4}$: $\dfrac{12}{40} - \dfrac{q}{40\times 10^{-6}} - \dfrac{3}{2}q' = i_3$. Substitute this and $\boxed{2}$ into $\boxed{1}$:

$12 - q/10^{-6} - 20q' + 10^{-4}(-q'/40\times 10^{-6} - \dfrac{3}{2}q'') = 12.$ Then: $1.5q'' + 2.25\times 10^5 q' + 10^{10} q = 0$

$1.5m^2 + 2.25\times 10^5 m + 10^{10} = 0. \quad \therefore m = -75\,000 \pm 3.23\times 10^4 i.$ The solution for $q(t)$ is:

$q(t) = (c_1 \cos 32\,300t + c_2 \sin 32\,300t)e^{-75\,000t}$. From $\boxed{2}$: $q(0) = 12\times 0^{-6}. \quad \therefore c_1 = 12\times 10^{-6}$

Also, $i_2(0) = q'(0) = 0 = -75\,000c_1 + 32\,300c_2. \quad \therefore c_2 = 2.79\times 10^{-5}$. Using $\boxed{3}$

$i_2(t) = e^{-75\,000t}(-0.388 \sin 32\,300t + 0.9\cos 32\,300t - 2.09\sin 32\,300t - 0.9\cos 32\,300t)$

$\qquad = \underline{-2.48e^{-75\,000t} \sin 32\,300t}$

Section 1.10

4. The basic solutions are $u_1 = \sin x, \quad u_2 = \cos x, \quad g = 1/\cos x.$

$$W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1. \quad \therefore u_p(x) = -\sin x \int \frac{\cos x}{-1}\frac{1}{\cos x}dx + \cos x \int \frac{\sin x}{-1}\frac{1}{\cos x}dx$$

$$= \sin x \int dx - \cos x \int \frac{\sin x}{\cos x}dx = x\sin x + \cos x \ln\cos x. \quad \therefore \underline{u(x) = c_1 \sin x + c_2 \cos x + x\sin x + \cos x \ln \cos x}$$

6. Basic solutions: $u_1 = e^{2x}, \quad u_2 = xe^{2x}, \quad g = e^x/x. \quad W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x}.$

$$\therefore u_p(x) = -e^x \int \frac{xe^{2x}e^x/x}{e^{4x}}dx + xe^{2x}\int \frac{e^{2x}e^x/x}{e^{4x}}dx = -e^{2x}(-e^{-x}) + xe^{2x}\int \frac{1}{x}e^{-x}dx = e^x + xe^{2x}\int \frac{1}{x}e^{-x}dx$$

$$\therefore \underline{u(x) = (c_1 + c_2 x)e^{2x} + e^x + xe^{2x}\int \frac{1}{x}e^{-x}dx}$$

8. See Problem 7. Here $u_p(x) = -x\displaystyle\int \frac{(1/x)\times 2}{-2/x}dx + \frac{1}{x}\int \frac{x\times 2}{-2/x}dx = x^2 - \frac{x^2}{3}$

$\therefore \underline{u(x) = c_1 x + c_2/x + 2x^2/3}$

10. Basic solutions: $u_1 = 1, \quad u_2 = x^2, \quad g(x) = 1 + x. \quad W = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} = 2x.$

$$\therefore u_p(x) = -x^4 \int \frac{x^2(1+x)}{2x}dx + x^2\int \frac{1+x}{2x}dx = -\frac{1}{2}\left(\frac{x^2}{2} + \frac{x^3}{3}\right) + \frac{x^2}{2}(\ln x + x).$$

$$\therefore \underline{u(x) = c_1 + c_2 x^2 - \frac{x^2}{4} + \frac{x^3}{3} + \frac{x^2}{2}\ln x}$$

Section 1.11

2. $m^2+8m+12=0. \quad \therefore m=-6, \quad -2$ and $\underline{u(x)=c_1x^{-6}+c_2x^{-2}}$

4. $m^2+m-12=0. \quad \therefore m=3, \quad -4$ and $u(x)=c_1x^3+c_2x^{-4}$. Let $u_p=A$. Then $-12A=24$
$\therefore A=-2$ and $\underline{u(x)=c_1x^3+c_2x^{-4}-2}$

6. Refer to No.4 and let $A=-1$. Then $u(x)=c_1x^{-4}+c_2x^3-1$.

$$\left.\begin{aligned} u(1)&=0=c_1+c_2-1 \\ u'(1)&=0=-4c_1+3c_2 \end{aligned}\right\} \quad c_1=\frac{3}{7}, \quad c_2=\frac{4}{7}. \qquad \therefore \underline{u(x)=\frac{1}{7}(3x^{-4}+4x^3)}$$

8. The C-E eqn of order 1 is $xu'+a_1u=0$. With $u=x^m$, $xmx^{m-1}+a_1x^m=0$
or $mx^m+a_1x^m=0$ or $m=-a_1$. Thus, $\underline{u(x)=c_1x^{-a_1}}$

Section 1.12.1

2. e^{-2x} is a solution of $u''+4u'+4u=0$. Assume that ye^{-2x} is a solution of $u''+4u'+4u=e^{-2x}$.
Then substitute ye^{-2x}: $ye^{-2x}y''+4ye^{-2x}-4y'e^{-2x}+4(y'e^{-2x}-2ye^{-2x})+4ye^{-2x}=e^{-2x}$.
$\therefore y''=1$. This has the general solution $y(x)=x^2/2+c_1x+c_2$.
Hence, $\underline{u(x)=y(x)e^{-2x}=\frac{1}{2}x^2e^{-2x}+(c_1x+c_2)e^{-2x}}$

4. Note that $u_1(x)=x$ is a solution. Substitute $u=xy$ and use Eq. 1.12.6:
$xy''+[2-xp(x)x]y'=0$ or $y''+\left(\frac{2}{x}-xp(x)\right)y'=0$. Let $F(x)=e^{-\int xp(x)dx}$.
Then $\frac{d}{dx}\left(e^{\int \frac{2}{x}-xp(x)dx}y'\right)=\frac{d}{dx}[x^2F(x)y']=0$. So, $x^2F(x)y'=c_1$. Hence, $y(x)=c_1\int\frac{dx}{x^2F(x)}+c_2$
The solution is $u(x)=xy(x)=\underline{c_2x+c_1x\int\frac{dx}{x^2F(x)}}$

Section 1.12.2

2. For the Legendre equation, $p_0=-\frac{2x}{1-x^2}$, $\quad p_1=\frac{n(n+1)}{1-x^2}$. Thus, we have

$$I=\frac{n(n+1)}{1-x^2}-\frac{1}{4}\frac{4x^2}{(1-x^2)^2}-\frac{1}{2}(-2)\frac{1-x^2-x(-2x)}{(1-x^2)^2}=\underline{\frac{n(n+1)}{1-x^2}+\frac{1}{(1-x^2)^2}}$$

4. For the Bessel equation, $p_0 = \frac{1}{x}, \quad p_1 = -\frac{\alpha}{x}$. Consequently,

$$I = 1 - \frac{n^2}{x^2} - \frac{1}{4}\frac{1}{x^2} - \frac{1}{2}(-\frac{1}{x^2}) = 1 - \frac{n^2}{x^2} + \frac{1}{4x^2} = \underline{1 - \frac{1}{4x^2}(4n^2 - 1)}$$

6. For the Hermite equation, $p_0 = -2x, \quad p_1 = 2n$ and hence,

$$I = 2n - \frac{1}{4}(-2x)^2 - \frac{1}{2}(2x)' = \underline{2n + 1 - x^2}$$

Section 1.12.3

2. Here $p_0 = \tan x,\ p_1 = \cos^2 x. \quad \therefore z = c\int \cos x\, dx = c \sin x$. With $c = 1,\ z' = \cos x,$

$z'' = -\sin x$. Hence, $\cos^2 x \frac{d^2u}{dz^2} + (-\sin x + \cos x \tan x)\frac{du}{dz} + \cos^2 x\, u = 0. \quad \therefore u'' + u = 0.$

Then, $u(z) = c_1 \cos z + c_2 \sin z$ from which $\underline{u(x) = c_1 \cos(\sin x) + c_2 \sin(\sin x)}$.

4. Here $p_0 = \frac{2x - 3}{x^2}, \quad p_1 = 2x^{-4}. \quad \therefore z = c\int \sqrt{2} x^{-2} dx = \frac{1}{x}. \quad \therefore z' = -\frac{1}{x^2}, \quad z'' = 2x^{-3}.$

Hence, $x^{-4}\frac{d^2u}{dz^2} + \left[2x^{-3} + (-x^{-2})\frac{1}{x^2}(2x - 3)\right]\frac{du}{dz} + 2x^{-4}u = 0.$ This simplifies to

$u'' + 3u' + 2u = 0. \quad \therefore u(z) = c_1 e^{-z} + c_2 e^{-2z}, \quad \underline{u(x) = c_1 e^{-1/x} + c_2 e^{-2/x}}$

6. Here $p_0 = \frac{8x^2 - 1}{x}, \quad p_1 = 20x^2. \quad \therefore z = c\int \sqrt{20}\, x dx = x^2$ with $c = \frac{2}{\sqrt{20}}$.

Then $z' = 2x, \quad z'' = 2$. Thus, $4x^2 u'' + \left[2 + 2x\frac{8x^2 - 1}{x}\right]u' + 20x^2 u = 0,$ or $u'' + 4u' + 5u = 0.$

$\therefore u(z) = e^{-2z}(c_1 \cos\frac{3}{2}z + c_2 \sin\frac{3}{2}z)$. Finally, $\underline{u(x) = e^{-2x^2}\left[c_1 \cos(\tfrac{3}{2}x^2) + c_2 \sin(\tfrac{3}{2}x^2)\right]}$

Chapter 2 Series Method

Section 2.2

2. $f(x)=e^x,\quad f'(x)=e^x,\quad f''(x)=e^x,\quad \ldots,\quad f^{(n)}(x)=e^x.$
   Hence,

$$\boxed{e^x=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots+\frac{1}{n!}x^n+\cdots}$$

4. $f(x)=\cos x,\quad f'(x)=\sin x,\quad f''(x)=-\cos x,\quad f'''(x)=-\sin x,\quad \ldots$
   $\therefore\quad f(0)=1,\quad f'(0)=0,\quad f''(0)=-1,\quad f'''(0)=0,\quad \ldots$
   Hence,

$$\boxed{\cos x=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\cdots}$$

6. $f(x)=\ln(1+x),\quad f'(x)=\frac{1}{1+x},\quad f''(x)=-\frac{-1}{(1+x)^2},\quad f'''(x)=-\frac{2!}{(1+x)^3}$
   The remaining derivatives follow the pattern in Exercise 1 with appropriate minus signs. Since $f(0)=0$, we see from Exercise 1 that

$$\boxed{\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots}$$

8. Since $\frac{1}{2+x}=\frac{1/2}{1+(x/2)}$, we can use $x/2$ in place of x in Exercise 7 to obtain

$$\frac{1}{2+x}=\frac{1}{2}(1-x/2+(x/2)^2-(x/2)^3+\cdots)=\boxed{\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\cdots}.$$

10. From $\frac{7}{x^2-x-12}=\frac{7}{(x-4)(x+3)}=-\frac{1}{3+x}+\frac{1}{x-4}=-\frac{1/3}{1+(x/3)}-\frac{1/4}{1-(x/4)}$, we have

$$\frac{7}{x^2-x-12}=\left(-\frac{1}{4}-\frac{x}{4^2}-\frac{x^2}{4^3}-\cdots\right)-\left(\frac{1}{3}-\frac{x}{3^2}+\frac{x^2}{3^3}-\cdots\right)=\boxed{-\frac{7}{12}+\frac{7x}{144}-\frac{91x^2}{172}+\cdots}$$

.

12. $e^{-x^2}=1+(-x^2)+\frac{1}{2!}(-x^2)^2+\frac{1}{3!}(-x^2)^3+\cdots=\boxed{1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots}$

14. Since,

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dots}$$

long division leads to $\boxed{\tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \cdots}$

In this case, there is no obvious pattern of the formation of the coefficients.

16. Here, $\ln\left(\dfrac{4-x^2}{4}\right) = \ln\left(1 + \dfrac{x^2}{4}\right) = \boxed{-\dfrac{x^2}{4} - \dfrac{x^4}{32}x - \dfrac{x^6}{192} - \cdots}$

18. Since $e^{-x}\sin x = \left(1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dots\right)\left(x - \dfrac{x^3}{3!} + \dots\right)$

we have $\boxed{e^{-x}\sin x = x - x^2 + \dfrac{x^3}{3} - \dfrac{x^5}{30} + \cdots}$

20. $\displaystyle\int_0^x \frac{dt}{4-t^2} = \frac{1}{4}\int_0^x \frac{dt}{1-(t/2)^2} = \frac{1}{4}\int_0^x (1 + \frac{t^2}{4} + \frac{t^4}{16} + \cdots)\,dt = \boxed{\frac{x}{4} + \frac{x^3}{48} + \frac{x^5}{320} + \cdots}$

22. $\displaystyle\int \sin^2 x\,dx = \int (x - \frac{x^3}{6} + \frac{x^5}{120} - \dots)^2\,dx = \int (x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \dots)\,dx$

$= \boxed{\dfrac{x^3}{3} - \dfrac{x^5}{15} + \dfrac{2x^7}{315} - \cdots + C}$

24. Since $\displaystyle\int \sin x \cos x\,dx = \frac{1}{2}\int \sin 2x\,dx = \frac{1}{2}\left(1 - \frac{1}{2}\cos 2x\right)$, it is only necessary to substitute the expansion of $\cos 2x$ into the above formula and get, after simplification, $\displaystyle\int \sin x \cos x\,dx = \boxed{\frac{x^2}{2} - \frac{x^4}{6} + \frac{2x^6}{45} + \cdots + C}$

26. No singular points so $\underline{R = \infty}$.

28. (0, 0), (0, 2) and (0, –2) $\quad \therefore \underline{R = 0}$.

30. (–1, 0) $\quad \therefore \underline{R = 1}$.

32. Since, $b_{n+1}/b_n = 1$, we have $\underline{R=1}$.

34. Here $b_{n+1}/b_n = \dfrac{n(n+1)2^n}{n(n-1)2^{n+1}} = \dfrac{n+1}{2(n-1)} \to \dfrac{1}{2}$, so $\underline{R=2.}$

36. Same as Exercise 33 and hence $\underline{R=\infty}$.

38. $\dfrac{1}{x} = \dfrac{1}{1+(x-1)} = \underline{1-(x-1)+(x-1)^2-(x-1)^3+\cdots}$

40. Here, $\dfrac{1}{x^2-4} = \dfrac{1}{(x-2)(x+2)} = \dfrac{1}{4}\left(\dfrac{1}{x-2} - \dfrac{1}{x+2}\right) = \dfrac{1}{4}\left(\dfrac{-1}{1-(x-1)} + \dfrac{-\frac{1}{3}}{1+\frac{x-1}{3}}\right)$. Now using the ideas developed in Exercise 39 and collecting terms, we have

$$\boxed{\frac{1}{x^2-4} = -\frac{1}{3} - \frac{2}{9}(x-1) - \frac{7}{27}(x-1)^2 - \frac{20}{81}(x-1)^3 - \cdots}.$$

Section 2.3

2. Set $t = kx$ and define $w(t) = u(t/k) = u(x)$. Then $\dfrac{du}{dx} = \dfrac{dw}{dt}\dfrac{dt}{dx} = k\dfrac{dw}{dt}$. Hence, $\dfrac{du}{dx} + ku = k\dfrac{dw}{dt} + kw = 0$. By Exercise 1,

$$w(t) = b_0\left(1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \cdots\right) = \boxed{b_0\left(1 - kx + \frac{k^2x^2}{2} - \frac{k^3x^3}{3!} + \cdots\right)}, \quad \underline{R=\infty}.$$

4. $x\,u(x) = \sum_{n=0}^{\infty} b_n x^{n+1}$, $u'(x) = \sum_{n=1}^{\infty} n b_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)b_{n+1}x^n$ and $R=\infty$. So,

$u' + xu = b_1 + \sum_{n=1}^{\infty}\left((n+1)b_{n+1} + b_{n-1}\right)x^n = 0$. Therefore $b_1 = 0$ and

$b_{n+1} = -\dfrac{b_{n-1}}{n+1}$., for all $n \ge 1$. Thus $b_1 = b_3 = b_5 = \ldots = 0$.

Also, $b_2 = -\dfrac{b_0}{2}$, $b_4 = -\dfrac{b_2}{4} = \dfrac{b_0}{8}$, $b_6 = -\dfrac{b_4}{6} = \dfrac{b_0}{6\cdot 8}$, …. And so

$$\boxed{u(x) = b_o\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{6\cdot 8} + \frac{x^8}{6\cdot 8\cdot 8} - \cdots\right)}.$$

10. $(4-x^2)\sum_{n=2}^{\infty} n(n-1)x^{n-2} + \sum_{n=0}^{\infty} 2b_n x^n = 0$ and $R=\infty$. Collecting terms leads to

$\sum_{n=0}^{\infty}\big(4(n+1)(n+2)b_{n+2} + 2b_n\big)x^n - \sum_{n=2}^{\infty} n(n-1)b_n x^{n-2} = 0$. Hence,

$4\cdot 2b_2 + 2b_0 = 0,\quad 4\cdot 3\cdot 2b_3 + 2b_1 = 0$ and $4(n+2)(n+1)b_{n+2} = (n-2)(n+1)b_n$ for all $n \ge 2$. Therefore, $b_{n+2} = \dfrac{(n-2)}{4(n+2)}b_n$ for all $n \ge 2$. Now we use the facts that $b_0 = 0$ and $b_1 = 1$. Since $b_0 = 0$, $b_2 = b_4 = \ldots = 0$ we consider only odd subscripted coefficients. So, $b_{2k+1} = \dfrac{2k-3}{4(2k+1)}b_{2k-1}$. Then,

$$b_{2k+1} = -\frac{1\cdot 3\cdot 5\ldots(2k-3)}{4^k 1\cdot 3\cdot 5\ldots(2k+1)} = -\frac{1}{(2k-1)(2k+1)}$$

So, $\boxed{u(x) = x - \sum_{k=1}^{\infty}\frac{x^{2k+1}}{4^k(2k-1)(2k+1)}}$

12 $\sum_{n=2}^{\infty} n(n-1)b_n x^{n-2} - x^2\sum_{n=1}^{\infty} nb_n x^n + \left(x - \dfrac{x^3}{3!} + \ldots\right)\sum_{n=0}^{\infty} b_{n-1}x^n = 0$ and $R=\infty$. Now $b_0 = 0$ and $b_1 = 1$ leads to $2b_0 = 0,\quad 6b_3 = 0,\quad 12b_4 + 1 = 0,\quad 20b_5 - 1 = 0,\quad \ldots$. So,

$\boxed{u(x) = x - \dfrac{x^2}{12} + \dfrac{x^5}{20} + \cdots}$.

14. Set $y = x-2$ and $u(y) = u(x-2) = f(x)$. Then the equation for u is given by $(1+y)u' + u = 0$. Therefore, $u(y) = \sum_{n=0}^{\infty}(b_{n+1} + b_n)y^n = 0$. So, $b_n = (-1)^n b_0$ and we deduce that $u(y) = b_0(1 - y + y^2 - \ldots$. Also $\boxed{R=1}$. Hence
$f(x) = b_0(1-(1-x)+(1-x)^2 - \ldots$.
Given that $f(2) = 1 = b_0$, we have $f(x) = \dfrac{1}{1+(x-2)} = \dfrac{1}{x-1}$. Using 4 terms of the series, $\underline{f(1.9) \approx 1.111}$. But $\underline{f(1.9) = 10/9}$ from the exact solution.

16. Use the same substitutions as in Exercise 15. Here $x^2 = y^2 + 2y + 1$. So,

$2b_2 + b_0 + (4b_2 + 6b_3 + b_1)y + \sum_{n=2}^{\infty}\big((n+2)(n+1)b_{n+2} + 2n(n+1)b_{n+1} + (n^2 - n + 1)\big)y^n = 0$.

Therefore, $b_2 = -\dfrac{b_0}{2},\quad b_3 = -\dfrac{b_1}{6} + \dfrac{b_0}{3},\quad b_4 = -\dfrac{b_3}{3} - \dfrac{b_2}{4} = \dfrac{b_1}{6} - \dfrac{5b_0}{24},\quad \ldots$

$$u(x) = w(x-1) = w(y) = \boxed{b_0\left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots\right) + b_1\left((x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{6} + \cdots\right)}$$

.

18. For this differential equation $R=\infty$ and $b_0=4,\quad b_1=-2$. Also

$$\sum_{n=0}^{\infty}(n+2)(n+1)b_{n+2}x^n+\sum_{n=2}^{\infty}b_{n-2}x^n=2b_2+6b_3x+\sum_{n=2}^{\infty}\big((n+2)(n+1)b_{n+2}+b_{n-2}\big)x^n=0.$$

Therefore, $b_2=0,\quad b_3=0,\quad b_{n+2}=-\dfrac{b_{n-2}}{(n+2)(n+1)}$. So, $b_2=b_6=b_{10}=b_{14}=\ldots=0$ and $b_3=b_7=b_{11}=b_{15}=\ldots=0$. For the nonzero coefficients, $b_{4k}=\dfrac{-b_{4k-4}}{(4k)(4k-1)}$ and $b_{4k+1}=\dfrac{-b_{4k-3}}{(4k)(4k+1)}$. So,

$$b_{4k}=\frac{(-1)^k b_0}{3\cdot 4\cdot 7\cdot 8\ldots(4k)(4k-1)} \quad\text{and}\quad b_{4k}=\frac{(-1)^k b_0}{3\cdot 4\cdot 7\cdot 8\ldots(4k)(4k-1)}.$$

Using the initial conditions $b_0=4,\quad b_1=-2$, we arrive at

$$\boxed{\begin{aligned}u(x)=4-2x-\frac{x^4}{3}+\frac{x^5}{10}+\frac{x^8}{3\cdot7\cdot8}-\frac{x^9}{2\cdot5\cdot8\cdot9}-\frac{x^{12}}{3\cdot7\cdot8\cdot11\cdot12}+\frac{x^{13}}{2\cdot5\cdot8\cdot9\cdot12\cdot13}\\ +\frac{x^{16}}{3\cdot7\cdot8\cdot11\cdot12\cdot15\cdot16}-\frac{x^{17}}{2\cdot5\cdot8\cdot9\cdot12\cdot13\cdot16\cdot17}+\cdots\end{aligned}}.$$

20. Again set $y=x-2$ and $u(x)=u(y+2)=w(y)$. Now $x-1=y+1$ and $xu''+(x-1)u=(y+2)w''+(y+1)w=0$. So, after the standard manipulation of series, $w(y)=4b_2+b_0+\sum_{n=1}^{\infty}\big(2(n+1)(n+2)b_{n+2}+n(n+1)b_{n+1}+b_n+b_{n-1}\big)y^n=0$.

So $b_2=-\dfrac{b_0}{6},\quad b_3=-\dfrac{b_2}{6}-\dfrac{b_1}{12}-\dfrac{b_0}{12}=-\dfrac{b_0}{24}-\dfrac{b_1}{12},\quad b_4=\dfrac{b_0}{48}-\dfrac{b_1}{48},\quad b_5=\dfrac{b_0}{960}+\dfrac{b_1}{120},\quad \ldots$.

Therefore, $w(y)=b_0\left(1-\dfrac{y^2}{4}-\dfrac{y^3}{24}+\dfrac{y^4}{48}+\dfrac{y^5}{960}+\cdots\right)+b_1\left(y+\dfrac{y^3}{12}-\dfrac{y^4}{48}+\dfrac{y^5}{120}+\cdots\right)$.

Now we use $w(0)=u(2)=10=b_0$ and $w'(0)=u'(2)=0=b_1$ to obtain

$$\boxed{u(x)=10\left(1-\frac{(x-2)^2}{4}-\frac{(x-2)^3}{24}+\frac{(x-2)^4}{48}+\frac{(x-2)^5}{960}+\cdots\right)}$$

Sections 2.3.2, 2.3.3 and 2.3.4

2. c) From Eq. 2.3.32, $P_8(x) = \sum_{n=0}^{4} (-1)^n \dfrac{(16-2n)!}{2^8 n!(8-n)!(8-2n)!} x^{8-2n}$. So,

$$\boxed{P_8(x) = \frac{1}{2^7}\left(35 - 1260x^2 + 6930x^4 - 12\,012x^6 + 6\,435x^8\right)}$$

8. Here $\lambda(\lambda+1) = 6$ so that $\lambda = 2$, $\lambda = -3$. We use $\lambda = 2$ and write the general solution of the homogeneous equation in the form $u_g(x) = c_1 P_2(x) + c_2 Q_2(x)$. A particular solution of the inhomogeneous equation is obtained by determining A and B so that $Ax + B$ is a solution. This leads to $A = 1/4$ and $B = 0$. Hence,

$$u(x) = \frac{1}{4}x + c_1 P_2(x) + c_2 Q_2(x) = \boxed{\frac{1}{4}x + c_1 P_2(x) + c_2\left(P_2(x)Q_0(x) - \frac{3}{2}x\right)}$$

10. From Exercise 8 we have $w(x) = c_1 P_2(x) + c_2\left(P_2(x)Q_0(x) - \frac{3}{2}x\right)$. Now set $x = \cos\phi$ and let $u(\phi) = w(\cos\phi) = w(x)$. Use $Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$ to write

$$u(\phi) = c_1 P_2(\cos\phi) + c_2\left(\frac{P_2(\cos\phi)}{2}\ln\frac{1+\cos\phi}{1-\cos\phi}) - \frac{3}{2}\cos\phi\right).$$

Now collect terms to obtain

$$\boxed{u(\phi) = \frac{1}{2}\left(3\cos^2\phi - 1\right)\left(A + B\ln\frac{1+\cos\phi}{1-\cos\phi}\right) + \frac{3}{2}B\cos\phi}$$

Section 2.3.5

6. $H_4(x) = 4!\sum_{k=0}^{2}(-1)^k \dfrac{(2x)^{4-2k}}{k!(4-2k)1} = \underline{16x^4 - 48x^2 + 12}$.

Section 2.4

6. As in earlier exercises it is necessary to combine summations and simplify. Here we have

$$[(2r+5)(r-2)]a_0x^r + \sum_{n=1}^{\infty}[(n+r-2)(2n-2r+5)a_n + a_{n-1}]x^{n+r} = 0 .$$

Set $a_0 = 1$. For $r = 2$ $a_n = -\dfrac{a_{n-1}}{n(2n+9)}$. So, $a_1 = -\dfrac{1}{11}, a_2 = \dfrac{1}{286}, \cdots$ and hence

$$\boxed{u_1(x) = x^2\left(1 - \frac{x}{11} + \frac{x^2}{286} + \cdots\right)}$$

Now choose $r = -5/2$. So, $a_n = -\dfrac{a_{n-1}}{n(2n-9)}$ and $a_1 = \dfrac{1}{7}, a_2 = \dfrac{1}{70}, \ldots.$ Hence,

$$\boxed{u_2(x) = x^{-5/2}\left(1 + \frac{x}{7} + \frac{x^2}{70} + \cdots\right)}$$

10. Here, collecting summations and simplifying leads to

$$r(2r+1)a_0x^{r-1}$$
$$+\sum_{n=1}^{\infty}\big(-(n+r)(2n+2r+1)a_n(r) + (n+r)(2n+2r-3)a_{n-1}(r)\big)x^{n+r-1} = 0$$

Thus $r = 0$, $r = -1/2$ and we set $a_0 = 1$. So, in general,

$$a_n(r) = \frac{2n+2r-3}{2n+2r+1}a_{n-1}(r)$$

For $r = 0$, $a_n(0) = \dfrac{2n-3}{2n+1}a_{n-1}(0)$ and for $r = -1/2$, $a_n(-1/2) = \dfrac{n-1}{n+1}a_{n-1}(-1/2)$.

For $r = 0$, we find that $a_k = \dfrac{-2}{4k^2-1}$. For $r = -1/2$, $a_1 = -1$, $a_2 = 0$ and thus $a_k = 0$ for all $k \geq 2$. Hence,

$$\boxed{u(x) = c_1\left(1 - 2\sum_{k=1}^{\infty}\frac{x^k}{4k^2-1}\right) + c_2\, x^{-1/2}(1-x)}$$

Section 2.5

2. Let $t = x^2$ then $2xdx = dt$. Then $2xdx = dt$ and

$$\int_0^{\infty} x^2e^{-x^2}dx = \frac{1}{2}\int_0^{\infty} e^{-t}\sqrt{t}dt = \frac{1}{2}\Gamma(3/2) = \underline{0.443}$$

4. Let $t = \sqrt{x}$ then $xdx = 2tdt$ and $\int_0^{\infty} x^{-4}e^{-\sqrt{x}}dx = 2\int_0^{\infty} t^{-7}e^{-t}dt = 2\Gamma(-6) = \underline{\infty}$ suggests that the given integral is divergent. This can be demonstrated rigorously by standard techniques.

6. Let $\sqrt{x} = t$. Then $\int_0^\infty (1-x)^3 e^{-\sqrt{x}} dx = 2\int_0^\infty (t - 3t^3 + 3t^5 - t^7) e^{-t} dt$. So,

$$\int_0^\infty (1-x)^3 e^{-\sqrt{x}} dx = 2(\Gamma(2) - 3\Gamma(4) + 3\Gamma(6) - \Gamma(8)) = \underline{-9394}\,.$$

8. Let $x^{1/3} = t$ so that $dx = 3t^2 dt$. Then,

$$\int_0^\infty x^3 e^{-x^{1/3}} dx = \int_0^\infty t^9 e^{-t} 3t^2 dt = 3 \cdot 10! = \underline{10\,886\,600}$$

10. From Example 2.5.2: $r = 1$, $h = 3$ and $\therefore$ $\boxed{3^n \dfrac{\Gamma(n+1/3)}{\Gamma(1/3)}}$.

12. Since $\Gamma(\lambda + 1) = \lambda\Gamma(\lambda)$, $1 = \Gamma(1) = 0 \cdot \Gamma(0)$. So if $\Gamma(0)$ is a number, $0 = 1$. Hence $\Gamma(0)$ does not exist.

Section 2.9.1

8. As we have seen in previous exercises, the differential equation leads to a combination of summations which when simplified leads to a collection of equalities. In this example, $(r-1)^2 = 0$ and $(n + r - 1)a_n = a_{n-1.}$ So, $a_k = \left(r(r+1)\ldots(r+k-1)\right)^{-1}$, From this,

$$u_1(x) = u_r(x)\big|_{r=1} = \boxed{x\left(1 + \sum_{k=1}^\infty \frac{x^k}{k!}\right) = xe^x}$$

Set $f(r) = \left(r(r+1)\ldots(r+k-1)\right)^{-1}$ and thus

$$\frac{f'(x)}{f(x)} = \frac{d}{dx} \ln f(x) = -\left(\frac{1}{r} + \frac{1}{r+1} + \ldots + \frac{1}{r+k-1}\right)$$

So, $f'(1) = -\dfrac{h_k}{k!}$ and therefore,

$$\boxed{u_2(x) = xe^x \ln x - x\sum_{k=1}^\infty \frac{h_k x^k}{k!}}$$

10. Here $r^2 = 0$ and $(n+r)a_n = a_{n-1}$. Now $a_k = f(r)$, where

$$f(r) = \left((r+1)\ldots(r+k)\right)^{-1}., \text{ and } f'(r) = -f(r)\left(\frac{1}{r+1} + \ldots + \frac{1}{r+k}\right)$$

So, $f(0) = \dfrac{1}{k!}$ and $f'(0) = -\dfrac{h_k}{k!}$. Thus, $u_r(x) = x^r\left(1 + \sum_{k=1}^{\infty} f(r)x^k\right)$. Therefore,

$$\boxed{u_1(x) = 1 + \sum_{k=1}^{\infty}\frac{x^k}{k!} = e^x} \quad \text{and} \quad \boxed{u_2(x) = e^x \ln x - \sum_{k=1}^{\infty}\frac{h_k x^k}{k!}}$$

Section 2.9.2

6. Here $r(r+2) = 0$ and $(n+r)(n+r+2)a_n = -2(n+r+1)a_{n-1}$. The fact that the coefficients of a_n and a_{n-1} vanish for some n when $r = -2$ leads to two linearly independent solutions with no logarithm term. To find these solutions, let $r = -2$. Then, $n(n-2)a_n = -2(n-1)a_{n-1}$. Hence $-a_1 = 0 \cdot a_0$ and $0 \cdot a_2 = -2a_1 = 0$ shows that a_0 and a_2 are arbitrary. So select $a_0 = 1$ and $a_2 = 0$ which leads to $\underline{u_1(x) = x^{-2}}$ since $a_2 = 0$ implies $a_k = 0$ for all $k \geq 2$. Now set $a_0 = 0$ and $a_2 = 1$. Then

$$a_3 = \frac{-2\cdot 2}{1\cdot 3}, \quad a_4 = \frac{-2\cdot 3}{2\cdot 4}a_3, \quad \ldots, \quad a_{k+2} = \frac{-2(k+1)}{k(k+2)}a_{k+1}$$

Therefore,

$$a_{k+2} = \frac{(-2)^k(k+1)!}{(k+2)!/2} = (-1)^k\frac{2^{k+1}}{(k+2)k!}, \text{ for all } k \geq 0$$

And finally,

$$u_2(x) = x^{-2}\sum_{j=0}^{\infty}(-1)^j\frac{2^{j+1}}{(j+2)j!}x^{j+2} = \boxed{2\sum_{j=0}^{\infty}(-1)^j\frac{2^j}{(j+2)j!}x^j}$$

Section 2.10.1

2. We find $\boxed{J_0(x) \approx 1 - \dfrac{x^2}{4} + \dfrac{x^4}{64} - \dfrac{x^6}{2304}}$ and $\boxed{J_1(x) \approx \dfrac{x}{2} - \dfrac{x^3}{16} + \dfrac{x^5}{384} - \dfrac{x^7}{18\,432}}$.

6. $\lambda = 1/4$ and $\underline{u(x) = c_1 J_{1/4}(x) + c_2 J_{-1/4}(x)}$.

8. $\lambda = 1/2$ and $\underline{u(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)}$.

12. Since $i^{2k} = (-1)^k$,

$$J_n(ix) = \sum_{k=0}^{\infty}\frac{(-1)^k(ix)^{2k+n}}{2^{2k+n}\Gamma(n+k+1)} = i^n\sum_{k=0}^{\infty}\frac{x^{2k+n}}{2^{2k+n}\Gamma(n+k+1)} = i^n I_n(x)$$

Section 2.11

2. As in the previous exercise $u(0) = u'(0) = 0$ we obtain, at $x = 0$:

$$u^{(2)} + (1-x)u = 4x \qquad \Rightarrow u^{(2)}(0) = -1$$
$$u^{(3)} + (1-x)u^{(1)} - u = 4 \qquad \Rightarrow u^{(3)}(0) = 5$$
$$u^{(4)} + (1-x)u^{(2)} - 2u^{(1)} = 0 \qquad \Rightarrow u^{(4)}(0) = 1$$

So, $\boxed{u(x) = 1 - \frac{x^2}{2} + \frac{5x^3}{6} + \frac{x^4}{24} + \cdots}$

4. Given $f(0) = 6 \Rightarrow f'(0) = 6$, then

$$(1-x)f^{(1)} - f = 2x$$
$$(1-x)f^{(2)} - 2f^{(1)} = 2 \qquad \Rightarrow f^{(2)}(0) = 14$$
$$(1-x)f^{(3)} - 3f^{(2)} = 0 \qquad \Rightarrow f^{(3)}(0) = 3 \cdot 14$$
$$(1-x)f^{(4)} - 4f^{(3)} = 0 \qquad \Rightarrow f^{(4)}(0) = 4 \cdot 3 \cdot 14$$

Thus, $\underline{f(x) = 6 + 6x + 7x^2 + 7x^3 + 7x^4 + \cdots}$

6. (See Exercise 1, Section 2.3.) Here, $R = \infty$ and we use $u(0) = 1$:

$$u^{(1)} + u = x^2 \qquad \Rightarrow u^{(1)}(0) = -1$$
$$u^{(2)} + u^{(1)} = 2x \qquad \Rightarrow u^{(2)}(0) = 1$$
$$u^{(3)} + u^{(2)} = 2 \qquad \Rightarrow u^{(3)}(0) = 1$$
$$u^{(4)} + u^{(3)} = 0 \qquad \Rightarrow u^{(4)}(0) = -1$$

So,

$$\boxed{u(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} - \cdots}$$

8. See Exercies 4, Section2.3. We search for $u_p(x)$ satisfying $u' + xu = (\sin x)/x$ with $u(0) = 0$ and $u'(0) = 1$. So, as above,

$$u' + xu = \sin x / x = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$$
$$u^{(2)} + xu^{(1)} + u = -\frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \ldots$$
$$u^{(3)} + xu^{(2)} + 2u^{(1)} = -\frac{2}{3!} + \frac{4 \cdot 3x^2}{5!} - \frac{6 \cdot 5x^4}{7!} + \ldots$$

Thus $R = \infty$ and $u^{(2)}(0) = 0$, $u^{(3)}(0) = -7/3$, $u^{(4)}(0) = 0$, and $u^{(5)}(0) = 143/15$. Thus,

$$u_p(x) = x - \frac{7x^3}{3 \cdot 3!} + \frac{143x^5}{3 \cdot 5 \cdot 5!} + \ldots$$

$$\therefore \boxed{u(x) = b_0\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \cdots\right) + x - \frac{7x^3}{3 \cdot 3!} + \frac{143x^5}{3 \cdot 5 \cdot 5!} + \cdots}$$

10. See Exercise 8, Section 2.3. We find $u_p(x)$ satisfying $u^{(2)} + 6u^{(1)} + 5u = x^2 + 2\sin x$, $u_p(0) = 0$ and $u_p^{(1)}(0) = 0$. So, $u^{(2)}(0) = 0$ and by repeated differentiation,

$$u^{(3)} + 6u^{(2)} + 5u^{(1)} = 2x + 2\cos x \qquad \Rightarrow u^{(3)}(0) = 2$$
$$u^{(4)} + 6u^{(3)} + 5u^{(2)} = 2 - 2\sin x \qquad \Rightarrow u^{(4)}(0) = -16$$
$$u^{(5)} + 6u^{(4)} + 5u^{(3)} = -2\cos x \qquad \Rightarrow u^{(5)}(0) = 84$$

Then, $R = \infty$ and

$$u_p(x) = \frac{x^3}{3} - \frac{2x^4}{3} + \frac{7x^5}{10} + \dots$$

$$\therefore \boxed{u(x) = b_0\left(1 - \frac{5}{2}x^2 + 5x^3 - \frac{155}{24}x^4 + \cdots\right) + b_1\left(x - 3x^2 + \frac{31}{6}x^3 + \cdots\right) + \frac{x^3}{3} - \frac{2x^4}{3} + \frac{7x^5}{10} + \cdots}$$

3. The Laplace Transform

Section 3.2

2. $\int_0^\infty (t-3)e^{-st}dt = \int_0^\infty te^{-st}dt - 3\int_0^\infty e^{-st}dt = \boxed{\frac{1}{s^2} - \frac{3}{s}}$

$u = t \quad dv = e^{-st}dt$

$du = dt \quad v = -\frac{1}{s}e^{-st}$

4. $\int_0^\infty 2\sin t\, e^{-st}dt \qquad = \frac{2}{s}\int_0^\infty \cos t\, e^{-st}dt \qquad = \frac{2}{s}\left(\frac{1}{s} - \frac{1}{s}\int_0^\infty \sin t\, e^{-st}dt\right)$

$u = \sin t \quad dv = e^{-st}dt \qquad u = \cos t \quad dv = e^{-st}dt$

$du = \cos t\, dt \quad v = -\frac{1}{s}e^{-st} \qquad du = -\sin t dt \quad v = -\frac{1}{s}e^{-st}$

$\therefore \left(1 + \frac{1}{s^2}\right)\int_0^\infty 2\sin te^{-st}dt = \frac{2}{s^2}. \qquad \therefore \int_0^\infty 2\sin te^{-st}dt = \boxed{\frac{2}{s^2+1}}$

6. $\int_0^\infty t^{1/2}e^{-st}dt = \frac{1}{s\sqrt{s}}\int_0^\infty x^{1/2}e^{-x}dx = \underline{s^{-3/2}\Gamma(3/2)}$

Let $st = x, \ sdt = dx$ See Eq.2.5.1

8. $\int_0^\infty (4t^2-3)e^{-st}dt \qquad = 4\int_0^\infty t^2e^{-st}dt - 3\int_0^\infty e^{-st}dt \qquad = \frac{8}{s}\int_0^\infty te^{-st}dt - \frac{3}{s} = \boxed{\frac{8}{s^2} - \frac{3}{s}}$

$u = t^2 \quad dv = e^{-st}dt \qquad u = t \quad dv = e^{-st}dt$

$du = 2t\, dt \quad v = -\frac{1}{s}e^{-st} \qquad du = dt \quad v = -\frac{1}{s}e^{-st}$

10. $\int_0^\infty (t-2)^2 e^{-st}dt \qquad = \int_0^\infty t^2e^{-st}dt \qquad -4\int_0^\infty te^{-st}dt \qquad +4\int_0^\infty e^{-st}dt$

$u = t^2 \quad dv = e^{-st}dt \qquad u = t \quad dv = e^{-st}dt$

$du = 2t\, dt \quad v = -\frac{1}{s}e^{-st} \qquad du = dt \quad v = -\frac{1}{s}e^{-st}$

$= \frac{2}{s}\int_0^\infty te^{-st}dt - 4 + \frac{4}{s} = \left(\frac{2}{s} - 4\right)\frac{1}{s^2} + \frac{4}{s} = \boxed{\frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s}}$

12. $\int_0^\infty e^{2t-1}e^{-st}\,dt = e^{-1}\int_0^\infty e^{(2-s)t}\,dt = \boxed{\dfrac{e^{-1}}{s-2}}$

14. $f(t) = \begin{cases} t/2 & 0 < t < 4 \\ 0 & 4 < t \end{cases}$ $\qquad \int_0^\infty f(t)e^{-st}\,dt = \int_0^4 \frac{t}{2}e^{-st}\,dt = \boxed{\dfrac{1-4s^2}{2s^2}e^{-4s} + \dfrac{1}{2s^2}}$

$u = t \qquad dv = e^{-st}\,dt$

$du = dt \qquad v = -\frac{1}{s}e^{-st}$

16. $\mathcal{L}[e^{3t}(3t)] = \boxed{\dfrac{3}{(s-3)^2}}$

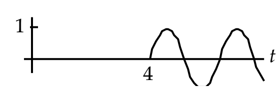
18. $\mathcal{L}[e^{-2t}(\cos 4t)] = \dfrac{s+2}{(s+2)^2+16} = \boxed{\dfrac{s+2}{s^2+4s+20}}$

20. $\mathcal{L}[e^{-t}(3\sin 2t)] = \dfrac{3\times 2}{(s+1)^2+4} = \boxed{\dfrac{6}{s^2+2s+5}}$

22. $\mathcal{L}[e^{-t}(\cos 4t)] - 2\mathcal{L}[e^{-t}(\sin 4t)] = \dfrac{s+1}{(s+1)^2+16} - \dfrac{2\times 4}{(s+1)^2+16} = \boxed{\dfrac{s-7}{s^2+2s+17}}$

24. $\mathcal{L}[e^{-2t}(t^2)] + 4\mathcal{L}[e^{-2t}(t)] + 5\mathcal{L}[e^{-2t}] = \dfrac{2}{(s+2)^3} + + \dfrac{4}{(s+2)^2} + \dfrac{5}{s+2} = \boxed{\dfrac{5s^2+24s+30}{(s+2)^3}}$

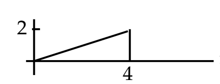
26. $\mathcal{L}[u_4(t)\sin \pi t] = \boxed{\dfrac{\pi e^{-4s}}{s^2+\pi^2}}$ Note: $\sin \pi(t-4) = \sin \pi t$



28. $\mathcal{L}[t/2] - \mathcal{L}[u_4(t)\,t/2]$

$\mathcal{L}[t/2] - \mathcal{L}\{u_4(t)[\tfrac{1}{2}(t-4)+2\}$

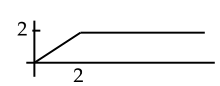
$= \dfrac{1}{2s^2} - \dfrac{e^{-4s}}{2s^2} - \dfrac{e^{-4s}}{s} = \boxed{\dfrac{1}{2s^2}(1-e^{-4s}) - \dfrac{1}{s}e^{-4s}}$



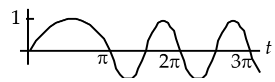
30. $f(t) = t - u_2(t)t + 2u_2(t)$

$\mathcal{L}[t] - \mathcal{L}\{u_2(t)[(t-2)+2]\} + 2\mathcal{L}[u_2(t)]$

$= \dfrac{1}{s^2} - \dfrac{e^{-2s}}{s^2} - \dfrac{2e^{-2s}}{s} + \dfrac{2e^{-2s}}{s} = \boxed{\dfrac{1}{s^2}(1-e^{-2s})}$



32. $f(t)=\sin t-u_\pi(t)\sin t+u_\pi(t)\sin 2t$



$\mathcal{L}[\sin t]-\mathcal{L}[u_\pi(t)\sin t]+\mathcal{L}[u_\pi(t)\sin 2t]$

$$=\frac{1}{s^2+1}-\frac{e^{-\pi s}}{s^2+1}+\frac{2e^{-\pi s}}{s^2+4}=\boxed{\frac{1}{s^2+1}(1-e^{-\pi s})+\frac{2}{s^2+4}e^{-\pi s}}$$

34. Use $\sinh t=(e^t-e^{-t})/2$. $\quad\therefore 2\sinh 3t\cos 2t=e^{3t}\cos 2t-e^{-3t}\cos 2t$

$$\mathcal{L}[e^{3t}\cos 2t]-\mathcal{L}[e^{-3t}\cos 2t]=\frac{s-3}{(s-3)^2+4}-\frac{s+3}{(s+3)^2+4}=\boxed{\frac{s-3}{s^2-6s+13}-\frac{s+3}{s^2+6s+13}}$$

36. See No.34: $\quad 6\sinh t\cos t=3(e^t-e^{-t})\cos t$

$$\mathcal{L}[3e^t\cos t]+\mathcal{L}[3e^{-t}\cos t]=\frac{3(s-1)}{(s-1)^2+1}-\frac{3(s+1)}{(s+1)^2+1}=\boxed{\frac{3(s-1)}{s^2-2s+2}-\frac{3(s+1)}{s^2+2s+2}}$$

38. See No.33: $\quad 2\cosh t\cos 2t=(e^t-e^{-t})\cos 2t$

$$\mathcal{L}[e^t\cos 2t]+\mathcal{L}[e^{-t}\cos 2t]=\frac{s-1}{(s-1)^2+4}+\frac{s+1}{(s+1)^2+4}=\boxed{\frac{3(s-1)}{s^2-2s+5}+\frac{3(s+1)}{s^2+2s+5}}$$

40. $F(s)=\frac{3}{s^3}+\frac{2}{s^2}$. $\quad\therefore \underline{f(t)=\frac{3}{2}t^2+2t}$

42. $F(s)=\frac{(s+1)-1}{(s+1)^3}=\frac{1}{(s+1)^2}-\frac{1}{(s+1)^3}$. $\quad\therefore \underline{f(t)=\frac{1}{2}t(2-t)e^{-t}}$

44. $F(s)=\frac{A}{s^2}+\frac{B}{s}+\frac{C}{s-2}=\frac{-1/2}{s^2}+\frac{-1/4}{s}+\frac{1/4}{s-2}$. $\quad\therefore \underline{f(t)=-\frac{1}{4}(2t+1-e^{2t})}$

46. $F(s)=\frac{-1/3}{s+2}+\frac{1/3}{s-1}$. $\quad\therefore \underline{f(t)=\frac{1}{3}(e^t-e^{-2t})}$

48. $F(s)=\frac{1}{s+1}e^{-s}$. $\quad\therefore \underline{f(t)=u_1(t)e^{1-t}}$

50. $F(s)=\frac{4}{(s+1)^2+4}$. $\quad\therefore \underline{f(t)=2e^{-t}\sin 2t}$

52. $F(s)=\frac{2}{(s-1)^2-4}$. $\quad\therefore \underline{f(t)=e^{-t}\sinh 2t}$

54. $F(s) = \dfrac{4(s+1)-4}{(s+1)^2+4}e^{-2\pi s}$. $\quad \therefore \underline{f(t) = 2e^{-(t-2\pi)}(2\cos 2t - \sin 2t)u_{2\pi}(t)}$

56. $F(s) = \dfrac{2(s+2)-1}{[(s+2)^2+9]}$. $\quad \therefore \underline{f(t) = \left(\dfrac{3}{2}t\sin 3t - \dfrac{1}{54}\sin 3t + \dfrac{1}{18}t\cos 3t\right)e^{-2t}}$

Section 3.3

2. $\mathcal{L}f'] = \int_0^a f'(t)e^{-st}dt + \int_a^b f'(t)e^{-st}dt + \int_b^\infty f'(t)e^{-st}dt$

$= s\mathcal{L}(f) - f(0) - [f(a^+) - f(a^-)]e^{-as} - [f(b^+) - f(b^-)]e^{-bs}$

4. $f''(t) = -\omega^2 \sin\omega t$. $\quad -\omega^2\mathcal{L}(\sin\omega t) = s^2\mathcal{L}(\sin\omega t) - s\times 0 - \omega\times 1$

$\therefore (s^2+\omega^2)\mathcal{L}(\sin\omega t) = \omega. \quad \therefore \mathcal{L}(\sin\omega t) = \boxed{\dfrac{\omega}{s^2+\omega^2}}$

6. $f''(t) = a^2 \sinh at$. $\quad a^2\mathcal{L}(\sinh at) = s^2\mathcal{L}(\sinh at) - s\times 0 - a\times 1$

$\therefore (s^2-a^2)\mathcal{L}(\sinh at) = a. \quad \therefore \mathcal{L}(\sinh at) = \boxed{\dfrac{a}{s^2-a^2}}$

8. $f''(t) = 4e^{2t}$. $\quad 4\mathcal{L}(e^{2t}) = s^2\mathcal{L}(e^{2t}) - s\times 1 - 2\times 1$

$\therefore (s^2-a^2)\mathcal{L}(e^{2t}) = s+2. \quad \therefore \mathcal{L}(e^{2t}) = \boxed{\dfrac{1}{s-2}}$

10. $f(t) = t - u_1(t)t + u_1(t) = t - u_1(t)[(t-1)+1] + u_1(t)$

$\mathcal{L}(f) = \dfrac{1}{s^2} - (\dfrac{1}{s^2} - \dfrac{1}{s})e^{-s} + \dfrac{1}{s}e^{-s} = \boxed{\dfrac{1}{s^2}(1-e^{-s})}$

$f'(t) = 1 - u_1(t)$. $\quad \mathcal{L}(f') = \boxed{\dfrac{1}{s} - \dfrac{1}{s}e^{-s}} = s\mathcal{L}(f) - 0. \quad \therefore \mathcal{L}(f) = \dfrac{1}{s^2}(1-e^{-s})$

12. $f'(t) = e^t + te^t$. $\quad \therefore \mathcal{L}(f') = \mathcal{L}(e^t) + \mathcal{L}(te^t) = s\mathcal{L}(te^t) + 0.$

$\therefore \mathcal{L}(te^t) = \dfrac{\mathcal{L}(e^t)}{s-1} = \boxed{\dfrac{1}{(s-1)^2}}$

14. $f'(t) = \cos t - t\sin t. \quad f''(t) = -\sin t - t\cos t. \quad \therefore \mathcal{L}(f'') = -2\mathcal{L}(\sin t) - \mathcal{L}(t\cos t)$

$$= s^2\mathcal{L}(t\cos t) - s\times 0 - 1. \quad \therefore \mathcal{L}(t\cos t) = \frac{1-2\mathcal{L}(\sin t)}{s^2+1} = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2} = \boxed{\frac{s^2-1}{(s^2+1)^2}}$$

16. $f'(t) = e^t\sin t + te^t\sin t + te^t\cos t. \quad \mathcal{L}(e^t\sin t) + \mathcal{L}(te^t\sin t) + \mathcal{L}(te^t\cos t) = s\mathcal{L}(te^t\sin t) = 0$

$\therefore \mathcal{L}(te^t\sin t) = \frac{1}{s-1}[\mathcal{L}(e^t\sin t) + \mathcal{L}(te^t\cos t)$. Use the 1st shifting property and the result

of No.14: $\therefore \mathcal{L}(te^t\sin t) = \frac{1}{s-1}\left\{\frac{1}{(s-1)^2+1}\right\} = \boxed{\frac{2}{s-1}\frac{s^2-2s+1}{(s^2-2s+2)^2}}$

18. $f'(t) = \cosh 2t + 2t\sinh 2t. \quad f''(t) = 4\sinh 2t + 4t\cosh 2t$

$\mathcal{L}(f'') = 4\mathcal{L}(\sinh 2t) + 4\mathcal{L}(t\cosh 2t) = s^2\mathcal{L}(t\cosh 2t) - s\times 0 - 1$

$$\therefore \mathcal{L}(t\cosh 2t) = \frac{4\mathcal{L}(\sinh 2t)+1}{s^2-4} = \frac{1}{s^2-4}\left[\frac{8}{s^2-4}+1\right] = \boxed{\frac{s^2+4}{(s^2-4)^2}}$$

20. $f'(t) = \sinh 2t + 2t\cosh 2t. \quad f''(t) = 4\cosh 2t + 4t\sinh 2t$

$\mathcal{L}(f'') = 4\mathcal{L}(\cosh 2t) + 4\mathcal{L}(t\sinh 2t) = s^2\mathcal{L}(t\sinh 2t) - s\times 0 - 0$

$$\therefore \mathcal{L}(t\sinh 2t) = \frac{4\mathcal{L}(\cosh 2t)}{s^2-4} = \boxed{\frac{4s}{(s^2-4)^2}}$$

22. $F(s) = \frac{1}{(s-\frac{1}{2})^2 - \frac{1}{4}}. \qquad \therefore f(t) = \underline{4e^{t/2}\sinh t/2}$ or $\underline{2(e^t-1)}$

24. $F(s) = \frac{4}{s^2(s^2+4)} = \frac{1}{s^2} - \frac{1}{s^2+4}. \qquad \therefore f(t) = \underline{t - \frac{1}{2}\sin 2t}$

26. $F(s) = \frac{6}{s^2(s^2-9)} = -\frac{2}{3}\left[\frac{1}{s^2} - \frac{1}{s^2-9}\right]. \qquad \therefore f(t) = \underline{-\frac{2}{3}(t - \frac{1}{3}\sinh 3t)}$

Section 3.4

2. $f(t) = \cos 2t. \qquad F(s) = \frac{s}{s^2+4}. \qquad \therefore -F'(s) = \boxed{\frac{s^2-4}{(s^2+4)^2}}$

4. $f(t) = \sinh t. \qquad F(s) = \frac{1}{s^2-1}. \qquad F'(s) = \frac{-2s}{(s^2-1)^2}$

$$\therefore F''(s) = \frac{(s^2-1)^2(-2) + 2s\cdot 2(s^2-1)2s}{(s^2-1)^4} = \boxed{\frac{2(3s^2+1)}{(s^2-1)^3}}$$

6. $f(t)=e^t-e^{-t}$. $\quad F(s)=\dfrac{1}{s-1}-\dfrac{1}{s+1}=\dfrac{2}{s^2-1}$. $\quad -F'(s)=\boxed{\dfrac{4s}{(s^2-1)^2}}$

8. $f(t)=e^{-t}\sin t$. $\quad F(s)=\dfrac{1}{(s+1)^2+1}$. $\quad -F'(s)=\boxed{\dfrac{2(s+1)}{(s^2+2s+2)^2}}$

10. $f(t)=\cosh t$. $\quad F(s)=\dfrac{1}{s^2-1}$. $\quad -F'(s)=\boxed{\dfrac{s^2+1}{(s^2-1)^2}}$

12. $f(t)=2-2\cosh 2t$. $\quad F(s)=\dfrac{2}{s}-\dfrac{2s}{s^2-4}$. $\quad \int_s^\infty\left(\dfrac{2}{s}-\dfrac{2s}{s^2-4}\right)ds=\boxed{\ln\dfrac{s^2-4}{s^2}}$

14. $f(t)=e^{2t}-1$. $\quad F(s)=\dfrac{1}{s-2}-\dfrac{1}{s}$. $\quad \int_s^\infty\left(\dfrac{1}{s-2}-\dfrac{1}{s}\right)ds=\boxed{\ln\dfrac{s}{s-2}}$

16. $F(s)=\dfrac{1}{(s+2)^2}$. $\quad \int_s^\infty (s+2)^{-2}\,ds=\dfrac{1}{s+2}=G(s)$. $\quad \therefore g(t)=e^{-2t}$. $\quad \therefore f(t)=tg(t)=\underline{te^{-2t}}$

18. $F(s)=\dfrac{s}{(s^2-4)^2}$. $\quad \dfrac{1}{2}\int_s^\infty (s-4)^{-2}\,2s\,ds=\dfrac{1/2}{s^2-4}$. $\quad \therefore g(t)=\dfrac{1}{4}\sinh 2t$. $\quad \therefore f(t)=\underline{\dfrac{t}{4}\sinh 2t}$

20. $G(s)=\ln(s-2)-\ln(s+3)$. $\quad H(s)=-G'(s)=\dfrac{1}{s+3}-\dfrac{1}{s-2}$

$\therefore h(t)=e^{-3t}-e^{2t}$. $\quad \therefore f(t)=\dfrac{1}{t}h(t)=\underline{\dfrac{1}{t}(e^{-3t}-e^{2t})}$

22. $G(s)=\ln(s^2+1)-\ln(s^2+4)$. $\quad H(s)=-G'(s)=\dfrac{2s}{s^2+4}-\dfrac{2s}{s^2+1}$

$\therefore h(t)=2\cos 2t-2\cos t$. $\quad \therefore f(t)=\dfrac{1}{t}h(t)=\underline{\dfrac{2}{t}(\cos 2t-\cos t)}$

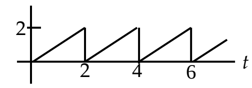
24. $G(s)=\ln(s^2+4s+5)-\ln(s^2+2s+5)$. $\quad H(s)=\dfrac{2s+2}{s^2+2s+5}-\dfrac{2s+4}{s^2+4s+5}=\dfrac{2(s+1)}{(s+1)^2+4}-\dfrac{2(s+2)}{(s+2)^2+1}$

$\therefore h(t)=2e^{-t}\cos 2t-2e^{-2t}\cos t$. $\quad \therefore f(t)=\dfrac{1}{t}h(t)=\underline{\dfrac{2}{t}(e^{-t}\cos 2t-e^{-2t}\cos t)}$

Section 3.5

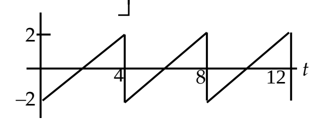
2. $\mathcal{L}(f) = \frac{1}{1-e^{-2s}} \int_0^2 t\, e^{-st} dt.\quad \int_0^2 t\, e^{-st} dt = -\frac{2}{s} e^{-2s} - \frac{1}{s^2}(e^{-2s} - 1)$

$\therefore \mathcal{L}(f) = \boxed{\frac{1}{s^2(1-e^{-2s})}\left(1 - e^{-2s} - 2se^{-2s}\right)}$

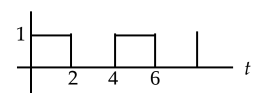


4. $\mathcal{L}(f) = \frac{1}{1-e^{-4s}} \int_0^4 (t-2)e^{-st} dt = \frac{1}{1-e^{-4s}}\left[\frac{1}{s^2}(1 - 2s - e^{-4s} - 2se^{-4s})\right]$

$= \boxed{\frac{1}{s^2(1-e^{-4s})}[1 - 2s - (1+2s)e^{-4s}}$

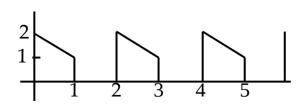


6. $\mathcal{L}(f) = \frac{1}{1-e^{-4s}} \int_0^2 e^{-st} dt = \boxed{\frac{1-e^{-2s}}{s^2(1-e^{-4s})}}$



8. $\mathcal{L}(f) = \frac{1}{1-e^{-2s}} \int_0^1 (2-t)e^{-st} dt = \frac{1}{1-e^{-4s}}\left\{-\frac{2}{3}(e^{-s} - 1) - \left[\frac{-1}{s}e^{-s} - \frac{1}{s^2}(e^{-s} - 1)\right]\right\}$

$= \boxed{\frac{1}{s^2(1-e^{-2s})}\left[2s - 1 - (s-1)e^{-s}\right]}$



Section 3.6

2. $F(s) = \frac{-5}{s} + \frac{5/2}{s-1} - \frac{2}{s+4} + \frac{25/6}{s+1}.\qquad \therefore f(t) = \underline{-5 + \frac{5}{2}e^{t} - 2e^{-4t} + \frac{25}{6}e^{-t}}$

4. $F(s) = \frac{1/4}{s+3} + \frac{3/4}{s-1} + \frac{1}{(s-1)^2}.\qquad \therefore f(t) = \underline{\frac{1}{4}e^{-3t} + \frac{3}{4}e^{t} + te^{t}}$

6. $F(s) = \frac{7/8}{s} + \frac{1/2}{s^2} + \frac{8/3}{s-1} + \frac{1/3}{(s-1)^2} + \frac{1/72}{s-4}.\qquad \therefore f(t) = \underline{\frac{7}{8} + \frac{1}{2}t + \frac{1}{3}(8+t)e^{t} + \frac{1}{72}e^{4t}}$

8. $F(s) = \frac{B_1 s + B_2}{s^2 + 400} + \frac{A_1 s + A_2}{s^2 + 441}.$

$B_1(20i) + B_2 = \left.\frac{5}{s^2 + 441}\right|_{s=20i} = \frac{5}{41}.\quad \therefore B_1 = 0,\quad B_2 = 5/41$

$A_1(21i) + A_2 = \left.\frac{5}{s^2 + 400}\right|_{s=21i} = -\frac{5}{41}.\quad \therefore A_1 = 0,\quad A_2 = -5/41$

$\therefore f(t) = \underline{5\left(\frac{1}{820}\sin 20t - \frac{1}{861}\sin 21t\right)}$

10. $F(s)=\dfrac{A_1}{s+1}+\dfrac{A_2}{(s+1)^2}+\dfrac{B_1 s+B_2}{s^2+4}$. $B_1(2i)+B_2=\left.\dfrac{s^2+1}{(s+1)^2}\right|_{s=2i}=-\dfrac{9}{25}-\dfrac{12}{25}i$

$\therefore B_1=-6/25,\quad B_2=-9/25,\quad A_1=-6/25,\quad A_2=2/5.\quad \therefore f(t)=\underline{(\frac{2}{5}t-\frac{6}{25})e^{-t}-\frac{6}{25}\cos 2t-\frac{9}{50}\sin 2t}$

12. $F(s)=\dfrac{C_1 s+C_2}{(s+4)^2}+\dfrac{B_1 s+B_2}{s^2+4}+\dfrac{D_1 s+D_2}{(s^2+1)^2}+\dfrac{A_1 s+A_2}{s^2+1}$

$C_1(2i)+C_2=\left.\dfrac{10}{(s^2+1)^2}\right|_{2i}=\dfrac{10}{9},\quad B_1(2i)+B_2=\left.\dfrac{d}{ds}\dfrac{10}{(s^2+1)^2}\right|_{2i}=\dfrac{80}{27}i.\quad \therefore C_1=0,\quad B_2=0$

$D_1(i)+D_2=\left.\dfrac{10}{(s^2+4)^2}\right|_{i}=\dfrac{10}{9},\quad A_1(i)+A_2=\left.\dfrac{d}{ds}\dfrac{10}{(s^2+4)^2}\right|_{i}=-\dfrac{40}{27}i.\quad \therefore D_1=0,\quad A_2=0$

$$\therefore f(t)=\underline{\frac{40}{27}(\cos 2t-\cos t)+\frac{5}{72}(\sin 2t-2t\cos 2t)+\frac{5}{9}(\sin t-t\cos t)}$$

Section 3.7

8. $e^{at}*e^{at}=\int_0^t e^{a(t-\tau)}e^{a\tau}\,d\tau=\int_0^t e^{at}d\tau=te^{at}$.

10. $\sin\omega t*\sin\omega t=\int_0^t \sin\omega(t-\tau)\sin\omega\tau\,d\tau=\dfrac{1}{2\omega}\left(\sin\omega t-t\omega\cos\omega t\right)$.

12. Since $L[1]=\dfrac{1}{s}$ and $L[\sinh at]=\dfrac{a}{s^2-a^2}$, the Convolution Theorem gives,

$\dfrac{1}{s(s^2+a^2)}=1*\dfrac{\sinh at}{a}$. But, $1*\dfrac{\sinh at}{a}=\int_0^t\dfrac{\sinh a\tau}{a}\,d\tau=\dfrac{1}{a^2}\left(-1+\cosh at\right)$. Thus,

$$L^{-1}\left[\frac{1}{s(s^2-a^2)}\right]=\frac{1}{a^2}\left(-1+\cosh at\right).$$

Section 3.8

2. $s^2Y(s)-sy(0)-\cancel{y'(0)}-4Y(s)=0.\quad \therefore Y(s)=\dfrac{2s}{s^2-4}.\qquad \therefore y(t)=\underline{2\cosh 2t}$

4. $s^2Y(s)+4Y(s)=\dfrac{2s}{s^2+1}.\quad \therefore Y(s)=\dfrac{2s}{(s^2+1)(s^2+4)}=\dfrac{2s/3}{s^2+1}+\dfrac{-4s/3}{s^2+4}.\quad \therefore y(t)=\underline{\frac{2}{3}\cos t-\frac{4}{3}\cos 2t}$

6. $s^2Y(s)+Y(s)=\frac{1}{s-1}+\frac{2}{s}$. $\therefore Y(s)=\frac{1}{(s-1)(s^2+1)}+\frac{2}{s(s^2+1)}=\frac{1/2}{s-1}+\frac{-(s+1)/2}{s^2+1}+\frac{2}{s}+\frac{-2s}{s^2+1}$

$$\therefore y(t)=\underline{2+\frac{1}{2}e^t-\frac{5}{2}\cos t-\frac{1}{2}\sin t}$$

8. $s^2Y(s)-sy(0)+4sY(s)-4y(0)+4Y(s)=0$. $\therefore Y(s)=\frac{s+4}{s^2+4s+4}=\frac{s+4}{(s+2)^2}=\frac{(s+2)+2}{(s+2)^2}$

$$=\frac{1}{s+2}+\frac{2}{(s+2)^2}. \qquad \therefore y(t)=\underline{e^{-2t}(1+2t)}$$

10. $s^2Y(s)-y'(0)+5sY(s)+6Y(s)=\frac{12}{s}$. $\therefore Y(s)=\frac{10s+12}{s(s^2+5s+6)}=\frac{2}{s}-\frac{6}{s+3}+\frac{4}{s+2}$

$$\therefore y(t)=\underline{2-6e^{-3t}+4e^{-2t}}$$

12. $s^2Y(s)-sy(0)+4sY(s)-4y(0)+4Y(s)=\frac{8}{s^2+4}$. $\therefore Y(s)=\frac{8}{(s+2)^2(s^2+4)}+\frac{4+s}{(s+2)^2}$

$$=\frac{1/2}{s+2}+\frac{1}{(s+2)^2}+\frac{-s/2}{s^2+4}+\frac{1}{s+2}+\frac{2}{(s+2)^2}. \qquad \therefore y(t)=\underline{\left(\frac{3}{2}+3t\right)e^{-2t}-\frac{1}{2}\cos 2t}$$

15. a) $s^2Y(s)+36Y(s)=\frac{1}{s}$. $\therefore Y(s)=\frac{1}{s(s^2+36)}=\frac{1/36}{s}-\frac{s/36}{s^2+36}$. $\therefore y(t)=\underline{\frac{1}{36}(1-\cos 6t)}$

c) $s^2Y(s)+36Y(s)=\frac{30}{s^2+36}$. $\therefore Y(s)=\frac{30}{(s^2+36)^2}$. $\therefore y(t)=\underline{\frac{5}{72}(\sin 6t-6t\cos 6t)}$

Note: Only solutions to selected parts of the following problems are provided.

16. a) $s^2Y(s)+sY(s)+36Y(s)=\frac{1}{s}$. $Y(s)=\frac{1}{s(s^2+s+36)}=\frac{1}{36}\left[\frac{1}{s}+\frac{-s-1}{(s+\frac{1}{2})^2+35.75}\right]$

$$\therefore y(t)=\underline{\frac{1}{36}-\frac{1}{36}(\cos 5.98t+\frac{1}{12}\sin 5.98t)e^{-t/2}}$$

c) $s^2Y(s)+sY(s)+36Y(s)=\frac{30}{s^2+36}$. $Y(s)=\frac{30}{(s^2+36)(s^2+s+36)}=\frac{1}{6}\left[\frac{-5s}{s^2+36}+\frac{5(s+1)}{(s+\frac{1}{2})^2+35.75}\right]$

$$\therefore y(t)=\underline{\frac{5}{6}(-\cos 6t+\cos 5.98t+\frac{1}{12}\sin 5.98t)e^{-t/2}}$$

17. b) $s^2Y(s)+12sY(s)+36Y(s)=\dfrac{10}{s^2+4}$. $Y(s)=\dfrac{10}{(s^2+4)(s^2+12s+36)}$

$$=\frac{1}{40}\left[\frac{-3s+8}{s^2+4}+\frac{-3}{s+6}+\frac{10}{(s+6)^2}\right].\quad \therefore y(t)=\underline{\frac{1}{40}\left[4\sin 2t-3\cos 2t-(3-10t)e^{-6t}\right]}$$

f) $s^2Y(s)+12sY(s)+36Y(s)=50$. $Y(s)=\dfrac{50}{(s+6)^2}$. $\therefore y(t)=\underline{50te^{-6t}}$

18. c) $s^2Y(s)+20sY(s)+36Y(s)=\dfrac{30}{s^2+36}$. $Y(s)=\dfrac{30}{(s^2+36)(s^2+20s+36)}$

$$=\frac{1}{192}\left[\frac{-48s}{s^2+36}+\frac{-1}{s+18}+\frac{9}{s+2}\right].\quad \therefore y(t)=\underline{\frac{1}{192}\left[-48\cos 6t-e^{-18t}+9e^{-2t}\right]}$$

19. The differential equation is $L\ddot{q}+R\dot{q}+\dfrac{1}{C}q=v(t),\quad i=\dot{q}$

a) $s^2Q(s)+100Q(s)=\dfrac{10}{s}$. $\therefore Q(s)=\dfrac{10}{s(s^2+100)}=\dfrac{1/10}{s}-\dfrac{s/10}{s^2+100}$

$\therefore q(t)=\dfrac{1}{10}-\dfrac{1}{10}\cos 10t$ and $\underline{i(t)=\sin 10t}$

c) $s^2Q(s)+100Q(s)=\dfrac{50}{s^2+100}$. $\therefore Q(s)=\dfrac{50}{(s^2+100)^2}$

$\therefore q(t)=\dfrac{1}{40}\sin 10t-\dfrac{1}{4}t\cos 10t$ and $\underline{i(t)=\dfrac{1}{4}t\sin 10t}$

e) $s^2Q(s)+100Q(s)=10$. $\therefore Q(s)=\dfrac{10}{s^2+100}$. $\therefore q(s)=\sin 10t$ and $\underline{i(t)=10\cos 10t}$

Note: Only solutions to selected parts of the following problems are provided.

20. e) $s^2Q(s)+16sQ(s)+100Q(s)=10$. $\therefore Q(s)=\dfrac{10}{(s+8)^2+36}$

$\therefore q(t)=\dfrac{5}{3}e^{-8t}\sin 6t$. $\therefore \underline{i(t)=\dfrac{10}{3}e^{-8t}(3\cos 6t-4\sin 6t)}$

21. b) $s^2Q(s)+20sQ(s)+100Q(s)=\dfrac{200}{s^2+400}$

$$\therefore Q(s)=\frac{200}{(s^2+400)(s^2+20s+100)}=\frac{-4s/250-6/25}{s^2+400}+\frac{4/250}{s+10}+\frac{2/5}{(s+10)^2}$$

$$\therefore q(t)=-\frac{1}{250}(4\cos 20t+3\sin 20t)+\frac{4}{250}e^{-10t}+\frac{2}{5}e^{-10t}$$

and $\underline{i(t)=\dfrac{1}{25}(8\sin 20t-6\cos 20t+6e^{-10t}-100te^{-10t})}$

d) $s^2Q(s)+20sQ(s)+100Q(s)=\dfrac{10}{s}-\dfrac{10}{s}e^{-2\pi s}$

$$\therefore Q(s)=\frac{10(1-e^{-2\pi s})}{s(s+10)^2}=\left(\frac{1/10}{s}+\frac{-1/10}{s+10}+\frac{-1}{(s+10)^2}\right)(1-e^{-2\pi s})$$

$$\therefore q(t)=\frac{1}{10}-\frac{1}{10}e^{-10t}-te^{-10t}-\left[\frac{1}{10}-\frac{1}{10}e^{-10(t-2\pi)}-(t-2\pi)e^{-10(t-2\pi)}\right]u_{2\pi}(t)$$

and $\underline{i(t)=10te^{-10t}-10(t-2\pi)e^{-10(t-2\pi)}u_{2\pi}(t)}$

22. e) $s^2Q(s)+25sQ(s)+100Q(s)=10.\quad \therefore Q(s)=\dfrac{10}{(s+20)(s+5)}=\dfrac{2/3}{s+5}-\dfrac{2/3}{s+20}$

$\therefore q(t)=-\dfrac{3}{2}(e^{-20t}-e^{-5t})$ and $\underline{i(t)=\dfrac{10}{3}(4e^{-20t}-e^{-5t})}$

f) $s^2Q(s)+25sQ(s)+100Q(s)=\dfrac{20}{s+1}.\quad \therefore Q(s)=\dfrac{20}{(s+1)(s+20)(s+5)}=\dfrac{5/19}{s+1}+\dfrac{4/57}{s+20}-\dfrac{1/3}{s+5}$

$\therefore q(t)=\dfrac{5}{19}e^{-t}+\dfrac{4}{57}e^{-20t}-\dfrac{1}{3}e^{-5t}.\qquad \therefore \underline{i(t)=\dfrac{5}{3}e^{-5t}-\dfrac{5}{19}e^{-t}-\dfrac{80}{57}e^{-20t}}$

23. The differential equation is $My''+Ky=F(t)$.

b) $s^2Y(s)+25Y(s)=\dfrac{1}{s}e^{-2\pi s}.\quad \therefore Y(s)=\dfrac{e^{-2\pi s}}{s(s^2+25)}=\left(\dfrac{1/25}{s}-\dfrac{s/25}{s^2+25}\right)e^{-2\pi s}$

$\therefore \underline{y(t)=\dfrac{1}{25}(1-\cos 5t)u_{2\pi}(t)}$

c) $s^2Y(s)+25Y(s)=\dfrac{5}{2\pi s^2}\left[1-e^{-2\pi s}\right]-\dfrac{5}{s}e^{-2\pi s}.\quad \therefore Y(s)=\dfrac{5(1-e^{-2\pi s})/2\pi}{s^2(s^2+25)}-\dfrac{5e^{-2\pi s}}{s(s^2+25)}$

$$=\frac{1}{10\pi}\left(\frac{1}{s^2}-\frac{1}{s^2+25}\right)(1-e^{-2\pi s})-\left(\frac{1/5}{s}-\frac{s/5}{s^2+25}\right)e^{-2\pi s}$$

$\therefore \underline{y(t)=\dfrac{1}{10\pi}[t-(t-2\pi)u_{2\pi}(t)]-\dfrac{1}{50\pi}\sin 5t[1-u_{2\pi}(t)]-\dfrac{1}{5}(1-\cos 5t)u_{2\pi}(t)}$

f) $s^2Y(s)+25Y(s)=5. \quad \therefore Y(s)=\dfrac{5}{s^2+25}. \qquad \therefore \underline{y(t)=\sin 5t}$

24. a) $s^2Q(s)+100Q(s)=\dfrac{1}{s}\left[e^{-2\pi s}-e^{-4\pi s}\right]. \quad \therefore Q(s)=\left(\dfrac{1/100}{s}-\dfrac{s/100}{s^2+100}\right)(e^{-2\pi s}-e^{-4\pi s})$

$\therefore q(t)=\dfrac{1}{100}(1-\cos 10t)[u_{2\pi}(t)-u_{4\pi}(t)]. \qquad \therefore \underline{i(t)=\dfrac{1}{10}\sin 10t[u_{2\pi}(t)-u_{4\pi}(t)]}$

b) $s^2Q(s)+100Q(s)=\dfrac{1}{s}e^{-2\pi s}. \quad \therefore Q(s)=\left(\dfrac{1/100}{s}-\dfrac{s/100}{s^2+100}\right)e^{-2\pi s}$

$\therefore q(t)=\dfrac{1}{100}(1-\cos 10t)u_{2\pi}(t). \qquad \therefore \underline{i(t)=\dfrac{1}{10}u_{2\pi}(t)\sin 10t}$

e) $s^2Q(s)+100Q(s)=\dfrac{1}{s}+\dfrac{1}{s}e^{-4\pi s}. \quad \therefore Q(s)=\dfrac{1}{s(s^2+100)}(1+e^{-4\pi s})=\left(\dfrac{1/100}{s}-\dfrac{s/100}{s^2+100}\right)(1+e^{-4\pi s})$

$q(t)=\dfrac{1}{100}(1-\cos 10t[1+u_{4\pi}(t)] \qquad \therefore \underline{i(t)=\dfrac{1}{10}\sin 10t[1+u_{4\pi}(t)]}$

g) $s^2Q(s)+100Q(s)=5+\dfrac{5}{s}e^{-2\pi s}. \qquad \therefore Q(s)=\dfrac{5}{s^2+100}+\left(\dfrac{1/20}{s}-\dfrac{s/20}{s^2+100}\right)e^{-2\pi s}$

$\therefore q(t)=\dfrac{1}{2}\sin 10t+\dfrac{1}{20}(1-\cos 10t)u_{2\pi}(t). \qquad \therefore \underline{i(t)=5\cos 10t+\dfrac{1}{2}u_{2\pi}(t)\sin 10t}$

25. For the system $Cy'+Ky=F(t)$.

b) $4sY(s)+100Y(s)=\dfrac{2}{s}e^{-2\pi s}. \quad \therefore Y(s)=\dfrac{1}{50}\left(\dfrac{1}{s}-\dfrac{1}{s+25}\right)e^{-2\pi s}. \quad \therefore \underline{y(t)=\dfrac{1}{50}[1-e^{-25(t-2\pi)}]u_{2\pi}(t)}$

f) $4sY(s)+100Y(s)=5. \quad \therefore Y(s)=\dfrac{10}{4(s+25)}. \qquad \therefore \underline{y(t)=\dfrac{5}{2}e^{-25t}}$

4. The Theory of Matrices

Section 4.2

2. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

8. Coefficient matrix: $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. Augmented matrix: $\begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}$

10. Coefficient matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Augmented matrix: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

12. (a), (b), (d), (g), (i), (j), (k)

Section 4.3

2. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \to \begin{bmatrix} c & b+d \\ c & d \end{bmatrix} \to \begin{bmatrix} c & b+d \\ 0 & -b \end{bmatrix}$

6. $\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 - x_3 &= 0 \\ 0 &= 1 \end{aligned}$ Augmented matrix: $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

8. The augmented matrix can be reduced as follows:

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix} \to \dots \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The given system is thus equivalent to $x_1 + x_2 = 0, \; x_3 = 1$. We set $x_2 = t$ and thus $x_1 = -t, \; x_2 = t, \; x_3 = -1$ is a solution for all t.

10. The reduction of the augmented matrix is given by:

$$\begin{bmatrix} 1 & -1 & 6 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 6 \\ 0 & 2 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 6 \\ 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -3 \end{bmatrix}$$

Thus $x = 3, \; y = -3$.

12. Here the augmented matrix reduces as follows:

`

$$\begin{bmatrix} 3 & 4 & 7 \\ 2 & -5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 4/3 & 7/3 \\ 2 & -5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 4/3 & 7/3 \\ 0 & -23/3 & -8/3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 4/3 & 7/3 \\ 0 & 1 & 8/23 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 43/23 \\ 0 & 1 & 8/23 \end{bmatrix}.$$

Thus, $x = 43/23, \; y = 8/23$.

14. $$\begin{bmatrix} 1 & -3 & 1 & -2 \\ 1 & -3 & -1 & 0 \\ 0 & -3 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1/3 & 0 \end{bmatrix}$$

The first row corresponds to the equation $0x + 0y + 0z = -2$, a contradiction. Hence the system has no solutions.

16. $$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}. \quad \text{So, } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

18. $\begin{bmatrix}1 & 2 & 1 & -2\\ 1 & 1 & 0 & 3\\ 1 & 0 & 1 & 4\end{bmatrix} \to \begin{bmatrix}1 & 2 & 1 & -2\\ 0 & -1 & -1 & 5\\ 0 & -2 & 0 & 6\end{bmatrix} \to \begin{bmatrix}1 & 2 & 1 & -2\\ 0 & 1 & 1 & -5\\ 0 & -2 & 0 & 6\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & 8\\ 0 & 1 & 1 & -5\\ 0 & 0 & 2 & -4\end{bmatrix}$

$\to \begin{bmatrix}1 & 0 & -1 & 8\\ 0 & 1 & 1 & -5\\ 0 & 0 & 1 & -2\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 6\\ 0 & 1 & 0 & -3\\ 0 & 0 & 1 & -2\end{bmatrix}$. So, $\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}6\\ -3\\ -2\end{bmatrix}$

Section 4.4

2. a) No; add (–1) times row 2 to row 1. b) No; multiply by 1/2.

   c) Yes. d) Yes. e) No; interchange rows 1 & 2. f) Yes.

   g) Yes. h) Yes. i) Yes; if * = 1; no otherwise. j) Yes.

   k) No; add (–3) times row 2 to row 1. l) Yes.

.Section 4.5

10. $\mathbf{A} - \mathbf{B} = \begin{bmatrix}1 & 0 & -1\\ 1 & -1 & -2\\ 2 & 1 & 3\end{bmatrix}$, $\quad \mathbf{B} - \mathbf{A} = \begin{bmatrix}-1 & 0 & 1\\ -1 & 1 & 2\\ -2 & -1 & -3\end{bmatrix}$

12. $4\mathbf{A} + 4\mathbf{B} = \begin{bmatrix}12 & 8 & 14\\ 4 & -4 & -8\\ 24 & 12 & -12\end{bmatrix} = 4(\mathbf{A} + \mathbf{B})$

14. $\mathbf{A}^{\mathrm{T}} = \begin{bmatrix}2 & 1 & 4\\ 1 & -1 & 2\\ 0 & -2 & 0\end{bmatrix}$ 15. Show it to be true.

16. $\mathbf{A}+\mathbf{A}^{\mathrm{T}}=\begin{bmatrix}2&1&0\\1&-1&-2\\4&2&0\end{bmatrix}+\begin{bmatrix}2&1&4\\1&-1&2\\0&-2&0\end{bmatrix}=\begin{bmatrix}4&2&4\\2&-2&0\\4&0&0\end{bmatrix}$

$\mathbf{A}-\mathbf{A}^{\mathrm{T}}=\begin{bmatrix}2&1&0\\1&-1&-2\\4&2&0\end{bmatrix}-\begin{bmatrix}2&1&4\\1&-1&2\\0&-2&0\end{bmatrix}=\begin{bmatrix}0&0&-4\\0&0&-4\\4&4&0\end{bmatrix}$

18. $\mathbf{B}^{\mathrm{T}}=\begin{bmatrix}0&-2&2\\-8&0&-4\\6&2&4\end{bmatrix}$, $\mathbf{B}+\mathbf{B}^{\mathrm{T}}=\begin{bmatrix}0&-8&6\\-2&0&2\\2&-4&4\end{bmatrix}+\begin{bmatrix}0&-2&2\\-8&0&-4\\6&2&4\end{bmatrix}=\begin{bmatrix}0&-10&8\\-10&0&-2\\8&-2&8\end{bmatrix}$

$\mathbf{B}-\mathbf{B}^{\mathrm{T}}=\begin{bmatrix}0&-8&6\\-2&0&2\\2&-4&4\end{bmatrix}-\begin{bmatrix}0&-2&2\\-8&0&-4\\6&2&4\end{bmatrix}=\begin{bmatrix}0&-6&4\\6&0&6\\-4&-6&0\end{bmatrix}$

Section 4.6

12. $\begin{bmatrix}1&0&4\\0&2&4\\0&0&1\end{bmatrix}$. 4. $\left[a^2+b^2+c^2\right]$. 6. $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$.

10. $\mathbf{A}^2=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}=\begin{bmatrix}1&2\\0&1\end{bmatrix}$.

12. $\mathbf{A}=\begin{bmatrix}1&2\\0&1\end{bmatrix}$, $\mathbf{A}^2=\begin{bmatrix}1&4\\0&1\end{bmatrix}$, $3\mathbf{A}^2=\begin{bmatrix}3&12\\0&3\end{bmatrix}$, $-9\mathbf{A}=\begin{bmatrix}9&18\\0&9\end{bmatrix}$, $6\mathbf{I}=\begin{bmatrix}6&0\\0&6\end{bmatrix}$.

Thus, $3\mathbf{A}^2+6\mathbf{A}-9\mathbf{I}=\begin{bmatrix}0&-6\\0&0\end{bmatrix}$.

12. $\mathbf{A}-\mathbf{I}=\begin{bmatrix}0&2\\0&0\end{bmatrix}$, $\mathbf{A}-2\mathbf{I}=\begin{bmatrix}-1&2\\0&-1\end{bmatrix}$. So, $3(\mathbf{A}-\mathbf{I})(\mathbf{A}-2\mathbf{I})=\begin{bmatrix}0&-6\\0&0\end{bmatrix}=3\mathbf{A}^2+6\mathbf{A}-9\mathbf{I}$.

18. $[x_1,\ x_2,\ x_3]\begin{bmatrix}0 & -1 & 0\\ 0 & 1 & 1\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = [x_1,\ x_2,\ x_3]\begin{bmatrix}x_1 - x_2\\ x_2 + x_3\\ x_2 - x_3\end{bmatrix}$

$= x_1(x_1 - x_2) + x_2(x_2 + x_3) + x_3(x_2 - x_3) = x_1^2 + x_2^2 - x_3^2 - x_1x_2 + 2x_2x_3\,.$

22. $\mathbf{A} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$, $\mathbf{B} - \mathbf{C} = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$. So, for any $\mathbf{C}$, define $\mathbf{B} = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} + \mathbf{C}$.

A second example uses $\mathbf{A} = \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$, $\mathbf{B} - \mathbf{C} = \begin{bmatrix}-2 & -2 & -2\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$.

30. $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{A}$. $(\mathbf{A}\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \mathbf{A}\mathbf{A}^{\mathrm{T}}$.

32. $\mathbf{BA} = -4$

34. $\begin{bmatrix}2 & 4 & -1\\ -2 & -4 & 1\\ 4 & 8 & -2\end{bmatrix}\begin{bmatrix}3 & 2 & 1\\ -2 & 0 & -1\\ 1 & 0 & 1\end{bmatrix} = \begin{bmatrix}-3 & 4 & -3\\ 3 & -4 & 3\\ -6 & 8 & -6\end{bmatrix}$.

36. $\begin{bmatrix}3 & 2 & 1\\ -2 & 0 & -1\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0 & 2\\ 1 & 2 & 1\\ 2 & -1 & -1\end{bmatrix} = \begin{bmatrix}1 & 3 & 6\\ 0 & 1 & -3\\ 1 & -1 & 1\end{bmatrix}$.

38. $\begin{bmatrix}-1 & 0 & 2\\ 1 & 2 & 1\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ -1\\ 2\end{bmatrix} = \begin{bmatrix}3\\ 1\\ 1\end{bmatrix}$.

40. $\mathbf{A} + \mathbf{B} = \begin{bmatrix}1 & 3 & 1\\ -2 & 4 & 1\\ 3 & 1 & 1\end{bmatrix}$, $(\mathbf{A} + \mathbf{B})\mathbf{C} = \begin{bmatrix}1 & 3 & 1\\ -2 & 4 & 1\\ 3 & 1 & 1\end{bmatrix}\begin{bmatrix}2\\ 0\\ -1\end{bmatrix} = \begin{bmatrix}1\\ -3\\ 5\end{bmatrix}$.

42. $\mathbf{D}(\mathbf{A} + \mathbf{B}) = \mathbf{DA} + \mathbf{DB} = [1,\ 2,\ 0]\begin{bmatrix}1 & 3 & 1\\ -2 & 4 & 1\\ 3 & 1 & 1\end{bmatrix} = [-3,\ 11,\ 3]$.

44. $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 0 & -1 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 3 & 1 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 13 & 3 \\ 0 & 3 & 2 \end{bmatrix}$ and

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 0 & 3 & 1 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 1 \\ 6 & 5 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

46. $\mathbf{A}^2 = \begin{bmatrix} -3 & 6 & 1 \\ -2 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. $\mathbf{A}^3 = \begin{bmatrix} -3 & 6 & 1 \\ -2 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 3 & 1 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 3 & 2 \\ -1 & -4 & -3 \\ 0 & 0 & 1 \end{bmatrix}$.

48. $\mathbf{A} + \mathbf{C}$ is undefined.

50. Undefined.

51. $\mathbf{A}\,\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ -1 & 1 & -2 \\ 3 & 9 & 6 \end{bmatrix}$. Also, $\mathbf{B}\mathbf{A} = \begin{bmatrix} 4 & 1 & 9 \\ 2 & 1 & 6 \\ 2 & -3 & 6 \end{bmatrix}$. Hence, they are unequal.

52. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$.

Section 4.7

2. $\mathbf{A}_{n\mathrm{x}n}\mathbf{C}_{n\mathrm{x}k} = \mathbf{B}_{n\mathrm{x}k}$. But $\mathbf{A}^{-1}$ is $n \times n$ so $\mathbf{A}^{-1}_{n\mathrm{x}n}\mathbf{B}_{n\mathrm{x}k}$ is defined. However, $\mathbf{B}_{n\mathrm{x}k}\mathbf{A}^{-1}_{n\mathrm{x}n}$ is undefined unless $n = k$.

4. $\mathbf{A}^{\mathrm{T}}(\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{-1}\mathbf{A})^{\mathrm{T}} = \mathbf{I}^{\mathrm{T}} = \mathbf{I}$ and. $(\mathbf{A}^{-1})^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = (\mathbf{A}\mathbf{A}^{-1})^{\mathrm{T}} = \mathbf{I}^{\mathrm{T}} = \mathbf{I}$. So, $(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$.

6. Let $\mathbf{A} = -\mathbf{B}$. Then, even if $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist, $\mathbf{A} + \mathbf{B}$ is singular.

Section 4.8

2. $\begin{bmatrix} 2 & 0 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 3 & 4 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 7 & \vdots & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/2 & \vdots & 1/2 & 0 & 0 \\ 0 & 1 & 4/3 & \vdots & 0 & 1/3 & 0 \\ 0 & 0 & 1 & \vdots & 0 & 0 & 1/7 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 0 & \vdots & 1/2 & 0 & -1/14 \\ 0 & 1 & 0 & \vdots & 0 & 1/3 & -4/21 \\ 0 & 0 & 1 & \vdots & 0 & 0 & 1/7 \end{bmatrix}$. So $\mathbf{A}^{-1} = \begin{bmatrix} 1/2 & 0 & -1/14 \\ 0 & 1/3 & -4/21 \\ 0 & 0 & 1/7 \end{bmatrix}$.

4. $\begin{bmatrix} 0 & 0 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \vdots & 0 & 1 & 0 \\ 1 & 0 & 1 & \vdots & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 & 0 & 1 \\ 0 & 1 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 1 & 0 & 0 \end{bmatrix}$, so $\mathbf{A}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

6. $\begin{bmatrix} 2 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 4 & -1 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 1 & -1 & \vdots & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \vdots & 1/2 & 0 & 0 \\ 0 & -1 & 0 & \vdots & -2 & 1 & 0 \\ 0 & 1 & -1 & \vdots & 0 & 0 & 1 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 0 & \vdots & 1/2 & 0 & 0 \\ 0 & 1 & 0 & \vdots & 2 & -1 & 0 \\ 0 & 0 & -1 & \vdots & -2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \vdots & 1/2 & 0 & 0 \\ 0 & 1 & 0 & \vdots & 2 & -1 & 0 \\ 0 & 0 & 1 & \vdots & 2 & -1 & -1 \end{bmatrix}$.

So, $\mathbf{A}^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 2 & -1 & \\ 2 & -1 & -1 \end{bmatrix}$.

14. $\begin{bmatrix} 3 & 1 & 2 & \vdots & 1 & 0 & 0 \\ 1 & 0 & 1 & \vdots & 0 & 1 & 0 \\ 2 & 1 & -1 & \vdots & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 1 & 0 \\ 3 & 1 & 2 & \vdots & 1 & 0 & 0 \\ 2 & 1 & -1 & \vdots & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 1 & 0 \\ 0 & 1 & -1 & \vdots & 1 & -3 & 0 \\ 0 & 1 & -3 & \vdots & 0 & -2 & 1 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 1 & 0 \\ 0 & 1 & -1 & \vdots & 1 & -3 & 0 \\ 0 & 0 & -2 & \vdots & -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 1 & 0 \\ 0 & 1 & -1 & \vdots & 1 & -3 & 0 \\ 0 & 0 & 1 & \vdots & -1/2 & 1/2 & 1/2 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 0 & \vdots & -1/2 & 3/2 & 1/2 \\ 0 & 1 & 0 & \vdots & 3/2 & -7/2 & -1/2 \\ 0 & 0 & 1 & \vdots & -1/2 & 1/2 & 1/2 \end{bmatrix}$. So $\mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 & 1 \\ 3 & -7 & -1 \\ 1 & -1 & -1 \end{bmatrix}$

16. $\begin{bmatrix} 2 & 1 & 1 & 1 & \vdots & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & \vdots & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 2 & \vdots & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 & -1 & \vdots & 1 & 0 & -2 & 0 \\ 1 & 2 & 0 & 0 & \vdots & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 2 & \vdots & 0 & 0 & -1 & 1 \end{bmatrix}$

$$\to \begin{bmatrix} 0 & 1 & 1 & -1 & \vdots & 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1/2 & \vdots & 0 & 1/2 & -1/2 & 0 \\ 1 & 0 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & \vdots & 0 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 & -1/2 & \vdots & 1 & -1/2 & -3/2 & 0 \\ 0 & 1 & 0 & -1/2 & \vdots & 0 & 1/2 & -1/2 & 0 \\ 1 & 0 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & \vdots & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & 0 & -3/2 & \vdots & 1 & -1/3 & 1/2 & -1 \\ 0 & 1 & 0 & -1/2 & \vdots & 0 & 1/2 & -1/2 & 0 \\ 1 & 0 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & \vdots & 0 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 1 & \vdots & -2/3 & 1/3 & 1/3 & 2/3 \\ 0 & 1 & 0 & -1/2 & \vdots & 0 & 1/2 & -1/2 & 0 \\ 1 & 0 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & \vdots & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & 0 & 1 & \vdots & -2/3 & 1/3 & 1/3 & 2/3 \\ 0 & 1 & 0 & 0 & \vdots & -1/3 & 2/3 & -1/3 & 1/3 \\ 1 & 0 & 0 & 0 & \vdots & 2/3 & -1/3 & -4/3 & -2/3 \\ 0 & 0 & 1 & 0 & \vdots & 2/3 & -1/3 & -4/3 & 1/3 \end{bmatrix}. \text{ So, } \mathbf{A}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 & -2 \\ -1 & 2 & -1 & 1 \\ 2 & -1 & -4 & 1 \\ -2 & 1 & 1 & 2 \end{bmatrix}.$$

18. $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$ is singular because the first row is twice the second.

20. $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ is singular.

22. $\begin{bmatrix} 1 & 0 & 2 & \vdots & 1 & 0 & 1 \\ 2 & 1 & 1 & \vdots & 0 & 1 & 0 \\ 1 & 1 & 1 & \vdots & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & \vdots & 1 & 0 & 1 \\ 0 & 1 & -3 & \vdots & -2 & 1 & 0 \\ 0 & 1 & -1 & \vdots & -1 & 0 & 1 \end{bmatrix}$

$$\to \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 & 1 & -1 \\ 0 & 1 & 0 & \vdots & -1/2 & -1/2 & 3/2 \\ 0 & 0 & 2 & \vdots & 1 & -1 & 1 \end{bmatrix}. \text{ So, } \mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 & -2 \\ -1 & -1 & 3 \\ 1 & -1 & 1 \end{bmatrix}.$$

24. $\begin{bmatrix} 3 & 1 & 2 & \vdots & 1 & 0 & 0 \\ -1 & 2 & 1 & \vdots & 0 & 1 & 0 \\ 0 & 1 & 1 & \vdots & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 2 & 1 & \vdots & 0 & 1 & 0 \\ 3 & 1 & 2 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 1 & \vdots & 0 & 0 & 1 \end{bmatrix}$

$\to \begin{bmatrix} -1 & 2 & 1 & \vdots & 0 & 1 & 0 \\ 0 & 7 & 5 & \vdots & 1 & 3 & 0 \\ 0 & 1 & 1 & \vdots & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & \vdots & 0 & 1 & 0 \\ 0 & 1 & 1 & \vdots & 0 & 0 & 1 \\ 0 & 7 & 5 & \vdots & 1 & 3 & 1 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 1 & 0 \\ 0 & 1 & 1 & \vdots & 0 & 0 & 1 \\ 0 & 0 & -1 & \vdots & 1/2 & 3/2 & -7/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \vdots & 1/2 & 1/2 & -3/2 \\ 0 & 1 & 0 & \vdots & 1/2 & 3/2 & -3/2 \\ 0 & 0 & -1 & \vdots & 1/2 & 3/2 & -7/2 \end{bmatrix}$

Thus, $\mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -3 \\ 1 & 3 & -5 \\ -1 & -3 & 7 \end{bmatrix}$.

Section 4.9

2. $\det \mathbf{A}^n = \det \mathbf{A} \det \mathbf{A} \ldots \det \mathbf{A} = (\det \mathbf{A})^n$.

4. There are $n!$ permutations of the integers, $1, 2, \ldots n$.

10. $\det\begin{vmatrix} 2 & 0 \\ -1 & 3 \end{vmatrix} = 6 - 0 = 6$. 11. $\det\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3 - 2 = 1$. 12. $\det\begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 2 - 2 = 0$.

14. $\det\begin{bmatrix} 4 & -1 & 3 \\ 2 & 2 & 2 \\ 1 & -2 & 4 \end{bmatrix} = 32 - 2 - 12 - 6 + 16 + 8 = 36.$

22. $\det\begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} = -\det\begin{bmatrix} -1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix} = -\det\begin{bmatrix} -1 & 0 & 3 \\ 0 & 3 & 10 \\ 0 & 2 & 6 \end{bmatrix} = -\det\begin{bmatrix} -1 & 0 & 3 \\ 0 & 3 & 10 \\ 0 & 0 & -2/3 \end{bmatrix} = -2$.

24. $\det\begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ * & * & * \end{bmatrix} = \det\begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix} = 0.$

26. $\det\begin{bmatrix} 3 & 1 & -1 & 0 \\ 2 & 2 & 2 & 1 \\ -1 & 3 & 0 & 4 \\ 8 & 6 & -2 & 2 \end{bmatrix} = \det\begin{bmatrix} 1 & -3 & 0 & -4 \\ 2 & 2 & 2 & 1 \\ 3 & 1 & -1 & 0 \\ 8 & 6 & -2 & 2 \end{bmatrix} = \det\begin{bmatrix} 1 & -3 & 0 & -4 \\ 0 & 8 & 2 & 9 \\ 0 & 10 & -1 & 12 \\ 0 & 30 & -2 & 34 \end{bmatrix}$

$$= \det\begin{bmatrix} 1 & -3 & 0 & -4 \\ 0 & 8 & 2 & 9 \\ 0 & 0 & -7/2 & 3/4 \\ 0 & 0 & -19/2 & 1/4 \end{bmatrix} = \det\begin{bmatrix} 1 & -3 & 0 & -4 \\ 0 & 8 & 2 & 9 \\ 0 & 0 & -7/2 & 3/4 \\ 0 & 0 & 0 & -25/4 \end{bmatrix} = 50.$$

28. $\det\begin{bmatrix} 2 & -1 & 6 & 3 \\ -2 & 4 & 5 & -1 \\ 3 & 4 & 3 & 2 \\ 1 & -1 & 2 & 3 \end{bmatrix} = -\det\begin{bmatrix} 1 & -1 & 2 & 3 \\ -2 & 4 & 5 & -1 \\ 3 & 4 & 3 & 2 \\ 2 & -1 & 6 & 3 \end{bmatrix} = -\det\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & 9 & -5 \\ 0 & 7 & -3 & -7 \\ 0 & 1 & 2 & -3 \end{bmatrix}$

$$= -\det\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 9 & -5 \\ 0 & 7 & -3 & -7 \end{bmatrix} = -\det\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & -17 & 14 \end{bmatrix} = -\det\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 87/5 \end{bmatrix} = -87.$$

Section 4.9.2

2. $-3\det\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} - 3\det\begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} = -12 - 24 = -36.$

4. $-2\det\begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 3 & -1 \\ 3 & 3 \end{bmatrix} = -6 - 24 = -36.$

6. $\det\begin{bmatrix} 2 & 8 & 6 \\ -1 & 2 & 0 \\ 3 & 7 & 3 \end{bmatrix} + 3\det\begin{bmatrix} 2 & 0 & 6 \\ -1 & 3 & 0 \\ 5 & 7 & 3 \end{bmatrix} = -42 + 3\cdot(-78) = -276.$

8. $-6\det\begin{bmatrix} -1 & 4 & 2 \\ 0 & -1 & 3 \\ 3 & 5 & 7 \end{bmatrix} + 3\det\begin{bmatrix} 2 & 0 & 6 \\ -1 & 4 & 2 \\ 0 & -1 & 3 \end{bmatrix} = -6\cdot 64 + 3\cdot 36 = -276.$

10. $1 \cdot 0 = 0.$

12. By the first row: $\det\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + 2\det\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix} = 0 + 2 = 2.$

14. By the third row: $-\det\begin{bmatrix}3 & 2\\-1 & 1\end{bmatrix} + \det\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix} = -5 + 7 = 2.$

16. By the first column: 1

Section 4.9.3

2. $\mathbf{A}^+ = \begin{bmatrix}3 & -6\\-1 & 2\end{bmatrix}$. **A** is singular.

4. $\mathbf{A}^+ = \begin{bmatrix}0 & -2\\0 & 1\end{bmatrix}$. **A** is singular.

6. $\mathbf{A}^+ = \begin{bmatrix}0 & -1 & 1\\2 & -1 & -1\\-2 & 3 & 1\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}0 & 2 & -2\\-1 & -1 & 3\\1 & -1 & 1\end{bmatrix}$, $\mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix}0 & 2 & -2\\-1 & -1 & 3\\1 & -1 & 1\end{bmatrix}$.

8. $\mathbf{A}^+ = \begin{bmatrix}1 & 1 & -1\\1 & 3 & -3\\-3 & -5 & 7\end{bmatrix} = \begin{bmatrix}1 & 1 & -3\\1 & 3 & -5\\-1 & -3 & 7\end{bmatrix}$. **A** is singular.

10. $\det\mathbf{A} = \det\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} = 2.$ $x = \frac{1}{2}\det\begin{bmatrix}6 & -1\\0 & 1\end{bmatrix} = 3,$ $y = \frac{1}{2}\det\begin{bmatrix}1 & 6\\1 & 0\end{bmatrix} = -3.$

12. $\det\mathbf{A} = \det\begin{bmatrix}3 & 4\\2 & -5\end{bmatrix} = -23.$ $x = \frac{-1}{23}\det\begin{bmatrix}7 & 4\\2 & -5\end{bmatrix} = \frac{43}{23},$ $y = \frac{-1}{23}\det\begin{bmatrix}3 & 7\\2 & 2\end{bmatrix} = \frac{8}{23}.$

14. $\det\mathbf{A} = \det\begin{bmatrix}1 & -3 & 1\\1 & -3 & -1\\0 & -3 & -1\end{bmatrix} = -6.$ $x = \frac{-1}{6}\det\begin{bmatrix}-2 & -3 & 1\\0 & -3 & -1\\0 & -3 & -1\end{bmatrix} = -2,$

$$y = \frac{-1}{6}\det\begin{bmatrix}1 & -2 & 1\\1 & 0 & -1\\0 & 0 & 1\end{bmatrix} = -\frac{1}{3},\quad z = \frac{-1}{6}\det\begin{bmatrix}1 & -3 & -2\\1 & -3 & 0\\0 & -3 & 0\end{bmatrix} = -1.$$

Section 4.10

2. Same as Exercise 1. The computation here is more tedious:

$$\det\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & 0 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & 0 \end{bmatrix} = \det\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & -1 & 0 \end{bmatrix} = 2.$$

Hence the vectors are linearly independent.

4. $\det\begin{bmatrix} k & 0 & 1 \\ 1 & k & -1 \\ -1 & 1 & k \end{bmatrix} = k^3 + 2k + 1$. So the determinant vanishes if and only if k is a root of this cubic. The only real root is seen to be negative between $-1/2$ and 0.

6. Not necessarily. If $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 = \mathbf{x}_4$, then $\mathbf{x}_1, \mathbf{x}_2\ \mathbf{x}_3, \mathbf{x}_4$ is dependent regardless of whether $\mathbf{x}_1, \mathbf{x}_2\ \mathbf{x}_3$ is.

8. $2\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3 = \mathbf{0}$ from the work in Exercise 7.

10. $\begin{bmatrix} 1 & 1 & 0 & 1 & \vdots & \mathbf{x}_1 \\ 1 & 0 & 0 & 1 & \vdots & \mathbf{x}_2 \\ 0 & 0 & 1 & 0 & \vdots & \mathbf{x}_3 \\ 1 & -1 & 0 & 1 & \vdots & \mathbf{x}_4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 & \vdots & \mathbf{x}_1 \\ 1 & 0 & 0 & 1 & \vdots & \mathbf{x}_2 \\ 0 & 0 & 1 & 0 & \vdots & \mathbf{x}_3 \\ 2 & 0 & 0 & 2 & \vdots & \mathbf{x}_4 \end{bmatrix}$, so $\mathbf{x}_1 - 2\mathbf{x}_2 + \mathbf{x}_4 = \mathbf{0}$.

12. At least one scalar is non-zero, say, $a_1 \neq 0$. Then

$$-a_1\mathbf{x}_1 = a_2\mathbf{x}_2 + a_3\mathbf{x}_3 + \dots + a_k\mathbf{x}_k.$$

Since $-a_1\mathbf{x}_1 \neq \mathbf{0}$, at least one of the scalars on the rhs. Is also non-zero. Therefore, at least two scalars are not zero.

Section 4.11

8. $\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{x}=\mathbf{0}$ is equivalent to $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{x}=\mathbf{0}$. Thus $r = 2$, $n = 4$, $\eta = 2$.

   As in Exercise 2, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and the general solution is $\alpha_1\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

10. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has $r = 3$, $n = 3$, so $\eta = 0$. Thus, there is only the trivial solution $\mathbf{x} = \mathbf{0}$.

12. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Here, $r = 2$, $n = 3$, so $\eta = 1$. Thus, $\mathbf{x} = \alpha\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is the general solution.

14. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Here, $r = 2$, $n = 3$, $\eta = 1$. Thus $\mathbf{x} = \alpha\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

16. Assume there are n columns in $[1, 1, \ldots, 1]$. Then $r = 1$ and $\eta = n - 1$. So, the general solution is $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_{n-1}\mathbf{x}_{n-1}$, where

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ \cdots, \ \mathbf{x}_{n-1} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \quad \mathbf{x}_g = \sum^{n-1} \alpha_k \mathbf{x}_k \ .$$

18. Suppose $\mathbf{u} = [u_1,\ u_2,\ \ldots,\ u_n]$ and $u_1 \neq 0$. Then,

$$\mathbf{u}\mathbf{u}^{\mathrm{T}} = \begin{bmatrix} u_1^2 & u_1u_2 & \cdots & u_1u_n \\ u_2u_1 & u_2^2 & \cdots & u_2u_n \\ \vdots & \vdots & \vdots & \vdots \\ u_nu_1 & u_nu_2 & \vdots & u_n^2 \end{bmatrix} \to \cdots \to \begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, $r = 1$, $\eta = n-1$. The general solution is $\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \cdots + \alpha_{n-1} \mathbf{x}_{n-1}$, where

$$\mathbf{x}_1 = \begin{bmatrix} -\frac{u_2}{u_1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ \mathbf{x}_2 = \begin{bmatrix} -\frac{u_2}{u_1} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix},\ \mathbf{x}_{n-1} = \begin{bmatrix} -\frac{u_2}{u_1} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

20. $\mathbf{u}\mathbf{v}^{\mathrm{T}} = [u_1v_1,\ u_2v_2,\ \cdots,\ u_nv_n]_{1\times 1} \neq 0$. Hence $\mathbf{x} = \mathbf{0}$ is the only solution.

Section 4.12

2. $\begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 1 & -1 & 0 & \vdots & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3/2 & \vdots & 1 \\ 0 & 1 & -1/2 & \vdots & 0 \end{bmatrix}$. Set $x_3 = 0$, then $x_1 = 1$, and $x_2 = 0$.

Hence $\mathbf{x}_p^{\mathrm{T}} = [1, 0, 0]$

4. $\begin{bmatrix} 1 & 1 & 1 & \vdots & -1 \\ 1 & -1 & -1 & \vdots & 0 \\ 2 & 0 & 0 & \vdots & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & -1 \end{bmatrix}$ so there are no solutions.

6. $\begin{bmatrix} 1 & -1 & 1 & -1 & \vdots & -1 \\ 1 & -1 & 2 & -1 & \vdots & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -1 & 0 & -1 & \vdots & -3 \\ 0 & 0 & 1 & 0 & \vdots & 2 \end{bmatrix}$. Set $x_2 = x_4 = 0$.

Then, $x_1 = -3$, $x_3 = 2$ and $\mathbf{x}_p = [-3, 0, 2, 0]^{\mathrm{T}}$.

8. $[1, 1, -1 \vdots 1]$. Set $x_2 = x_3 = x_4 = 0$. Then, $x_1 = 1$, $\mathbf{x}_p = [1, 0, 0, 0]^{\mathrm{T}}$, $\mathbf{x}_1 = [1, -1, 0, 0]^{\mathrm{T}}$, $\mathbf{x}_2 = [1, 0, 1, 0]^{\mathrm{T}}$, $\mathbf{x}_3 = [1, 0, 0, -1]^{\mathrm{T}}$. So $\mathbf{x} = \mathbf{x}_p + \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \alpha_3 \mathbf{x}_3$.

10. $\begin{bmatrix} 1 & -1 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 1 & 2 & \vdots & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 & \vdots & -2 \\ 0 & 0 & 1 & -2 & \vdots & 2 \end{bmatrix}$. So, $\mathbf{x}_p = [-2, 0, 2, 0]^{\mathrm{T}}$.

$\mathbf{x}_1 = [1, 1, 0, 0]^{\mathrm{T}}$, $\mathbf{x}_2 = [-3, 0, 2, 1]^{\mathrm{T}}$.

12. $\begin{bmatrix} 1 & -1 & 1 & \vdots & 1 \\ 1 & 2 & -1 & \vdots & 1 \end{bmatrix} \rightarrow \dots \rightarrow \begin{bmatrix} 1 & 0 & 1/3 & \vdots & 1 \\ 0 & 1 & -2/3 & \vdots & 0 \end{bmatrix}$. Thus, $\mathbf{x}_p = [1, 0, 0]^{\mathrm{T}}$.

$\mathbf{x}_1 = [-1/3, 2/3, 1]^{\mathrm{T}}$, and $\mathbf{x} = [1, 0, 0]^{\mathrm{T}} + \alpha[-1, 2, 3]^{\mathrm{T}}$.

5. Matrix Applications

Section 5.2

6. $\|\mathbf{0}\| = \sqrt{0^2 + 0^2 + \ldots 0^2} = 0.$

8. $\|\mathbf{e}_k\| = \left\| \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\| = \sqrt{0^2 + \ldots + 1 + 0 + \ldots + 0} = 1.$

10. $\left\langle \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\rangle = 1 \cdot 0 + (-1) \cdot 1 + 1 \cdot 1 = 0.$

12. Since $i \neq j$, $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0 + 0 + \ldots + 0 = 0$

14. $\langle \mathbf{x}, \mathbf{x} \rangle = \sum x_i^2.$

16. $\langle \mathbf{0}, \mathbf{x} \rangle = 0 \cdot x_1 + 0 \cdot x_2 + \ldots + 0 \cdot x_n = 0.$

18. $\left\langle \begin{bmatrix} \sqrt{y} \\ \sqrt{x} \end{bmatrix}, \begin{bmatrix} \sqrt{x} \\ \sqrt{y} \end{bmatrix} \right\rangle = 2\sqrt{xy}.$

20. No. Let $\mathbf{x} = \mathbf{z}$, for the counterexample.

22. Yes. $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle = 0 + 0 = 0.$

24. Since $\mathbf{u}^{\mathrm{T}}\mathbf{u} = 1$, $(\mathbf{u}\mathbf{u}^{\mathrm{T}})(\mathbf{u}\mathbf{u}^{\mathrm{T}}) = \mathbf{u}(\mathbf{u}^{\mathrm{T}}\mathbf{u})\mathbf{u}^{\mathrm{T}} = \mathbf{u}\mathbf{u}^{\mathrm{T}}.$

26. $\langle \mathbf{u}, \mathbf{b} \rangle = \langle \mathbf{u}, \sum \alpha_i x_i \rangle = \sum \left(\alpha_i \langle \mathbf{u}, x_i \rangle \right) = 0.$

Section 5.3

2. Here, $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. So, $r_1 = \sqrt{3}$ and $\mathbf{q}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Thus, $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right\}$. is the required orthogonal list.

4. Here $\mathbf{v}_1 = \mathbf{a}_1 = \mathbf{q}_1$ and $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. So, $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ = $\{\mathbf{q}_1, \mathbf{q}_2\}$. To obtain the third orthogonal vector, we note $\mathbf{a}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and use, Hence, the orthogonal list is

$$\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{v}_3\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

6. Here $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. As in the previous exercises, $r_1 = 1$, $\mathbf{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. So,

$$\mathbf{v}_3 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} - \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \right\rangle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left\langle \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \right\rangle \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} - \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}(\alpha_2 + \alpha_3) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ \alpha_2 - \alpha_3 \\ \alpha_3 - \alpha_2 \end{bmatrix}.$$

For simplicity, choose $\varepsilon = \frac{1}{2}(\alpha_2 - \alpha_3)$. Then, $\{\mathbf{v}_1, \mathbf{v}_2\, \mathbf{v}_3\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \varepsilon \\ -\varepsilon \end{bmatrix}\right\}$ is an orthogonal set, provided $\varepsilon \neq 0$.

8. From Exercise 3, $r_1 = \sqrt{2}, \quad r_2 = 1, \quad r_{12} = \frac{1}{\sqrt{2}}\langle \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\rangle = \frac{2}{\sqrt{2}}$.

So,
$$\begin{bmatrix}1 & 1\\1 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}\sqrt{2}/2 & 0\\ \sqrt{2}/2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{2} & \sqrt{2}\\ 0 & 1\end{bmatrix}.$$

10. From Exercise 5, $r_1 = r_2 = \sqrt{2},\ r_3 = 1,\ \mathbf{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \mathbf{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$, and $\mathbf{q}_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}$.

Also, $r_{12} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}\bullet\begin{bmatrix}1\\0\\-1\end{bmatrix} = 0$, $r_{13} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}\bullet\begin{bmatrix}1\\1\\1\end{bmatrix} = \sqrt{2}$, $r_{23} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}\bullet\begin{bmatrix}1\\1\\1\end{bmatrix} = 0$.

Thus, $$\begin{bmatrix}1 & 1 & 1\\0 & 0 & 1\\1 & -1 & 1\end{bmatrix} = \begin{bmatrix}\sqrt{2}/2 & \sqrt{2}/2 & 0\\ 0 & 0 & 1\\ \sqrt{2}/2 & -\sqrt{2}/2 & 0\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0 & \sqrt{2}\\ 0 & \sqrt{2} & 0\\ 0 & 0 & 1\end{bmatrix}.$$

12. We consider $\sum c_i \mathbf{q}_i = \mathbf{0}$ and take the dot product on both sides with $\mathbf{q}_j$. If $i \neq j$ then $\mathbf{q}_i \bullet \mathbf{q}_j = 0$. If $i = j$ then $\mathbf{q}_i \bullet \mathbf{q}_i = \|\mathbf{q}_i\|^2 > 0$. Since,
$$\mathbf{q}_j \bullet \sum c_i \mathbf{q}_i = \sum c_i(\mathbf{q}_j \bullet \mathbf{q}_i) = c_j\|\mathbf{q}_j\|^2 = \mathbf{q}_j \bullet \mathbf{0} = \mathbf{0},$$
it follows that $c_i = 0$ for all j. Thus, orthogonal sets are linearly independent.

Clearly, one such solution is $[1,\ -2,\ 1]^{\mathrm{T}}$ which is clearly orthogonal to the rows of $\mathbf{A}$.

16. $\mathbf{AA}^{\mathrm{T}}$ is $k \times k$. Now, rank $\mathbf{A} \le \min(k, n) = n$, and rank $\mathbf{AA}^{\mathrm{T}} \le$ rank $\mathbf{A}$. But $n < k$ by hypothesis. Hence, rank $\mathbf{AA}^{\mathrm{T}} < k$ which implies that $\mathbf{AA}^{\mathrm{T}}$ is singular.

18. In general, $(\mathbf{P}_1\mathbf{P}_2)^2 = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1\mathbf{P}_2$. If $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$ both are projections, then
$$(\mathbf{P}_1\mathbf{P}_2)^2 = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_1\mathbf{P}_2\mathbf{P}_2 = (\mathbf{P}_1)^2(\mathbf{P}_2)^2 = \mathbf{P}_1\mathbf{P}_2.$$
Next, note that $(\mathbf{P}_1\mathbf{P}_2)^{\mathrm{T}} = \mathbf{P}_2^{\mathrm{T}}\mathbf{P}_1^{\mathrm{T}} = \mathbf{P}_2\mathbf{P}_1$. So we have proved that $\mathbf{P}_1\mathbf{P}_2$ is a projection provided, $\mathbf{P}_1$ and $\mathbf{P}_2$ are projections AND they commute. If they do not commute, then the second equality in the displayed expression does not follow.

20. Let $\mathbf{u} = \frac{1}{\sqrt{n}}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$. Then $\mathbf{u}\mathbf{u}^{\mathrm{T}} = \frac{1}{n}\begin{bmatrix}1 & 1 & \cdots & 1\\1 & 1 & \cdots & 1\\\vdots & \vdots & \vdots & \vdots\\1 & 1 & 1 & 1\end{bmatrix} = \frac{1}{n}\mathbf{J}_n$.

22. Let $\mathbf{A} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$. Then $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$. We use the following row operations to compute $\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}$:

$$\begin{bmatrix}1 & 1 & \vdots & 1 & 0\\1 & 2 & \vdots & 0 & 1\end{bmatrix} \to \cdots \to \begin{bmatrix}1 & 0 & \vdots & 2 & -1\\0 & 1 & \vdots & -1 & 1\end{bmatrix}$$

So $\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1} = \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$. Therefore, $\mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} = \mathbf{I}_2 = \mathbf{P}$.

.

24. $\mathbf{A} = \mathbf{QR}$; $\mathbf{Ax} = \mathbf{b}$ becomes

$$\frac{1}{2}\begin{bmatrix}1 & 1 & -1\\1 & -1 & 1\\1 & -1 & -1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & -1\\0 & 1 & 1\\0 & 0 & 2\end{bmatrix} = \begin{bmatrix}2\\-1\\1\\0\end{bmatrix}.$$

From Eq.. 5.3.10, Eq. 5.3.21, and Eq. 5.3.22:

$$\begin{bmatrix}2 & 1 & -1\\0 & 1 & 1\\0 & 0 & 2\end{bmatrix}\mathbf{x} = \frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\1 & -1 & -1 & 1\\-1 & 1 & -1 & 1\end{bmatrix}\begin{bmatrix}2\\-1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\-2\end{bmatrix}.$$

Section 5.4

4. Let $\mathbf{A} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{bmatrix}$ be the coefficient matrix in $\mathbf{Ax} = \mathbf{b}$. So the normal equations are

$$\mathbf{AA}^{\mathrm{T}}\mathbf{x} = \begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix}\mathbf{x} = \mathbf{A}^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix}.$$

Section 5.5.

2. $\det\begin{bmatrix} -\lambda & 4 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 4$. So $\underline{\lambda_1 = 2}$ and $\underline{\lambda_2 = -2}$

4. $\det\begin{bmatrix} -\lambda & 3 \\ 3 & 8-\lambda \end{bmatrix} = (\lambda - 9)(\lambda + 1)$. So, $\underline{\lambda_1 = -1}$ and $\underline{\lambda_2 = 9.}$

6. $\det\begin{bmatrix} 5-\lambda & 4 \\ 4 & -1-\lambda \end{bmatrix} = \lambda^2 - 4\lambda - 21$. So, $\underline{\lambda_1 = 7}$ and $\underline{\lambda_2 = -3.}$

8. $\det[1-\lambda] = 1 - \lambda$ and $\underline{\lambda_1 = 1.}$

10. $\det\begin{bmatrix} 1-\lambda & 0 & \dots & 0 \\ 0 & 1-\lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1-\lambda \end{bmatrix} = (1-\lambda)^n$. Here $\underline{\lambda_i = 1}$ for all i.

12. $\det\begin{bmatrix} -1-\lambda & 2 & 0 \\ 3 & 7-\lambda & 0 \\ 0 & 0 & 6-\lambda \end{bmatrix} = (6-\lambda)(\lambda^2 - 6\lambda - 16)$. So, $\underline{\lambda_1 = 6}$, $\underline{\lambda_2 = 8,}$ $\underline{\lambda_3 = -2.}$

14. We compute $\mathbf{A}^{-1} = \dfrac{1}{16}\begin{bmatrix} -7 & 3 & 0 \\ 8 & 1 & 0 \\ 0 & 0 & 8/3 \end{bmatrix}.$

Hence,

$$\det(\mathbf{A}^{-1} - \lambda\mathbf{I}) = \begin{bmatrix} -7/16-\lambda & 3/16 & 0 \\ 3/16 & 1/16-\lambda & 0 \\ 0 & 0 & 1/6-\lambda \end{bmatrix}$$

$$= \left(\frac{1}{6}-\lambda\right)\left(\lambda^2 - \frac{3}{8}\lambda - \frac{1}{16}\right) = \left(\frac{1}{6}-\lambda\right)\left(\lambda - \frac{1}{8}\right)\left(\lambda + \frac{1}{2}\right)$$

and the eigenvalues are the reciprocals of those in Exercise 13.

18. $\det\begin{bmatrix} 1-\lambda & 2 & 4 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^3.$ Thus $\underline{\lambda_1 = \lambda_2 = \lambda_3 = 1.}$

20. $\det\begin{bmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 5 & 0 & -1-\lambda \end{bmatrix} = (2-\lambda)(\lambda^2 - 2\lambda - 8).$ So, $\underline{\lambda_1 = 2,}$ $\underline{\lambda_2 = 4,}$ and $\underline{\lambda_3 = -2.}$

22. $\det\begin{bmatrix} 1-\lambda & 1 & -1 & 2 \\ 0 & 2-\lambda & 0 & 1 \\ 0 & 0 & -1-\lambda & 1 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} = -\lambda(-1-\lambda)(2-\lambda)(1-\lambda).$

$$\therefore \underline{\lambda_1 = 0}, \quad \underline{\lambda_2 = -1}, \quad \underline{\lambda_3 = 2}, \quad \underline{\lambda_4 = 1}.$$

24. $\det\begin{bmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{bmatrix} = \lambda^2 + a\lambda + b.$ So $\underline{\lambda = \pm\dfrac{-a \pm \sqrt{a^2 - 4b}}{2}}.$

26. $\det\begin{bmatrix} \cos\theta - \lambda & \sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix} = (\cos\theta - \lambda)^2 - \sin^2\theta.$ So $\underline{\lambda = \cos\theta \pm \sin\theta.}$ Compare this with the answer to Exercise 23.

If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then so is $k\mathbf{x}$ for every nonzero k. In the Exercises to follow we will choose k so that $\mathbf{x}$ has convenient (usually integer) entries.

28. $\lambda_1 = 2$ leads to the equations $\begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}\mathbf{x} = \mathbf{0}$ and thus to the eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The eigenvalue $\lambda_2 = -1$ leads to $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0}$ and the eigenvector $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

30. For $\lambda_1 = 2 + 2i$ we solve $\begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix}\mathbf{x} = \mathbf{0}$ for the eigenvector $\mathbf{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$. For $\lambda_2 = 2 - 2i$ solve $\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix}\mathbf{x} = \mathbf{0}$ and get $\mathbf{x}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

32. Using $\lambda_i = 4$ we solve $\begin{bmatrix} -1 & 1 \\ 5 & -5 \end{bmatrix}\mathbf{x} = \mathbf{0}$ to obtain $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For $\lambda_2 = -2$ we solve $\begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$ to obtain $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$.

34. For $\lambda_1 = 1$, $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0}$ and thus, $\mathbf{x}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$. For $\lambda_2 = 0$, $\begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Finally, for $\lambda_3 = 4$, $\begin{bmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\mathbf{x} = \mathbf{0}$ and $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

36. For $\lambda_1 = 2$, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 5 & 0 & -3 \end{bmatrix}\mathbf{x} = \mathbf{0}$ leads to $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. For $\lambda_2 = 4$, $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 5 & 0 & -5 \end{bmatrix}\mathbf{x} = \mathbf{0}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. For $\lambda_3 = -2$, $\begin{bmatrix} 5 & 0 & 1 \\ 0 & 4 & 0 \\ 5 & 0 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$. Thus $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$.

38. For $\lambda_1 = 1$, we solve $\begin{bmatrix} 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{x} = \mathbf{0}$ to obtain $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. For $\lambda_2 = 2$, we

solve $\begin{bmatrix} -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}\mathbf{x} = \mathbf{0}$ to obtain $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Likewise, $\lambda_3 = -1$, and we solve

$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$ to obtain $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$. Finally, $\lambda_4 = 0$, $\mathbf{x}_4 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 2 \end{bmatrix}$.

58. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ since $a_1 = 0$ and $a_2 = 1$.

60. Here $n = 4$, $a_1 = -1$, $a_2 = a_3 = a_4 = 0$. So, $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

62. Here $n = 3$ and $C_3(\lambda) = (-1)^3(\lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3$.
Thus, $a_3 = -\lambda_1\lambda_2\lambda_3$, $a_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ and $a_1 = -\lambda_1 - \lambda_2 - \lambda_3$. Hence,

$$\mathbf{A} = -\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Section 5.6

2. $\mathbf{x}^{\mathrm{T}}\bar{\mathbf{x}} = \begin{bmatrix} 1, & i \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = 2$. $\mathbf{x}^{\mathrm{T}}\mathbf{x} = \begin{bmatrix} 1, & i \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = 0$.

Section 5.7

2. $\det\begin{bmatrix}1-\lambda & 1\\ 3 & -1-\lambda\end{bmatrix} = \lambda^2 - 4 = 0$. Therefore, $\lambda = \pm 2$. Thus,

$$\lambda_1 = 2, \qquad \begin{bmatrix}-1 & 1\\ 3 & -3\end{bmatrix}\mathbf{x} = \mathbf{0}, \quad \mathbf{x}_1 = \begin{bmatrix}1\\1\end{bmatrix}.$$

And

$$\lambda_2 = -2, \begin{bmatrix}3 & 1\\ 3 & 1\end{bmatrix}\mathbf{x} = \mathbf{0}, \qquad \mathbf{x}_2 = \begin{bmatrix}1\\-3\end{bmatrix}.$$

Hence, $\mathbf{x}(t) = a\begin{bmatrix}1\\1\end{bmatrix}e^{2t} + b\begin{bmatrix}1\\-3\end{bmatrix}e^{-2t}$.

4. $\det\begin{bmatrix}2-\lambda & 1\\ 2 & 3-\lambda\end{bmatrix} = \lambda^2 - 5\lambda + 4 = 0$. Therefore,

$$\lambda_1 = 4, \qquad \begin{bmatrix}-2 & 1\\ 2 & -1\end{bmatrix}\mathbf{x} = \mathbf{0}, \quad \mathbf{x}_1 = \begin{bmatrix}1\\2\end{bmatrix}.$$

And

$$\lambda_2 = 1, \qquad \begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}\mathbf{x} = \mathbf{0}, \qquad \mathbf{x}_2 = \begin{bmatrix}-2\\1\end{bmatrix}.$$

Hence, $\mathbf{x}(t) = a\begin{bmatrix}1\\2\end{bmatrix}e^{4t} + b\begin{bmatrix}-2\\1\end{bmatrix}e^{t}$.

6. $\det\begin{bmatrix}1-\lambda & 1 & 1\\ 0 & 2-\lambda & 1\\ 0 & 0 & 3-\lambda\end{bmatrix} = (1-\lambda)(2-\lambda)(3-\lambda)$.

$$\lambda_1 = 1, \qquad \begin{bmatrix}0 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 2\end{bmatrix}\mathbf{x} = \mathbf{0}, \qquad \mathbf{x}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

$$\lambda_2 = 2, \qquad \begin{bmatrix}-1 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}\mathbf{x} = \mathbf{0}, \qquad \mathbf{x}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}.$$

$$\lambda_3 = 3, \qquad \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}, \qquad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus,

$$\mathbf{x}(t) = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t}.$$

12. Since, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, we need to solve $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

So, $\begin{bmatrix} 1 & 1 & 1 & \vdots & 0 \\ 0 & -1 & -1 & \vdots & 1 \\ 0 & 1 & 2 & \vdots & 0 \end{bmatrix} \to \dots \to \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & -2 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$. shows that $a = 1$, $b = -2$, and

$c = 1$. Thus,

$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} - 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-2t}.$$

14. As in Exercise 12: $\begin{bmatrix} 1 & 1 & 1 & \vdots & 0 \\ 0 & -1 & -1 & \vdots & 0 \\ 0 & 1 & 2 & \vdots & 1 \end{bmatrix} \to \dots \to \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$. Hence $a = 0$, $b = -1$, and $c = 1$. Thus,

$$\mathbf{x}(t) = -\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-2t}.$$

16. $L_1 i_1' + q_1 / C_1 = q_2 / C_2 \Rightarrow L_1 i_1'' = (i_1 + i_2) / C_2 - i_1 / C_1 \Rightarrow \dfrac{C_1 - C_2}{L_1 C_1 C_2} i_1 + \dfrac{1}{L_1 C_2} i_2$

$$L_2 i_2' = q_2 / C_2 \Rightarrow L_2 i_2'' = (i_1 + i_2) / C_2 \Rightarrow i_2'' = \frac{1}{L_2 C_2} i_1 + \frac{1}{L_2 C_2} i_2 .$$

18. Let $\mathbf{i} = \mathbf{x} e^{mt}$. Then, $\mathbf{i}'' = m^2 \mathbf{x} e^{mt}$. Therefore, $m^2 \mathbf{x} = \mathbf{A}\mathbf{x}$ or $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$, where $\lambda = m^2$.

19. With $L_1 = 1,\ L_2 = 2,\ C_1 = .02,\ C_2 = .01,$ we have $\mathbf{A} = \begin{bmatrix} 50 & 100 \\ 50 & 50 \end{bmatrix}$. Hence,

$\det\begin{bmatrix} 50-\lambda & 100 \\ 50 & 50-\lambda \end{bmatrix} = \lambda^2 - 100\lambda - 2500$. Therefore, $\lambda_1 = 120.7,\ \lambda_2 = -20.7$. So,

$$\left.\begin{aligned} -70.7x_1 + 100x_2 &= 100 \\ x_1^2 + x_2^2 &= 1 \end{aligned}\right\} \Rightarrow \mathbf{x}_1 = \begin{bmatrix} .816 \\ .577 \end{bmatrix}$$

$$\left.\begin{aligned} 70.7x_1 + 100x_2 &= 100 \\ x_1^2 + x_2^2 &= 1 \end{aligned}\right\} \Rightarrow \mathbf{x}_2 = \begin{bmatrix} .816 \\ -.577 \end{bmatrix}$$

$$\begin{aligned} i_1(t) &= 0.25(e^{11t} - e^{-11t}) + 0.5\cos 4.55t, \\ i_1(t) &= 0.25(e^{11t} - e^{-11t}) - 0.5\cos 4.55t. \end{aligned}$$

22. $$M_1\ddot{y}_1 = K_1\left(y_2 - y_1 + \frac{M_2}{K_2}g\right) + M_1 g - K_1\left(y_1 + \frac{M_1 + M_2}{K_1}g\right),$$

$$M_2\ddot{y}_2 = M_2 g - K_2\left(y_2 - y_1 + \frac{M_2}{K_2}g\right).$$

Simplifying these equations leads to this:

$$\ddot{y}_1 = -\frac{K_1 + K_2}{M_1}y_1 + \frac{K_2}{M_1}y_2, \quad \ddot{y}_2 = \frac{K_2}{M_2}y_1 - \frac{K_2}{M_2}y_2.$$

In Matrix-Vector form: $\mathbf{y}'' = \begin{bmatrix} -\dfrac{M_1 + M_2}{M_1} & \dfrac{K_2}{M_1} \\ \dfrac{K_2}{M_2} & -\dfrac{K_2}{M_2} \end{bmatrix}\mathbf{y}$.

So, $\mathbf{y}'' = \begin{bmatrix} -10 & 4 \\ 4 & -4 \end{bmatrix}\mathbf{y}$ and $C_2(\lambda) = \lambda^2 + 14\lambda + 24 = 0$.

For, $\lambda_1 = -12,\quad \mathbf{y}_1 = \frac{\sqrt{5}}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. For, $\lambda_2 = -2,\quad \mathbf{y}_2 = \frac{\sqrt{5}}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Both eigenvectors have been normalized to have length 1.

26. Refer to Exercise 24: $\ddot{\mathbf{y}} = \mathbf{A}\,\mathbf{y}$ where $\mathbf{A} = \begin{bmatrix} -60 & 40 \\ 10 & -10 \end{bmatrix}$. So, $C(\lambda) = \lambda^2 + 70\lambda + 200$ and hence, $\lambda_1 = -2.99$, $\lambda_2 = -67$. Therefore, $m_1 = \pm 1.729i$ and $m_2 = \pm 8.19i$. The normalized eigenvectors (corresponding to the eigenvalues) are:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1.425 \end{bmatrix},\ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0.175 \end{bmatrix}.$$

Thus, $\mathbf{y}(t) = \mathbf{x}_1(c_1 \sin 1.73t + c_2 \cos 1.73t) + \mathbf{x}_2(c_3 \sin 8.19t + c_4 \cos 8.19t)$

Thus, $y_1(t) = c_1 \sin 1.73t + c_2 \cos 1.73t + c_3 \sin 8.19t + c_4 \cos 8.19t,$

And, $y_2(t) = 1.425(c_1 \sin 1.73t + c_2 \cos 1.73t) - 0.175(c_3 \sin 8.19t + c_4 \cos 8.19t).$

We solve this system for the constants:

$$\begin{bmatrix} 0 & 1 & 0 & 1` \\ 0 & 1.425 & 0 & -0.175 \\ 1.73 & 0 & 8.19 & 0 \\ 2.465 & 0 & -1.43 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, $c_1 = c_3 = 0$, $c_2 = 0.219$, $c_4 = 1.78$. So, $y_1(t) = 0.219\cos 173t + 1.78\cos 8.12t,$ and $y_2(t) = 0.312(\cos 173t - 1.78\cos 8.12t).$

Section 5.8

For Exercises 2 and 4 we compute $C(\lambda) = \lambda^2 - 1$ so that $\lambda = \pm 1$. Next, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence, the general solution of the homogeneous system, $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{x}$ is given by, $\mathbf{x}_h(t) = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^t + b\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t}$. The general solution of $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \mathbf{f}(t)$ is written $\mathbf{x}_h(t) + \mathbf{x}_p(t)$. In each Exercise we will only compute $\mathbf{x}_p(t)$. Note that the method of variation of parameters requires $\Phi(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}$. We solve for $\mathbf{u}'$ using the system $\Phi(t)\mathbf{u}' = \mathbf{f}(t)$ by row reduction of the augmented matrix. The reduction is illustrated by $[\Phi(t) \vdots \mathbf{f}(t)] \to \ldots \to [\mathbf{I} \vdots \Phi(t)^{-1}\mathbf{f}(t)]$.

2. Here $\mathbf{f}(t)=\begin{bmatrix}0\\e^t\end{bmatrix}$. So, $\left[\begin{array}{cc:c}e^t & e^{-t} & 0\\ e^t & -e^{-t} & e^t\end{array}\right]\to\cdots\to\left[\begin{array}{cc:c}1 & 0 & 1/2\\ 0 & 1 & -e^{2t}/2\end{array}\right]$ Thus,

$\mathbf{u}(t)=\frac{1}{4}\begin{bmatrix}2t\\-e^{2t}\end{bmatrix}$. So, $\mathbf{x}_p(t)=\frac{1}{4}\begin{bmatrix}e^t & e^{-t}\\ e^t & -e^{-t}\end{bmatrix}\begin{bmatrix}2t\\-e^{2t}\end{bmatrix}=\frac{1}{4}\begin{bmatrix}2te^t-e^t\\2te^t+e^t\end{bmatrix}=\frac{e^t}{4}\begin{bmatrix}2t-1\\2t+1\end{bmatrix}$.

4. Since, $\begin{bmatrix}1\\1-e^t\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}0\\e^t\end{bmatrix}$, we obtain a particular solution by subtracting the particular solution in Exercise 2 from that in Exercise 3: $\mathbf{x}_p(t)=-\begin{bmatrix}1\\1\end{bmatrix}-\frac{e^t}{4}\begin{bmatrix}2t-1\\2t+1\end{bmatrix}$.

6. We find a particular solution by assuming the existence of one in the form $\mathbf{x}_p(t)=\mathbf{b}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$. Since $\mathbf{b}'=\mathbf{0}$ and $\mathbf{x}'(t)=\mathbf{A}\,\mathbf{x}+\mathbf{f}$, we see that $\mathbf{b}$ must satisfy, $\mathbf{A}\,\mathbf{b}+\mathbf{f}=\mathbf{0}$. Then $\begin{bmatrix}0 & 1\\-1 & 2\end{bmatrix}\mathbf{b}=-\mathbf{f}=\begin{bmatrix}0\\-1\end{bmatrix}$. $\therefore\mathbf{b}=\begin{bmatrix}0\\1\end{bmatrix}=\mathbf{x}_p(t)$. The general solution is then $\mathbf{x}(t)=e^t\begin{bmatrix}1 & -1+t\\1 & t\end{bmatrix}\mathbf{c}+\begin{bmatrix}1\\0\end{bmatrix}$. At $t=0$ we require $\begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}\mathbf{c}+\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$.

Thus, $\mathbf{c}=\begin{bmatrix}1\\2\end{bmatrix}$. $\therefore\mathbf{x}(t)=e^t\begin{bmatrix}1 & -1+t\\1 & t\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}=e^t\begin{bmatrix}2t-1\\2t+1\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}$.

8. For this Exercises we assume $\mathbf{x}_p(t)=e^{-t}\begin{bmatrix}b_1\\b_2\end{bmatrix}$. Then $\mathbf{x}'_p(t)=-e^{-t}\begin{bmatrix}b_1\\b_2\end{bmatrix}$. From this we obtain, after canceling e^{-t}, $-\mathbf{b}=\begin{bmatrix}0 & 1\\-1 & 2\end{bmatrix}\mathbf{b}+\begin{bmatrix}1\\-1\end{bmatrix}$. So, $\mathbf{b}=\begin{bmatrix}-1\\0\end{bmatrix}$. Thus

$\mathbf{x}'_p(t)=e^{-t}\begin{bmatrix}-1\\0\end{bmatrix}$ and $\mathbf{x}(t)=e^t\begin{bmatrix}1 & -1+t\\1 & t\end{bmatrix}\mathbf{c}+e^t\begin{bmatrix}-1\\0\end{bmatrix}$.

Since $\mathbf{x}(0)=\mathbf{0}=\begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}\mathbf{c}+\begin{bmatrix}-1\\0\end{bmatrix}$, we find $\mathbf{c}=\begin{bmatrix}0\\-1\end{bmatrix}$. Finally, then

$$\mathbf{x}(t)=e^t\begin{bmatrix}1 & -1+t\\1 & t\end{bmatrix}\begin{bmatrix}0\\-1\end{bmatrix}+\begin{bmatrix}-1\\0\end{bmatrix}e^{-t}=\begin{bmatrix}1-t\\-t\end{bmatrix}e^t+\begin{bmatrix}-1\\0\end{bmatrix}e^{-t}.$$

Exercises 16 and 18 involve finding a particular solution when the forcing function is a combination of functions of the form $\mathbf{f}(t) = \mathbf{c}e^{at} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{at}$. Here, a can be zero or complex. Our trial solution will be $\mathbf{x}_p(t) = \mathbf{b}e^{at} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{at}$. Since $\mathbf{x}'_p(t) = a\mathbf{b}e^{at}$, it will be necessary to find $\mathbf{b}$ from the equation, $\mathbf{x}'_p(t) = a\mathbf{b}e^{at} = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix} \mathbf{b}e^{at} + \mathbf{c}e^{at}$.

We simplify this express so that it reads, $\begin{bmatrix} 1-a & 0 \\ -1 & 3-a \end{bmatrix} \mathbf{b} = -\mathbf{c}$, since the exponential factor cancels in each term. Row reductions give the following solution:

$$\left[\begin{array}{cc:c} 1-a & 0 & -c_1 \\ -1 & 3-a & -c_2 \end{array}\right] \to \dots \to \left[\begin{array}{cc:c} 1 & 0 & \dfrac{-c_1}{1-a} \\ 0 & 1 & \dfrac{-c_2}{3-a} - \dfrac{c_1}{(3-a)(1-a)} \end{array}\right].$$

Hence,

$$\mathbf{x}_p(t) = \begin{bmatrix} \dfrac{-c_1}{1-a} \\ \dfrac{-c_2}{3-a} - \dfrac{c_1}{(3-a)(1-a)} \end{bmatrix} = \frac{-1}{(3-a)(1-a)} \begin{bmatrix} c_1(3-a) \\ c_1 + (1-a)c_2 \end{bmatrix}.$$

16. Here $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $a = 0$. Using the above expression gives the particular solution

$$\mathbf{x}_p(t) = -\frac{1}{3} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

18. In this case we see that $\mathbf{f}(t) = \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{it}$ so, $c_1 = -i = -a$, $c_2 = 1$. As in the two previous exercises, we substitute these values in our general formula to obtain:

$$\mathbf{x}_p(t) = \frac{-1}{2-4i} \begin{bmatrix} -1-3i \\ 1-2i \end{bmatrix} e^{it} = -\frac{1}{2} \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{it}.$$

We want $\operatorname{Re} \mathbf{x}_p(t)$. So use $e^{it} = \cos t + i \sin t$ and simplify:

$$\operatorname{Re} \mathbf{x}_p(t) = -\frac{1}{2} \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix}.$$

20. Since $\mathbf{f}(t)=\begin{bmatrix}\cos t\\ \sin t\end{bmatrix}=\begin{bmatrix}\cos t\\ 0\end{bmatrix}+\begin{bmatrix}0\\ \sin t\end{bmatrix}$, Write $\mathbf{f}_1(t)=\begin{bmatrix}\cos t\\ 0\end{bmatrix}$ and $\mathbf{f}_2(t)=\begin{bmatrix}0\\ \sin t\end{bmatrix}$. Then $\mathbf{f}(t)=\mathbf{f}_1(t)+\mathbf{f}_2(t)$. We have found an $\mathbf{x}_p(t)$ for $\mathbf{f}_1(t)$ in Exercise 19. We need only find a particular solution using $\mathbf{f}_2(t)$. As in Exercise 19, we try a solution in the form $\mathbf{b}e^{it}$ and write $\hat{\mathbf{f}}_2(t)=\begin{bmatrix}0\\ 1\end{bmatrix}e^{it}$. Then, again as in Exercise 19,

$$\left[\begin{array}{ccc} -i & -2 & \vdots & 0\\ 1 & -1-i & \vdots & -1\end{array}\right]\to\cdots\to\left[\begin{array}{ccc} 1 & 0 & \vdots & -1+i\\ 0 & 1 & \vdots & (1+i)/2\end{array}\right]$$

so $\hat{\mathbf{x}}_p(t)=\begin{bmatrix}-1+i\\ (1+i)/2\end{bmatrix}e^{it}$. Hence, after simplification:

$$\mathbf{x}_p(t)=\operatorname{Re}\hat{\mathbf{x}}_p(t)=\frac{1}{2}\begin{bmatrix}2\cos t-2\sin t\\ \cos t+\sin t\end{bmatrix}.$$

We add this (real) solution to the solution given in Exercise 19 to obtain the particular solution

$$\begin{bmatrix}2\cos t-\sin t\\ \cos t+\sin t\end{bmatrix}.$$

6. Vector Analysis

Section 6.2

2. $\|\mathbf{A}+\mathbf{B}\| = \sqrt{(15+7.07)^2 + 7.07^2} = 23.17, \quad \theta = \tan^{-1}\dfrac{7.07}{22.07} = 17.76^\circ$

$\|\mathbf{A}-\mathbf{B}\| = \sqrt{(-15+7.07)^2 + 7.07^2} = 10.62, \quad \theta = \tan^{-1}\dfrac{7.07}{-7.93} = 138.3^\circ$

4. $\|\mathbf{A}+\mathbf{B}\| = \sqrt{10^2 + (-5)^2} = 11.18, \quad \theta = \tan^{-1}\dfrac{-5}{10} = -26.57^\circ$

$\|\mathbf{A}-\mathbf{B}\| = \sqrt{(-10)^2 + (-5)^2} = 11.18, \quad \theta = \tan^{-1}\dfrac{-5}{-10} = -153.43^\circ$

6. $1 = \cos^2 60^\circ + \cos^2 120^\circ + \cos^2 \beta. \quad \therefore \cos^2\beta = 0.5. \quad \therefore \cos\beta = 0.707. \quad \therefore \beta = 45^\circ$

$\mathbf{A} = 20\cos 120^\circ\,\mathbf{i} + 20\cos 45^\circ\,\mathbf{j} + 20\cos 60^\circ\,\mathbf{k} = \underline{-10\mathbf{i} + 14.14\mathbf{j} + 10\mathbf{k}}$

$\mathbf{i}_A = \dfrac{\mathbf{A}}{A} = \underline{-0.5\mathbf{i} + 0.707\mathbf{j} + 0.5\mathbf{k}}$

8. $\mathbf{A}+\mathbf{B} = (2+4)\mathbf{i} + (-4+7)\mathbf{j} + (-4-4)\mathbf{k} = \underline{6\mathbf{i} + 3\mathbf{j} - 8\mathbf{k}}$

10. $\mathbf{A}+\mathbf{B}-\mathbf{C} = (2+4-3)\mathbf{i} + (-4+7-0)\mathbf{j} + (-4-4+4)\mathbf{k} = \underline{3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}}$

12. $\mathbf{A}\cdot\mathbf{B} = 2\times 4 + (-4)\times 7 + (-4)\times(-4) = \underline{-4}$

14. $\mathbf{A}\times\mathbf{B} = (16+28)\mathbf{i} + (-16+8)\mathbf{j} + (14+16)\mathbf{k} = \underline{44\mathbf{i} - 8\mathbf{j} + 30\mathbf{k}}$

$\therefore \|\mathbf{A}\times\mathbf{B}\| = \sqrt{44^2 + 8^2 + 30^2} = \underline{53.85}$

16. $\mathbf{A}\cdot\mathbf{A}\times\mathbf{B} = 2\times 44 + (-4)(-8) + (-4)\times 30 = \underline{0}$. (See No.14 for $\mathbf{A}\times\mathbf{B}$.)

18. $\mathbf{B}\times\mathbf{C} = (4\mathbf{i} + 7\mathbf{j} - 4\mathbf{k})\times(3\mathbf{i} - 4\mathbf{k}) = -28\mathbf{i} + 4\mathbf{j} - 21\mathbf{k}$

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (2\mathbf{i} - 4\mathbf{j} - 4\mathbf{k})\times(-28\mathbf{i} + 4\mathbf{j} - 21\mathbf{k}) = \underline{100\mathbf{i} + 154\mathbf{j} - 104\mathbf{k}}$

20. $\mathbf{A}\cdot\mathbf{B} = (-2\mathbf{j} + 4\mathbf{k})\cdot(-4\mathbf{i} - 4\mathbf{j}) = 0 + 8 + 0 = \underline{8}$

22. $\|(\mathbf{A}\times\mathbf{B})\times\mathbf{A}\| = \|(16\mathbf{i} - 16\mathbf{j} - 8\mathbf{k})\times(-2\mathbf{i} + 4\mathbf{k})\| = \|-80\mathbf{i} - 64\mathbf{j} - 32\mathbf{k}\| = \underline{107.3}$

28. $\mathbf{A}\cdot\mathbf{i}_B = (3\mathbf{i}-6\mathbf{j}+2\mathbf{k})\cdot(7\mathbf{i}-4\mathbf{j}+4\mathbf{k})/9 = (21+24+8)/9 = \underline{5.889}$

30. $\mathbf{A}\cdot\mathbf{i}_B = (4\mathbf{i}-3\mathbf{j}+7\mathbf{k})\cdot(2\mathbf{i}-5\mathbf{j}-7\mathbf{k})/8 = (8+15-49)/8 = \underline{-3.25}$

32. $\mathbf{A}\times\mathbf{B} = (3\mathbf{i}-2\mathbf{j})\times(\mathbf{i}-2\mathbf{j}-4\mathbf{k}) = -2\mathbf{i}-3\mathbf{j}-4\mathbf{k}. \quad \therefore \|\mathbf{A}\times\mathbf{B}\| = 5.385$

$$\mathbf{i}_C = \frac{\mathbf{A}\times\mathbf{B}}{\|\mathbf{A}\times\mathbf{B}\|} = \underline{-0.371\mathbf{i}-0.557\mathbf{j}-0.743\mathbf{k}}$$

Of course $-\mathbf{i}_C$ (all plus signs) is also perpendicular to both **A** and **B**.

34. Let **B** go from pt. 1 to pt. 2: $\therefore \mathbf{B} = 3\mathbf{i}-2\mathbf{j}+6\mathbf{k}-(\mathbf{i}+3\mathbf{j}+2\mathbf{k}) = 2\mathbf{i}-5\mathbf{j}+4\mathbf{k}$

$$\mathbf{A} = 15\left(\frac{\mathbf{i}}{3}+\frac{2\mathbf{j}}{3}-\frac{2\mathbf{k}}{3}\right) = 5\mathbf{i}+10\mathbf{j}-10\mathbf{k} \qquad \therefore \mathbf{i}_B = 0.298\mathbf{i}-0.745\mathbf{j}+0.596\mathbf{k}$$

$$\mathbf{A}\cdot\mathbf{i}_B = 5\times 0.298+10\times(-0.745)-10\times 0.596 = \underline{-11.92}$$

36. $W = \mathbf{F}\cdot\mathbf{d} = (3\mathbf{i}-10\mathbf{j})\cdot(2\mathbf{i}-2\mathbf{j}+10\mathbf{k}) = 6+20 = \underline{26\ \mathrm{J}}$

38. $\mathbf{V} = \boldsymbol{\omega}\times\mathbf{r} = 45(\left(\frac{7\mathbf{i}}{9}-\frac{4\mathbf{j}}{9}+\frac{4\mathbf{k}}{9}\right)\times(2\mathbf{i}+3\mathbf{j}-\mathbf{k}) = \underline{-40\mathbf{i}+75\mathbf{j}+145\mathbf{k}}$

40. a) $\mathbf{M} = \mathbf{r}\times\mathbf{F} = (4\mathbf{i}-2\mathbf{j}+4\mathbf{k})\times(100\mathbf{i}-100\mathbf{j}+50\mathbf{k})/3 = (300\mathbf{i}+200\mathbf{j}-200\mathbf{k})/3$

$\therefore \underline{M_x = 100}$

b) $\mathbf{M}\cdot\mathbf{i}_L = \frac{1}{3}(300\mathbf{i}+200\mathbf{j}-200\mathbf{k})\cdot(-2\mathbf{i}+2\mathbf{j}-\mathbf{k})/3 = -\frac{200}{3}+\frac{400}{9}+\frac{200}{9} = \underline{0.0}$

Section 6.3

2. $\frac{d\mathbf{u}}{dt} = 2\mathbf{i}+2t\mathbf{k} = \underline{2\mathbf{i}+4\mathbf{k}}$

4. $\frac{d(\mathbf{u}\cdot\mathbf{v})}{dt} = \frac{d}{dt}(2t\cos 5t-10t^2) = 2\cos 5t-10t\sin 5t-20t = 2\cos 10-20\sin 10-40 = \underline{-30.7}$

6. $\frac{d^2(\mathbf{u}\cdot\mathbf{v})}{dt^2} = -10\sin 10-10\sin 10-100\cos 10-20 = \underline{74.60}$

8. $\mathbf{A} = \cancel{\mathbf{A}_{ref}} + \mathbf{a}+2\boldsymbol{\omega}\times\mathbf{v}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})+\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}$

$$= -30\mathbf{i}+2(30\mathbf{k}\times 10\mathbf{i})+30\mathbf{k}\times(30\mathbf{k}\times 0.2\mathbf{i}) = \underline{-210\mathbf{i}+600\mathbf{j}}$$

10. $\boldsymbol{\omega} = \dfrac{2\pi}{24\times 3600}\mathbf{k} = 7.27\times 10^{-5}\mathbf{k}$. $\mathbf{v} = \dfrac{90\times 1000}{3600}(-0.707\mathbf{i} - 0.707\mathbf{k})$. $\mathbf{r} = 6\,400\,000(-0.707\mathbf{i} + 0.707\mathbf{k})$

$\therefore \boldsymbol{\omega}\times\mathbf{v} = -0.00257\mathbf{j}$. $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) = 0.707\times(7.27\times 10^{-5})^2\times 6\,400\,000\mathbf{i} = 0.0239\mathbf{i}$

$\|2\boldsymbol{\omega}\times\mathbf{v}\| = \underline{0.00514\ \text{m/s}^2}$. $\|\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\| = \underline{0.0239\ \text{m/s}^2}$

12. $\dfrac{DT}{Dt} = \dfrac{\partial T}{\partial t} + v_x\dfrac{\partial T}{\partial x} + v_y\dfrac{\partial T}{\partial y} + v_z\dfrac{\partial T}{\partial z} = 0.1e^{-0.1t}\sin 5x + 10e^{-0.1t}5\cos 5x$

$$= \frac{0.1}{e}\sin 10 + \frac{50}{e}\cos 10 = \underline{-15.44\ ^\circ\text{C/s}}$$

Section 6.4

2. $\nabla\phi = \underline{2y\mathbf{i} + 2x\mathbf{j}}$

4. $\nabla\phi = \underline{e^x(\sin 2y\,\mathbf{i} + 2\cos 2y\,\mathbf{j})}$

6. $\phi = \ln(x^2+y^2+z^2)^{1/2} = \dfrac{1}{2}\ln(x^2+y^2+z^2)$

$$\nabla\phi = \frac{1}{2}\left[\frac{2x}{x^2+y^2+z^2}\mathbf{i} + \frac{2y}{x^2+y^2+z^2}\mathbf{j} + \frac{2z}{x^2+y^2+z^2}\mathbf{k}\right] = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2} = \boxed{\frac{\mathbf{r}}{r^2}}$$

8. $\nabla\phi = \dfrac{-y}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j} = \underline{(-y\mathbf{i}+x\mathbf{j})/(x^2+y^2)}$

10. $\nabla\phi = 2x\mathbf{i} + 2y\mathbf{j} = 4\mathbf{i} + 2\mathbf{j}$. $\therefore \mathbf{i}_n = \dfrac{\nabla\phi}{\|\nabla\phi\|} = \underline{(2\mathbf{i}+\mathbf{j})/\sqrt{5}}$

12. $\nabla\phi = 4x\mathbf{i} - 2y\mathbf{j} = 8\mathbf{i} - 2\mathbf{j}$. $\therefore \underline{\mathbf{i}_n = 0.97\mathbf{i} - 0.243\mathbf{j}}$

14. $\nabla\phi = \mathbf{i} + 2y\mathbf{j} - 4z\mathbf{k} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$. $\therefore \underline{\mathbf{i}_n = 0.174\mathbf{i} + 0.696\mathbf{j} - 0.696\mathbf{k}}$

16. $\nabla\phi = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = 6\mathbf{i} + 8\mathbf{j}$. $\mathbf{r} - \mathbf{r}_0 = (x-2)\mathbf{i} + (y-4)\mathbf{j} + z\mathbf{k}$

$\nabla\phi\cdot(\mathbf{r}-\mathbf{r}_0) = 0 = 6(x-3) + 8(y-4)$. $\therefore \underline{3x + 4y = 25}$

18. $\nabla\phi = (2x-2y)\mathbf{i} - 2x\mathbf{j} = -4\mathbf{j}$. $\mathbf{r} - \mathbf{r}_0 = (x-2)\mathbf{i} + (y-2)\mathbf{j} + (z-1)\mathbf{k}$

$\nabla\phi\cdot(\mathbf{r}-\mathbf{r}_0) = 0 = -4(y-2)$. $\therefore \underline{y = 2}$

20. $\nabla T = (2x+y)\mathbf{i} + (x+z)\mathbf{j} + y\mathbf{k} = 5\mathbf{i} + 6\mathbf{j} + \mathbf{k}.\quad \mathbf{i}_n = \dfrac{\nabla T}{\|\nabla T\|} = \underline{0.635\mathbf{i} + 0.762\mathbf{j} + 0.127\mathbf{k}}$

21. $\nabla T = 5\mathbf{i} + 6\mathbf{j} + \mathbf{k}. \qquad \therefore \dfrac{\partial T}{\partial x} = \underline{5}$

22. $\nabla T \cdot \mathbf{i}_n = (5\mathbf{i} + 6\mathbf{j} + \mathbf{k}) \cdot \left(\dfrac{\mathbf{i}}{3} - \dfrac{2\mathbf{j}}{3} + \dfrac{2\mathbf{k}}{3} \right) = \dfrac{5}{3} - \dfrac{12}{3} + \dfrac{2}{3} = \underline{-5/3}$

24. $\nabla \cdot \mathbf{u} = 0 + 0 + 0 = \underline{0}$

26. $\nabla \cdot \mathbf{u} = y + 2y + 2z = \underline{1}$

28. $$\nabla \cdot \mathbf{u} = \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}}$$
$$= \frac{r^3 - x\frac{3}{2}r(2x)}{r^6} + \frac{r^3 - y\frac{3}{2}r(2y)}{r^6} + \frac{r^3 - z\frac{3}{2}r(2z)}{r^6} = \frac{3}{r^3} - \frac{3(x^2+y^2+z^2)r}{r^6} = \frac{3}{r^3} - \frac{3}{r^3} = \underline{0}$$

30. $\nabla \cdot \mathbf{v}$ must be zero for a liquid flow. For the first, $\nabla \cdot \mathbf{v} = 2x - 2y,$ and for the second $\nabla \cdot \mathbf{v} = 0 + 0 = 0.$ $\therefore$ The second is possible but <u>not the first</u> since the divergence must be zero everywhere in a flow.

32. $\nabla \times \mathbf{u} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + (0-0)\mathbf{k} = \underline{\mathbf{0}}$

34. $\nabla \times \mathbf{u} = (0-0)\mathbf{i} + (0-z)\mathbf{j} + (0-x)\mathbf{k} = \underline{-\mathbf{j} + 2\mathbf{k}}$

36. $\nabla \times \mathbf{u} = (0-0)\mathbf{i} + (0-e^x)\mathbf{j} + (e^x \cos y - e^x \cos y)\mathbf{k} = -e^{-2}\mathbf{j} = \underline{-0.1353\mathbf{j}}$

38. $\nabla \cdot \mathbf{u} = y + 2y + 1 = 3y + 1 = \underline{7}$

40. $\nabla \times \mathbf{u} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + (0-x)\mathbf{k} = \underline{\mathbf{k}}$

42. $\mathbf{u} \times \mathbf{v} = (y^3 z - xyz)\mathbf{i} + (x^2 z - xy^2 z)\mathbf{j} + \cancel{(x^2y^2 - x^2y^2)}\mathbf{k}$
$\therefore \nabla \cdot \mathbf{u} \times \mathbf{v} = -yz - 2xyz = -4 + 8 = \underline{4}$

44. $$\nabla \times (\mathbf{u} \times \mathbf{v}) = \nabla \times \left[(y^3 z - xyz)\mathbf{i} + (x^2 z - xy^2 z)\mathbf{j} \right]$$
$$= (-x^2 + xy^2)\mathbf{i} + (y^3 - xy)\mathbf{j} + (2xz - y^2 z - 3y^2 z + xz)\mathbf{k} = \underline{-5\mathbf{i} + 10\mathbf{j} - 38\mathbf{k}}$$

46. $\mathbf{u}\times\nabla=\left(y^2\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)\mathbf{i}+\left(z\frac{\partial}{\partial x}-xy\frac{\partial}{\partial z}\right)\mathbf{j}+\left(xy\frac{\partial}{\partial y}-y^2\frac{\partial}{\partial x}\right)\mathbf{k}$

$$\therefore (\mathbf{u}\times\nabla)\times\mathbf{v}=\left[(z\frac{\partial}{\partial x}-xy\frac{\partial}{\partial z})yz-(xy\frac{\partial}{\partial y}-y^2\frac{\partial}{\partial x})xy\right]\mathbf{i}$$
$$+\left[(xy\frac{\partial}{\partial y}-y^2\frac{\partial}{\partial x})x^2-(y^2\frac{\partial}{\partial z}-z\frac{\partial}{\partial y})yz\right]\mathbf{j}$$
$$+\left[(y^2\frac{\partial}{\partial z}-z\frac{\partial}{\partial y})xy-(z\frac{\partial}{\partial x}-xy\frac{\partial}{\partial z})x^2\right]\mathbf{k}$$
$$=(-xy^2-x^2y+y^3)\mathbf{i}+(-2xy^2-y^3+z^2)\mathbf{j}+(-xz-2xz)\mathbf{k}=\underline{10\mathbf{i}+4\mathbf{j}+6\mathbf{k}}$$

48. $\nabla(\mathbf{u}\cdot\mathbf{v})=\nabla(x^3y+xy^3+yz^2)=(3x^2y+y^3)\mathbf{i}+(x^3+3xy^2+z^2)\mathbf{j}+2zy\mathbf{k}=\underline{14\mathbf{i}-9\mathbf{j}+8\mathbf{k}}$

50. $\nabla\cdot\mathbf{u}=1+1+1=3.\quad \nabla\times\mathbf{u}=0.\quad \therefore$ Irrotational

52. $\nabla\cdot\mathbf{u}=0.\quad \nabla\times\mathbf{u}=(1-1)\mathbf{k}=0.\quad \therefore$ Irrotational and solenoidal

54. $\nabla\cdot\mathbf{u}=2x+2z.\quad \nabla\times\mathbf{u}=(\cos x-\cos y)\mathbf{k}.\quad \therefore$ Neither

56. $\nabla\cdot\mathbf{u}=e^z.\quad \nabla\times\mathbf{u}=(\cos x-\cos y)\mathbf{k}.\quad \therefore$ Neither

58. $\nabla\cdot\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}=0$ (See No.28). $\nabla\times\mathbf{u}=0$ (See No.37). $\therefore$ Irrotational and solenoidal

68. $\mathbf{u}=\nabla\phi.\quad \frac{\partial\phi}{\partial x}=x^2.\quad \therefore\phi=\frac{x^3}{3}+f(y,z).\quad \frac{\partial\phi}{\partial y}=y^2=\frac{\partial f}{\partial y}.\quad \therefore f=\frac{y^3}{3}+g(z)$

$\frac{\partial\phi}{\partial z}=z^2=\frac{dg}{dz}.\quad \therefore g=\frac{z^3}{3}+C.\qquad \therefore \underline{\phi=\frac{1}{3}(x^3+y^3+z^3)+C}$

70. $\mathbf{u}=\nabla\phi.\quad \frac{\partial\phi}{\partial x}=e^x\sin y.\quad \therefore\phi=e^x\sin y+f(y,z).\quad \frac{\partial\phi}{\partial y}=e^x\cos y=e^x\cos y+\frac{\partial f}{\partial y}$

$\therefore\frac{\partial f}{\partial y}=0$ and $f=f(z).\quad \frac{\partial\phi}{\partial z}=0=\frac{df}{dz}.\quad \therefore f=C.\qquad \therefore \underline{\phi=e^x\sin y+C}$

72. $\mathbf{u}=\nabla\phi.\quad \frac{\partial\phi}{\partial x}=2xz.\quad \therefore\phi=x^2z+f(y,z).\quad \frac{\partial\phi}{\partial y}=y^2=\frac{\partial f}{\partial y}.\quad \therefore f=\frac{y^3}{3}+g(z)$

$\frac{\partial\phi}{\partial z}=x^2=x^2+\frac{dg}{dz}.\quad \therefore g=C.\qquad \therefore \underline{\phi=x^2z+\frac{y^3}{3}+C}$

Section 6.5

2. $\mathbf{i}\cdot(\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j})=\underline{\cos\theta}$

4. $\mathbf{j}\cdot(\sin\phi\cos\theta\,\mathbf{i}+\sin\phi\sin\theta\,\mathbf{j}+\cos\theta\,\mathbf{k})=\underline{\sin\phi\sin\theta}$

6. $(-\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j})\cdot(\cos\phi\cos\theta\,\mathbf{i}+\cos\phi\sin\theta\,\mathbf{j}-\sin\theta\,\mathbf{k})=-\cos\phi\cos\theta\sin\theta+\cos\phi\cos\theta\sin\theta=\underline{0}$

8. $\mathbf{k}\cdot(\sin\phi\cos\theta\,\mathbf{i}+\sin\phi\sin\theta\,\mathbf{j}+\cos\phi\,\mathbf{k})=\underline{\cos\phi}$

10. $(\cos\phi\cos\theta\,\mathbf{i}+\cos\phi\sin\theta\,\mathbf{j}-\sin\phi\,\mathbf{k})\cdot(\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j})=\cos\phi\cos^2\theta+\cos\phi\sin^2\theta=\underline{\cos\phi}$

16. $$\mathbf{u}=r_S\mathbf{i}_\theta=r_S(-\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j})=(x^2+y^2+z^2)^{1/2}\left(-\frac{y}{\sqrt{x^2+y^2}}\mathbf{i}+\frac{x}{\sqrt{x^2+y^2}}\mathbf{j}\right)$$
$$=\boxed{\left(\frac{x^2+y^2+z^2}{x^2+y^2}\right)^{1/2}(-y\mathbf{i}+x\mathbf{j})}$$

22. $$\mathbf{u}=\left(A-\frac{B}{r^2}\right)\cos\theta\,\mathbf{i}_r-\left(A+\frac{B}{r^2}\right)\sin\theta\,\mathbf{i}_\theta=\frac{\partial\Phi}{\partial r}\mathbf{i}_r+\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{i}_\theta+\frac{\partial\Phi}{\partial z}\mathbf{i}_z$$
$$\frac{\partial\Phi}{\partial r}=\left(A-\frac{B}{r^2}\right)\cos\theta.\quad\therefore\Phi=Ar\cos\theta+\frac{B}{r}\cos\theta+f(\theta,\,z)$$
$$\frac{1}{r}\frac{\partial\Phi}{\partial\theta}=-\left(A+\frac{B}{r^2}\right)\sin\theta=-A\sin\theta-\frac{B}{r^2}\sin\theta+\frac{1}{r}\frac{\partial f}{\partial\theta}.\quad\therefore\frac{\partial f}{\partial\theta}=0.\quad\therefore f=f(z)$$
$$\frac{\partial\Phi}{\partial z}==0=\frac{\partial f}{\partial z}.\quad\therefore f=C.\qquad\underline{\therefore\Phi=Ar\cos\theta+\frac{B}{r}\cos\theta+C}$$

Section 6.6

2. $$\oiint_S\mathbf{u}\cdot\mathbf{n}\,dS=\iiint_V\nabla\cdot\mathbf{u}\,dV=\int_0^1\int_0^1\int_0^1(y-1)\,dy\,dx\,dz=\int_0^1\int_0^1(-1/2)\,dx\,dz=\underline{-1/2}$$

4. $$\oiint_S(x\mathbf{i}\cdot\mathbf{n}+2y\mathbf{j}\cdot\mathbf{n}+y^2\mathbf{k}\cdot\mathbf{n})dS=\oiint_S(x\mathbf{i}+2y\mathbf{j}+y^2\mathbf{k})\cdot\mathbf{n}\,dS=\iiint_V(1+2)\,dV=3\times\frac{4}{3}\pi\times2^2=\underline{32\pi}$$

6. $$\oiint_S z^2\mathbf{k}\cdot\mathbf{n}\,dS=\iiint_V 2z\,dV=\int_0^8 2z\times4\pi\,dz=\underline{256\pi}$$

But $\nabla\phi\cdot\mathbf{n}=\dfrac{\partial\phi}{\partial n}$. $\quad\therefore\boxed{\oiint_S\frac{\partial\phi}{\partial n}dS=\iiint_V\nabla^2\phi\,dV}$

12. $\iiint\limits_V \rho \frac{\partial e}{\partial t} dV = -\iiint\limits_V \nabla \cdot \mathbf{q}\, dV.$ $\therefore \iiint\limits_V \left[\rho \frac{\partial e}{\partial t} + \nabla \cdot \mathbf{q} \right] dV = 0$. Since V is arbitrary,

$\rho \frac{\partial e}{\partial t} + \nabla \cdot \mathbf{q} = 0.$ Use $\Delta e = C \Delta T$ so that $\frac{\partial e}{\partial t} = C \frac{\partial T}{\partial t}$ and $\mathbf{q} = -K \nabla T$. Then there results

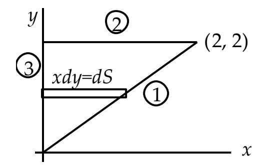
$$\boxed{\rho C \frac{\partial T}{\partial t} = K \nabla^2 T}$$

14. a) $\oint\limits_C (y^2 \mathbf{i} + xy \mathbf{j} + z^2 \mathbf{k}) \cdot (dx \mathbf{i} + dy \mathbf{j} + \cancel{dz} \mathbf{k})$

$$= \oint y^2 dx + xy dy$$

$$= \int\limits_1 y^2 dx + xy dy + \int\limits_2 y^2 dx + xy \cancel{dy} + \oint\limits_3 y^2 \cancel{dx} + \cancel{x} y dy$$

$$= \int_0^2 (x^2 dx + x^2 dx) + \int_2^0 2^2 dx = 2 \times \frac{8}{3} - 8 = \underline{-8/3}$$



16. $\Gamma = \iint\limits_S \nabla \times \mathbf{u} \cdot (-\mathbf{j}) dS = \iint\limits_S \left[\frac{\partial}{\partial z}(2z) - \frac{\partial x}{\partial x} \right] dS = \iint dS = \underline{\underline{2}}$

18. $\Gamma = \iint\limits_S \nabla \times \mathbf{u} \cdot \mathbf{k}\, dS = \iint\limits_S 0 dS = \underline{0}$

7. Fourier Series

Section 7.1

4. $a_n \cos\frac{n\pi t}{T} = \left(\cos\frac{n\pi t}{T}\right)\frac{1}{T}\int_{-T}^{T} f(s)\cos\frac{n\pi s}{T}ds.\quad b_n \sin\frac{n\pi t}{T} = \left(\sin\frac{n\pi t}{T}\right)\frac{1}{T}\int_{-T}^{T} f(s)\sin\frac{n\pi s}{T}ds$

$$\begin{aligned}\therefore a_n \cos\frac{n\pi t}{T} + b_n \sin\frac{n\pi t}{T} &= \frac{1}{T}\int_{-T}^{T} f(s)\left[\cos\frac{n\pi s}{T}\cos\frac{n\pi t}{T} + \sin\frac{n\pi s}{T}\sin\frac{n\pi t}{T}\right]ds \\ &= \frac{1}{T}\int_{-T}^{T} f(s)\cos\frac{n\pi}{T}(s-t)ds\end{aligned}$$

6. $-1 \sim -1 + \sum_{n=0}^{\infty} 0\times\cos nt + 0\times\sin nt = -1$

Section 7.3.1

2. Set $g(x) = x^n,\quad f(x) = \sin ax$. Then,

$$\begin{aligned}\int x^n \sin ax\,dx &= x^n\left(-\frac{\cos ax}{a}\right) - nx^{n-1}\left(\frac{-\sin ax}{a^2}\right) - \frac{n(n-1)x^{n-2}}{a^3}\cos ax - \cdots \\ &= \boxed{-\frac{x^n}{a}\cos ax + \frac{nx^{n-1}}{a^2}\sin ax - \frac{n(n-1)x^{n-2}}{a^3}\cos ax - \cdots}\end{aligned}$$

4. $\int x^n \sinh dx = \boxed{\frac{x^n}{b}\cosh bx - \frac{nx^{n-1}}{b^2}\sinh bx + \frac{n(n-1)x^{n-2}}{b^3}\cosh bx - \cdots}$

6. $\int x^n (ax+b)^\alpha dx = \boxed{\frac{x^n (ax+b)^{\alpha+1}}{a(\alpha+1)} - \frac{nx^{n-1}(ax+b)^{\alpha+2}}{a^2(\alpha+1)(\alpha+2)} + \frac{n(n-1)x^{n-2}(ax+b)^{\alpha+3}}{a^3(\alpha+1)(\alpha+2)(\alpha+3)} - \cdots}$

Section 7.3.2 The problems associated with Section 7.3.2 are under Section 7.3.3.

Section 7.3.3

2. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{2\pi^2}{8}$. $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos nt\, dt = \frac{1}{\pi}\left[t^2\left(\frac{\sin nt}{n}\right) - 2t\left(\frac{-\cos nt}{n^2}\right) + 2\left(\frac{-\sin nt}{n^3}\right)\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left(\frac{2\pi\cos n\pi}{n^2} + \frac{2\pi\cos n\pi}{n^2}\right) = \frac{4}{n^2}(-1)^n$. $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \sin nt\, dt = 0$ since $t^2 \sin nt$ is odd.

$$\therefore \boxed{f(t) = \frac{\pi^2}{8} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos nt}{n^2}}$$

4. $a_0 = \frac{1}{2\pi}\int_{-2\pi}^{2\pi}(t+2\pi)\,dt = 4\pi$. $a_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi}(t+2\pi)\cos\frac{nt}{2}dt = \frac{1}{2\pi}\left[\frac{t+2\pi}{n/2}\sin\frac{nt}{2} - \frac{1}{(n/2)^2}\cos\frac{nt}{2}\right]_{-2\pi}^{2\pi} = 0$

$b_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi}(t+2\pi)\sin nt/2\, dt = \frac{1}{2\pi}\left[-\frac{t+2\pi}{n/2}\cos\frac{nt}{2} + \frac{1}{(n/2)^2}\sin\frac{nt}{2}\right]_{-2\pi}^{2\pi} = -\frac{4}{n}(-1)^n$

$$\therefore \boxed{f(t) = 2\pi - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{nt}{2}}$$

6. $a_0 = \int_0^1 (1-t)\,dt = \frac{1}{2}$. $a_n = \int_0^1 (1-t)\cos n\pi t\, dt = \left[\frac{(1-t)\sin n\pi t}{n\pi}\right]_0^1 - \left[\frac{(-1)(-\cos n\pi t)}{(n\pi)^2}\right]_0^1 = \frac{1-(-1)^n}{(n\pi)^2}$

$b_n = \int_0^1 (1-t)\sin n\pi t\, dt = \left[\frac{-(1-t)\cos n\pi t}{n\pi} - \frac{(-1)(-\sin n\pi t)}{n^2\pi^2}\right]_0^1 = \frac{1}{n\pi}$

$$\therefore \boxed{f(t) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{1-(-1)^n}{(n\pi)^2}\cos n\pi t + \frac{1}{n\pi}\sin n\pi t}$$

8. $a_0 = \frac{1}{2}\int_{-2}^{0} 2t^2\, dt + \frac{1}{2}\int_0^2 4t\, dt = \frac{20}{3}$. $a_n = \frac{1}{2}\int_{-2}^{0} 2t^2 \cos\frac{n\pi t}{2}dt + \frac{1}{2}\int_0^2 4t\cos\frac{n\pi t}{2}dt = \frac{8}{n^2\pi^2}\left[3(-1)^2 - 1\right]$

$b_n = \frac{1}{2}\int_{-2}^{0} 2t^2 \sin\frac{n\pi t}{2}dt + \frac{1}{2}\int_0^2 4t\sin\frac{n\pi t}{2}dt = \frac{16}{n^3\pi^3}\left[1-(-1)^2\right]$

$$\therefore \boxed{f(t) = \frac{10}{3} + \sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}\left\{\left[3(-1)^n - 1\right]\cos\frac{n\pi t}{2} + \frac{2}{n\pi}\left[1-(-1)^n\right]\sin\frac{n\pi t}{2}\right\}}$$

10. $a_0 = \frac{1}{\pi}\int_0^{\pi} dt + \frac{1}{\pi}\int_{-\pi}^{0}(-1)\,dt = 0. \quad a_n = \frac{1}{\pi}\int_{-\pi}^{0} -\cos nt\,dt + \frac{1}{\pi}\int_0^{\pi}\cos nt\,dt = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} -\sin nt\,dt + \frac{1}{\pi}\int_0^{\pi}\sin nt\,dt = \frac{2}{\pi}\int_0^{\pi}\sin nt\,dt = \frac{2}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 0 & \text{if } n \text{ is even} \\ 4/n\pi & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \boxed{f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{2n-1}}$$

12. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin t|\,dt = \frac{2}{\pi}\int_{-\pi}^{\pi}\sin t\,dt = \frac{4}{\pi}. \quad a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin t|\cos t\,dt = \frac{2}{\pi}\int_0^{\pi}\sin t\cos t\,dt = \frac{1}{\pi}\left[\sin^2 t\right]_0^{\pi} = 0$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\sin t\cos t\,dt = \frac{1}{\pi}\int_0^{\pi}[\sin(n+1)t + \sin(n-1)t\,dt = \frac{-1}{\pi}\left[\frac{\cos(n+1)t}{n+1} + \frac{\cos(n-1)t}{n-1}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{(-1)^{n+1}}{n+1} - \frac{1}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1}\right] = \frac{1}{\pi}\left[\frac{2n(-1)^n}{n^2-1} + \frac{2n}{n^2-1}\right] = \frac{2n}{\pi(n^2-1)}\left[(-1)^n + 1\right]$$

Also, $b_n = 0$. $\therefore \boxed{f(t) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{n}{n^2-1}\left[(-1)^n + 1\right]\cos nt}$

Section 7.3.4

2. Sine series: $a_n = 0. \quad b_n = \frac{2}{\pi}\int_0^{\pi} 4t\sin nt\,dt = \frac{8}{\pi}\left[\frac{-t\cos nt}{n} + \frac{\sin nt}{n^2}\right]_0^{\pi} = -\frac{8}{n}(-1)^n$

$$\therefore \boxed{f(t) = -8\sum_{n=1}^{\infty}\frac{(-1)^n \sin nt}{n}}$$

Cosine series: $a_0 = \frac{2}{\pi}\int_0^{\pi} 4t\,dt = 4\pi. \quad a_n = \int_0^{\pi} 4t\cos nt\,dt = \frac{8}{\pi n^2}\left[(-1)^n - 1\right]$

$$\therefore \boxed{f(t) = 2\pi + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nt}$$

4. Sine series: $a_n = 0$. $b_n = \frac{2}{2\pi}\int_0^{\pi} \sin t \sin nt\, dt = 0$ for all n except $n = 1$.

$b_1 = 1$. $\therefore \boxed{f(t) = \sin t}$

Cosine series: $a_0 = \frac{2}{\pi}\int_0^{\pi} \sin t\, dt = \frac{4}{\pi}$. $a_n = \frac{2}{\pi}\int_0^{\pi} \sin t \cos nt\, dt = \frac{2}{\pi(n^2-1)}\left[(-1)^{n+1} - 1\right]$

$$\therefore \boxed{f(t) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}-1}{n^2-1}\cos nt}$$

6. $f(t) = \frac{t^2}{8}$. Sine series: $b_n = \frac{2}{4}\int_0^{4} \frac{t^2}{8} \sin\frac{n\pi t}{4}\, dt = -\frac{4(-1)^n}{n\pi} + \frac{8}{n^2\pi^2}\left[(-1)^n - 1\right]$

$$\therefore \boxed{f(t) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n} - 2\frac{(-1)^n - 1}{\pi^2 n^2}\right]\sin\frac{n\pi t}{4}}$$

Cosine series: $a_0 = \frac{2}{4}\int_0^{4} \frac{t^2}{8}\, dt = \frac{4}{3}$. $a_n = \frac{2}{4}\int_0^{4} \frac{t^2}{8}\cos\frac{n\pi t}{4}\, dt = \frac{8}{n^2\pi^2}(-1)^n$

$$\therefore \boxed{f(t) = \frac{2}{3} - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi t}{4}}$$

Section 7.3.5

2. $a_0 = 2\int_0^1 (t - t^2)\, dt = \frac{1}{3}$. $a_n = 2\int_0^1 (t-t^2)\cos n\pi t\, dt = -\frac{2}{n^2\pi^2}(\cos n\pi + 1)$

$$\therefore f(t) = \frac{1}{6} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^2 - 1\right]\cos n\pi t = \boxed{\frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 2n\pi t}$$

$$b_n = 2\int_0^1 (t-t^2)\sin n\pi t\, dt = \left[(t-t^2)\frac{-\cos n\pi t}{n\pi} - (1-2t)\frac{-\sin n\pi t}{\pi^2 n^2} - 2\frac{\cos n\pi t}{\pi^3 n^3}\right]_0^1 = -\frac{4}{n^3\pi^3}(\cos n\pi - 1)$$

$$\therefore \boxed{f(t) = \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi t}{(2n-1)^3}}$$

Section 7.3.6

2. We have $f(-t) = \dfrac{\pi}{4} + \sum_{n=1}^{\infty} \dfrac{(-1)^n - 1}{n^2\pi} \cos nt + \dfrac{(-1)^n}{n} \sin nt.$ $\therefore$ $\boxed{|t| \sim \dfrac{\pi}{2} + \dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^n - 1}{n^2} \cos nt}$

4. $\boxed{f(t) \sim \dfrac{1}{2} + \dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{\sin(2n-1)t}{2n-1}}$

6. It is the odd extension of *f*(*t*) provided $f(0) = f(n\pi) = 0$.

Section 7.4

For Exercises 2 and 4, the coefficients of $y'(t)$, namely C, is zero. We find a particular solution by solving this general equation:

$$M y'' + K y = a_n \cos[nt] + b_m \sin[mt]$$

This is a second-order equation with constant coefficients and we can use standard techniques to find solutions:

$$y_p(t) = \frac{a_m (K - n^2 M) \cos mt + b_n (K - m^2 M) \sin nt}{(K - n^2 M)(K - m^2 M)}$$

provided the denominator doe snot vanish. (See Exercise 5 below.) So in Exercises 1-5, we substitute the specific values of K, M, a_n, b_n, n, and m.

2. Here, $M = 2$, $K = 2$, $a_m = 1,$, $b_n = 0$, and $m = 2$. So, $y_p(t) = \dfrac{-1}{6} \cos 2t$.

4. $M = 1$, $K = 25$, $a_m = 1$, $b_n = 1/10$, $n = 2$, and $m = 1$. So,

$$y_p(t) = \frac{1}{21} \cos 2t + \frac{1}{90} \sin 4t$$

6. Proceed as in Exercises 1-4 except $C = 4$ and compute $y_p(t) = \dfrac{5}{116} \cos 2t + \dfrac{1}{58} \sin 2t.$

Section 7.5.1

8. $f(t)=\dfrac{4}{\pi}+\sum\limits_{n=1}^{\infty}\left[\dfrac{-2\cos(2n-1)t}{\pi(2n-1)^2}-\dfrac{(-1)^n}{n}\sin nt\right]$. Here $f(t)=\begin{cases}0 & -\pi<t<0\\ t & 0<t<\pi\end{cases}$

$$\therefore \int_0^t (s-\frac{\pi}{4})\,ds=\frac{t^2}{2}-\frac{\pi}{4}t=\sum_{n=1}^{\infty}\left[\frac{-2\sin(2n-1)t}{\pi(2n-1)^3}+\frac{(-1)^n}{n^2}\cos nt-\frac{(-1)^n}{n^2}\right]$$

$$=\boxed{\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\left[\frac{-2\sin(2n-1)t}{\pi(2n-1)^3}+\frac{(-1)^n}{n^2}\cos nt\right]}$$

10. Here $f(t)=t+2\pi$ and from the answer to No.4, Section 7.3.2, $f(t)=2\pi-\sum\limits_{n=1}^{\infty}\dfrac{4}{n}\sin\dfrac{nt}{2}$.

$$\therefore \frac{t^2}{2}=8\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{nt}{2}-8\sum_{n=1}^{\infty}\frac{1}{n^2}=\boxed{-\frac{4}{3}\pi^2+8\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{nt}{2}}$$

Section 7.5.3

2. $e^z=1+z+\dfrac{z^2}{2!}+\dfrac{z^3}{3!}+\cdots+\dfrac{z^n}{n!}+\cdots$. Set $z=\cos t+i\sin t$. Then,

$$e^{\cos t+i\sin t}=1+\cos t+\frac{\cos 2t}{2!}+\cdots+i\left(\sin t+\frac{\sin 2t}{2!}+\cdots\right)$$

Taking real parts, $\operatorname{Re} e^{\cos t+i\sin t}=e^{\cos t}\cos(\sin t)$.

$$\therefore \boxed{\cos(\sin t)e^{\cos t}=1+\cos t+\frac{\cos 2t}{2!}+\cdots+\frac{\cos nt}{n!}+\cdots}=1+\sum_{n=1}^{\infty}\frac{\cos nt}{n!}$$

4. $\sin z=z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\cdots$. $\therefore \sin(e^{it})=\sin(\cos t)\cosh(\sin t)+i\cos(\cos t)\sinh(\sin t)$

$$\therefore \boxed{\sin(\cos t)\cosh(\sin t)=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)!}\cos(2n-1)t}$$

6. $a\dfrac{a-\cos t}{a^2-2a\cos t+1}-\dfrac{1}{2}=\dfrac{a^2+1}{a^2-2a\cos t+1}=-\dfrac{1}{2}+\sum\limits_{n=0}^{\infty}a^{-n}\cos nt=\dfrac{1}{2}+\sum\limits_{n=1}^{\infty}a^{-n}\cos nt$

$$\therefore \boxed{\frac{1}{a^2-2a\cos t+1}=-\frac{1}{2(a^2+1)}+\sum_{n=0}^{\infty}\frac{a^{-n}}{a^2+1}\cos nt}$$

8. Partial Differential Equations

Section 8.1

2. Linear, 2nd order, homogeneous, partial, parabolic.

4. Linear, 2nd order, nonhomogeneous, partial, elliptic.

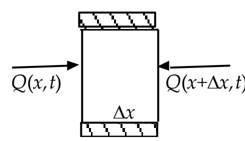
8. Nonlinear, homogeneous, partial.

Section 8.3

1. $\Delta E = Cm\Delta T = C\rho A\Delta x\Delta T$

$$\therefore C\rho A\Delta x\Delta T = \left[Q(x,t) + Q(x+\Delta x,t)\right]\Delta t$$

$$= -KA\Delta t\frac{\partial T}{\partial x}(x,t) + KA\Delta t\frac{\partial T}{\partial x}(x+\Delta x,t)$$



Divide by $C\rho A\Delta x\Delta t$:

$$\frac{\Delta T}{\Delta x} = \frac{K}{\rho C}\frac{\frac{\partial T}{\partial x}(x+\Delta x,t) - \frac{\partial T}{\partial x}(x,t)}{\Delta x}$$

Now, let $\Delta t \to 0$ and $\Delta x \to 0$. There results, with $\frac{K}{\rho C} = k$, $\boxed{\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}}$

2. To the derivation in No.1 add the heat generation term $\phi A\Delta x$. There results

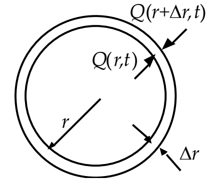
$$C\rho A\Delta x\Delta T = -KA\Delta t\frac{\partial T}{\partial x}(x,t) + KA\Delta t\frac{\partial T}{\partial x}(x+\Delta x,t) + \phi A\Delta x\Delta t$$

Divide by $A\Delta x\Delta t$ and let $\Delta t \to 0, \quad \Delta x \to 0 \ \ (A = \pi D^2/4)$:

$$\boxed{\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + \frac{\phi}{\rho C}}$$

4. $C\rho\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2}$ since the temperature depends only on x and t. After a long time the temperature ceases to depend on t. Hence, $\frac{\partial^2 T}{\partial x^2} = 0$ and $T(x) = Ax + B$. Let $T(0) = 0$ and $T(2) = 200°$. Then $B = 0$ and $A = 100$ so that $\underline{T(x) = 100x}$.

8. For the cylinder shown of length L:



$$Cm\Delta T = \left[Q(r,t) + Q(r+\Delta r,t)\right]\Delta t$$

$$C\rho 2\pi L\Delta r\Delta T = -K2\pi L\left(r\frac{\partial T}{\partial r}\right)_r \Delta t + K2\pi L\left(r\frac{\partial T}{\partial r}\right)_{r+\Delta r} \Delta t$$

Divide by $C\rho 2\pi L\Delta r\Delta t$:

$$\frac{\Delta T}{\Delta t} = k\frac{1}{r}\frac{\left(r\frac{\partial T}{\partial r}\right)_{r+\Delta r} - \left(r\frac{\partial T}{\partial r}\right)_r}{\Delta r}$$

Let $\Delta t \to 0, \quad \Delta r \to 0$:

$$\boxed{\frac{\partial T}{\partial t} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)}$$

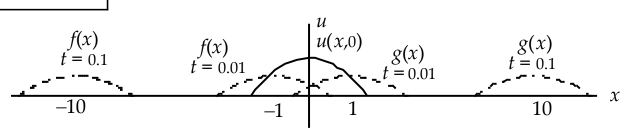
Surface area of cylinder
$= 4\pi r^2$

Surface area of sphere
$= 2\pi rL$

Section 8.5

2. The wave speed is $a = \sqrt{P/m} = \sqrt{300/0.03} = 100$ m/s. Using Eqs. 8.5.11 and 8.5.18:

$$\boxed{u(x,t) = \frac{1}{2}[\cos(x-100t) + \cos(x+100t)]}$$



Section 8.6

2. $\dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2}$. Assume $u(x,t) = X(x)T(t)$. Then $\dfrac{T''}{T} = a^2\dfrac{X''}{X}$.

Thus, $\left.\begin{array}{l} T'' = 0, \quad T(t) = c_1 t + c_2 \\ X'' = 0, \quad X(x) = c_3 x + c_4 \end{array}\right\}$ $\therefore u(x,t) = (c_1 t + c_2)(c_3 x + c_4)$. Let $u(0,t) = 0$. $\therefore c_4 = 0$

Let $u(L,t) = 0 = c_3 L(c_1 t + c_2)$. Since $L \neq 0$ we must let $c_3 = 0$. But, this gives $u(x,t) = 0$ which is unacceptable. Also, the general solution for $u(x,t)$ is not oscillatory which it must be. Therefore, the solution is no good as presented.

4. $T(t) = c_1 e^{i\beta at} + c_2 e^{-i\beta at} = c_1(\cos\beta at + i\sin\beta at) + c_2(\cos\beta at - i\sin\beta at)$

$X(x) = c_3 e^{i\beta x} + c_4 e^{-i\beta x} = c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)$

$\cos\beta at$: $\underline{B = c_1 + c_2}$ $\qquad$ $\cos\beta x$: $\underline{D = c_3 + c_4}$

$\sin\beta at$: $\underline{A = i(c_1 - c_2)}$ $\qquad$ $\sin\beta x$: $\underline{C = i(c_3 - c_4)}$

6. The initial velocity of a stationary wire is zero. Hence, in Eq. 8.6.25 $A = 0$. The left end is fixed, i.e., $u(0,t) = 0$. $\therefore D = 0$. Thus,

$$u(x,t) = K\cos\beta at\sin\beta x \quad \text{where} \quad K = BC$$

a) $u(x,0) = 0.1\sin\dfrac{\pi x}{2} = K\sin\beta x. \quad \therefore K = 0.1, \quad \beta = \pi/2$

$$\therefore \boxed{u(x,t) = 0.1\cos\frac{\pi at}{2}\sin\frac{\pi x}{2}}.$$ This satisfies $u(2,t) = 0$.

8. For zero initial displacement with fixed left end: $u(x,t) = A\sin 40\beta t\sin\beta x$.

Let $\dfrac{\partial u}{\partial t}(x,0) = 4\sin x = 40\beta A\sin\beta x. \quad \therefore \beta = 1, \quad A = 0.1. \quad \therefore u(x,t) = 0.1\sin 40t\sin x.$

$\therefore \underline{u_{\max} = 0.1 \text{ m}}$ at $x = \pi/2$ m, and at $t = \pi/80$ s, $\pi/16$ s, $9\pi/80$ s, $\cdots$

10. For zero initial velocity and fixed left end: $u(x,t) = A\cos 20\beta t\sin\beta x$.

$u(x,0) = 0.2\sin\dfrac{\pi x}{4} = A\sin\beta x. \quad \therefore A = 0.2, \quad \beta = \pi/4. \quad \therefore \underline{u(x,t) = 0.2\cos 5\pi t\sin\dfrac{\pi x}{4}}$

12. The general solution, with fixed left end, is $u(x,t) = (A\cos\beta at + B\sin\beta at)\sin\beta x$. Let

$u(x,0) = 0.1\sin\dfrac{\pi x}{4} = A\sin\beta x. \quad \therefore A = 0.1, \quad \beta = \dfrac{\pi}{4}.$ Let $\dfrac{\partial u}{\partial t}(x,0) = 10\sin\dfrac{\pi x}{4} = 40\beta B\sin\beta x$

$\therefore B = \dfrac{10}{40\beta} = \dfrac{1}{\pi}. \quad \therefore u(x,t) = (0.1\cos 10\pi t + \pi^{-1}\sin 10\pi t)\sin\dfrac{\pi x}{4}$

$u_{\max}$ occurs at $\underline{x = 2 \text{ m}}$ with magnitude $\sqrt{(1/\pi)^2 + 0.1^2} = \underline{0.334 \text{ m}}$

14. $u(0,t) = 0$ and $\dfrac{\partial u}{\partial t}(x,0) = 0. \qquad \therefore u(x,t) = A\cos\beta at\sin\beta x. \quad u(L,t) = 0 = A\cos\beta at\sin\beta L.$

$\therefore \beta = \dfrac{n\pi}{L}. \quad \therefore u(x,t) = \displaystyle\sum_{n=1}^{\infty} A_n\cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L}. \quad u(x,0) = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{L}.$

$$\therefore A_n = \frac{2}{L}\int_0^L k\sin\frac{n\pi x}{L}dx = \frac{2k}{L}\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L}\bigg|_0^L = \frac{2k}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right).$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty}\frac{2k}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)\cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L}$$

$$= \boxed{\frac{2k}{\pi}\cos\frac{\pi at}{L}\sin\frac{\pi x}{L} + \frac{2k}{\pi}\cos\frac{2\pi at}{L}\sin\frac{2\pi x}{L} + \frac{2k}{3\pi}\cos\frac{3\pi at}{L}\sin\frac{3\pi x}{L} + \cdots}$$

16. If $u(0,t) = 0$ and $u(x,0) = 0$, $u(x,t) = A\sin\beta at \sin\beta x$. $\quad u(\pi,t) = 0 = A\sin\beta at\sin\beta\pi$.

$\therefore \beta = \pi. \quad \therefore u(x,t) = \sum_{n=1}^{\infty} A_n \sin ant \sin nx.$ The initial condition is

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} A_n an \sin nx = \begin{cases} 0 & 0 < x < \pi/4 \\ 20 & \pi/4 < x < 3\pi/4 \\ 0 & 3\pi/4 < x < \pi \end{cases}$$

$$\therefore A_n = \frac{2}{\pi an}\int_{\pi/4}^{\pi} 20\sin nx\,dx = \frac{2\times 20}{60\pi n^2}\left(\cos\frac{n\pi}{4} - \cos\frac{3n\pi}{4}\right).$$

$$\therefore u(x,t) = \boxed{\sum_{n=1}^{\infty} \frac{2}{3\pi n^2}\left(\cos\frac{n\pi}{4} - \cos\frac{3n\pi}{4}\right)\sin 60nt\sin nx}$$

18. $a = \sqrt{\frac{40\,000}{10}} = 63.2$ m/s. The solution may become ill-behaved if $\frac{\omega L}{a} = (2n-1)\pi$ (see Example 8.6.5). The forcing function has $\omega = 21\pi$. Using $L = 15$, we have $\frac{21\pi\times 5}{63.2} = (2n-1)\pi$. This gives $n = 2.992$. Thus, for $n = 3$ the solution is nearly singular and near-resonance occurs.

Section 8.7

For problems 2, 4, 6, 8, the following general solution is used:

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}. \quad T = \Theta(t)X(x). \quad \therefore \frac{1}{k}\frac{\dot\Theta}{\Theta} = \frac{X''}{X} = -\beta^2. \quad \dot\Theta + k\beta^2\Theta = 0. \quad \therefore \Theta(t) = e^{-\beta^2 t}$$

$$X'' + \beta^2 X = 0. \quad \therefore X(x) = A\sin\beta x + B\cos\beta x. \quad \therefore T(x,t) = (A\sin\beta x + B\cos\beta x)e^{-k\beta^2 t}$$

The $\beta = 0$ solution is often of use in such problems. It would lead to $\dot\Theta = 0$ and $X'' = 0$ resulting in $\Theta = 1$ and $X = ax + b$. So, the most general solution results by adding the solutions together:

$$T(x,t) = ax + b + (A\sin\beta x + B\cos\beta x)e^{-k\beta^2 t}$$

2. Refer to the work before No.2. With $T(0,t) = 0$, $b = 0$, $B = 0$. $\therefore T(x,t) = ax + Ae^{-\beta^2 t}\sin\beta x$.

With $T(x,0) = 100\sin\frac{\pi x}{4} = ax + A\sin\beta x$, $a = 0$, A=100, and $\beta = \pi/4$.

$\therefore T(x,t) = 100e^{-k\pi^2 t/16}\sin\frac{\pi x}{4}$. At $x = 2$, $T(2,t) = 100e^{-k\pi^2 t/16} \neq 100$. $\therefore$ no good.

Try $T(2,t) = 100 = 2a + Ae^{-k\beta^2 t}\sin 2\beta$. $\therefore a = 50$, $2\beta = n\pi$ or $\beta = \frac{n\pi}{2}$, $n = 1, 2, 3, \cdots$

Thus, $T_n(x,t) = 50x + A_n e^{-kn^2\pi^2 t/4}\sin\frac{n\pi x}{2}$. The most general solution is

$$\boxed{T(x,t) = 50x + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/4}\sin\frac{n\pi x}{2}}$$

Try $T(x,0) = 100\sin\frac{\pi x}{4} = 50x + \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{2}$ or $100\sin\frac{\pi x}{4} - 50x = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{2}$.

Finally, $\boxed{A_n = \int_0^2 \left(100\sin\frac{\pi x}{4} - 50x\right)\sin\frac{n\pi x}{2}\,dx}$

4. If $T(0,t) = 200 = b + Be^{-k\beta^2 t}$, $b = 200$, $B = 0$. $\therefore T(x,t) = 200 + Ae^{-k\beta^2 t}\sin\beta x$.

$T(x,0) = 200(1 + \sin\pi x) = 200 + A\sin\beta x$, $A = 200$, $\beta = \pi$.

$\therefore \underline{T(x,t) = 200 + 200e^{-k\pi^2 t}\sin\pi x}$. At $x = 2$, $T(2,t) = 200$. $\therefore OK$.

6. For $T(0,t) = 0$, $b = 0$, $B = 0$. $\therefore T(x,t) = ax + Ae^{-k\beta^2 t}\sin\beta x$.

$T(2,t) = 100 = 2a + Ae^{-k\beta^2 t}\sin 2\beta$, $a = 50$, $\beta = n\pi/2$, $n = 1, 2, 3, \cdots$

$$\boxed{T(x,t) = 50x + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/4}\sin\frac{n\pi x}{2}}$$

Finally, $T(x,0) = 100 = 50x + \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{2}$. $\therefore A_n = \int_0^2 (100 - 50x)\sin\frac{n\pi x}{2}\,dx = \boxed{\frac{200}{n\pi}}$

8. $T(0,t) = 0$, $b = 0$, $B = 0$. $\therefore T(x,t) = ax + Ae^{-k\beta^2 t}\sin\beta x$

$T(2,t) = 0$. $\therefore a = 0$, $b = \frac{n\pi}{2}$, $n = 1, 2, 3, \cdots$. $\therefore \boxed{T(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/4}\sin\frac{n\pi x}{2}}$

$T(x,0) = 100(2x - x^2) = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{2}$, $\boxed{A_n = \int_0^2 100(2x - x^2)\sin\frac{n\pi x}{2}\,dx}$

10. $T(1,1000) = 100e^{-\pi^2 \times 0.025} = \underline{78.1°}$

12. $T(1,\infty) = 50 + \underbrace{\sum_{n=1}^{\infty} A_n e^{-\infty} \sin\frac{n\pi}{2}}_{\text{negligible}} = \underline{50°}$

14. $T(1,3600) = \dfrac{400}{\pi} e^{-\pi^2 \times 0.36/4} + \cancel{\dfrac{400}{3\pi} e^{-9\pi^2 \times 0.36/4}}(-1) = \underline{52.5°}$

16. $50 = 100e^{-\pi^2 t/40\,000} \times 1. \qquad \therefore \underline{t = 2812 \text{ s}}$

18. $50 = 50 + A_1 e^{-kn^2 t/4} \times 1. \qquad \therefore \underline{t = \infty \text{ s}}$

22. $\dot{Q} = -KA \left.\dfrac{\partial T}{\partial x}\right|_{x=0} = -800\pi \times 0.1^2 \times (200\pi) \times 1 = \underline{-15780 \text{ W}}$

24. $\dot{Q} = -KA \left.\dfrac{\partial T}{\partial x}\right|_{x=0} = -800\pi \times 0.1^2 \times \left(\dfrac{400}{\pi} \times \dfrac{\pi}{2}\right) \times 1 = \underline{-5024 \text{ W}}$

26. $\dot{Q} = -KA \left.\dfrac{\partial T}{\partial x}\right|_{x=0} = -800\pi \times 0.1^2 \times \left(50 + \dfrac{200}{\pi} \times \dfrac{\pi}{2} e^{-\pi^2/40} + \dfrac{100}{\pi} \times \pi e^{-\pi^2/10} + \dfrac{200}{3\pi} \times \dfrac{3\pi}{2} e^{-9\pi^2/40}\right) \times 1$
$= \underline{-4700 \text{ W}}$

32. $T(x,t) = (A\sin\beta x + B\cos\beta x)e^{-k\beta^2 t}. \qquad \dfrac{\partial T}{\partial x}(0,t) = 0. \qquad \therefore A = 0$

$\therefore T(x,t) = B\cos\beta x e^{-k\beta^2 t}. \qquad T(x,0) = 100\cos x = B\cos\beta x. \qquad \therefore B = 100, \quad \beta = 1$

$\therefore \underline{T(x,t) = 100e^{-kt}\cos x}. \qquad \text{At } x = \pi, \quad \dfrac{\partial T}{\partial x}(0,t) = 0. \quad \therefore \text{OK.}$

34. $T(x,t) = (A\sin\beta x + B\cos\beta x)e^{-k\beta^2 t}. \qquad T(0,t) = 0. \qquad \therefore B = 0$

$T(x,0) = 100\sin\dfrac{x}{2} = A\sin\beta x. \qquad \therefore A = 100, \quad \beta = \dfrac{1}{2}$

$\therefore \underline{T(x,t) = 100e^{-kt/4}\sin\dfrac{x}{2}}. \qquad \text{At } x = \pi, \quad \dfrac{\partial T}{\partial x} = 0. \quad \therefore \text{OK.}$

36. $T(x,t) = (A\sin\beta x + B\cos\beta x)e^{-k\beta^2 t}$. $\quad T(0,t) = 0$. $\quad \therefore B = 0$

$$\frac{\partial T}{\partial x}(\pi,t) = 0 = A\beta\cos\pi\beta e^{-k\beta^2 t}. \quad \therefore \beta = \frac{2n-1}{2}, \quad n = 1,\ 2,\ 3,\cdots$$

$$\therefore \boxed{T(x,t) = \sum_{n=1}^{\infty} A_n e^{-k(2n-1)^2 t/4} \sin\frac{2n-1}{2}x}$$

$$T(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{2n-1}{2}x. \quad \therefore A_n = \frac{2}{\pi}\int_0^{\pi/2} 100\sin\frac{2n-1}{2}dx = \boxed{\frac{400}{(2n-1)\pi}\left(1-\cos\frac{2n-1}{4}\pi\right)}$$

38. $T(\pi/2,1000) = A_1\sin\frac{\pi}{4}e^{-1/40} + A_2\sin\frac{3\pi}{4}e^{-9/40} + A_3\sin\frac{5\pi}{4}e^{-25/40} +$ negligible terms

$$= 25.7 + 40.9 - 23.3 - 1.1 = \underline{41.2°}$$

40. $T_p = g(x)$. Substitute into $\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + \frac{\phi}{\rho C}$: $\quad 0 = kg'' + \frac{2000}{9000\times 400}$ or $g'' = -5.556$.

$\therefore g(x) = -2.778x^2 + c_1 x + c_2$. If $T(2,t) = 0$ then $c_1 = 5.556$. Thus, since $T_h \to 0$

as $t \to \infty$, $\underline{T(x,\infty) = 2.778(2x - x^2)}$

42. See No.40. Again $g(x) = -2.778x^2 + c_1 x + c_2$. $\quad T(0,t) = 100 = c_2$.

$$\frac{\partial T}{\partial x}(2,t) = 0 = -5.556\times 2 + c_1. \quad \therefore \underline{T(x,\infty) = 100 + 2.778(4x - x^2)}$$

44. $T_h(x,t) = (A\sin\beta x + B\cos\beta x)e^{-k\beta^2 t}$. $\quad T(0,t) = 0$. $\quad \therefore B = 0$. $\quad \frac{\partial T}{\partial x}(2,t) = 0 = A\beta\cos 2\beta e^{-k\beta^2 t}$

$$\therefore \beta = \frac{2n-1}{4}\pi, \quad n = 1,\ 2,\ 3,\cdots. \quad \therefore T_h(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{2n-1}{4}\pi x e^{-k(2n-1)^2\pi^2 t/16}$$

Using T_p from No.41, $\boxed{T(x,t) = 2.778(4x - x^2) + \sum_{n=1}^{\infty} A_n \sin\frac{2n-1}{4}\pi x e^{-k(2n-1)^2\pi^2 t/16}}$

where $\boxed{A_n = \int_0^2 \left[100 + 2.778(x^2 - 4x)\right]\sin\frac{2n-1}{4}\pi x\, dx}$

46. Since we desire a Fourier series in x, we use $\frac{X''}{X} = \frac{-Y''}{Y} = -\beta^2$. Thus,

$T(x,t) = (c_1\sin\beta x + c_2\cos\beta x)(c_3 e^{\beta y} + c_4 e^{-\beta y})$. $T(0,y) = 0$. $\quad \therefore c_2 = 0$. $\quad T(x,0) = 0$. $\quad \therefore c_4 - c_3$

Let $c_1 c_3 = A$. $\quad T(x,1) = 100\sin\pi x = A\sin\beta x(e^{\beta} - e^{-\beta})$. $\therefore \beta = \pi$, $\quad 100 = A(e^{\pi} - e^{-\pi})$.

$\therefore A = 4.336$ and $\boxed{T(x,y) = 4.336\sin\pi x(e^{\pi y} - e^{-\pi y})}$. $\quad$ At $x = 1$, $\quad T = 0$. $\quad \therefore OK.$

48. We desire a Fourier series in x so we can meet the $T = 100^{o}$ condition. Hence,

$$\frac{X''}{X} = \frac{-Y''}{Y} = -\beta^2. \text{ Thus, } T(x,t) = (c_1 \sin\beta x + c_2 \cos\beta x)(c_3 e^{\beta y} + c_4 e^{-\beta y}). \ T(0,y) = 0$$

$$\therefore c_2 = 0. \quad T(x,0) = 0. \quad \therefore c_4 - c_3. \text{ Let } c_1 c_3 = A. \quad T(1,y) = 0 = A\sin\beta(e^{\beta y} - e^{-\beta y}).$$

$$\therefore \beta = n\pi, \quad n = 1,\ 2,\ 3,\cdots. \quad \therefore \boxed{T(x,y) = \sum_{n=1}^{\infty} A_n \sin n\pi x(e^{n\pi y} - e^{-n\pi y})}$$

$$T(x,1) = 100 = \sum_{n=1}^{\infty} A_n \sin n\pi x(e^{n\pi} - e^{-n\pi}). \quad \therefore A_n = \frac{200}{e^{n\pi} - e^{-n\pi}} \int_0^1 \sin n\pi x\, dx = \boxed{\frac{200}{n\pi(e^{n\pi} - e^{-n\pi})}(1 - \cos n\pi)}$$

50. Let $T = \theta + 100.$ Then all boundary conditions on θ are 0 except $\theta(1,y)$. This is the solution to No.48 with x and y interchanged. Simply add 100 to that solution and we have $T(x,y)$, i.e.,

$$\boxed{T(x,y) = 100 + \sum_{n=1}^{\infty} A_n \sin n\pi y(e^{n\pi x} - e^{-n\pi x})} \text{ where } \boxed{A_n = \frac{200}{n\pi(e^{n\pi} - e^{-n\pi})}(1 - \cos n\pi)}$$

Section 8.8

1. The describing equation is $\frac{\partial^2}{\partial r^2}\left(r^2 T\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial T}{\partial\phi}\right) = 0$. The solution is (see Eq. 8.8.11): $T(r,x) = \sum_{n=0}^{\infty}\left[A_n r^n P_n(x) + B_n r^{-(n+1)} P_n(x)\right]$ where $x = \cos\phi.$

Inside the surface: We must let $B_n = 0$ since at $r = 0$, $r^{-(n+1)} = \infty.$ Thus, $T(r,x) = \sum_{n=0}^{\infty} A_n r^n P_n(x)$. Now, we require $T(0.2,x) = 250 = \sum_{n=0}^{\infty} A_n 0.2^n P_n(x).$ Using Eq. 8.8.15: $A_n = \frac{2n+1}{2\times 0.2^n}\int_{-1}^{1} 250 P_n(x)\,dx.$ Hence, $T(x,r) = 250 r^0 P_0(x) = \underline{250°C}.$

$\therefore A_0 = 250, \quad A_n = 0, \quad n = 1,\ 2,\ 3,\cdots.$

Outside the surface: If $T(\infty,x)$ is finite, $A_n = 0$. Thus, we have for this region

$T(r,x) = \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(x).$ Use $T(0.2,x) = 250 = \sum_{n=0}^{\infty} B_n 0.2^{-(n+1)} P_n(x).$ Therefore,

$$B_n = \frac{2n+1}{2\times o.2^{-(n+1)}}\int_{-1}^{1} 250 P_n(x)\,dx. \quad \therefore B_0 = 50 \text{ and } B_n = 0, \quad n = 1,\ 2,\ 3,\cdots. \quad \therefore \underline{T(r,x) = \frac{50}{r}}$$

2. Use the results of No.1: $\boxed{T(r,x)=\sum_{n=0}^{\infty} A_n r^n P_n(x)}$ where $x=\cos\phi$. We have

$T(1,x)=100x=\sum_{n=0}^{\infty} A_n P_n(x)$ so that $\boxed{A_n = \frac{2n+1}{2}\int_{-1}^{1} 100xP_n(x)\,dx}$

Section 8.9

2. $\frac{\partial T}{\partial t}=k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$. Let $T(r,t)=R(r)\theta(t)$. Substitute in and $\frac{1}{k}\frac{\dot{\theta}}{\theta}=\frac{1}{R}\left(\frac{1}{r}R'+R''\right)=-\mu^2$.

There results: $\dot{\theta}=-k\mu^2\theta$. $\quad\therefore \theta(t)=c_1e^{-k\mu^2 t}$

$R''+\frac{1}{r}R'+\mu^2R=0.\quad \therefore R(r)=c_2J_0(\mu r)+c_3Y_0(\mu r)$

$\therefore T(r,t)=[AJ_0(\mu r)+BY_0(\mu r)]e^{-k\mu^2 t}.\quad Y_0(0)=\infty.\quad \therefore B=0.$

$\therefore T(r_0,t)=0=AJ_0(\mu r_0)e^{-k\mu^2 t}.\quad J_0(\mu r_0)=0.$ Let μ_n be the roots. Then,

$\boxed{T(r,t)=\sum_{n=1}^{\infty} A_n J_0(\mu_n r)e^{-k\mu_n^2 t}}$. $T(r,0)=f(r)=\sum_{n=1}^{\infty} A_n J_0(\mu_n r)$. $\therefore \boxed{A_n = \frac{2}{r_0^2 J_1^2(\mu_n r_0)}\int_0^{r_0} rf(r)J_0(\mu_n r)\,dr}$

4. The solution, using separation of variables, is $T(r,t)=[AJ_0(\mu r)+BY_0(\mu r)]e^{-k\mu^2 t}$.
$Y(0)=\infty.\quad \therefore B=0.$ We then have $T(r,t)=AJ_0(\mu r)e^{-k\mu^2 t}$. $\frac{\partial T}{\partial r}(1,t)=0=AJ_0'(\mu)e^{-k\mu^2 t}$
so that $J_0'(\mu)=0$. The first three roots are $\mu_1=0,\quad \mu_2=3.832,\quad \mu_3=7.016,$ using the values given in Example 8.9.1. The general solution is then $\boxed{T(r,t)=\sum_{n=1}^{\infty} A_n J_0(\mu_n r)e^{-k\mu_n^2 t}}$

Finally, $T(r,0)=100r=\sum_{n=1}^{\infty} A_n J_0(\mu_n r)$. Thus, $A_n=\frac{2}{J_0^2(\mu_n)}\int_0^1 100r^2J_0(\mu_n r)\,dr$. We find that

$$A_1=2\int_0^1 100r^2dr=\frac{200}{3},\quad A_2=1230\int_0^1 r^2J_0(3.832r)\,dr,\quad A_3=2220\int_0^1 r^2J_0(7.016r)\,dr$$

9. Numerical Methods

Section 9.2

6. Following Example 9.2.1, $\Delta = \dfrac{\delta^2}{2} + \delta\sqrt{1+\delta^2/4}$. use the binomial theorem with n = ½:

$$\Delta = \frac{\delta^2}{2} + \delta\left[1 + \frac{1}{2}\frac{\delta^2}{4} - \frac{1}{4}\frac{(\delta^2/4)^2}{2!} + \cdots\right] = \frac{\delta^2}{2} + \delta + \frac{\delta^3}{8} - \frac{\delta^5}{128} + \cdots = \underline{\delta + \frac{\delta^2}{2} + \frac{\delta^3}{8} - \cdots}$$

Section 9.4

12. $$Df_i = \frac{1}{h}\left(\Delta - \Delta^2/2 + \Delta^3/3 - \overset{\text{neglect}}{\cancel{\cdots}}\right) f_i = \frac{1}{h}\left[f_{i+1} - f_i - (f_{i+2} - 2f_{i+1} + f_i)/2 + (f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i)\right]$$
$$= \underline{\frac{1}{6h}(2f_{i+3} - 9f_{i+2} + 18f_{i+1} - 11f_i)}$$

14. $$D^3 f_i = \frac{1}{h^3}\left(\nabla^3 + 3\nabla^4/2 + 7\nabla^5/4 - \overset{\text{neglect}}{\cancel{\cdots}}\right) f_i$$
$$= \frac{1}{4h^3}[f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3} + 3(f_i - 4f_{i-1} + 6f_{i-2} - 4f_{i-3} + f_{i-4})/2 + 7(f_i - 5f_{i-1}$$
$$+ 10f_{i-2} - 10f_{i-3} + 5f_{i-4} - f_{i-5})/4] = \underline{\frac{1}{4h^3}(17f_i - 71f_{i-1} + 118f_{i-2} - 98f_{i-3} + 41f_{i-4} - 7f_{i-5})}$$

16. $$D(erfx) = \frac{1}{h}(f_{i+1} - f_i) = \frac{1}{0.02}(0.978038 - 0.976348) = \underline{0.08450}$$

From No.15, the exact value = 0.08723.

18. $$D(erfx) = \frac{1}{2h}(3f_i - 4f_{i-1} + f_{i-2}) = \frac{1}{2\times 0.02}(3\times 0.976348 - 4\times 0.974547 + 0.972628) = \underline{0.08710}$$

From No.15, the exact value = 0.08723.

20. $$D^2 J_1 = \frac{1}{h^2}(2f_i - 5f_{i-1} + 4f_{i-2} - f_{i-3})$$
$$= \frac{1}{0.1^2}(2\times 0.57672 - 5\times 0.58116 + 4\times 0.58152 - 0.57777) = \underline{-0.405}$$

22. $D^2 J_0 = \frac{1}{h^2}(-f_{i+3} + 4f_{i+2} - 5f_{i+1} + 2f_i)$

$$= \frac{1}{0.1^2}(-0.97763 + 4\times 0.99002 - 5\times 0.99750 + 2\times 1.000) = \underline{-0.505}$$

Section 9.5

2. a) $\int_0^6 f(x)\,dx = \frac{1}{2}(0 + 2\times 1^2 + 2\times 2^2 + 2\times 3^2 + 2\times 4^2 + 2\times 5^2 + 6^2) = \underline{73}$

b) $\int_0^6 f(x)\,dx = \frac{1}{3}(0 + 4\times 1^2 + 2\times 2^2 + 4\times 3^2 + 2\times 4^2 + 4\times 5^2 + 6^2) = \underline{72}$

$\int_0^6 x^2\,dx = \frac{6^3}{3} = 72.$ Since Simpson's one-third rule is approximating a parabola with a parabola, it is exact. $f'' = 2.$ $\frac{h^3}{12}Nf'' = \frac{1^3}{12}\times 6\times 2 = 1.$ $73 - 72 = 1.$ $\therefore e = \frac{h^3}{12}Nf''.$

4. a) $\int_0^2 J_0(x)\,dx = \frac{0.2}{2}\big(1 + 2\times 0.990 + 2\times 0.960 + 2\times 0.912 + 2\times 0.846 + 2\times 0.765 + 2\times 0.671$
$+2\times 0.567 + 2\times 0.455 + 2\times 0.340 + 0.224\big) = \underline{1.424}$

b) $\int_0^2 J_0(x)\,dx = \frac{0.2}{3}\big(1 + 4\times 0.990 + 2\times 0.960 + 4\times 0.912 + 2\times 0.846 + 4\times 0.765 + 2\times 0.671$
$+4\times 0.567 + 2\times 0.455 + 4\times 0.340 + 0.224\big) = \underline{1.426}$

6. $\int_0^2 y(t)\,dt = \frac{0.2}{3}\big(9.6 + 4\times 9.1 + 2\times 7.4 + 4\times 6.8 + 2\times 7.6 + 4\times 8.8 + 12.2\big) = \underline{10.04}$

Section 9.6

2. $f_{i+n} = 0.29507 + 0.4(0.28822 - 0.29507) + \frac{0.4(-0.6)}{2}(0.27860 - 2\times 0.28822 + 0.29507) = \underline{0.29267}$

4. $f_{i-n} = \text{sum from No.3} - \frac{0.6(-0.4)(-1.4)}{6}(0.28822 - 3\times 0.29507 + 3\times 0.29905 - 0.30008)$
$= 0.29267 - 0.0000045 = \underline{0.29267}$

10. $erf(2.01) = .995720 - .5(.995720 - .995323) + \frac{.5(-.5)}{2}(.995720 - 2\times .995323 + .994892) = \underline{.995526}$

12. $erf(1.51) = .966105 - .5(.968413 - .966105) + \frac{.5(-.5)}{2}(.970586 - 2\times .968413 + .966105) = \underline{.96728}$

$\frac{0.5(-0.5)(-1.5)}{6}(0.972628 - 3\times 0.970586 + 3\times 0.968413 - 0.966105) = 0.0000003.$ The fourth term does not influence the answer.

14. $J_1(1.51) = .55794 + 0.1(.56990 - .55794) + \frac{0.1(-0.9)}{2}(.57777 - 2\times .56990 + .55794) = \underline{0.55932}$

Section 9.7

2. $x_1 = 1 - \frac{1}{-1} = 2, \quad x_2 = 2 - \frac{-1}{-2} = 1.5, \quad x_3 = 1.5 - \frac{-0.125}{-2.25} = 1.556, \quad x_4 = 1.556 - \frac{-.0056}{-2.259} = \underline{1.553}$

4. $x_1 = 2 - \frac{6}{14} = 1.57, \qquad x_2 = 1.57 - \frac{1.01}{9.41} = 1.46, \qquad x_3 = 1.46 - \frac{0.032}{8.42} = \underline{1.46}$

6. $f(x) = x^4 - 4x - 2. \quad x_1 = 2 - \frac{6}{28} = 1.79, \quad x_2 = 1.79 - \frac{1.11}{18.9} = 1.73, \quad x_3 = 1.73 - \frac{0.037}{16.7} = \underline{1.728}$

8. $f(x) = x^5 + 10x - 4.$ Try $x_0 = 0, \quad f = -4. \quad x_0 = 1, \quad f = 7.$
Try $x_0 = 0.3, \quad x_1 = 0.3 - \frac{-0.97}{10.3} = 0.394, \quad x_2 = 0.394 - \frac{0.0012}{10.5} = \underline{0.394}$

10. $f(x) = x + \ln x - 10. \quad f'(x) = 1 + \frac{1}{x}.$ Try $x_0 = 1, \quad f = -9. \quad x_0 = 10, \quad f = 2.3.$
$\therefore$ Try $x_0 = 8. \quad x_1 = 8 - \frac{0.079}{1.125} = 7.93. \quad x_2 = 7.93 - \frac{0.00065}{1.126} = \underline{7.93}$

Section 9.8

2. $y_{i+1} = y_i + 0.05\dot{y}_i + 0.00125\ddot{y}_i \qquad \ddot{y} = -2y - 2\dot{y}t$

$t = 0.05 \qquad y_1 = .2 + .05\times 4 + .00125\times(-.4) = 0.3995$

$t = 0.01 \qquad y_2 = .3995 + .05\times(4 - 2\times .3995\times .05) + .00125\times(-2\times .3995 - 2\times 3.96\times .05) = 0.596$

$t = 0.015 \qquad y_3 = .596 + .05\times(4 - 2\times .596\times .1) + .00125\times(-2\times .596 - 2\times 3.88\times .1) = 0.789$

$t = 0.02 \qquad y_4 = .789 + .05\times(4 - 2\times .789\times .15) + .00125\times(-2\times .789 - 2\times 3.76\times .15) = \underline{0.973}$

This compares with $y(0.2) = 0.976$ in Example 9.8.1.

4. $y_{i+1} = y_i + h\dot{y}_i + \dfrac{h^2}{2}\ddot{y}_i \qquad 2\dot{y}y = t^2 \qquad 2\dot{y}^2 + 2\ddot{y}y = 2t \qquad \ddot{y} = (t - \dot{y}^2)/y \qquad y(0) = 2$

$$\begin{array}{ll} t = 0.4 & y_1 = 2 + 0.4 \times 0 + 0.08 \times 0 = 2 \\ t = 0.8 & y_2 = 2 + 0.4 \times 0.4^2/(2\times 2) = 2.03 \\ t = 1.2 & y_3 = 2.03 + 0.4 \times 0.8^2/(2 \times 2.03) + 0.8(0.8 - 0.158^2)/2.03 = 2.12 \\ t = 1.6 & y_4 = 2.12 + 0.4 \times 1.2^2/(2 \times 2.12) + 0.8(1.2 - 0.34^2)/2.12 = 2.30 \\ t = 2.0 & y_5 = 2.30 + 0.4 \times 1.6^2/(2 \times 2.30) + 0.8(1.6 - 0.557^2)/2.30 = \underline{2.57} \end{array}$$

This compares with $y(2) = 2.58$ from the exact solution.

6. $y_{i+1} = y_i + 0.4\eta_i \qquad \eta_i = f(y_i + 0.2 f_i,\ t_i + 0.2) \qquad f = \dot{y} = \dfrac{t^2}{2y} \qquad y_0 = 2$

$$\begin{array}{ll} t = 0.4 & y_1 = 2 + 0.4[0.2^2/2 \times 2] = 2.004 \\ t = 0.8 & y_2 = 2.004 + 0.4[0.6^2/2(2.004 + 0.2 \times 0.4^2/2 \times 2.004)] = 2.040 \\ t = 1.2 & y_3 = 2.04 + 0.4[1.0^2/2(2.04 + 0.2 \times 0.8^2/2 \times 2.04)] = 2.137 \\ t = 1.6 & y_4 = 2.137 + 0.4[1.4^2/2(2.137 + 0.2 \times 1.2^2/2 \times 2.137)] = 2.315 \\ t = 2.0 & y_5 = 2.315 + 0.4[1.8^2/2(2.315 + 0.2 \times 1.6^2/2 \times 2.315)] = \underline{2.58} \end{array}$$

This compares with $y(2) = 2.58$ from the exact solution.

8. $\dot{y} = t^2 - 4yt \qquad \ddot{y} = 2t - 4y - 4\dot{y}t \qquad y_{i+1} = y_i + 0.2\dot{y}_i + 0.02\ddot{y}_i \qquad y_0 = 2$

$$\begin{array}{ll} t = 0.2 & y_1 = 2 + .2 \times 0 + .02(-8) = \underline{1.84} \\ t = 0.4 & y_2 = 1.84 + .2(.2^2 - 4 \times 1.84 \times .2) + .02[2 \times .2 - 4 \times 1.84 - 4(-1.432) \times .2] = \underline{1.437} \\ t = 0.6 & y_3 = 1.437 + .2(.4^2 - 4 \times 1.437 \times .4) + .02[2 \times .4 - 4 \times 1.437 - 4(-2.139) \times .4] = \underline{0.979} \\ t = 0.8 & y_4 = .979 + .2(.6^2 - 4 \times .979 \times .6) + .02[2 \times .6 - 4 \times .979 - 4(-1.990) \times .6] = \underline{0.622} \\ t = 1.0 & y_5 = .622 + .2(.8^2 - 4 \times .622 \times .8) + .02[2 \times .8 - 4 \times .622 - 4(-1.35) \times .8] = \underline{0.421} \end{array}$$

10. $\dot{y}_i = \dfrac{1}{2 \times 0.2}(-y_{i+2} + 4y_{i+1} - 3y_i) = t_i^2 - 4y_i t_i. \qquad \therefore y_{i+2} = 4y_{i+1} - 3y_i - 0.4t_i^2 + 1.6y_i t_i$

Use Taylor's method for $y_1 = 1.84$ (see No.8). Then, with $y_0 = 2$:

$$\begin{array}{l} y_2 = 4 \times 1.84 - 3 \times 2 - 0.4 \times 0 + 1.6 \times 2 \times 0 = \underline{1.36} \\ y_3 = 4 \times 1.36 - 3 \times 1.84 - 0.4 \times 0.2^2 + 1.6 \times 1.84 \times 0.2 = \underline{0.493} \\ y_4 = 4 \times 0.493 - 3 \times 1.36 - 0.4 \times 0.4^2 + 1.6 \times 1.36 \times 0.4 = \underline{-1.30} \\ y_5 = 4 \times (-1.3) - 3 \times 0.493 - 0.4 \times 0.6^2 + 1.6 \times 0.493 \times 0.6 = \underline{-6.35} \end{array}$$

12. $\dot{y}^2 = 4 - 2y, \quad \ddot{y} = -1. \quad \dddot{y} = 0. \quad \therefore y_1 = \underline{0.38}, \quad y_2 = \underline{0.72}$

14. $y_{i+1} = y_i + \frac{0.2}{6}(\xi_i + 4\eta_i + \zeta_i). \qquad \xi_0 = 4^{1/2} = 2, \quad \eta_0 = [4 - 2(0 + 0.1 \times 2)]^{1/2} = 1.897,$

$$\xi_0 = [4 - 2(0 + 2 \times 0.2 \times 1.897 - 0.2 \times 2)]^{1/2} = 1.812$$

$$\therefore y_1 = 0 + \frac{0.2}{6}(2 + 4 \times 1.897 + 1.812) = \underline{0.380}$$

$$\xi_1 = (4 - 2 \times 0.38)^{1/2} = 1.80, \quad \eta_1 = [4 - 2(0.38 + 0.1 \times 1.8)]^{1/2} = 1.697$$

$$\zeta_1 = [4 - 2(0.38 + 2 \times 0.2 \times 1.697 - 0.2 \times 1.8)]^{1/2} = 1.613$$

$$\therefore y_2 = 0.38 + \frac{0.2}{6}(1.8 + 4 \times 1.697 + 1.613) = \underline{0.720}$$

16. Truncate the series in Eq. 9.8.20 omitting the ∇^3 terms. Then, there results

$$y_{i+1} = y_i + h\left[1 + \frac{1}{2}\left(\nabla + \frac{\nabla^2}{2}\right) + \frac{1}{6}\nabla^2\right]\dot{y}_i \qquad e = O(h^4)$$

$$= y_i + h\left(1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2\right)\dot{y}_i = y_i + h\left[\dot{y}_i + \frac{1}{2}(\dot{y}_i - \dot{y}_{i-1}) + \frac{5}{12}(\dot{y}_i - 2\dot{y}_{i-1} + \dot{y}_{i-2})\right]$$

$$= \underline{y_i + \frac{h}{12}(23\dot{y}_i - 16\dot{y}_{i-1} + 5\dot{y}_{i-2})}$$

Section 9.9

2. $\frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) - 2t_i y_i = 5. \qquad \therefore y_{i+1} = 2y_i - y_{i-1} + 2h^2 t_i y_i + 5h^2.$ Use $y_0 = 2, \quad y_1 = 2.1$:

	From Ex. 9.9.2
$y_2 = 2 \times 2.1 - 2 + 2 \times 0.2^2 \times 0.2 \times 2.1 + 5 \times 0.2^2 = \underline{2.43}$	2.43
$y_3 = 2 \times 2.43 - 2.1 + 2 \times 0.2^2 \times 0.4 \times 2.43 + 5 \times 0.2^2 = \underline{3.04}$	3.02
$y_4 = 2 \times 3.04 - 2.43 + 2 \times 0.2^2 \times 0.6 \times 3.04 + 5 \times 0.2^2 = \underline{4.00}$	3.90

4. $\ddot{y} = -y, \quad \dddot{y} = -\dot{y} \qquad y_{i+1} = y_i + 0.4\dot{y}_i + 0.08\ddot{y}_i \qquad \dot{y}_{i+1} = \dot{y}_i + 0.4\ddot{y}_i + 0.08\dddot{y}_i$

$y_1 = 0 + 0.4 \times 4 + 0.8 \times 0 = \underline{1.6}$ $\qquad \dot{y}_1 = 4 + 0.4(0) + 0.08(-4) = 3.68$

$y_2 = 1.6 + 0.4 \times 3.68 + 0.8 \times (-1.6) = \underline{2.94}$ $\qquad \dot{y}_2 = 3.68 + 0.4(-1.6) + 0.08(-3.68) = 2.75$

$y_3 = 2.94 + 0.4 \times 2.75 + 0.8 \times (-2.94) = \underline{3.80}$ $\qquad \dot{y}_3 = 2.75 + 0.4(-2.94) + 0.08(-2.75) = 1.354$

$y_4 = 3.80 + 0.4 \times 1.354 + 0.8 \times (-3.8) = \underline{4.04}$ $\qquad \dot{y}_4 = 1.354 + 0.4(-3.8) + 0.08(-1.354) = -0.274$

$y_5 = 4.04 + 0.4(-0.274) + 0.8 \times (-4.04) = \underline{3.61}$

6. Use central differences: $\frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) = -y_i$. Use $y_1 = 1.6$ (see No.4).

$\therefore y_{i+1} = (2 - h^2)y_i - y_{i-1}$ $\quad y_2 = (2 - 0.4^2) \times 1.6 - 0 = \underline{2.94}$

$y_3 = (2 - 0.4^2) \times 2.94 - 1.6 = \underline{3.87}$ $\quad y_4 = (2 - 0.4^2) \times 3.87 - 2.94 = \underline{4.26}$

$y_5 = (2 - 0.4^2) \times 4.26 - 3.87 = \underline{4.05}$

8. $\ddot{y} = -10y - 2\dot{y}^2, \quad \dddot{y} = -10\dot{y} - 4\dot{y}\ddot{y}. \qquad y_{i+1} = y_i + 0.2\dot{y}_i + 0.02\ddot{y}_i, \quad \dot{y}_{i+1} = \dot{y}_i + 0.2\ddot{y}_i + 0.02\dddot{y}_i$

$y_1 = 0 + .2 \times 1 + .02(-10 \times 0 - 2 \times 1^2) = \underline{.16} \qquad \dot{y}_1 = 1 + .2(-2) + .02(-10 \times 1 - 4 \times 1 \times -2) = .56$

$y_2 = .16 + .2 \times .56 + .02(-10 \times .16 - 2 \times .56^2) = \underline{0.227}$

$\dot{y}_2 = .56 + .2(-2.23) - .02(10 + 4 \times 2.23) \times .56 = -.098$

$y_3 = .227 + .2(-.098) - .02(10 \times .227 + 2 \times .098^2) = \underline{0.162}$

$\dot{y}_3 = -.098 + .2(-2.29) + .02(10 + 4 \times 2.29) \times .098 = -.518$

$y_4 = .162 + .2(-.518) - .02(10 \times .162 + 2 \times .518^2) = \underline{0.0153}$

$\dot{y}_4 = -.518 + .2(-2.16) + .02(10 + 4 \times 2.16) \times .518 = -.757$

$y_5 = .0153 + .2(-.757) - .02(10 \times .0153 + 2 \times .757^2) = \underline{-0.162}$

Section 9.12

2. At t = 24 ks we follow the example to find:

$$T_{i,6} = \frac{1}{5}(T_{i+1,5} - 2T_{i,5} + T_{i-1,5}) + T_{i,5}. \quad \therefore T_{0,6} = 500, \quad T_{1,6} = 296, \quad T_{2,6} = 155, \text{ etc.}$$

Interpolate to find the time when $T_{2,j} = 145$:

$$\text{time } = \frac{145 - 138}{155 - 138} \times (24 - 20) + 20 = \underline{21.65 \text{ ks}}$$

4. Continue with the table of Problem 3:

Time\Distance	**0**	**0.5**	**1.0**	**1.5**	**2.0**	**2.5**
25	600	331	156	82	63	60

At x = 1.25, the center = 119°. $\quad \therefore \underline{t = 22.4 \text{ ks}}$

6. Continue with the table of Problem 3:

Time\Distance	**0**	**0.5**	**1.0**	**1.5**	**2.0**	**2.5**
35	600	365	194	104	70	60

At x = 1.25, the center = 149°. $\quad \therefore \underline{t = 31.8 \text{ ks}}$

8. $ak/h^2 = 4\times10^{-6}\times20\,000/0.4^2 = 1/2.$ $\quad T_{i,j} = \frac{1}{2}(T_{i+1,j} - 2T_{i,j} + T_{i-1,j})$

A table is constructed displaying the calculations:

Time/Distance	**0**	**0.4**	**0.8**	**1.2**	**1.6**	**2.0**
0	200	0	0	0	0	0
20	200	100	0	0	0	0
40	200	100	50	0	0	0
60	200	125	50	25	0	0
80	**200**	**125**	**75**	**25**	**12.5**	**12.5**

Note: At x = 2.0 we must have $T_{4,j} = T_{5,j}$ since the right end is insulated. $Q = -KA\frac{\partial T}{\partial x} = 0.$ $\therefore \frac{\partial T}{\partial x} = 0$.

10. With $ak/h^2 = 1/2$ and an insulated right end (Q = 0) so that $T_{4,j} = T_{5,j}$ we have:

Time/Distance	**0**	**0.4**	**0.8**	**1.2**	**1.6**	**2.0**
0	200	100	100	100	100	100
20	200	150	100	100	100	100
40	200	150	125	100	100	100
60	200	162	125	112	100	100
80	**200**	**162**	**137**	**112**	**106**	**106**

12. $ak/h = 40\times0.025/1 = 1.$ With $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 4$ or $u_{i,1} - u_{i,0} = 0.025\times4 = 0.1.$ $\therefore u_{i,1} = u_{i,0} + 0.1$

Time/Distance	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	0.1	0.2	0.3	0.4	0.2	0
0.025	0	0.2	0.3	0.4	0.5	0.3	0
0.050	0	0.2	0.4	0.5	0.3	0.3	0
0.075	0	0.2	0.4	0.3	0.3	0	0
0.100	0	0.2	0.1	0.2	0	0	0
0.125	**0**	**–0.1**	**0**	**–0.2**	**–0.1**	**0**	**0**

14. $ak/h = 50\times0.004/0.2 = 1.$ $\quad \therefore u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$

For an initial velocity of zero $u_{i,0} = u_{i,1}$. Display the results as follows:

Time/Distance	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
0	0	0.02	0.04	0.04	0.02	0
0.004	0	0.02	0.04	0.04	0.02	0
0.008	0	0	0.02	0.02	0.02	0
0.012	0	–0.02	0	0	0	0
0.016	0	–0.02	–0.02	–0.02	–0.02	0
0.020	**0**	**–0.02**	**–0.04**	**–0.04**	**–0.02**	**0**

At t = 0.02 s, one-half a cycle has elapsed.

16. $ak/h = 40 \times 0.025/1 = 1.$ With $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 10$ then $u_{i,1} - u_{i,0} = 0.004 \times 10 = 0.04.$

Time/Distance	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
0	0	0.02	0.04	0.04	0.02	0
0.004	0	0.06	0.08	0.08	0.06	0
0.008	0	0.06	0.10	0.10	0.06	0
0.012	0	0.04	0.08	0.08	0.04	0
0.016	0	0.02	0.02	0.02	0.02	0
0.020	**0**	**–0.02**	**–0.04**	**–0.04**	**–0.02**	**0**

At t = 0.02 s, one-half a cycle has elapsed.

10. Complex Variables

Section 10.2

2. 143.1^{o}, 2.498 rad

4. 216.9^{o}, 3.785 rad

6. $(3-4i)(3+4i)=9+16=\underline{25}$

8. $\left|\dfrac{4-4i}{2-4i}\right|=\dfrac{|4-4i|}{|2-4i|}=\dfrac{\sqrt{16+16}}{\sqrt{4+16}}=\underline{1.265}$

10. $|3-4i|^2=|3-4i||3-4i|=5\times 5=\underline{25}$

12. $(3-4i)^4=(-7-24i)^2=\underline{-527+336i}$

14. $(3-4i)^{1/3}=5^{1/3}\left(\cos\dfrac{5.356+2k\pi}{3}+i\sin\dfrac{5.356+2k\pi}{3}\right)$

$$\begin{aligned}
k=0: &\quad 1.710(-0.2129+0.9771i)=\underline{-0.3641+1.671i}\\
k=1: &\quad 1.710(-0.7397-0.6729i)=\underline{-1.265-1.151i}\\
k=2: &\quad 1.710(0.9526-0.3042i)=\underline{1.629-0.5202i}
\end{aligned}$$

16. $z^2/z^{1/2}=z^{3/2}$. Using the results of No.13, we cube each part:

$$(-2+i)^3=(3-4i)(-2+i)=\underline{-2+11i}$$
$$(2-i)^3=(3-4i)(2-i)=\underline{2-11i}$$

18. $(-16)^{1/4}=2\left(\cos\dfrac{\pi+2k\pi}{4}+i\sin\dfrac{\pi+2k\pi}{4}\right)$. $\quad k=0:\ (-16)^{1/4}=\underline{\sqrt{2}(1+i)}$

$k=1:\ (-16)^{1/4}=\underline{\sqrt{2}(-1+i)}$. $\quad k=2:\ (-16)^{1/4}=\underline{\sqrt{2}(-1-i)}$. $\quad k=3:\ (-16)^{1/4}=\underline{\sqrt{2}(1-i)}$.

20. $9^{1/2}=3\left(\cos\dfrac{2k\pi}{2}+i\sin\dfrac{2k\pi}{2}\right)$. $\quad k=0:\ 9^{1/2}=\underline{3}$. $\quad k=1:\ 9^{1/2}=\underline{-3}$

22. $|z-2|=2$. $\quad |x-2+iy|=\sqrt{(x-2)^2+y^2}=2$. $\quad \therefore \underline{(x-2)^2+y^2=4}$, a circle.

24. $\left|\dfrac{z-1}{z+1}\right|=4$. $\ |z-1|=4|z+1|$. $\ |x-1+iy|=4|x+1+iy|$. Thus, $(x-1)^2+y^2=16[(x+1)^2+y^2]$

or $x^2+\dfrac{34}{15}x+y^2=-1$ or $\underline{(x+34/30)^2+y^2=\dfrac{64}{225}}$, a circle.

Section 10.3

2. $-2 = 2e^{\pi i}$

4. $-2i = 2e^{3\pi i/2}$

6. $5-12i = 13e^{5.107i}$

8. $-5-12i = 13e^{1.966i}$

12. $e^{2i} = \cos 2 + i\sin 2 = \underline{-0.416+0.909i}$

14. $e^{4\pi i} = \cos 4\pi + i\sin 4\pi = \underline{1}$

16. $e^{-1-\pi i/4} = e^{-1}(\cos\pi/4 - i\sin\pi/4) = \underline{e^{-1}\sqrt{2}(1-i)/2}$

18. $(1-i) = \sqrt{2}e^{7\pi i/4} = \sqrt{2}e^{15\pi i/4} = \sqrt{2}e^{23\pi i/4} = \sqrt{2}e^{31\pi i/4}$

$$(1-i)^{1/4} = \sqrt{2}(\cos 7\pi i/4 + i\sin 7\pi i/4) = \underline{0.276+1.39i}$$
$$(1-i)^{1/4} = \sqrt{2}(\cos 15\pi i/4 + i\sin 15\pi i/4) = \underline{-1.39+0.276i}$$
$$(1-i)^{1/4} = \sqrt{2}(\cos 23\pi i/4 + i\sin 23\pi i/4) = \underline{-0.276-1.39i}$$
$$(1-i)^{1/4} = \sqrt{2}(\cos 31\pi i/4 + i\sin 31\pi i/4) = \underline{1.39-0.276i}$$

20. $2+i = \sqrt{5}e^{0.4636i}$. $\quad \therefore (2+i)^3 = 5^{3/2}e^{1.3908i} = 11.2(\cos 1.3908 + i\sin 1.3908) = \underline{2+11i}$

22. $2-i = \sqrt{5}e^{5.82i}$. $\quad \therefore (2-i)^{1/2} = 5^{1/4}e^{2.91i} = 1.495(\cos 2.91 + i\sin 2.91) = \underline{-1.455+0.3431i}$

24. $\sin\left(\frac{\pi}{2} - \frac{\pi}{4}i\right) = \sin\frac{\pi}{2}\cosh\left(-\frac{\pi}{4}\right) = \underline{1.324}$

26. $\sinh\left(\frac{\pi}{2} - \frac{\pi}{4}i\right) = \frac{1}{2}\left(e^{\frac{\pi}{2}-\frac{\pi}{4}i} - e^{-\frac{\pi}{2}+\frac{\pi}{4}i}\right) = \frac{1}{2}e^{\pi/2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) - \frac{1}{2}e^{-\pi/2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$= 2.405\frac{\sqrt{2}}{2}(1-i) - 0.1039\frac{\sqrt{2}}{2}(1+i) = \underline{1.63-1.77i}$$

28. See Problem 24. Then $|\sin z| = \underline{1.324}$

30. $\ln i = \ln 1 + i\frac{\pi}{2} = \underline{i\pi/2}$

32. $\ln(4-3i)=\ln 5+5.64i=\underline{1.609+5.64i}$

34. $\ln(ei)=\ln e+i\pi/2=\underline{1+i\pi/2}$

36. $\ln(e^{i})=i\ln e=\underline{i}$

38. $(3+4i)^{1-i}=e^{(1-i)\ln(3+4i)}=e^{(1-i)(1.609+0.9273i)}=e^{2.536-0.6817i}=e^{2.536}(\cos 0.6817-i\sin 0.6817)$
$=\underline{9.807-7.958i}$

40. $(1+i)^{1+i}=e^{(1+i)\ln(1+i)}=e^{(1+i)(0.3466+0.7854i)}=e^{-0.4388+1.132i}=e^{-0.4388}(\cos 1.132-i\sin 1.132)$
$=\underline{0.2739+0.5837i}$

42. $\sin z=\sin x\cosh y+i\cos x\sinh y=2$

$$\left.\begin{aligned}\sin x\cosh y&=2\\ \cos x\sinh y&=0\end{aligned}\right\}\qquad \therefore x=\frac{\pi}{2},\quad \cosh y=\frac{1}{2}(e^{y}+e^{-y})=2,\quad \text{or}\quad e^{2y}-4e^{y}+1=0.$$

$\therefore y=1.317,\ -1.317.\qquad \therefore \underline{z=1.571+1.317i,\quad 1.571-1.317i}$

44. $e^{z}=e^{x}e^{iy}=e^{x}(\cos y+i\sin y)=-3.\qquad \left.\begin{aligned}e^{x}\cos y&=-3\\ e^{x}\sin y&=0\end{aligned}\right\}\quad \therefore y=\pi,\quad e^{x}=3.\quad \therefore x=1.099$

$\underline{z=1.099+\pi i}$

46. $\cos z=\cos x\cosh y-i\sin x\sinh y=-2$

$$\left.\begin{aligned}\cos x\cosh y&=-2\\ \sin x\sinh y&=0\end{aligned}\right\}\qquad \therefore x=\pi,\quad \cosh y=\frac{1}{2}(e^{y}+e^{-y})=2,\quad \text{or}\quad e^{2y}-4e^{y}+1=0.$$

$\therefore y=1.317,\ -1.317.\qquad \therefore \underline{z=\pi+1.317i,\quad \pi-1.317i}$

50. $\tan^{-1}(2-i)=\dfrac{i}{2}\ln\dfrac{1-i(2-i)}{1+i(2-i)}=\dfrac{1}{2}\ln\left(-\dfrac{1}{2}-\dfrac{i}{2}\right)=\dfrac{i}{2}(\ln 0.707+3.927)=\underline{-1.964-0.1734i}$

Section 10.4

2. $\bar{z}=x-iy.\qquad u=x,\quad v=-y.\quad \therefore f'(z)=\underline{1}$ and $f'(z)=\underline{-1}.$ $\bar{z}$ is not analytic.

4. $(z-1)^{2}=(x-1)^{2}-y^{2}+2(x-1)yi.\qquad u=(x-1)^{2}-y^{2},\quad v=2(x-1)y$
$\therefore f'(z)=\underline{2(x-1)+2yi}$ and $f'(z)=-i(-2y)+2(x-1)=\underline{2(x-1)+2yi}$

6. $e^z = e^x(\cos y + i\sin y)$. $\quad u = e^x \cos y, \quad v = e^x \sin y$

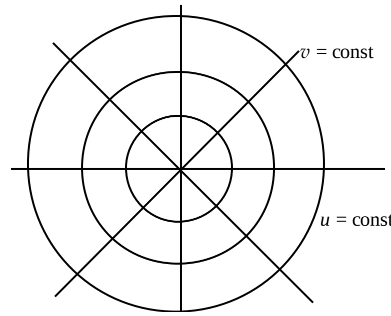
$\therefore f'(z) = \underline{e^x \cos y + ie^x \sin y}$ and $f'(z) = -i(-e^x \sin y) + e^x \cos y = \underline{e^x \cos y + ie^x \sin y}$

10. If $u = \ln r, \quad \dfrac{\partial u}{\partial r} = \dfrac{1}{r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$

$\therefore \dfrac{\partial v}{\partial r} = 1$ and $v(r,\theta) = \theta + g(r)$

But $\dfrac{\partial v}{\partial r} = \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = 0. \quad \therefore g(r) = C$

$\therefore \underline{v(r,\theta) = \theta + C}$



12. $u(x,y) = x^2 - y^2. \quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 2 - 2 = 0. \quad \therefore u(x,y)$ is harmonic.

$\dfrac{\partial u}{\partial x} = 2x = \dfrac{\partial v}{\partial y}. \quad \therefore v = 2xy + g(x). \quad \dfrac{\partial v}{\partial x} = \dfrac{dg}{dx} + 2y = -\dfrac{\partial u}{\partial y} = 2y$

$\therefore \dfrac{dg}{dx} = 0. \quad \therefore g(x) = C = 0. \quad \therefore v = 2xy$ and $\underline{f(z) = x^2 - y^2 + 2xyi}$

Section 10.5

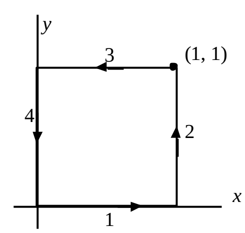
2. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Let $x = a\cos t, \quad y = b\sin t$. Then $\cos^2 t + \sin^2 t = 1$.

4. $\oint u\,dx - v\,dy$

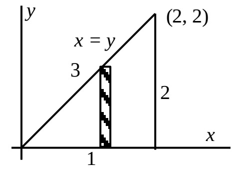
$$= \oint_1 \cancel{u}dx - v\,\cancel{dy} + \oint_2 u\,\cancel{dx} - v\,dy + \oint_3 u\,dx - v\,\cancel{dy}$$

$$+ \oint_4 u\,\cancel{dx} - \cancel{v}dy = -\int_0^1 1dy + \int_1^0 1dx = \underline{-2}$$

$$-\iint\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dxdy = -\int_0^1\int_0^1 2dxdy = \underline{-2}$$



6. $\oint udx - vdy = \oint_1 udx - v\cancel{dy} + \oint_2 u\cancel{dx} - vdy + \oint_3 udx - vdy$



$$= \int_0^2 x^2dx - \int_0^2 2xydy + \int_2^0 \cancel{(x^2 - y^2)}dx + \int_2^0 2xydy$$

$$= \frac{8}{3} + 8 + \int_0^2 2y^2dy = \frac{8}{3} + 8 - \frac{16}{3} = \underline{16/3}$$

$$-\iint\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dxdy = -\int_0^2\int_0^x(-2y - 2y)dydx = \underline{-16/3}$$

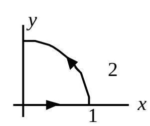
8. See No.5 for a helpful sketch. $\oint udx - vdy = \int_0^{2\pi}[r^2\sin^2\theta(-r\sin\theta d\theta) + r^2\cos^2\theta(r\cos\theta d\theta)]$

$$= \int_0^{2\pi} r^3(\cos^3\theta - \sin^3\theta)d\theta = \frac{1}{3}\cos^2\theta\sin^2\theta\Big|_0^{2\pi} + \frac{2}{3}\sin\theta\Big|_0^{2\pi} + \frac{1}{3}\sin^2\theta\cos\theta\Big|_0^{2\pi} + \frac{2}{3}\cos\theta\Big|_0^{2\pi} = \underline{0}$$

$$-\iint\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dxdy = -\int\int(-2x + 2y)dxdy - \int_0^1\int_0^{2\pi}(2r\cos\theta - 2r\sin\theta)rd\theta dr$$

$$= \int_0^1 (2r^2\sin\theta\Big|_0^{2\pi} + 2r^2\cos\theta\Big|_0^{2\pi})dr = \underline{0}$$

10. $\int_{(0,0)}^{(0,2)}(x^2 + y^2)(dx + idy) = \int_0^2 \cancel{(x^2 + y^2)}dx + \int_0^2(\cancel{x^2} + y^2)idy = \underline{8i/3}$



$\int_1 (x^2 + \cancel{y^2})(dx + i\cancel{dy}) + \int_2 (x^2 + y^2)(dx + idy) = \int_0^2 x^2dx + \int_2(4dx + 4idy)$

$x^2 + y^2 = 4$

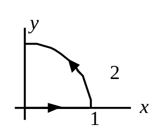
$x = 2\cos\theta$

$y = 2\sin\theta$

$$= \frac{8}{3} + 4\int_0^{\pi/2}(-2\sin\theta d\theta + 2i\cos\theta d\theta) = \frac{8}{3} + 8\cos\theta\Big|_0^{\pi/2} + i8\sin\theta\Big|_0^{\pi/2}$$

$$= \underline{-\frac{16}{3} + 8i}$$

12. $\int_1 (x^2 - \cancel{y^2} + 2x\cancel{y}i)(dx + i\cancel{dy}) + \int_2 4e^{2i\theta}2ie^{i\theta}d\theta$



$$= \int_0^2 x^2dx + \int_0^{\pi/2} 8ie^{3i\theta}d\theta = \frac{8}{3} + \frac{8i}{3i}e^{3i\theta}\Big|_0^{\pi/2} = \underline{-8i/3}$$

$z = 2e^{i\theta}$

$dz = 2ie^{i\theta}d\theta$

$$\int_{(0,0)}^{(0,2)}(\cancel{x^2} - y^2 + 2\cancel{x}yi)(\cancel{dx} + idy) = -\int_0^2 iy^2dy = \underline{-8i/3}$$

14. Yes 16. No 18. Yes 20. Yes

Section 10.6

2. $\oint \sin z dz = \underline{0}$ since $\sin z$ is analytic.

4. $\oint \frac{dz}{z-2} = \underline{0}$ since $f(z) = \frac{1}{z-2}$ is analytic in the unit circle.

6. $\oint \frac{dz}{(z-3)(z-2)} = \underline{0}$ since $f(z) = \frac{1}{(z-3)(z-2)}$ is analytic in the circle.

8. $\oint \frac{dz}{(z-3)(z-2)} = \oint \frac{dz}{z-3} - \oint \frac{dz}{z-2}$ using partial fractions. In the first integral, let $z-3 = \varepsilon_1 e^{i\theta}$ where ε_1 is small (see Example 10.6.2). In the second integral, let $z-2 = \varepsilon_2 e^{i\theta}$ where ε_2 is small. $\therefore \oint \frac{dz}{z^2-5z+6} = \int_0^{2\pi} \frac{\varepsilon_1 i e^{i\theta} d\theta}{\varepsilon_1 e^{i\theta}} - \int_0^{2\pi} \frac{\varepsilon_2 i e^{i\theta} d\theta}{\varepsilon_2 e^{i\theta}} = 2\pi i - 2\pi i = \underline{0}$

10. $\oint \frac{dz}{(z-2)(z+2)} = \frac{1}{4}\oint \frac{dz}{z-2} - \frac{1}{4}\oint \frac{dz}{z+2} = \frac{1}{4}\int_0^{2\pi} \frac{\varepsilon_1 i e^{i\theta} d\theta}{\varepsilon_1 e^{i\theta}} - \frac{1}{4}\int_0^{2\pi} \frac{\varepsilon_2 i e^{i\theta} d\theta}{\varepsilon_2 e^{i\theta}} = \frac{1}{4}(2\pi i - 2\pi i) = \underline{0}$

12. $\oint \frac{zdz}{z-1}$. Let $z-1 = \varepsilon e^{i\theta}, \quad dz = \varepsilon i e^{i\theta} d\theta.$

$$\oint \frac{zdz}{z-1} = \int_0^{2\pi} \frac{1+\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}} \varepsilon i e^{i\theta} d\theta = \int_0^{2\pi} i d\theta + \int_0^{2\pi} \varepsilon i e^{i\theta} d\theta = 2\pi i + \frac{\varepsilon i}{i} e^{i\theta}\Big|_0^{2\pi} = \underline{2\pi i}$$

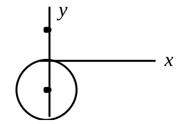
Section 10.7

2. $\oint \frac{e^z}{z-1} dz = 2\pi i f(1) = \underline{2\pi e i}$. $\qquad f(z) = e^z$

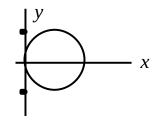
4. $\oint \frac{z+1}{(z-1)(z^2+9)} dz = 2\pi i f(1) = \underline{2\pi i/5}$. $\qquad f(z) = \frac{z+1}{z^2+9}$

6. $\oint \frac{z^2}{z+i} dz = 2\pi i f(-i) = \underline{-2\pi i}$. $\qquad f(z) = z^2$

8. $\oint \frac{z-1}{(z+i)(z-i)}dz = 2\pi i f(i) = 2\pi i \frac{-i-1}{-2i} = \underline{\pi(i+1)}.$ $\quad f(z) = \frac{z-1}{z-i}$



10. $\oint \frac{z-1}{(z+i)(z-i)}dz = \underline{0}$ since $f(z)$ is analytic in the circle.



12. $\oint \frac{z-1}{(z+i)(z-i)}dz = 2\pi i f(-i) = 2\pi i \frac{-i-1}{-2i} = \underline{\pi(i+1)}$

14. $\oint \frac{z-1}{(z+1)^2}dz = 2\pi i f'(-1) = \underline{2\pi i}.$ $\quad f(z) = z-1$

16. $\oint \frac{\cos z}{(z-1)^2}dz = 2\pi i f'(1) = 2\pi i(-\sin 1) = \underline{-5.287i}.$ $\quad f(z) = \cos z$

18. $\oint \frac{\sinh z}{z^4}dz = \frac{\pi i}{3} f'''(0) = \pi i \cosh 0 = \underline{\pi i/3}.$ $\quad f(z) = \sinh z$

Section 10.8

2. $f(z) = \frac{1}{1+z}, \quad f'(z) = \frac{-1}{(1+z)^2}, \quad f''(z) = \frac{2!}{(1+z)^3}, \quad \cdots.$

$\therefore \frac{1}{1+z} = 1 - z + \frac{2!}{2!}z^2 - \frac{3!}{3!}z^3 + \cdots = \underline{1 - z + z^2 - z^3 + \cdots} \qquad R = 1$

4. $f(z) = \frac{z-1}{z+1}, \quad f'(z) = \frac{2}{(z+1)^2}, \quad f''(z) = \frac{-4}{(z+1)^3}, \quad f'''(z) = \frac{12}{(z+1)^4} \quad \cdots.$

$\therefore \frac{z-1}{z+1} = -1 + 2z - \frac{4}{2!}z^2 + \frac{12}{3!}z^3 - \cdots = \underline{-1 + 2z - 2z^2 + 2z^3 - \cdots} \qquad R = 1$

6. $\sinh z = \sinh 0 + z\cosh 0 + \frac{z^2}{2!}\sinh 0 + \frac{z^3}{3!}\cosh 0 + \cdots = \underline{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots} \qquad R = \infty$

8. $\dfrac{1}{z-2}=\dfrac{-1}{1-(z-1)}=-\left[1+(z-1)+(z-1)^2+\cdots\right]=\underline{-z-(z-1)^2-(z-1)^3-\cdots}\qquad R=1$

10. $\dfrac{1}{z-2}=\dfrac{-1}{2-z}=\dfrac{-1}{3-(z+1)}=\dfrac{-1/3}{1-(z+1)/3}=-\dfrac{1}{3}\left[1+\dfrac{z+1}{3}+\left(\dfrac{z+1}{3}\right)^2+\left(\dfrac{z+1}{3}\right)^3+\cdots\right]$

$=\boxed{-\dfrac{1}{9}\left[4+z+\dfrac{1}{3}(z+1)^2+\dfrac{1}{9}(z+1)^3+\cdots\right]}\qquad R=3$

12. $\dfrac{1}{z-2}=\dfrac{-1}{2+2i-(z+2i)}=\dfrac{-1/(2+2i)}{1-(z+2i)/(2+2i)}=-\dfrac{1}{2+2i}\left[1+\dfrac{z+2i}{2+2i}+\left(\dfrac{z+2i}{2+2i}\right)^2+\cdots\right]$

$=\boxed{-\dfrac{1-i}{4}\left[1+\dfrac{1-i}{4}(z+2i)-\dfrac{i}{8}(z+2i)^2+\dfrac{1+i}{32}(z+2i)^3+\cdots\right]}\qquad R=2\sqrt{2}$

14. $\dfrac{z-1}{1+z^3}=\dfrac{z}{1+z^3}-\dfrac{1}{1+z^3}=z(1-z^3+z^6-\cdots)-(1-z^3+z^6+\cdots)=\underline{-1+z+z^3-z^4-z^6+\cdots}$

16. $\dfrac{1}{z^2-3z-4}=\dfrac{1/5}{z-4}-\dfrac{1/5}{z+1}=\dfrac{1}{20}\left(1+\dfrac{z}{4}+\dfrac{z^2}{16}+\cdots\right)-\dfrac{1}{5}\left(1-z+z^2-\cdots\right)=\boxed{-\dfrac{1}{4}+\dfrac{3}{16}z-\dfrac{13}{64}z^2+\cdots}$

18. $e^{2-z}=e^2e^{-z}=\boxed{e^2\left(1-z+\dfrac{z^2}{2}-\dfrac{z^3}{6}+\cdots\right)}$

20. $\sin z^2=\boxed{z^2-\dfrac{z^6}{6}+\dfrac{z^{10}}{120}-\cdots}$

22. $e^z\cos z=\left(+1+z-\dfrac{z^2}{2!}\cdots\right)(1-\dfrac{z^2}{2}+\dfrac{z^4}{24}+\cdots)=\boxed{1+z-\dfrac{1}{3}z^3+\dfrac{1}{12}z^4-\cdots}$

26. $\displaystyle\int_0^z\sin w^2dw=\int_0^z(w^2-w^6/6+w^{10}/120-\cdots)dw=\boxed{\frac{1}{3}z^3-\frac{1}{42}z^7+\frac{1}{1320}z^{11}-\cdots}$

28. $\displaystyle\int_0^z\cos w^2dw=\int_0^z(1-w^4/2+w^8/24-\cdots)dw=\boxed{z-\frac{1}{10}z^5+\frac{1}{216}z^9-\cdots}$

30. $e^z=e^{z-1}e^1=e\left[1+(z-1)+(z-1)^2/2+(z-1)^3/6+\cdots\right]=\boxed{e\left[z+\dfrac{1}{2}(z-1)^2+\dfrac{1}{6}(z-1)^3+\cdots\right]}$

32. $\sin z = \sin[(z-\pi/2)+\pi/2] = \sin(z-\pi/2)\cos\pi/2 + \sin\pi/2\cos(z-\pi/2)$

$$= \cos(z-\pi/2) = \boxed{1 - \frac{1}{2}(z-\pi/2)^2 + \frac{1}{24}(z-\pi/2)^4 - \cdots}$$

34. $\dfrac{1}{z^2-z-2} = \dfrac{1/3}{z-2} - \dfrac{1/3}{z+1} = \dfrac{-1/6}{1-z/2} - \dfrac{1/3}{1+z} = -\dfrac{1}{6}\left(1+\dfrac{z}{2}+\dfrac{z^2}{4}+\cdots\right) - \dfrac{1}{3}\left(1-z+z^2-\cdots\right)$

$$= \boxed{-\frac{1}{2} + \frac{1}{4}z - \frac{3}{8}z^2 + \frac{5}{16}z^3 - \cdots}$$

Section 10.9

2. $\dfrac{1}{z}\dfrac{1}{z-2} = \dfrac{1}{z}\dfrac{-1/2}{1-z/2} = \dfrac{1}{z}\left(-\dfrac{1}{2}\right)\left(1+\dfrac{z}{2}+\dfrac{z^2}{4}+\cdots\right) = \boxed{-\dfrac{1}{2}\left(\dfrac{1}{z}+\dfrac{1}{2}+\dfrac{z}{4}+\dfrac{z^2}{8}+\cdots\right)}$ $\quad R = 2$

4. $\dfrac{1}{z}e^z e^{-1} = \dfrac{1}{ez}\left(1+z+\dfrac{z^2}{2}+\dfrac{z^3}{6}+\cdots\right) = \boxed{\dfrac{1}{e}\left(\dfrac{1}{z}+1+\dfrac{z}{2}+\dfrac{z^2}{6}+\cdots\right)}$ $\quad R = \infty$

6. $e^{1/z} = \boxed{1+\dfrac{1}{z}+\dfrac{1}{2z^2}+\dfrac{1}{6z^3}+\cdots}$ $\quad |z| > 0$

8. $\dfrac{1}{1-z} = \dfrac{1}{2-(z+1)} = \dfrac{1/2}{1-(z+1)/2} = \boxed{\dfrac{1}{2}\left[1+\dfrac{z+1}{2}+\dfrac{(z+1)^2}{4}+\cdots\right]}$ $\quad 0 \le |z+1| < 2$

$= \dfrac{1}{z+1}\dfrac{-1}{1-\dfrac{2}{z+1}} = -\dfrac{1}{z+1}\left[1+\dfrac{2}{z+1}+\dfrac{4}{(z+1)^2}+\cdots\right] = \boxed{-\dfrac{1}{z+1}-\dfrac{2}{(z+1)^2}-\dfrac{4}{(z+1)^3}-\cdots}$ $\quad 2 < |z+1|$

10. $\dfrac{1}{z(z-1)} = \dfrac{1}{z}(-1-z-z^2-\cdots) = \boxed{-\dfrac{1}{z}-1-z-z^2-\cdots}$ $\quad 0 \le |z| < 1$

$$= \frac{1}{z^2}\frac{1}{1-1/z} = \boxed{\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots} \quad |z| > 1$$

12. $$\frac{1}{z^2+1}=\frac{1}{z+i}\,\frac{1}{z-i}=\frac{1}{z-i}\,\frac{1}{2i+z-i}=\frac{1}{z-i}\,\frac{-i/2}{1+(z-i)/2i}=\frac{-i/2}{z-i}\left[1-\frac{z-i}{2i}+\frac{(z-i)^2}{4}-\cdots\right]$$

$$=\boxed{-\frac{i}{2}\left[\frac{1}{z-i}+\frac{i}{2}-\frac{z-i}{4}+\frac{(z-i)^2}{8}i+\cdots\right]}\qquad 0<|z-i|<2$$

$$\frac{1}{z^2+1}=\frac{1}{(z-i)^2}\,\frac{1}{1+2i/(z-i)}=\frac{1}{(z-i)^2}\left[1-\frac{2i}{z-i}+\frac{4}{(z-i)^2}+\cdots\right]$$

$$=\boxed{\frac{1}{(z-i)^2}-\frac{2i}{(z-i)^3}-\frac{4}{(z-i)^4}+\cdots}\qquad 2<|z-i|$$

14. $$\frac{1}{(z+1)(z-2)}=\frac{1}{z+1}\,\frac{1}{z+1}\,\frac{1}{1-3/(z+1)}=\frac{1}{(z+1)^2}\left[1+\frac{3}{z+1}+\frac{9}{(z+1)^2}+\cdots\right]$$

$$=\boxed{\frac{1}{(z+1)^2}+\frac{3}{(z+1)^3}+\frac{9}{(z+1)^4}+\cdots}\qquad 3<|z+1|$$

$$\frac{1}{(z+1)(z-2)}=\frac{1}{z+1}\,\frac{-1/3}{1-(z+1)/3}=-\frac{1}{3(z+1)}\left[1+\frac{z+1}{3}+\frac{(z+1)^2}{9}+\cdots\right]$$

$$=\boxed{-\frac{1}{3(z+1)}-\frac{1}{9}-\frac{z+1}{27}-\frac{(z+1)^2}{81}+\cdots}\qquad 0<|z+1|<3$$

Section 10.10

2. $$\frac{z}{z^2+4}=\frac{z+2i-2i}{z+2i}\,\frac{-1/4i}{1-(z+2i)/4i}=\left(1-\frac{2i}{z+2i}\right)\left(\frac{-1}{4i}\right)\left[1+\frac{z+2i}{4i}+\cdots\right].\quad \therefore\ \underline{b_1=1/2 \text{ at } z=-2i}$$

$$\frac{z}{z^2+4}=\frac{z-2i+2i}{z-2i}\,\frac{1/4i}{1+(z-2i)/4i}=\left(1+\frac{2i}{z-2i}\right)\frac{1}{4i}\left[1-\frac{z-2i}{4i}+\cdots\right].\quad \therefore\ \underline{b_1=1/2 \text{ at } z=2i}$$

4. $$\frac{e^z}{(z-1)^2}=ee^{z-1}\,\frac{1}{(z-1)^2}=\frac{e}{(z-1)^2}[1+(z-1)+\cdots].\qquad \therefore\ \underline{b_1=e \text{ at } z=1}$$

6. $$\frac{z^2+1}{z^2+3z+2}=\frac{z^2+1}{(z+2)(z+1)}=\frac{(z+1)^2-2(z+1)+2}{(z+1)}\,\frac{1}{1+z+1}=\left(z+1-2+\frac{2}{z+1}\right)\frac{1}{1+(z+1)}$$

$$=\left[(z+1)-2+\frac{2}{z+1}\right][1-(z+1)+\cdots].\qquad \therefore\ \underline{b_1=2 \text{ at } z=-1}$$

8. $\dfrac{\sin z}{z^3} = \dfrac{1}{z^3}\left(z - \dfrac{z^3}{6} + \cdots\right)$. $\quad \therefore b_1 = 0. \quad \therefore \oint \dfrac{\sin z}{z^3}\,dz = \underline{0}$

10. $\dfrac{z+1}{z+i} = \dfrac{z+i+1-i}{z+i} = 1 + \dfrac{1-i}{z+i}$. $\quad \therefore b_1 = 1-i. \quad \therefore \oint \dfrac{z+1}{z+i}\,dz = \underline{2\pi(1-i)}$

12. $\dfrac{1}{4z^2+9} = \dfrac{1}{2z+3i}\,\dfrac{1}{2z-3i} = \dfrac{1/2}{z+3i/2}\,\dfrac{-1/6i}{1-(2z+3i)/6i} = \dfrac{1}{2}\,\dfrac{1}{z+3i/2}\left(\dfrac{-1}{6i}\right)\left(1 - \dfrac{z+3i/2}{3i} + \cdots\right)$

$\therefore b_1 = -1/12i \text{ at } z = -3i/2$

$\dfrac{1}{4z^2+9} = \dfrac{1}{2z+3i}\,\dfrac{1}{2z-3i} = \dfrac{1/2}{z-3i/2}\,\dfrac{1/6i}{1+(2z-3i)/6i} = \dfrac{1}{2}\,\dfrac{1}{z-3i/2}\left(\dfrac{1}{6i}\right)\left(1 - \dfrac{z-3i/2}{3i} + \cdots\right)$

$\therefore b_1 = 1/12i \text{ at } z = 3i/2. \quad \therefore \oint \dfrac{dz}{4z^2+9} = 2\pi i\left(b_1\big|_{z=3i/2} + b_1\big|_{z=-3i/2}\right) = 2\pi i\left(\dfrac{1}{12i} - \dfrac{1}{12i}\right) = \underline{0}$

14. $\dfrac{\sin z}{z^2(z-1)} = \dfrac{1}{z^2}\left(z - \dfrac{z^3}{3!} + \cdots\right)(-1 - z - z^2 - \cdots). \quad \therefore b_1 = -1 \text{ at } z = 0$

$= \dfrac{\sin[(z-1)+1]}{[(z-1)+1]^2(z-1)} = \dfrac{\sin(z-1)\cos 1 + \cos(z-1)\sin 1}{(z-1)[1+(z-1)]^2}$

$= \dfrac{1}{z-1}[1-(z-1)+\cdots]^2\left\{.54[(z-1)-\cdots] + .841\left[1 - \dfrac{(z-1)^2}{2} + \cdots\right]\right\}$

$\therefore b_1 = 0.841 \text{ at } z = 1. \quad \therefore \oint \dfrac{\sin z}{z^3 - z^2}\,dz = 2\pi i(-1+0.841) = \underline{-i}$

16. $\dfrac{z^2+1}{z(z+1)^3} = \dfrac{z^2+1}{z}\left[1 - z + z^2 - \cdots\right]. \quad \therefore b_1 = 1 \text{ at } z = 0$

$\dfrac{z^2+1}{z(z+1)^3} = \dfrac{z^2+1}{(z+1)^3}\,\dfrac{-1}{1-(z-1)} = \dfrac{(z+1)^2 - 2(z+1) + 2}{(z+1)^3}(-1)\left[1 + (z+1) + (z+1)^2 + \cdots\right]$

$= \left[-\dfrac{1}{z+1} + \dfrac{2}{(z+1)^2} - \dfrac{2}{(z+1)^3}\right]\left[1 + (z+1) + (z+1)^2 + \cdots\right]$

$\therefore b_1 = -1 + 2 - 2 = -1 \text{ at } z = -1. \quad \therefore \oint \dfrac{z^2+1}{z(z+1)^3}\,dz = 2\pi i(1-1) = \underline{0}$

18. $\dfrac{\cosh \pi z}{z(z+1)} = \dfrac{e^{\pi z} + e^{-\pi z}}{2z(1+z)} = \dfrac{1}{2z}(1 - z + \cdots)(1 + \pi z + \cdots + 1 - \pi z + \cdots). \quad \therefore b_1 = 1 \text{ at } z = 0.$

$\dfrac{\cosh \pi z}{z(z+1)} = \dfrac{-1}{2(z+1)}\,\dfrac{-e^{\pi(z+1)} - e^{-\pi(z+1)}}{1-(z+1)} = \dfrac{1}{2(z+1)}[1 + (z+1) + \cdots][1 + \cdots + 1 - \cdots]$

$\therefore b_1 = 1 \text{ at } z = -1. \quad \therefore \oint \dfrac{\cosh \pi z}{z(z+1)}\,dz = \underline{4\pi i}$

20. $\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2} = \oint \frac{4z^2}{(z^2+4z+1)^2}\,\frac{dz}{iz} = \oint \frac{-4zi}{(z+.268)^2(z+3.73)^2}\,dz$

$$\frac{z}{(z+.268)^2[3.46+(z+.268)]^2} = \frac{(z+.268)-.268}{(z+.268)^2}\,\frac{1/3.46^2}{[1+(z+.268)/3.46]^2}$$

$$= \left[\frac{1}{z+.268} - \frac{.268}{(z+.268)^2}\right]\frac{1}{3.46^2}\left[1-\frac{z+.268}{3.46}+\cdots\right]$$

$\therefore b_1 = \frac{1}{3.46^2}\left(1+\frac{2\times 0.268}{3.46}\right) = 0.0965.$ $\qquad \therefore \int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2} = -4i(0.0965)\times 2\pi i = \underline{2.43}$

22. $\int_0^{2\pi} \frac{d\theta}{2+2\sin\theta} = \oint \frac{1}{2+(z^2-1)/iz}\,\frac{dz}{iz} = \oint \frac{dz}{z^2+2iz-1} = \oint \frac{dz}{(z+i)^2}.$ There is no zero in the unit circle. $\therefore \int_0^{2\pi} \frac{d\theta}{2+2\sin\theta} = \underline{0}$

24. $\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5-4\cos\theta} = \oint \frac{1/2\left(z^2+1/z^2\right)}{5-4(z^2+1)/2z}\,\frac{dz}{iz} = -\frac{1}{4i}\oint \frac{z^2+1/z^2}{(z-2)(z-1/2)}\,dz$

$$\frac{z^2+1/z^2}{\left(z-\frac{1}{2}\right)\left[-\frac{3}{2}+\left(z-\frac{1}{2}\right)\right]} = \frac{-2/3}{z-1/2}\,\frac{1}{1-\frac{z-1/2}{3/2}}\left\{\left[(z-1/2)+1/2\right]^2 + \frac{4}{\left[(z-1/2)-1/2\right]^2}\right\}$$

$$= \frac{-2/3}{z-1/2}\left\{\frac{1}{4}+\cdots+4\left[1+\cdots\right]\right\}\left[1+\frac{z-1/2}{3/2}+\cdots\right]$$

$\therefore b_1 = \left(\frac{1}{4}+4\right)\left(-\frac{2}{3}\right) = -\frac{17}{6}.$ $\qquad \therefore \int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta}\,d\theta = -\frac{17}{6}\left(-\frac{1}{4i}\right)\times 2\pi i = \underline{4.45}$

26. The zero of $1+z^2$ in the upper half-plane is at $z=i.$ It is a pole of order two. Hence,

$$b_1 = \frac{d}{dz}\left[(z-i)^2\,\frac{z^2}{(1+z^2)^2}\right]_{z=i} = \frac{d}{dz}\left[\frac{z^2}{(z+i)^2}\right]_{z=i} = -\frac{i}{4}$$

$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}\,dx = -\frac{i}{4}(2\pi i) = \frac{\pi}{2}.$ $\qquad \therefore \int_0^{\infty} \frac{x^2}{(1+x^2)^2}\,dx = \underline{\pi/4}$

28. $z^4 + z^2 + 1$ has zeros at $z^2 = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ or $z^2 = e^{2\pi i/3}$ and $e^{4\pi i/3}$. $\therefore z = e^{\pi i/3} = \frac{1+\sqrt{3}i}{2}$ and $z = e^{2\pi i/3} = \frac{-1+\sqrt{3}i}{2}$ in the upper half-plane. The residues are

$$b_1 = \left(z - \frac{1-\sqrt{3}i}{2} \right) \frac{z^2}{z^4 + z^2 + 1} \bigg|_{z=\frac{1+\sqrt{3}i}{2}} = \frac{\left(\frac{1+\sqrt{3}i}{2} \right)^2}{(\sqrt{3}i)2\frac{1+\sqrt{3}i}{2}} = 0.25 - 0.144i$$

$$b_2 = \left(z + \frac{1-\sqrt{3}i}{2} \right) \frac{z^2}{z^4 + z^2 + 1} \bigg|_{z=\frac{-1+\sqrt{3}i}{2}} = \frac{\left(\frac{-1+\sqrt{3}i}{2} \right)^2}{(-\sqrt{3}i)2\frac{-1+\sqrt{3}i}{2}} = -0.25 - 0.144i$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 dx}{x^4 + x^2 + 1} = 2\pi i(0.25 - 0.144i - 0.25 - 0.144i) = \underline{1.81}$$

30. $z^4 + 5z^2 + 4 = (z+2i)(z-2i)(z+i)(z-i)$. There are two zeros in the upper half-plane at $z = 2i,\ i$. The residues are

$$\left.\begin{aligned} b_1 &= (z-2i)\frac{1}{z^4 + 5z^2 + 4}\bigg|_{z=2i} = \frac{1}{(-3)\times 4i} = \frac{i}{12} \\ b_1 &= (z-i)\frac{1}{z^4 + 5z^2 + 4}\bigg|_{z=i} = \frac{1}{3\times 2i} = -\frac{i}{6} \end{aligned}\right\} \qquad \therefore \int_{-\infty}^{\infty} \frac{dx}{x^4 + 5x + 4} = 2\pi i\left(-\frac{i}{12}\right) = \underline{\pi / 6}$$

32. $(1+z^2)^2$ has a zero of order two at $z = i$. $b_1 = \frac{d}{dz} \frac{(z-i)^2 e^{iz}}{(z-i)^2 (z+i)^2}\bigg|_{z=i}$

$$= \frac{i(z+i)^2 e^{iz} - e^{iz}(z+i)2}{(z+i)^4}\bigg|_{z=i} = \frac{i(2i)^2 e^{-1} - e^{-1} 2i(2)}{(2i)^4} = -\frac{i}{2e}.$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{ix}}{(1+x^2)^2} dx = 2\pi i\left(-\frac{i}{2e}\right) = \frac{\pi}{e}. \qquad \therefore \int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2} dx = \underline{\pi / e}$$

34. $1+z^4$ has zeros at $z=\dfrac{1+i}{\sqrt{2}},\quad \dfrac{-1+i}{\sqrt{2}}$ in the upper half-plane:

$$b_1 = \left(z-\frac{1+i}{\sqrt{2}}\right)\frac{e^{iz}}{1+z^4}\bigg|_{z=(1+i)/\sqrt{2}} = \frac{e^{(1+i)/\sqrt{2}}}{2i\left(\dfrac{1+i}{\sqrt{2}}+\dfrac{1+i}{\sqrt{2}}\right)} = -0.00958-0.123i$$

$$b_1 = \left(z-\frac{-1+i}{\sqrt{2}}\right)\frac{e^{iz}}{1+z^4}\bigg|_{z=(-1+i)/\sqrt{2}} = \frac{e^{(-1+i)/\sqrt{2}}}{-2i\left(\dfrac{-1+i}{\sqrt{2}}+\dfrac{-1+i}{\sqrt{2}}\right)} = 0.00958-0.123i$$

$$\therefore \int_{-\infty}^{\infty}\frac{e^{ix}}{1+x^4}dx = 2\pi i(-0.246i) = 1.546. \qquad \therefore \int_0^{\infty}\frac{\cos x}{1+x^4}dx = \underline{0.773}$$

36. $(z^2+1)^2$ has a zero of order two at $z=i$. Thus,

$$b_1 = \frac{d}{dz}\frac{e^{4iz}}{(z+i)^2}\bigg|_{z=i} = \frac{(z+i)^2 4ie^{4iz}-2(z+i)e^{4iz}}{(z+i)^4}\bigg|_{z=i} = -0.0229i$$

$$\therefore \int_{-\infty}^{\infty}\frac{e^{4ix}}{(x^2+1)^2}dx = -0.0229i(2\pi i) = 0.144. \qquad \therefore \int_0^{\infty}\frac{\cos 4x}{(x^2+1)^2}dx = \underline{0.072}$$

Chapter 11 Wavelets

Section 11.2

2. $\int_0^{\infty} (1/t)^2 dt$ does not converge, so the function does not have finite energy.

4. $\int_0^{\pi/2} \tan^2 t\, dt$ does not converge, so the function does not have finite energy.

9. $f(t) = 3\,\phi(t) + \psi(t) - 10\,\psi_{1,0}(t) - 5\,\psi_{1,1}(t)$

Section 11.3

5. One example is the function that equals 1 on the interval (0, 8], and zero elsewhere.

7. The second basis property is satisfied, but the third and fourth are not.

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