CHAPTER 6

Section 6.1

1. The joint pmf of X_1 and X_2 appears below. Each probability is calculated assuming independence. For the original distribution, $\mu = 25(.2) + 40(.5) + 65(.3) = 44.5$ oz and $\sigma^2 = 212.25$.

$x_2 \setminus x_1$	25	40	65
25	.04	.10	.06
40	.10	.25	.15
65	.06	.15	.09

a. Calculate $\overline{x} = (x_1 + x_2)/2$ for each of the nine pairs above and record the associated probabilities.

$\overline{x}$	25	32.5	40	45	52.5	65
$p(\overline{x})$	.04	.20	.25	.12	.30	.09

From this pmf, $E(\overline{X}) = (25)(.04) + 32.5(.20) + \cdots + 65(.09) = 44.5 = \mu$.

b. Compute s^2 for each pair using $s^2 = (x_1 - \overline{x})^2 + (x_2 - \overline{x})^2$. Again record the probabilities.

s^2	0	112.5	312.5	800
$p(s^2)$	.38	.20	.30	.12

From this pmf, $E(S^2) = 0(.38) + 112.5(.20) + 312.5(.30) + 800(.12) = 212.25 = \sigma^2$.

3. X is a binomial random variable with $n = 10$ and $p = .8$. Thus $P(X/n = x/n) = P(X = x) = b(x; 10, .8)$.

x	0	1	2	3	4	5	6	7	8	9	10
x/n	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$p(x/n)$	.000	.000	.000	.001	.005	.027	.088	.201	.302	.269	.107

5. All 16 possible pairs of outcomes, their probabilities, and the resulting $\bar{x}$ and r values appear below.

Outcome	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4
Probability	.16	.12	.08	.04	.12	.09	.06	.03
$\bar{x}$	1	1.5	2	2.5	1.5	2	2.5	3
r	0	1	2	3	1	0	1	2

Outcome	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
Probability	.08	.06	.04	.02	.04	.03	.02	.01
$\bar{x}$	2	2.5	3	3.5	2.5	3	3.5	4
r	2	1	0	1	3	2	1	2

a. From the preceding table, the pmf of $\bar{X}$ is as follows:

$\bar{x}$	1	1.5	2	2.5	3	3.5	4
$p(\bar{x})$	.16	.24	.25	.20	.10	.04	.01

b. From **a**, $P(\bar{X} \le 2.5) = .16 + .24 + .25 + .20 = .85$.

c. From the earlier table, the pmf of R is as follows:

r	0	1	2	3
$p(r)$	.30	.40	.22	.08

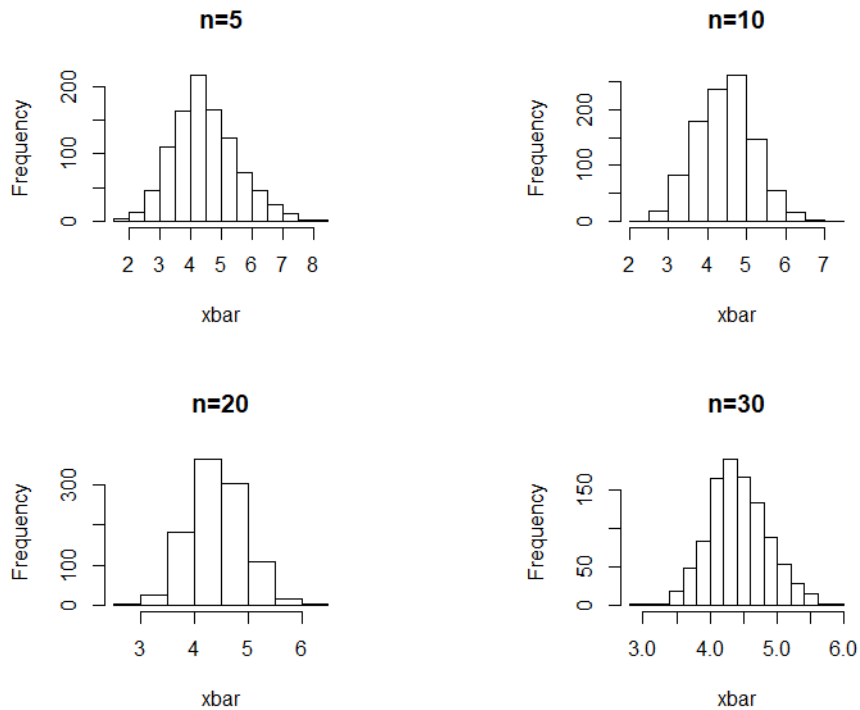
d. $P(\bar{X} \le 1.5) = P(1,1,1,1) + P(2,1,1,1) + \ldots + P(1,1,1,2) + P(1,1,2,2) + \ldots + P(2,2,1,1) + P(3,1,1,1) + \ldots + P(1,1,1,3) = (.4)^4 + 4(.4)^3(.3) + 6(.4)^2(.3)^2 + 4(.4)^3(.2) = .2400$.

7. The mgf of each X_i is $\exp(2(e^t - 1))$, so the mgf of their sum is the product of these 5 mgf's, i.e., $\exp(10(e^t - 1))$. That is to say, ΣX_i is Poisson with parameter 10. The possible values in the sampling distribution of $\bar{X}$ are $\{k/5 : k = 0,1,2,\ldots\}$, and the exact sampling distribution of $\bar{X}$ for all its possible values 0, .2, .4, … can be computed by $P(\bar{X} = k/5) = P(\Sigma X_i = k) = \dfrac{e^{-10}10^k}{k!}$.

9. The following R code demonstrates how the simulation can be performed. To change the sample size, simply replace the value of n at the top.

```
xbar = NULL; n=5
for (i in 1:1000){
  x=rweibull(n,shape=2,scale=5)
  xbar[i]=mean(x)
}
```

The histograms below show the resulting $\bar{X}$ simulation distributions for n = 5, 10, 20, 30. Even for n = 10, the simulated distribution of $\bar{X}$ looks approximately normal.



Section 6.2

11.

a. The sampling distribution of $\bar{X}$ is centered at $E(\bar{X}) = \mu = 12$ cm, and the standard deviation of the $\bar{X}$ distribution is $\sigma_{\bar{X}} = \frac{\sigma_X}{\sqrt{n}} = \frac{.04}{\sqrt{16}} = .01$ cm.

b. With $n = 64$, the sampling distribution of $\bar{X}$ is still centered at $E(\bar{X}) = \mu = 12$ cm, but the standard deviation of the $\bar{X}$ distribution is $\sigma_{\bar{X}} = \frac{\sigma_X}{\sqrt{n}} = \frac{.04}{\sqrt{64}} = .005$ cm.

c. $\bar{X}$ is more likely to be within .01 cm of the mean (12 cm) with the second, larger, sample. This is due to the decreased variability of $\bar{X}$ that comes with a larger sample size.

13.

a. No, it doesn't seem plausible that waist size distribution is approximately normal. The normal distribution is symmetric; however, for this data the mean is 86.3 cm and the median is 81.3 cm (these should be nearly equal). Likewise, for a symmetric distribution the lower and upper quartiles should be equidistant from the mean (or median); that isn't the case here.

If anything, since the upper percentiles stretch much farther than the lower percentiles do from the median, we might suspect a right-skewed distribution, such as the exponential distribution (or gamma or Weibull or …) is appropriate.

b. Irrespective of the population distribution's shape, the Central Limit Theorem tells us that $\bar{X}$ is (approximately) normal, with a mean equal to μ = 85 cm and a standard deviation equal to $\sigma/\sqrt{n} = 15/\sqrt{277} = .9$ cm. Thus,

$$P(\bar{X} \geq 86.3) = P\left(Z \geq \frac{86.3-85}{.9}\right) = 1-\Phi(1.44) = .0749$$

c. Replace 85 with 82 in (b):

$$P(\bar{X} \geq 86.3) = P\left(Z \geq \frac{86.3-82}{.9}\right) = 1-\Phi(4.77) \approx 1-1 = 0$$

That is, if the population mean waist size is 82 cm, there would be almost no chance of observing a sample mean waist size of 86.3 cm (or higher) in a random sample of 277 men. Since a sample mean of 86.3 was actually observed, it seems incredibly implausible that μ would equal 82 cm.

15.

a. Let $\bar{X}$ denote the sample mean tip percentage for these 40 bills. By the Central Limit Theorem, $\bar{X}$ is approximately normal, with $E(\bar{X}) = \mu = 18$ and $\text{SD}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{6}{\sqrt{40}}$. Hence,

$$P(16 \leq \bar{X} \leq 19) \approx \Phi\left(\frac{19-18}{6/\sqrt{40}}\right) - \Phi\left(\frac{16-18}{6/\sqrt{40}}\right) = \Phi(1.05) - \Phi(-2.11) = .8357.$$

b. According to the common convention, n should be greater than 30 in order to apply the C.L.T., thus using the same procedure for $n = 15$ as was used for $n = 40$ would not be appropriate.

17. We have $X \sim N(10,1)$, $n = 4$, $\mu_T = n\mu = (4)(10) = 40$ and $\sigma_T = \sigma\sqrt{n} = 2$. Hence, $T \sim N(40, 2)$. We desire the 95th percentile of T: 40 + (1.645)(2) = 43.29 hours.

19.

a. Let $\bar{X}$ denote the sample mean fracture angle of our $n = 4$ specimens. Since the individual fracture angles are normally distributed, $\bar{X}$ is also normal, with mean $E(\bar{X}) = \mu = 53$ but with standard deviation $\text{SD}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{4}} = .5$. Hence,

$$P(\bar{X} \leq 54) = \Phi\left(\frac{54-53}{.5}\right) = \Phi(2) = .9772, \text{ and}$$

$$P(53 \leq \bar{X} \leq 54) = \Phi(2) - \Phi(0) = .4772.$$

b. Replace 4 with n, and set the probability expression equal to .999:

$.999 = P(\bar{X} \leq 54) = \Phi\left(\frac{54-53}{1/\sqrt{n}}\right) = \Phi(\sqrt{n}) \Rightarrow \sqrt{n} \approx 3.09 \Rightarrow n \approx 9.5$. Since n must be a whole number, round up: the least such n is $n = 10$.

21. With Y = # of tickets ~ Poisson(50), the Central Limit Theorem implies that Y has approximately a normal distribution with $\mu = 50$ and $\sigma = \sqrt{\mu} = \sqrt{50}$. (We saw this normal approximation in Chapter 4, but now we know it's justified by C.L.T.)

a. $P(35 \le Y \le 70) \approx \Phi\left(\frac{70.5-50}{\sqrt{50}}\right) - \Phi\left(\frac{34.5-50}{\sqrt{50}}\right) = \Phi(2.90) - \Phi(-2.19) = .9838$.

b. Now Y is the sum of 5 Poisson rvs, so Y is still Poisson but with $E(Y) = 5(50) = 250$ and $\sigma = \sqrt{\mu} = \sqrt{250}$. Hence, $P(225 \le Y \le 275) \approx \Phi\left(\frac{275.5-250}{\sqrt{250}}\right) - \Phi\left(\frac{224.5-250}{\sqrt{250}}\right) = \Phi(1.61) - \Phi(-1.61) =$ .8926.

c. From software, **a** = .9862 and **b** = .8934. Both normal approximations are quite close.

23. The law of large numbers says that $\overline{X}$ converges to μ; or, equivalently, that ($\overline{X} - \mu$) converges to zero as $n \to \infty$. The central limit theorem says that if you multiply ($\overline{X} - \mu$) by the fraction $\frac{\sqrt{n}}{\sigma}$, the result is a standard normal random variable as $n \to \infty$. That is, the inflation factor $\frac{\sqrt{n}}{\sigma}$ "balances out" the convergence of ($\overline{X} - \mu$).

Another way to look at the two theorems is this: roughly, CLT says that (for large n) the sampling distribution of $\overline{X}$ is approximately normal with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$. As n increases to infinity, this fraction converges to zero, and so the distribution of $\overline{X}$ degenerates into a distribution with mean μ and standard deviation zero, analogous to saying $\overline{X}$ converges (in some sense) to μ.

25. $P(|Y_n - \theta| \ge \varepsilon) = P(Y_n \ge \theta + \varepsilon) + P(Y_n \le \theta - \varepsilon) = 0 + P(Y_n \le \theta - \varepsilon)$, since Y_n obviously can't be greater than θ. Using the pdf of Y_n provided in the hint,
$P(Y_n \le \theta - \varepsilon) = \int_{\theta-\varepsilon}^{\theta} ny^{n-1}/\theta^n\,dy = \left(\frac{\theta-\varepsilon}{\theta}\right)^n$. Since $\frac{\theta-\varepsilon}{\theta} < 1$, $\left(\frac{\theta-\varepsilon}{\theta}\right)^n \to 0$ as $n \to \infty$, which proves that $P(|Y_n - \theta| \ge \varepsilon) \to 0$ as $n \to \infty$, as claimed.

27. Assume you have a random sample $X_1, \ldots, X_n$ from an exponential distribution with parameter λ. Let $\overline{X}$ denote their sample average. Then by the law of large numbers, $\overline{X} \to E(X) = \frac{1}{\lambda}$ as $n \to \infty$. But our goal is a consistent estimator of λ, i.e. a quantity that converges to λ itself as $n \to \infty$.
The solution is obvious: let $h(t) = 1/t$, which is continuous for all $t > 0$. Then by the theorem cited in the exercise, $h(\overline{X}) \to h(1/\lambda)$. In other words, the consistent estimator is

$Y_n = \frac{1}{\overline{X}} \to \frac{1}{1/\lambda} = \lambda$ as $n \to \infty$.

Section 6.3

29. If $X \sim \chi^2_\nu$, then X is distributed as the sum of ν iid χ^2_1 random variables. By the Central Limit Theorem, X is then approximately normal for large ν.

31. Recall from calculus that the maximum of f and $\ln(f)$ occur at the same x-value. If f is the χ^2_ν pdf, then $\ln(f) = C + (\nu/2 - 1)\ln(x) - x/2$, where C is a constant. Take the first derivative and set that equal to zero: $0 + (\nu/2 - 1)/x - 1/2 = 0 \rightarrow x = \nu - 2$. This is only a valid value for x if $\nu - 2 > 0$; i.e., $\nu > 2$.

33.

a. If X_1 and X_2 are independent, then $M_3(t) = M_1(t)M_2(t)$, and so $M_2(t) = M_3(t)/M_1(t)$. Substitute in the given distributions, and $M_2(t) = (1-2t)^{-\nu_3/2} / (1-2t)^{-\nu_1/2} = (1-2t)^{-(\nu_3-\nu_1)/2}$, which is the mgf of the chi-square distribution with $\nu_3 - \nu_1$ df. Therefore, by the uniqueness of mgfs, $X_2 \sim \chi^2_{\nu_3-\nu_1}$.

b. If X_1 and X_2 are independent, then $V(X_1 + X_2) = V(X_1) + V(X_2)$. Under the assumed distributions, we have $V(X_1) = V(\chi^2_{\nu_1}) = 2\nu_1$ and $V(X_1 + X_2) = V(\chi^2_{\nu_3}) = 2\nu_3$. Thus $V(X_2) = V(X_1 + X_2) - V(X_1)$ = $2\nu_3 - 2\nu_1 = 2(\nu_3 - \nu_1)$. But variances can never be negative, so it must be the case that $2(\nu_3 - \nu_1) \geq 0$, i.e., $\nu_3 \geq \nu_1$.

35.

a. From the t table, $t_{.005,10} = 3.2$.

b. From the F table, $F_{.01,1,10} = 10.04 \approx 3.2^2$. This should be, since $t^2_{\alpha/2,df} = F_{\alpha,1,df}$.

c. Minitab gives the following:

Inverse Cumulative Distribution Function

```
F distribution with 1 DF in numerator and 10 DF in denominator
P( X <= x )        x
       0.99  10.0443
```

37. $E(T)$ exists iff $E(|T|) < \infty$. But $E(|T|) = \int_{-\infty}^{\infty} \frac{|t|}{\pi(1+t^2)} dt = 2\int_0^{\infty} \frac{t}{\pi(1+t^2)} dt = \frac{1}{\pi}\ln(1+t^2)\Big|_0^{\infty} = \infty$. That is, $E(|T|)$ diverges, so $E(T)$ does not exist.

39.

a. From the F table, $F_{.1,2,4} = 4.32$.

b. The $F_{2,4}$ pdf is $\frac{\Gamma(\frac{1}{2}(2+4))(2/4)^{2/2} x^{(2-2)/2}}{\Gamma(\frac{1}{2}(2))\Gamma(\frac{1}{2}(4))[1+(2/4)x]^{(2+4)/2}} = \frac{1}{[1+x/2]^3}$. Let $c = F_{.1,2,4}$; then .1 = $\int_c^{\infty} \frac{1}{[1+x/2]^3} dx = \frac{1}{[1+c/2]^2}$. Solving, $[1+c/2]^2 = 10 \rightarrow c = 2(\sqrt{10} - 1) = 4.3246$.

c. Minitab gives the following:

Inverse Cumulative Distribution Function

```
F distribution with 2 DF in numerator and 4 DF in denominator
P( X <= x )        x
        0.9  4.32456
```

41. Let X and Y be independent chi-square rvs with ν_1 and ν_2 df, respectively.

a. $E[F_{\nu_1,\nu_2}] = E[(X/\nu_1) \div (Y/\nu_2)] = \nu_2/\nu_1\, E[X]\, E[1/Y]$. $E[X] = \nu_1$ and, from Equation (6.7),

$E[1/Y] = \dfrac{2^{-1}\Gamma(-1+\nu_2/2)}{\Gamma(\nu_2/2)} = \dfrac{2^{-1}\Gamma(\nu_2/2-1)}{(\nu_2/2-1)\Gamma(\nu_2/2-1)} = \dfrac{1}{\nu_2-2}$. Canceling gives a final answer of

$E[F_{\nu_1,\nu_2}] = \dfrac{\nu_2}{\nu_2-2}$. This only holds, obviously, if $\nu_2 > 2$.

b. By the same process, $E[F^2_{\nu_1,\nu_2}] = (\nu_2/\nu_1)^2 E[X^2]E[1/Y^2]$.

From Equation (6.7), $E[X^2] = \dfrac{2^2\,\Gamma(2+\nu_1/2)}{\Gamma(\nu_1/2)} = \nu_1(\nu_1+2)$ and $E[1/Y^2] = \ldots = \dfrac{1}{(\nu_2-2)(\nu_2-4)}$.

Put together, $E[F^2_{\nu_1,\nu_2}] = \dfrac{\nu_2^2(\nu_1+2)}{\nu_1(\nu_2-2)(\nu_2-4)}$, and finally $V(F_{\nu_1,\nu_2}) = \dfrac{\nu_2^2(\nu_1+2)}{\nu_1(\nu_2-2)(\nu_2-4)} - \left(\dfrac{\nu_2}{\nu_2-2}\right)^2$

$= \ldots = \dfrac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$, for $\nu_2 > 4$.

43. Let X and Y be independent chi-square rvs with ν_1 and ν_2 df, respectively. Let $c = F_{p,\nu_1,\nu_2}$. Then, by definition, $p = P((X/\nu_1) \div (Y/\nu_2) > c) = P((Y/\nu_2) \div (X/\nu_1) < 1/c) = 1 - P((Y/\nu_2) \div (X/\nu_1) > 1/c) \rightarrow$ $P((Y/\nu_2) \div (X/\nu_1) > 1/c) = 1 - p$. Since $(Y/\nu_2) \div (X/\nu_1) \sim F_{\nu_2,\nu_1}$ by definition, we have $1/c = F_{1-p,\nu_2,\nu_1}$. Take reciprocals of both sides to get the desired result.

45. Use properties of mgfs. If $X \sim \text{Gamma}(\alpha,\beta)$, then $M_X(t) = (1-\beta t)^{-\alpha}$. Hence, the mgf of cX is $M_X(ct) = (1-\beta[ct])^{-\alpha} = (1-[\beta c]t)^{-\alpha}$, which we can identify as the Gamma($\alpha,\beta c$) mgf.
In particular, if $X \sim \chi^2_\nu = \text{Gamma}(\nu/2,2)$, then $cX \sim \text{Gamma}(\nu/2,2c)$.

47. There isn't a unique solution, but here's one approach. An $F_{3,2}$ rv has the form $[\chi^2_3/3] \div [\chi^2_2/2]$. The denominator chi-squared is easy to construct: $Z_1^2 + Z_2^2$. For the numerator, the X's must be standardized:

$\left(\dfrac{X_1-0}{5}\right)^2 + \left(\dfrac{X_2-0}{5}\right)^2 + \left(\dfrac{X_3-0}{5}\right)^2 = \dfrac{X_1^2+X_2^2+X_3^2}{25} \sim \chi^2_3$. Therefore, an example of an $F_{3,2}$ rv is

$\dfrac{(X_1^2+X_2^2+X_3^2)/25/3}{(Z_1^2+Z_2^2)/2} = \dfrac{2}{75}\dfrac{X_1^2+X_2^2+X_3^2}{Z_1^2+Z_2^2}$.

49.

a. Using the fact that the χ^2_{50} distribution is approximately normal with mean 50 and variance 2(50) = 100, $P(\chi^2_{50} > 70) \approx P(Z > [70-50]/10) = 1 - \Phi(2) = .0228$.

b. Substitute $\nu = 50$ to get $\chi^2_{50} \approx 50(1 - 1/225 + Z/15)^3$. Then

$P(\chi^2_{50} > 70) \approx P(50(1 - 1/225 + Z/15)^3 > 70) = P(Z > 1.847) = 1 - \Phi(1.847) = .03237$.

Software gives an answer of .032374, suggesting the approximation in (b) is more accurate.

Section 6.4

51. According to the main theorem of this section, $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$. Exercise 45 showed that if $X \sim \chi^2_\nu$ then $cX \sim$ Gamma(ν/2,2c). Apply this with $c = \sigma^2/(n-1)$ and ν = n –1, and we have

$$S^2 = \frac{\sigma^2}{n-1}\cdot\frac{(n-1)S^2}{\sigma^2} = \frac{\sigma^2}{n-1}\cdot\chi^2_{n-1} \sim \text{Gamma}\left(\frac{n-1}{2},\frac{2\sigma^2}{n-1}\right).$$

53.

a. Since the X's are normal, $\bar{X}$ is also normal, with mean $\mu = 5$ and standard deviation $\sigma/\sqrt{n} = 8/\sqrt{13}$. Thus $P(\bar{X} < 9.13) = \Phi\left(\frac{9.13-5}{8/\sqrt{13}}\right) = \Phi(1.86) = .9686$.

b. Since $S^2 = \sum(X_i - \bar{X})^2/(n-1)$, $\sum(X_i-\bar{X})^2/\sigma^2 = (n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$. Thus $P\left(\sum(X_i-\bar{X})^2 < 1187\right) = P\left(\sum(X_i-\bar{X})^2/\sigma^2 < 1187/\sigma^2\right) = P(\chi^2_{13-1} < 1187/8^2) = P(\chi^2_{12} < 18.55)$, which from the chi-squared table equals .90.

c. Per a theorem in this section, $\bar{X}$ and $\sum(X_i-\bar{X})^2$ are independent rvs, so the compound probability is $P\left(\bar{X} < 9.13 \cap \sum(X_i-\bar{X})^2 < 1187\right) = P\left(\bar{X} < 9.13\right)P\left(\sum(X_i-\bar{X})^2 < 1187\right) = (.9686)(.90) = .87174$.

d. Use Gosset's Theorem: $\frac{\bar{X}-\mu}{S/\sqrt{n}} = \frac{\bar{X}-5}{\sqrt{(n-1)\sum(X_i-\bar{X})^2/n}} = \frac{\bar{X}-5}{\sqrt{12\sum(X_i-\bar{X})^2/13}}$ has a t distribution with df = n – 1 = 12.

55. The trick here is to create a t-distributed rv via Gosset's Theorem:

$P(|\bar{X}-5| > 0.4S) = P\left(\left|\frac{\bar{X}-5}{S/\sqrt{n}}\right| > \frac{0.4S}{S\sqrt{n}}\right) = P(|T| > 0.4\sqrt{n})$, where $T \sim t_{27-1}$ (because the X's are normal and $\mu = 5$). Continuing, $0.4\sqrt{n} = 0.4\sqrt{27} = 2.078$, and $P(|T| > 2.078) = 2P(T > 2.078) \approx 2(.024) = .048$.

57.

a. $Z \sim N(0, 1)$ regardless of n, so $P(-2 \le Z \le 2) = \Phi(2) - \Phi(-2) = .9772 - .0228 = .9544$ for all n.

b. $T \sim t_{n-1}$. For $n = 5$, $P(-2 \le t_4 \le 2) \approx .8839$ from software. For $n = 10$, $P(-2 \le t_9 \le 2) \approx .9234$. For $n = 15$, $P(-2 \le t_{14} \le 2) \approx .9347$. As n increases, this t probability approaches the corresponding standard normal probability, i.e. the answer from part **a**.

Supplementary Exercises

59.

a. From the distribution provided, $E(X) = .05(0) + .15(1) + .25(2) + .25(3) + .30(4) = 2.6$ tickets. Similarly, $SD(X) = 1.2$ tickets.

b. $T = X_1 + \ldots + X_{150}$, which we assume to be an iid sum. Thus, with $n = 150$, $E(T) = 150(2.6) = 390$ and $SD(T) = \sqrt{150}(1.2) = 14.7$.

c. By the Central Limit Theorem, T is approximately normal, so $P(T \le 500) \approx \Phi\left(\frac{500-390}{14.7}\right) = \Phi(7.48)$ ≈ 1. In other words, it's nearly certain that the gym will be able to accommodate all requests.

61. $X \sim \text{Bin}(200, .45)$ and $Y \sim \text{Bin}(300, .6)$. Because both n's are large, both X and Y are approximately normal, so $X + Y$ is approximately normal with mean $(200)(.45) + (300)(.6) = 270$, variance $200(.45)(.55) + 300(.6)(.4) = 121.40$, and standard deviation 11.02. Thus,

$$P(X + Y \ge 250) = 1 - \Phi\left(\frac{249.5-270}{11.02}\right) = 1 - \Phi(-1.86) = .9686.$$

63. The total number T of claims filed is the sum of 500 independent Poisson(2.3) rvs, so T is also Poisson but with mean $500(2.3) = 1150$. By the central limit theorem, Poisson is approximately normal (with mean 1150 and also variance 1150), so

$$P(T \ge 1200) = P(T > 1199.5) \approx 1 - \Phi\left(\frac{1199.5-1150}{\sqrt{1150}}\right) = 1 - \Phi(1.46) = .0722.$$

65.

a. The "center" of a t_2 distribution is 0. With the aid of software, $P(-1 \le t_2 \le 1) = .5774$, $P(-2 \le t_2 \le 2) = .8165$, $P(-3 \le t_2 \le 3) = .9045$. Notice these are all somewhat less than the standard normal probabilities (.68, .95, .997), because the t distributions have heavier tails.

b. For the first part, we desire the value c such that $P(-c \le t_2 \le c) = .68$. The symmetry of the t distribution implies that $-c$ and c divide the distribution into areas of .16, .68, and .16 (the two ends are equal and the three must sum to 1). Hence, c itself is the .16 + .68 = .84 quantile of the t_2 distribution. With the aid of R software, c = `qt(.84,df=2)` = 1.312. That is, 68% of a t_2 distribution lines within ± 1.312 of center.
Similarly, $P(-c \le t_2 \le c) = .95$ implies c is the .95 + (1 – .95)/2 = .975 quantile, and software provides $c = 4.303$. Finally, $P(-c \le t_2 \le c) = .997$ implies c is the .997 + (1 – .997)/2 = .9985 quantile of the t_2 distribution, and software gives $c = 18.216$. (That's a lot bigger than 3!)

67.

a. Divide all terms by σ_1^2/σ_2^2: $P\left(2.90\frac{\sigma_1^2}{\sigma_2^2} \le \frac{S_1^2}{S_2^2} \le 8.12\frac{\sigma_1^2}{\sigma_2^2}\right) = P\left(2.90 \le \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \le 8.12\right)$. By one of the theorems in Section 6.4, the rv in this middle of this expression has an F distribution, with degrees of freedom $\nu_1 = 10 - 1 = 9$ and $\nu_1 = 12 - 1 = 11$. From the F table, 8.12 is the .999 quantile of the $F_{9,11}$ distribution and 2.90 is the .95 quantile. Therefore, $P(2.90 \le F_{9,11} \le 8.12) = .999 - .95 = .049$.

b. Start the same way: $P\left(2.19\frac{\sigma_1^2}{\sigma_2^2} \le \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} \le 4.30\frac{\sigma_1^2}{\sigma_2^2}\right) = P\left(2.19 \le \frac{\hat{\sigma}_1^2/\sigma_1^2}{\hat{\sigma}_2^2/\sigma_2^2} \le 4.30\right)$. Observe that

$\frac{\hat{\sigma}_1^2}{\sigma_1^2} = \frac{\frac{1}{10}\sum(X_i - \mu_1)^2}{\sigma_1^2} = \frac{1}{10}\sum\left(\frac{X_i - \mu_1}{\sigma_1}\right)^2 = \sum Z_i^2/10$, the sum of squares of 10 independent standard normal rvs divided by 10. By definition $\sum Z_i^2 \sim \chi_{10}^2$, so $\hat{\sigma}_1^2/\sigma_1^2 \sim \chi_{10}^2/10$. Similarly,

$\hat{\sigma}_2^2/\sigma_2^2 \sim \chi_{12}^2/12$, from which $\frac{\hat{\sigma}_1^2/\sigma_1^2}{\hat{\sigma}_2^2/\sigma_2^2} \sim \frac{\chi_{10}^2/10}{\chi_{12}^2/12} = F_{10,12}$.

From the F table, 4.30 and 2.19 are the .99 and .9 quantiles, respectively, of the $F_{10,12}$ distribution. Therefore, $P(2.19 \le F_{10,12} \le 4.30) = .99 - .9 = .09$.