CHAPTER 5

Section 5.1

1.

a. $P(X = 1, Y = 1) = p(1,1) = .20$.

b. $P(X \le 1 \text{ and } Y \le 1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = .42$.

c. At least one hose is in use at both islands. $P(X \ne 0 \text{ and } Y \ne 0) = p(1,1) + p(1,2) + p(2,1) + p(2,2) = .70$.

d. By summing row probabilities, $p_X(x) = .16, .34, .50$ for $x = 0, 1, 2$, By summing column probabilities, $p_Y(y) = .24, .38, .38$ for $y = 0, 1, 2$. $P(X \le 1) = p_X(0) + p_X(1) = .50$.

e. $p(0,0) = .10$, but $p_X(0) \cdot p_Y(0) = (.16)(.24) = .0384 \ne .10$, so X and Y are not independent.

3.

a. $p(1,1) = .15$, the entry in the 1st row and 1st column of the joint probability table.

b. $P(X_1 = X_2) = p(0,0) + p(1,1) + p(2,2) + p(3,3) = .08 + .15 + .10 + .07 = .40$.

c. $A = \{X_1 \ge 2 + X_2 \cup X_2 \ge 2 + X_1\}$, so $P(A) = p(2,0) + p(3,0) + p(4,0) + p(3,1) + p(4,1) + p(4,2) + p(0,2) + p(0,3) + p(1,3) = .22$.

d. $P(X_1 + X_2 = 4) = p(1,3) + p(2,2) + p(3,1) + p(4,0) = .17$.
$P(X_1 + X_2 \ge 4) = P(X_1 + X_2 = 4) + p(4,1) + p(4,2) + p(4,3) + p(3,2) + p(3,3) + p(2,3) = .46$.

e. $p_1(0) = P(X_1 = 0) = p(0,0) + p(0,1) + p(0,2) + p(0,3) = .19$
$p_1(1) = P(X_1 = 1) = p(1,0) + p(1,1) + p(1,2) + p(1,3) = .30$, etc.

x_1	0	1	2	3	4
$p_1(x_1)$	.19	.30	.25	.14	.12

f. $p_2(0) = P(X_2 = 0) = p(0,0) + p(1,0) + p(2,0) + p(3,0) + p(4,0) = .19$, etc.

x_2	0	1	2	3
$p_2(x_2)$	.19	.30	.28	.23

g. $p(4,0) = 0$, yet $p_1(4) = .12 > 0$ and $p_2(0) = .19 > 0$, so $p(x_1, x_2) \ne p_1(x_1) \cdot p_2(x_2)$ for every (x_1, x_2), and the two variables are <u>not</u> independent.

5. Let X_1 = the number of freshmen in the sample of 10; define X_2, X_3, X_4, analogously for sophomores, juniors, and seniors, respectively. Then the joint distribution of (X_1, X_2, X_3, X_4) is multinomial with n = 10 and (p_1, p_2, p_3, p_4) = (.20, .18, .21, .41).

a. $P((X_1, X_2, X_3, X_4) = (2, 2, 2, 4)) = \dfrac{10!}{2!2!2!4!}(.20)^2(.18)^2(.21)^2(.41)^4 = .0305.$

b. Let $Y = X_1 + X_2$ = the number of underclassmen in the sample. Then Y meets the conditions of a binomial rv, with n = 10 and p = .20 + .18 = .38. Hence, the probability the sample is evenly split among under- and upper-classmen is $P(Y = 5) = \binom{10}{5}(.38)^5(.62)^5 = .1829$.

c. The marginal distribution of X_1 is Bin(10, .20), so $P(X_1 = 0) = (.80)^{10} = .1073$.
If selections were truly random from the population of all students, there's about a 10.7% chance that no freshmen would be selected. If we consider this a low probability, then we have evidence that something is amiss; otherwise, we might ascribe this occurrence to random chance alone ("bad luck").

7.

a. $p(3, 3) = P(X = 3, Y = 3) = P$(3 customers, each with 1 package)
$= P$(each has 1 package | 3 customers) · P(3 customers) = $(.6)^3 \cdot (.25) = .054$.

b. $p(4, 11) = P(X = 4, Y = 11) = P$(total of 11 packages | 4 customers) · P(4 customers).
Given that there are 4 customers, there are four different ways to have a total of 11 packages: 3, 3, 3, 2 or 3, 3, 2, 3 or 3, 2, 3, 3 or 2, 3, 3, 3. Each way has probability $(.1)^3(.3)$, so $p(4, 11) = 4(.1)^3(.3)(.15)$ = .00018.

9.

a. $p(1,1) = .030$.

b. $P(X \le 1 \text{ and } Y \le 1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = .120$.

c. $P(X = 1) = p(1,0) + p(1,1) + p(1,2) = .100$; $P(Y = 1) = p(0,1) + \ldots + p(5,1) = .300$.

d. $P(\text{overflow}) = P(X + 3Y > 5) = 1 - P(X + 3Y \le 5) = 1 - P((X,Y)$=(0,0) or …or (5,0) or (0,1) or (1,1) or (2,1)) = 1 – .620 = .380.

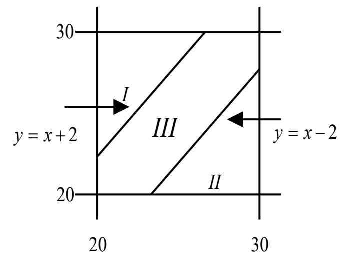
e. The marginal probabilities for X (row sums from the joint probability table) are $p_X(0) = .05$, $p_X(1) = .10$, $p_X(2) = .25$, $p_X(3) = .30$, $p_X(4) = .20$, $p_X(5) = .10$; those for Y (column sums) are $p_Y(0) = .5$, $p_Y(1) = .3$, $p_Y(2) = .2$. It is now easily verified that for every (x,y), $p(x,y) = p_X(x) \cdot p_Y(y)$, so X and Y are independent.

11.

a. $$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_{20}^{30}\int_{20}^{30} k(x^2+y^2)dxdy = k\int_{20}^{30}\int_{20}^{30} x^2 dydx + k\int_{20}^{30}\int_{20}^{30} y^2 dxdy$$
$$= 10k\int_{20}^{30} x^2 dx + 10k\int_{20}^{30} y^2 dy = 20k \cdot \left(\frac{19{,}000}{3}\right) \Rightarrow k = \frac{3}{380{,}000}.$$

b. $P(X < 26 \text{ and } Y < 26) = \int_{20}^{26}\int_{20}^{26} k(x^2+y^2)\,dxdy = k\int_{20}^{26}\left[x^2y+\frac{y^3}{3}\right]_{20}^{26} dx = k\int_{20}^{26}(6x^2+3192)\,dx =$ $k \cdot (38{,}304) = .3024$.

c. The region of integration is labeled *III* below.



$P(|X-Y| \le 2) = \iint_{III} f(x,y)dxdy = 1 - \iint_{I} f(x,y)dxdy - \iint_{II} f(x,y)dxdy =$
$1 - \int_{20}^{28}\int_{x+2}^{30} f(x,y)dydx - \int_{22}^{30}\int_{20}^{x-2} f(x,y)dydx = .3593$ (after much algebra).

d. $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{20}^{30} k(x^2+y^2)dy = kx^2y + k\frac{y^3}{3}\Big|_{20}^{30} = 10kx^2 + .05$, for $20 \le x \le 30$.

e. $f_Y(y)$ can be obtained by substituting y for x in (d); clearly $f(x,y) \ne f_X(x) \cdot f_Y(y)$, so X and Y are not independent.

13.

a. Since X and Y are independent, $p(x,y) = p_X(x)\cdot p_Y(y) = \frac{e^{-\mu_1}\mu_1^x}{x!}\cdot\frac{e^{-\mu_2}\mu_2^y}{y!} = \frac{e^{-\mu_1-\mu_2}\mu_1^x\mu_2^y}{x!\,y!}$
for $x = 0, 1, 2, \ldots$; $y = 0, 1, 2, \ldots$.

b. $P(X+Y \le 1) = p(0,0) + p(0,1) + p(1,0) = \ldots = e^{-\mu_1-\mu_2}\left[1+\mu_1+\mu_2\right]$.

c. $P(X+Y=m) = \sum_{k=0}^{m} P(X=k, Y=m-k) = e^{-\mu_1-\mu_2}\sum_{k=0}^{m}\frac{\mu_1^k}{k!}\frac{\mu_2^{m-k}}{(m-k)!} = \frac{e^{-\mu_1-\mu_2}}{m!}\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}\mu_1^k\mu_2^{m-k} =$
$\frac{e^{-\mu_1-\mu_2}}{m!}\sum_{k=0}^{m}\binom{m}{k}\mu_1^k\mu_2^{m-k} = \frac{e^{-\mu_1-\mu_2}}{m!}(\mu_1+\mu_2)^m$ by the binomial theorem. We recognize this as the pmf of a Poisson random variable with parameter $\mu_1+\mu_2$. Therefore, the total number of errors, $X+Y$, also has a Poisson distribution, with parameter $\mu_1+\mu_2$.

15.

a. $f(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} e^{-x-y} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

b. By independence, $P(X \leq 1 \text{ and } Y \leq 1) = P(X \leq 1) \cdot P(Y \leq 1) = (1 - e^{-1})\,(1 - e^{-1}) = .400$.

c. $P(X + Y \leq 2) = \int_0^2 \int_0^{2-x} e^{-x-y}\,dydx = \int_0^2 e^{-x}\left[1 - e^{-(2-x)}\right]dx = \int_0^2 (e^{-x} - e^{-2})dx = 1 - e^{-2} - 2e^{-2} = .594$.

d. $P(X + Y \leq 1) = \int_0^1 e^{-x}\left[1 - e^{-(1-x)}\right]dx = 1 - 2e^{-1} = .264$,
so $P(1 \leq X + Y \leq 2) = P(X + Y \leq 2) - P(X + Y \leq 1) = .594 - .264 = .330$.

17.

a. Each X_i has cdf $F(x) = P(X_i \leq x) = 1 - e^{-\lambda x}$. Using this, the cdf of Y is
$F(y) = P(Y \leq y) = P(X_1 \leq y \cup [X_2 \leq y \cap X_3 \leq y])$
$= P(X_1 \leq y) + P(X_2 \leq y \cap X_3 \leq y) - P(X_1 \leq y \cap [X_2 \leq y \cap X_3 \leq y])$
$= (1 - e^{-\lambda y}) + (1 - e^{-\lambda y})^2 - (1 - e^{-\lambda y})^3$ for $y > 0$.

The pdf of Y is $f(y) = F'(y) = \lambda e^{-\lambda y} + 2(1 - e^{-\lambda y})\left(\lambda e^{-\lambda y}\right) - 3(1 - e^{-\lambda y})^2\left(\lambda e^{-\lambda y}\right) = 4\lambda e^{-2\lambda y} - 3\lambda e^{-3\lambda y}$
for $y > 0$.

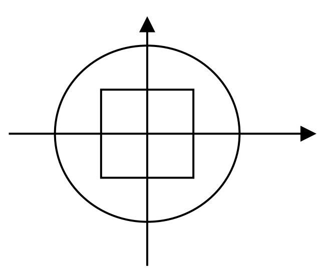
b. $E(Y) = \int_0^\infty y \cdot \left(4\lambda e^{-2\lambda y} - 3\lambda e^{-3\lambda y}\right)dy = 2\left(\frac{1}{2\lambda}\right) - \frac{1}{3\lambda} = \frac{2}{3\lambda}$.

19.

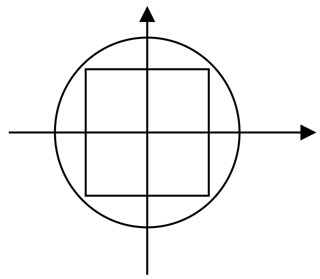
a. Let A denote the disk of radius $r/2$. Then $P((X,Y) \text{ lies in } A) = \iint_A f(x,y)dxdy$

$= \iint_A \frac{1}{\pi r^2}dxdy = \frac{1}{\pi r^2}\iint_A dxdy = \frac{\text{area of } A}{\pi r^2} = \frac{\pi(r/2)^2}{\pi r^2} = \frac{1}{4} = .25$. Notice that, since the joint pdf of X and Y is a constant (i.e., (X,Y) is <u>uniform</u> over the disk), it will be the case for any subset A that $P((X,Y)$ lies in $A) = \frac{\text{area of } A}{\pi r^2}$.

b. By the same ratio-of-areas idea, $P\left(-\frac{r}{2} \leq X \leq \frac{r}{2}, -\frac{r}{2} \leq Y \leq \frac{r}{2}\right) = \frac{r^2}{\pi r^2} = \frac{1}{\pi}$. This region is the square depicted in the graph below.



c. Similarly, $P\left(-\frac{r}{\sqrt{2}} \le X \le \frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}} \le Y \le \frac{r}{\sqrt{2}}\right) = \frac{2r^2}{\pi r^2} = \frac{2}{\pi}$. This region is the slightly larger square depicted in the graph below, whose corners actually touch the circle.



d. $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2}dy = \frac{2\sqrt{r^2-x^2}}{\pi r^2}$ for $-r \le x \le r$.

Similarly, $f_Y(y) = \frac{2\sqrt{r^2-y^2}}{\pi r^2}$ for $-r \le y \le r$. X and Y are <u>not</u> independent, since the joint pdf is not the product of the marginal pdfs: $\frac{1}{\pi r^2} \ne \frac{2\sqrt{r^2-x^2}}{\pi r^2} \cdot \frac{2\sqrt{r^2-y^2}}{\pi r^2}$.

21. Picture an inscribed equilateral triangle with one vertex at A, so the other two vertices are 120° away from A in either direction. Clearly chord AB will exceed the side length of this triangle if and only if point B is "between" the other two vertices (i.e., "opposite" A). Since that arc between the other two vertices spans 120° and the points were selected uniformly, the probability is clearly $\frac{120}{360} = \frac{1}{3}$.

Section 5.2

23.

a. $P(X > Y) = \sum\sum_{x>y} p(x,y) = p(1, 0) + p(2, 0) + p(3, 0) + p(2, 1) + p(3, 1) + p(3, 2) = .03 + .02 + .01 + .03 + .01 + .01 = .11$.

b. Adding down the columns gives the probabilities associated with the x-values:

x	0	1	2	3
$p_X(x)$	.78	.12	.07	.03

Similarly, adding across the rows gives the probabilities associated with the y-values:

y	0	1	2
$p_Y(y)$	.77	.14	.09

c. Test a coordinate, e.g. (0, 0): $p(0, 0) = .71$, while $p_X(0) \cdot p_Y(0) = (.78)(.77) = .6006 \ne .71$. Therefore, X and Y are <u>not</u> independent.

d. The average number of syntax errors is $E(X) = 0(.78) + 1(.12) + 2(.07) + 3(.03) = 0.35$, while the average number of logic errors is $E(Y) = 0(.77) + 1(.14) + 2(.09) = 0.32$.

e. By linearity of expectation, $E(100 - 4X - 9Y) = 100 - 4E(X) - 9E(Y) = 100 - 4(.35) - 9(.32) = 95.72$.

25. $E(X_1 - X_2) = \sum_{x_1=0}^{4}\sum_{x_2=0}^{3}(x_1 - x_2)\cdot p(x_1, x_2) = (0-0)(.08) + (0-1)(.07) + \ldots + (4-3)(.06) = .15.$

Or, by linearity of expectation, $E(X_1 - X_2) = E(X_1) - E(X_2)$, so in this case we could also work out the means of X_1 and X_2 from their marginal distributions: $E(X_1) = 1.70$ and $E(X_2) = 1.55$, so $E(X_1 - X_2) = E(X_1) - E(X_2) = 1.70 - 1.55 = .15$.

27. The expected value of X, being uniform on $[L - A, L + A]$, is simply the midpoint of the interval, L. Since Y has the same distribution, $E(Y) = L$ as well. Finally, since X and Y are independent,
$E(\text{area}) = E(XY) = E(X) \cdot E(Y) = L \cdot L = L^2$.

29. The amount of time Annie waits for Alvie, if Annie arrives first, is $Y - X$; similarly, the time Alvie waits for Annie is $X - Y$. Either way, the amount of time the first person waits for the second person is $h(X, Y) = |X - Y|$. Since X and Y are independent, their joint pdf is given by $f_X(x) \cdot f_Y(y) = (3x^2)(2y) = 6x^2y$. From these, the expected waiting time is

$$E[h(X,Y)] = \int_0^1\int_0^1 |x-y|\cdot f(x,y)dxdy = \int_0^1\int_0^1 |x-y|\cdot 6x^2ydxdy$$

$$= \int_0^1\int_0^x (x-y)\cdot 6x^2ydydx + \int_0^1\int_x^1 (x-y)\cdot 6x^2ydydx = \frac{1}{6} + \frac{1}{12} = \frac{1}{4} \text{ hour, or 15 minutes.}$$

31. $\text{Cov}(X,Y) = -\frac{2}{75}$ and $\mu_X = \mu_Y = \frac{2}{5}$.

$E(X^2) = \int_0^1 x^2 \cdot f_X(x)dx = 12\int_0^1 x^3(1 - x^2dx) = \frac{12}{60} = \frac{1}{5}$, so $V(X) = \frac{1}{5} - \left(\frac{2}{5}\right)^2 = \frac{1}{25}$.

Similarly, $V(Y) = \frac{1}{25}$, so $\rho_{X,Y} = \dfrac{-\frac{2}{75}}{\sqrt{\frac{1}{25}}\cdot\sqrt{\frac{1}{25}}} = -\frac{50}{75} = -\frac{2}{3}$.

33. $E(X) = \int_{20}^{30} xf_X(x)dx = \int_{20}^{30} x\left[10kx^2 + .05\right]dx = \frac{1925}{76} = 25.329$; by symmetry, $E(Y) = 25.329$ also;

$E(XY) = \int_{20}^{30}\int_{20}^{30} xy\cdot k(x^2 + y^2)dxdy = \frac{24375}{38} = 641.447 \Rightarrow$
$\text{Cov}(X, Y) = 641.447 - (25.329)^2 = -.1082$.

$E(X^2) = \int_{20}^{30} x^2\left[10kx^2 + .05\right]dx = \frac{37040}{57} = 649.8246 \Rightarrow V(X) = 649.8246 - (25.329)^2 = 8.2664$; by symmetry, $V(Y) = 8.2664$ as well; thus, $\rho = \dfrac{-.1082}{\sqrt{(8.2664)(8.2664)}} = -.0131$.

35. $E(XY) = (0)(0)(.71) + \ldots + (3)(2)(.01) = .35 \Rightarrow \text{Cov}(X, Y) = E(XY) - E(X)E(Y) = .35 - (.35)(.32) = .238$. Next, from the marginal distributions, $V(X) = E(X^2) - [E(X)]^2 = .67 - (.35)^2 = .5475$ and, similarly, $V(Y) = .3976$. Thus, $\text{Corr}(X, Y) = \dfrac{.238}{\sqrt{(.5475)(.3976)}} = .51$. There is a moderate, direct association between the number of syntax errors and the number of logic errors in a program. A direct association indicates that programs with a higher-than-average number of syntax errors also tend to have a higher-than-average number of logic errors, and vice versa.

37.

a. Let $H = h(X, Y)$. The variance shortcut formula states that $V(H) = E(H^2) - [E(H)]^2$. Applying that shortcut formula here yields $V(h(X,Y)) = E(h^2(X,Y)) - [E(h(X,Y))]^2$. More explicitly, if X and Y are discrete, $V(h(X,Y)) = \sum\sum[h(x,y)]^2 \cdot p(x,y) - \left[\sum\sum h(x,y)\cdot p(x,y)\right]^2$; if X and Y are continuous, $V(h(X,Y)) = \iint [h(x,y)]^2 \cdot f(x,y)dA - \left[\iint h(x,y)\cdot f(x,y)dA\right]^2$.

b. $E[h(X, Y)] = E[\max(X, Y)] = 9.60$, and $E[h^2(X, Y)] = E[(\max(X, Y))^2] = (0)^2(.02) + (5)^2(.06) + \ldots + (15)^2(.01) = 105.5$, so $V(\max(X, Y)) = 105.5 - (9.60)^2 = 13.34$.

39. First, by linearity of expectation, $\mu_{aX+bY+c} = a\mu_X + b\mu_Y + c$. Hence, by definition,

$$\begin{aligned}\text{Cov}(aX+bY+c, Z) &= E[(aX+bY+c-\mu_{aX+bY+c})(Z-\mu_Z)] = E[(aX+bY+c-[a\mu_X+b\mu_Y+c])(Z-\mu_Z)]\\ &= E[(a(X-\mu_X)+b(Y-\mu_Y))(Z-\mu_Z)]\end{aligned}$$

Apply linearity of expectation a second time:

$$\begin{aligned}\text{Cov}(aX+bY+c, Z) &= E[a(X-\mu_X)(Z-\mu_Z)+b(Y-\mu_Y)(Z-\mu_Z)]\\ &= aE[(X-\mu_X)(Z-\mu_Z)]+bE[(Y-\mu_Y)(Z-\mu_Z)]\\ &= a\text{Cov}(X,Z)+b\text{Cov}(Y,Z)\end{aligned}$$

41. Use the previous exercise: $\text{Cov}(X, Y) = \text{Cov}(X, aX + b) = a\text{Cov}(X, X) = aV(X) \Rightarrow$ so $\text{Corr}(X,Y) = \dfrac{a\text{Var}(X)}{\sigma_X \cdot \sigma_Y} = \dfrac{a\sigma_X^2}{\sigma_X \cdot |a|\sigma_X} = \dfrac{a}{|a|} = 1$ if $a > 0$, and -1 if $a < 0$.

Section 5.3

43.

a. $E(27X_1 + 125X_2 + 512X_3) = 27E(X_1) + 125E(X_2) + 512E(X_3)$
$= 27(200) + 125(250) + 512(100) = 87{,}850$ ft^3.
$V(27X_1 + 125X_2 + 512X_3) = 27^2\, V(X_1) + 125^2\, V(X_2) + 512^2\, V(X_3)$
$= 27^2\,(10)^2 + 125^2\,(12)^2 + 512^2\,(8)^2 = 19{,}100{,}116 \Rightarrow \text{SD}(27X_1 + 125X_2 + 512X_3) = 4370.37$ ft^3.

b. The expected value is still correct, but the variance is not because the covariances now also contribute to the variance.

c. Let V = volume. From **a**, $E(V) = 87{,}850$ and $V(V) = 19{,}100{,}116$ assuming the X's are independent. If they are also normally distributed, then V is normal, and so

$$P(V > 100{,}000) = 1 - \Phi\left(\frac{100{,}000 - 87{,}850}{\sqrt{19{,}100{,}116}}\right) = 1 - \Phi(2.78) = .0027.$$

45. Y is normally distributed with $\mu_Y = \frac{1}{2}(\mu_1 + \mu_2) - \frac{1}{3}(\mu_3 + \mu_4 + \mu_5) = -5$, and

$\sigma_Y^2 = \frac{1}{4}\sigma_1^2 + \frac{1}{4}\sigma_2^2 + \frac{1}{9}\sigma_3^2 + \frac{1}{9}\sigma_4^2 + \frac{1}{9}\sigma_5^2 = 7.445 \Rightarrow \sigma_Y = 2.73$.

Thus, $P(Y \geq 0) = P\left(Z \geq \frac{0-(-5)}{2.73}\right) = 1 - \Phi(1.83) = .0336$ and

$P(-3 \leq Y \leq 3) = P\left(\frac{-3-(-5)}{2.73} \leq Z \leq \frac{3-(-5)}{2.73}\right) = P(0.73 \leq Z \leq 2.93) = \Phi(2.93) - \Phi(0.73) = .2310.$

47. $E(X_1 + X_2 + X_3) = E(X_1) + E(X_2) + E(X_3) = 15 + 30 + 20 = 65$ min, and
$V(X_1 + X_2 + X_3) = 1^2 + 2^2 + 1.5^2 = 7.25 \Rightarrow \text{SD}(X_1 + X_2 + X_3) = 2.6926$ min.
Thus, $P(X_1 + X_2 + X_3 \leq 60) = P\left(Z \leq \frac{60-65}{2.6926}\right) = P(Z \leq -1.86) = .0314.$

49. Let $X_1, \ldots, X_5$ denote morning times and $X_6, \ldots, X_{10}$ denote evening times.

a. $E(X_1 + \ldots + X_{10}) = E(X_1) + \ldots + E(X_{10}) = 5E(X_1) + 5E(X_6) = 5(4) + 5(5) = 45$ min.

b. $V(X_1 + \ldots + X_{10}) = V(X_1) + \ldots + V(X_{10}) = 5V(X_1) + 5V(X_6) = 5\left[\frac{64}{12} + \frac{100}{12}\right] = \frac{820}{12} = 68.33$.

c. $E(X_1 - X_6) = E(X_1) - E(X_6) = 4 - 5 = -1$ min, while
$V(X_1 - X_6) = V(X_1) + (-1)^2V(X_6) = \frac{64}{12} + \frac{100}{12} = \frac{164}{12} = 13.67$.

d. $E[(X_1 + \ldots + X_5) - (X_6 + \ldots + X_{10})] = 5(4) - 5(5) = -5$ min, while
$V[(X_1 + \ldots + X_5) - (X_6 + \ldots + X_{10})] = V(X_1 + \ldots + X_5) + (-1)^2V(X_6 + \ldots + X_{10}) = 68.33$, the same variance as for the sum in (b).

51.

a. With $M = 5X_1 + 10X_2$, $E(M) = 5(2) + 10(4) = 50$,
$V(M) = 5^2 (.5)^2 + 10^2 (1)^2 = 106.25$ and $\sigma_M = 10.308$.

b. $P(75 < M) = P\left(\frac{75-50}{10.308} < Z\right) = P(2.43 < Z) = .0075$.

c. $M = A_1X_1 + A_2X_2$ with the A_i and X_i all independent, so
$E(M) = E(A_1X_1) + E(A_2X_2) = E(A_1)E(X_1) + E(A_2)E(X_2) = 50$.

d. $V(M) = E(M^2) - [E(M)]^2$. Recall that for any rv Y, $E(Y^2) = V(Y) + [E(Y)]^2$.
Thus, $E(M^2) = E\left(A_1^2X_1^2 + 2A_1X_1A_2X_2 + A_2^2X_2^2\right)$
$= E\left(A_1^2\right)E\left(X_1^2\right) + 2E(A_1)E(X_1)E(A_2)E(X_2) + E\left(A_2^2\right)E\left(X_2^2\right)$ (by independence)
$= (.25 + 25)(.25 + 4) + 2(5)(2)(10)(4) + (.25 + 100)(1 + 16) = 2611.5625$, so
$V(M) = 2611.5625 - (50)^2 = 111.5625$.

e. $E(M) = 50$ still, but now $\text{Cov}(X_1, X_2) = (.5)(.5)(1.0) = .25$, so
$V(M) = a_1^2V(X_1) + 2a_1a_2\text{Cov}(X_1, X_2) + a_2^2V(X_2) = 6.25 + 2(5)(10)(.25) + 100 = 131.25.$

53. Let X_1 and X_2 denote the (constant) speeds of the two planes.

a. After two hours, the planes have traveled $2X_1$ km and $2X_2$ km, respectively, so the second will not have caught the first if $2X_1 + 10 > 2X_2$, i.e. if $X_2 - X_1 < 5$.
$X_2 - X_1$ has a mean 500 – 520 = –20, variance 100 + 100 = 200, and standard deviation 14.14. Thus,
$$P(X_2 - X_1 < 5) = P\left(Z < \frac{5-(-20)}{14.14}\right) = P(Z < 1.77) = .9616.$$

b. After two hours, #1 will be $10 + 2X_1$ km from where #2 started, whereas #2 will be $2X_2$ from where it started. Thus, the separation distance will be at most 10 if
$|2X_2 - 10 - 2X_1| \le 10$, i.e. $-10 \le 2X_2 - 10 - 2X_1 \le 10$ or $0 \le X_2 - X_1 \le 10$. The corresponding probability is $P(0 \le X_2 - X_1 \le 10) = P(1.41 \le Z \le 2.12) = .9830 - .9207 = .0623$.

55.

a. $E(Y_i) = p = \dfrac{1}{2}$, so $E(W) = \displaystyle\sum_{i=1}^{n} i \cdot E(Y_i) = \frac{1}{2}\sum_{i=1}^{n} i = \frac{n(n+1)}{4}$.

b. $V(Y_i) = p(1-p) = \dfrac{1}{4}$, so $V(W) = \displaystyle\sum_{i=1}^{n} V(i \cdot Y_i) = \sum_{i=1}^{n} i^2 \cdot V(Y_i) = \frac{1}{4}\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{24}$.

57. The total elapsed time between leaving and returning is $T = X_1 + X_2 + X_3 + X_4$, with $E(T) = 15 + 5 + 8 + 12 = 40$ minutes and $V(T) = 4^2 + 1^2 + 2^2 + 3^2 = 30$. T is normally distributed, and the desired value t is the 99th percentile of the lapsed time distribution added to 10a.m.:
$.99 = P(T \le t) = \Phi\left(\dfrac{t-40}{\sqrt{30}}\right) \Rightarrow t = 40 + 2.33\sqrt{30}$ = 52.76 minutes past 10a.m., or 10:52.76a.m.

59. Note: $\exp(u)$ will be used as alternate notation for e^u throughout this solution.

a. Using the theorem from this section,
$$f_W(w) = f_X \star f_Y = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(w-x)^2}{2}\right) dx =$$
$\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \exp\left(-\frac{x^2+(w-x)^2}{2}\right) dx$. Complete the square inside the exponential function to get
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2+(w-x)^2}{2}\right) dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-(x-w/2)^2 - w^2/4\right) dx = \frac{e^{-w^2/4}}{2\pi}\int_{-\infty}^{\infty} e^{-(x-w/2)^2}\,dx =$$
$\dfrac{e^{-w^2/4}}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-u^2}\,dx$ under the substitution $u = x - w/2$.

This last integral is Euler's famous integral, which equals $\sqrt{\pi}$; equivalently, $\dfrac{1}{\sqrt{\pi}} e^{-u^2}$ is the pdf of a normal rv with mean 0 and variance 1/2, which establishes the same result (because pdfs must integrate to 1). Either way, at long last we have $f_W(w) = \dfrac{e^{-w^2/4}}{2\pi}\sqrt{\pi} = \dfrac{1}{\sqrt{4\pi}} e^{-w^2/4}$. This is the normal pdf with $\mu = 0$ and $\sigma^2 = 2$, so we have proven that $W \sim N(0, \sqrt{2})$.

b. Since X and Y are independent and normal, $W = X + Y$ is also normal. The mean of W is $E(X) + E(Y) = 0 + 0 = 0$ and the variance of W is $V(X) + V(Y) = 1^2 + 1^2 = 2$. This is obviously <u>much</u> easier than convolution in this case!

61.

a. Since the conditions of a binomial experiment are clearly met, $X \sim$ Bin(10, 18/38).

b. Similarly, $Y \sim$ Bin(15, 18/38). Notice that X and Y have different n's but the same p.

c. $X + Y$ is the combined number of times Matt and Liz won. They played a total of 25 games, all independent and identical, with $p = 18/38$ for every game. So, it appears that the conditions of a binomial experiment are met again and that, in particular, $X + Y \sim$ Bin(25, 18/38).

d. The mgf of a Bin(n, p) rv is $M(t) = (1 - p + pe^t)^n$. Using the proposition from this section and the independence of X and Y, the mgf of $X + Y$ is $M_{X+Y}(t) = M_X(t)\cdot M_Y(t) = \left(\frac{20}{38}+\frac{18}{38}e^t\right)^{10}\cdot\left(\frac{20}{38}+\frac{18}{38}e^t\right)^{15} = \left(\frac{20}{38}+\frac{18}{38}e^t\right)^{25}$. This is the mgf of a Bin(25, 18/38) rv. Hence, by uniqueness of mgfs, $X + Y \sim$ Bin(25, 18/38) as predicted.

e. Let $W = X_1 + \ldots + X_k$. Using the same technique as in **d**, $M_W(t) = M_{X_1}(t)\cdots M_{X_k}(t) = (1-p+pe^t)^{n_1}\cdots(1-p+pe^t)^{n_k} = (1-p+pe^t)^{n_1+\cdots+n_k}$. This is the mgf of a binomial rv with parameters $n_1+\cdots+n_k$ and p. Hence, by uniqueness of mgfs, $W \sim$ Bin($\sum n_i$, p).

f. No, for two (equivalent) reasons. Algebraically, we cannot combine the terms in **d** or **e** if the p's differ. Going back to **c**, the combined experiment of 25 trials would not meet the "constant probability" condition of a binomial experiment if Matt and Liz's success probabilities were different. Hence, $X + Y$ would not be binomially distributed.

63. This is a simple extension of the previous exercise. The mgf of X_i is $\left(\frac{pe^t}{1-(1-p)e^t}\right)^{r_i}$. Assuming the X's are independent, the mgf of their sum is

$$M_{X_1+\cdots+X_n}(t) = M_{X_1}(t)\cdots M_{X_n}(t) = \left(\frac{pe^t}{1-(1-p)e^t}\right)^{r_1}\cdots\left(\frac{pe^t}{1-(1-p)e^t}\right)^{r_n} = \left(\frac{pe^t}{1-(1-p)e^t}\right)^{r_1+\cdots+r_n}.$$

This is the mgf of a negative binomial rv with parameters $r_1+\cdots+r_n$ and p. Hence, by uniqueness of mgfs, $X_1+\cdots+X_n \sim$ NB($\sum r_i$, p).

65.

a. The pdf of X is $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$, and this is also the pdf of Y. The pdf of $W = X + Y$ is $f_W(w) = f_X \star f_Y = \int_{-\infty}^{\infty} f_X(x)f_Y(w-x)dx = \int \lambda e^{-\lambda x}\cdot\lambda e^{-\lambda(w-x)}dx = \int \lambda^2 e^{-\lambda w}dx$, where the limits of integration are determined by the pair of constraints $x > 0$ and $w - x > 0$. These are clearly equivalent to $0 < x < w$, so $f_W(w) = \int_0^w \lambda^2 e^{-\lambda w}dx = \lambda^2 w e^{-\lambda w}$ for $w > 0$ (since $x + y > 0$). Matching terms with the gamma pdf, we identify this as a gamma distribution with $\alpha = 2$ and $\beta = 1/\lambda$.

b. If $X \sim$ Exponential(λ) = Gamma(1, $1/\lambda$) and $Y \sim$ Exponential(λ) = Gamma(1, $1/\lambda$) as well, then by the previous exercise (with $\alpha_1 = \alpha_2 = 1$ and $\beta = 1/\lambda$) we have $X + Y \sim$ Gamma(1 + 1, $1/\lambda$) = Gamma(2, $1/\lambda$).

c. More generally, if $X_1, \ldots, X_n$ are independent Exponential(λ) rvs, then their sum has a Gamma(n, $1/\lambda$) distribution. This can be established through mgfs:

$M_{X_1+\cdots+X_n}(t) = M_{X_1}(t)\cdots M_{X_n}(t) = \frac{\lambda}{\lambda-t}\cdots\frac{\lambda}{\lambda-t} = \left(\frac{\lambda}{\lambda-t}\right)^n = \frac{1}{(1-(1/\lambda)t)^n}$, which is precisely the mgf of the Gamma(n, $1/\lambda$) distribution.

67.

a. $V(X) = E[(X-0)^2] = E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2}e^{-|x|}\,dx = 2\int_0^{\infty} x^2 \cdot \frac{1}{2}e^{-x}\,dx = \int_0^{\infty} x^2 e^{-x}\,dx$. To evaluate this integral, use Expression (4.5): $\int_0^{\infty} x^2 e^{-x}\,dx = 1^3\Gamma(3) = 2! = 2$.

b. See Chapter 4, Exercise 34.

c. By linearity of expectation $E(Y_n) = E(X_1) + \cdots + E(X_n) = 0 + \cdots + 0 = 0$.
By independence, $V(Y_n) = V(X_1) + \cdots + V(X_n) = 2 + \cdots + 2 = 2n$.
Again by independence, $M_{Y_n}(t) = M_{X_1}(t)\cdots M_{X_n}(t) = \frac{1}{1-t^2}\cdots\frac{1}{1-t^2} = \frac{1}{(1-t^2)^n}$

d. Apply the rescaling property with $a = 1/\sigma_{Y_n} = 1/\sqrt{2n}$ and $b = -\mu_{Y_n}/\sigma_{Y_n} = 0$:

$M_{Z_n}(t) = M_{Y_n}(at) = \frac{1}{(1-[at]^2)^n} = \frac{1}{(1-t^2/2n)^n}$.

e. Recall that $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$. Rewrite the denominator and take the limit:

$(1-t^2/2n)^n = \left(1+\frac{-t^2/2}{n}\right)^n \to e^{-t^2/2}$. Thus $M_{Z_n}(t) \to \frac{1}{e^{-t^2/2}} = e^{t^2/2}$, the $N(0, 1)$ mgf.

Section 5.4

69.

a. $f_X(x) = \int_0^x f(x,y)\,dy = \int_0^x 2\,dy = 2x,\ 0 < x < 1$.

b. $f_{Y|X}(y|x) = f(x, y)/f_X(x) = 2/2x = 1/x$, $0 < y < x$. That is, $Y|X=x$ is Uniform on $(0, x)$. We will use this repeatedly in what follows.

c. From (b), $P(0 < Y < .3 | X = .5) = .3/.5 = .6$.

d. No, the conditional distribution $f_{Y|X}(y|x)$ actually depends on x.

e. From (b), $E(Y|X=x) = (0 + x)/2 = x/2$.

f. From (b), $V(Y|X=x) = (x - 0)^2/12 = x^2/12$.

71.

a. $f_X(x) = \int_x^\infty f(x,y)dy = \int_x^\infty 2e^{-(x+y)}dy = 2e^{-2x}$, $x > 0$.

b. $f_{Y|X}(y|x) = f(x, y)/f_X(x) = 2e^{-(x+y)}/2e^{-2x} = e^{x-y}$, $0 < x < y$.

c. $P(Y > 2 \mid X = 1) = \int_2^\infty f_{Y|X}(y \mid 1)dy = \int_2^\infty e^{1-y}dy = e^{-1} = .3679$.

d. <u>No</u>, since $f_{Y|X}(y|x)$ actually depends on x.

e. $E(Y|X=x) = \int_x^\infty y \cdot e^{x-y}dy = e^x \int_x^\infty ye^{-y}dy = e^x(1+x)e^{-x}$ using integration by parts $= (1 + x)$.

f. Using integration by parts and proceeding as in (e), $E(Y^2|X=x) = \ldots = x^2 + 2x + 2$.
Thus, $V(Y|X=x) = x^2 + 2x + 2 - (1 + x)^2 = 1$.

73.

a. $Y|X=x$ is Uniform$(0, x)$, so $E(Y|X=x) = (0 + x)/2 = x/2$ and $V(Y|X=x) = (x - 0)^2/12 = x^2/12$.

b. $f(x, y) = f_X(x) \cdot f_{Y|X}(y|x) = 1/(1 - 0) \cdot 1/(x - 0) = 1/x$ for $0 < y < x < 1$.

c. $f_Y(y) = \int_y^1 f(x,y)dx = \int_y^1 (1/x)dx = \ln(1) - \ln(y) = -\ln(y)$, $0 < y < 1$. [Note: since $0 < y < 1$, $\ln(y)$ is actually negative, and the pdf is indeed positive.]

d. $E(Y) = \int_0^1 y(-\ln y)dy$. Make the substitution $u = -\ln y$, $y = e^{-u}$, $dy = -e^{-u}\,du$:
$E(Y) = \int_\infty^0 e^{-u}u(-e^{-u})du = \int_0^\infty ue^{-2u}\,du = (1/2)^2\Gamma(2) = 1/4$ using Expression (4.5). Similarly,
$E(Y^2) = 1/9 \Rightarrow V(Y) = 1/9 - (1/4)^2 = 7/144$.

e. From **a**, $E(Y \mid X) = X/2$. Thus $E[Y] = E[E(Y \mid X)] = E[X/2] = E[X]/2 = (1/2)/2 = 1/4$. (The mean of X is 1/2 because $X \sim$ Unif[0, 1].) Next, applying the law of total variance, $V(Y) = V[E(Y \mid X)] + E[V(Y \mid X)] = V(X/2) + E(X^2/12) = (1/4)V(X) + (1/12)E(X^2)$. Using the uniform distribution, $V(X) = (1-0)^2/12 = 1/12$ and $E(X^2) = V(X) + [E(X)]^2 = 1/12 + 1/4 = 1/3$. Put it all together, and $V(Y) = (1/4)(1/12) + (1/12)(1/3) = 7/144$. This is *a lot* easier than dealing with the integrals in part **d**.

75.

a. $p_{Y|X}(y|1)$ results from dividing each entry in the $x = 1$ row of the joint probability table by $p_X(1) = .34$:
$p_{Y|X}(0\,|\,1) = \dfrac{.08}{.34} = .2353$, $p_{Y|X}(1\,|\,1) = \dfrac{.20}{.34} = .5882$, $p_{Y|X}(2\,|\,1) = \dfrac{.06}{.34} = .1765$.

b. $p_{Y|X}(y|2)$ is requested; to obtain this divide each entry in the $x = 2$ row by $p_X(2) = .50$:

y	0	1	2
$p_{Y\|X}(y\|2)$	.12	.28	.60

c. $P(Y \le 1 \mid X = 2) = p_{Y|X}(0|2) + p_{Y|X}(1|2) = .12 + .28 = .40$.

d. $p_{X|Y}(x|2)$ is requested; to obtain this divide each entry in the $y = 2$ column by $p_Y(2) = .38$:

x	0	1	2
$p_{X\|Y}(x\|2)$	.0526	.1579	.7895

77.

a. $Y|X=x$ is Unif$[0,x^2]$. So, $E(Y|X=x) = (0+x^2)/2 = x^2/2$ and $V(Y|X=x) = (x^2 - 0)^2/12 = x^4/12$.

b. $f(x,y) = f_X(x) \cdot f_{Y|X}(y|x) = 1/(1-0) \cdot 1/(x^2 - 0) = 1/x^2$ for $0 < y < x^2 < 1$.

c. $f_Y(y) = \displaystyle\int f(x,y)dx = \int_{\sqrt{y}}^{1} (1/x^2)dx = \frac{1}{\sqrt{y}} - 1$, $0 < y < 1$.

79.

a. By considering all 9 possible pairs of numbers David and Peter could select, we find the joint pmf $p(x,y)$ displayed in the table below.

$x \setminus y$	1	2	3
1	1/9	0	0
2	2/9	1/9	0
3	2/9	2/9	1/9

b. Adding across the rows, $p_X(1) = 1/9$, $p_X(2) = 3/9$, $p_X(3) = 5/9$.

c. $p_{Y|X}(y\,|\,x) = p(x,y)/p_X(x)$, so divide each row by its total from part **b**.

y	1	2	3
$p(y\,\|\,1)$	1	0	0
$p(y\,\|\,2)$	2/3	1/3	0
$p(y\,\|\,3)$	2/5	2/5	1/5

d. Take the weighted averages from the rows in part **c**. $E(Y \mid X = 1) = 1$, $E(Y \mid X = 2) = 4/3 = 1.33$, and $E(Y \mid X = 3) = 9/5 = 1.8$.

e. Clearly $V(Y \mid X = 1) = 0$. From the rows of part **c**, $V(Y \mid X = 2) = 2/9$, and $V(Y \mid X = 3) = 14/25$.

81. Computations here are similar to the previous two exercises.

a. Add down the columns of the joint pmf.

x	$p(x \mid 1)$	$p(x \mid 2)$	$p(x \mid 3)$
1	1/5	0	0
2	2/5	1/3	0
3	2/5	2/3	1

b. From **a**, $E(X \mid Y = 1) = 1(1/5) + 2(2/5) + 3(2/5) = 11/5$; $E(X \mid Y = 2) = 1(0) + 2(1/3) + 3(2/3) = 8/3 = 2.6667$, and $E(X \mid Y = 3) = 3$.

c. Similarly, $V(X \mid Y = 1) = 14/25$, $V(X \mid Y = 2) = 2/9$, and obviously $V(X \mid Y = 3) = V(3) = 0$.

83.

a. Since all ten digits are equally likely, $p_X(x) = 1/10$ for $x = 0,1,\ldots,9$.
Next, $p_{Y|X}(y|x) = 1/9$ for $y = 0,1,\ldots,9$, $y \neq x$. (That is, any of the 9 remaining digits are equally likely.)
Combining, $p(x, y) = p_X(x) \cdot p_{Y|X}(y|x) = 1/90$ for (x,y) satisfying $x,y = 0,1,\ldots,9$, $y \neq x$.

b. $E(Y|X = x) = \sum_{y\neq x} y\, p_{Y|X}(y|x) = (1/9) \sum_{y\neq x} y = (1/9)\,[0 + 1 + \ldots + 9 - x] = (1/9)(45 - x) = 5 - x/9$.

85. We have $X \sim$ Poisson(100) and $Y|X=x \sim$ Bin(x, .6).

a. $E(Y|X=x) = np = .6x$, and $V(Y|X=x) = np(1 - p) = x(.6)(.4) = .24x$.

b. From **a**, $E(Y|X) = .6X$. Then, from the Law of Total Expectation, $E(Y) = E[E(Y|X)] = E(.6X) = .6E(X) = .6(100) = 60$. This is the common-sense answer given the specified parameters.

c. From **a**, $E(Y|X) = .6X$ and $V(Y|X) = .24X$. Since X is Poisson(100), $E(X) = V(X) = 100$. By the Law of Total Variance,
$V(Y) = E(V(Y \mid X)) + V(E(Y \mid X)) = E(.24X) + V(.6X) = .24E(X) + .6^2V(X) = .24(100) + .36(100) = 60$.

87. We're give $E(Y|X) = 4X - 104$ and SD$(Y|X) = .3X - 17$. Down the road we'll need $E(X^2) = V(X) + [E(X)]^2 = 3^2 + 70^2 = 4909$.
By the Law of Total Expectation, the unconditional mean of Y is
$E(Y) = E[E(Y|X)] = E(4X - 104) = 4E(X) - 104 = 4(70) - 104 = 176$ pounds.
By the Law of Total Variance, the unconditional variance of Y is
$V(Y) = V(E(Y \mid X)) + E(V(Y \mid X)) = V(4X - 104) + E[(.3X - 17)^2] = 4^2V(X) + E[.09X^2 - 10.2X + 289] = 16(9) + .09(4909) - 10.2(70) + 289 = 160.81$.
Thus, SD$(Y) = 12.68$ pounds.

89.

a. $E(X) = E(1 + N) = 1 + E(N) = 1 + 4p$. $V(X) = V(1 + N) = V(N) = 4p(1 - p)$.

b. Let W denote the winnings from one chip. Using the pmf, $\mu = E(W) = 0(.39) + \ldots + 10{,}000(.23) = \$2598$ and $\sigma^2 = V(W) = 16{,}518{,}196$.

c. By the Law of Total Expectation, $E(Y) = E[E(Y|X)] = E[\mu X] = \mu\, E(X) = 2598(1 + 4p)$. By the Law of Total Variance, $V(Y) = V(E(Y \mid X)) + E(V(Y \mid X)) = V(\mu X) + E(\sigma^2 X) = \mu^2 V(X) + \sigma^2 E(X) = (2598)^2 \cdot 4p(1 - p) + 16{,}518{,}196(1 + 4p)$. Simplifying and taking the square root gives $\mathrm{SD}(Y) = \sqrt{16518196 + 93071200p - 26998416p^2}$.

d. When $p = 0$, $E(Y) = \$2598$ and $\mathrm{SD}(Y) = \$4064$. If the contestant always guesses incorrectly, s/he will get exactly 1 chip and the answers from **b** apply.
When $p = .5$, $E(Y) = \$7794$ and $\mathrm{SD}(Y) = \$7504$.
When $p = 1$, $E(Y) = \$12{,}990$ and $\mathrm{SD}(Y) = \$9088$.
As the ability to acquire chips improves, so does the contestant's expected payout. The variability around that expectation also increases (since the set of options widens), but the standard deviation does not quite increase linearly with p.

Section 5.5

91.

a. Since X and W are bivariate normal, $X + W$ has a (univariate) normal distribution, with mean $E(X+W) = E(X) + E(W) = 496 + 488 = 984$ and variance given by
$V(X + W) = V(X) + V(W) + 2\,\mathrm{Cov}(X, W) = V(X) + V(W) + 2\,\mathrm{SD}(X)\,\mathrm{SD}(W)\,\mathrm{Corr}(X, W) = 114^2 + 114^2 + 2(114)(114)(.5) = 38{,}988$. Equivalently, $\mathrm{SD}(X + W) = \sqrt{38{,}988} = 197.45$.
That is, $X + W \sim N(984, 197.45)$.

b. $P(X + W > 1200) = 1 - \Phi\left(\dfrac{1200-984}{197.45}\right) = 1 - \Phi(1.09) = .1379$.

c. We're looking for the 90th percentile of the $N(984, 197.45)$ distribution:
$.9 = \Phi\left(\dfrac{x-984}{197.45}\right) \Rightarrow \dfrac{x-984}{197.45} = 1.28 \Rightarrow x = 984 + 1.28(197.45) = 1237$.

93. As stated in the section, $Y|X=x$ is normal with mean $\mu_2 + \rho\sigma_2\left(\dfrac{x-\mu_1}{\sigma_1}\right)$ and variance $(1-\rho^2)\sigma_2^2$.

a. Substitute into the above expressions: mean $= 170 + .9(20)\left(\dfrac{68-70}{3}\right) = 158$ lbs, variance $= (1 - .9^2)(20)^2 = 76$, sd $= 8.72$ lbs. That is, the weight distribution of 5'8" tall American males is $N(158, 8.72)$.

b. Similarly, $x = 70$ returns mean $= 170$ lbs and sd $= 8.72$ lbs, so the weight distribution of 5'10" tall American males is $N(170, 8.72)$. These two conditional distributions are both normal and both have standard deviation equal to 7.82 lbs, but the average weight differs by height.

c. Plug in $x = 72$ as above to get $Y|X=72 \sim N(182, 8.72)$. Thus, $P(Y < 180 \mid X = 72) = \Phi\left(\dfrac{180-182}{8.72}\right) = \Phi(-0.23) = .4090$.

95.

a. The mean is $\mu_2 + \rho\sigma_2(x - \mu_1)/\sigma_1 = 70 + (.71)(15)(x - 73)/12 = .8875x + 5.2125$.

b. The variance is $\sigma_2^2(1 - \rho^2) = 15^2(1 - .71^2) = 111.5775$.

c. From **b**, sd =10.563.

d. From **a**, the mean when $x = 80$ is 76.2125. So, $P(Y > 90|X = 80) = 1 - \Phi\left(\frac{90 - 76.2125}{10.563}\right) = 1 - \Phi(1.31)$ $= .0951$.

97.

a. The mean is $\mu_2 + \rho\sigma_2(x - \mu_1)/\sigma_1 = 30 + (.8)(5)(x - 20)/2 = 2x - 10$.

b. The variance is $\sigma_2^2(1 - \rho^2) = 5^2(1 - .8^2) = 9$.

c. From **b**, sd = 3.

d. From **a**, the mean when $x = 25$ is 40. So, $P(Y > 46|X = 25) = 1 - \Phi\left(\frac{46 - 40}{3}\right) = 1 - \Phi(2) = .0228$.

99.

a. $P(50 < X < 100, 20 < Y < 25) = P(X < 100, Y < 25) - P(X < 50, Y < 25) - P(X < 100, Y < 20) +$ $P(X < 50, Y < 20) = .3333 - .1274 - .1274 + .0625 = .1410$.

b. If X and Y are independent, then $P(50 < X < 100, 20 < Y < 25) = P(50 < X < 100) \cdot P(20 < Y < 25) =$ $[\Phi(0) - \Phi(-1)]\,[\Phi(0) - \Phi(-1)] = (.3413)^2 = .1165$. This is smaller than (a). When $\rho > 0$, it's more likely that the event $X < 100$ (its mean) coincides with $Y < 25$ (its mean).

Section 5.6

101. If X_1 and X_2 are independent, standard normal rvs, then $f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2) = \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-x_2^2/2}$ $= \frac{1}{2\pi} e^{-(x_1^2 + x_2^2)/2}$.

a. Solve the given equations for X_1 and X_2: by adding, $Y_1 + Y_2 = 2X_1 \Rightarrow X_1 = (Y_1 + Y_2)/2$. Similarly, subtracting yields $X_2 = (Y_1 - Y_2)/2$. Hence, the Jacobian of this transformation is

$$\det(M) = \begin{vmatrix} \partial x_1/\partial y_1 & \partial x_1/\partial y_2 \\ \partial x_2/\partial y_1 & \partial x_2/\partial y_2 \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = (1/2)(-1/2) - (1/2)(1/2) = -1/2$$

Also, the sum of squares in the exponent of the joint pdf above can be re-written:

$$x_1^2 + x_2^2 = \left(\frac{y_1 + y_2}{2}\right)^2 + \left(\frac{y_1 - y_2}{2}\right)^2 = \frac{y_1^2 + 2y_1y_2 + y_2^2 + y_1^2 - 2y_1y_2 + y_2^2}{4} = \frac{y_1^2 + y_2^2}{2}$$

Finally, the joint pdf of Y_1 and Y_2 is

$$f(y_1, y_2) = \frac{1}{2\pi} e^{-((y_1^2 + y_2^2)/2)/2} \cdot |-1/2| = \frac{1}{4\pi} e^{-(y_1^2 + y_2^2)/4}$$

b. To obtain the marginal pdf of Y_1, integrate out Y_2:

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} \frac{1}{4\pi} e^{-(y_1^2+y_2^2)/4}\,dy_2 = \frac{1}{\sqrt{4\pi}} e^{-y_1^2/4} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-y_2^2/4}\,dy_2$$

The integrand is the pdf of a normal distribution with $\mu = 0$ and $\sigma^2 = 2$ (so $2\sigma^2 = 4$). Since it's a pdf, its integral is 1, and we're left with $f_{Y_1}(y_1) = \frac{1}{\sqrt{4\pi}} e^{-y_1^2/4}$. This is (also) a normal pdf with mean 0 and variance 2, which we know to be the distribution of the sum of two independent $N(0, 1)$ rvs.

c. Yes, Y_1 and Y_2 are independent. Repeating the math in **b** to obtain the marginal pdf of Y_2 yields $f_{Y_2}(y_2) = \frac{1}{\sqrt{4\pi}} e^{-y_2^2/4}$, from which we see that $f(y_1, y_2) = f_{Y_1}(y_1) \cdot f_{Y_2}(y_2)$. Thus, by definition, Y_1 and Y_2 are independent.

103. Let $Y = X_1 + X_2$ and $W = X_2 - X_1$, so $X_1 = (Y - W)/2$ and $X_2 = (Y + W)/2$. We will find their joint distribution, and then their marginal distributions to answer **a** and **b**.

The Jacobian of the transformation is $\det\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = 1/2$.

Graph the triangle $0 \le x_1 \le x_2 \le 1$ and transform this into the (y, w) plane. The result is the triangle bounded by $y = 0$, $w = y$, and $w = 2 - y$. Therefore, on this triangle, the joint pdf of Y and W is

$$f(y, w) = 2\left(\frac{y-w}{2} + \frac{y+w}{2}\right) \cdot \left|\frac{1}{2}\right| = y\,.$$

a. For $0 \le y \le 1$, $f_Y(y) = \int_0^y y\,dw = y^2$; for $1 \le y \le 2$, $f_Y(y) = \int_0^{2-y} y\,dw = y(2 - y)$.

b. For $0 \le w \le 1$, $f_W(w) = \int_w^{2-w} y\,dy = \ldots = 2(1 - w)$.

105. Solving for the X's gives $X_1 = Y_1$, $X_2 = Y_2/Y_1$, and $X_3 = Y_3/Y_2$. The Jacobian of the transformation is $\det\begin{bmatrix} 1 & - & - \\ 0 & 1/y_1 & - \\ 0 & 0 & 1/y_2 \end{bmatrix} = 1/y_1y_2$; the dashes indicate partial derivatives that don't matter. Thus, the joint pdf of the Y's is

$f(y_1, y_2, y_3) = 8y_3 \cdot |1/y_1y_2| = \frac{8y_3}{y_1y_2}$ for $0 < y_3 < y_2 < y_1 < 1$. The marginal pdf of Y_3 is

$$f_{Y_3}(y_3) = \int_{y_3}^1 \int_{y_2}^1 \frac{8y_3}{y_1y_2}\,dy_1 dy_2 = \int_{y_3}^1 -\frac{8y_3}{y_2}\ln(y_2)\,dy_2 = \int_{\ln(y_3)}^0 -8y_3 u\,du = 4y_3[\ln(y_3)]^2 \text{ for } 0 < y_3 < 1.$$

107. If $U \sim$ Unif(0, 1), then $Y = -2\ln(U)$ has an exponential distribution with parameter $\lambda = ½$ (mean 2); see the section on one-variable transformations in the previous chapter. Likewise, $2\pi U$ is Uniform on $(0, 2\pi)$. Hence, Y_1, and Y_2 described here are precisely the random variables that result in the previous exercise, and the transformations $z_1 = \sqrt{y_1}\cos(y_2)$, $z_2 = \sqrt{y_1}\sin(y_2)$ restore the original independent standard normal random variables in that exercise.

Section 5.7

109.

a. $f(x) = 1/10 \to F(x) = x/10 \to g_5(y) = 5[y/10]^4[1/10] = 5y^4/10^5$ for $0 < y < 10$. Hence, $E(Y_5) = \int_0^{10} y \cdot 5y^4/10^5\,dy = 50/6$, or 8.33 minutes.

b. By the same sort of computation as in **a**, $E(Y_1) = 10/6$, and so $E(Y_5 - Y_1) = 50/6 - 10/6 = 40/6$, or 6.67 minutes.

c. The median waiting time is Y_3; its pdf is $g_3(y) = \dfrac{5!}{2!1!2!}[F(y)]^2 f(y)[1 - F(y)]^2 = 30y^2(10 - y)^2/10^5$ for $0 < y < 10$. By direct integration, or by symmetry, $E(Y_3) = 5$ minutes (which is also the mean and median of the original Unif[0, 10] distribution).

d. $E(Y_5^2) = \int_0^{10} y^2 \cdot 5y^4/10^5\,dy = 500/7$, so $V(Y_5) = 500/7 - (50/6)^2 = 125/63 = 1.984$, from which $SD(Y_5) = 1.409$ minutes.

111. The joint pdf of the sample minimum and maximum is

$$g_{1,5}(y_1, y_5) = \frac{5!}{(1-1)!(5-1-1)!(5-5)!}[F(y_1)]^{1-1}[F(y_5) - F(y_1)]^{5-1-1}[1 - F(y_5)]^{5-5} f(y_1) f(y_5)$$

$$= \frac{20}{10^5}(y_5 - y_1)^3, \quad 0 < y_1 < y_5 < 10$$

The pdf of Y_1 is $g_1(y_1) = 5\left(1 - \frac{y_1}{10}\right)^4 \cdot \frac{1}{10} = \frac{5}{10^5}(10 - y_1)^4$. So, the conditional pdf of Y_5 given $Y_1 = 4$ is

$$g(y_5 \mid y_1 = 4) = \frac{g_{1,5}(4, y_5)}{g_1(4)} = \cdots = \frac{4}{6^4}(y_5 - 4)^3, \quad 4 < y_5 < 10.$$

The conditional expectation is

$$E(Y_5 \mid Y_1 = 4) = \int y_5 g(y_5 \mid y_1 = 4)\,dy_5 = \int_4^{10} y_5 \frac{4}{6^4}(y_5 - 4)^3\,dy_5 = \cdots = 8.8 \text{ minutes.}$$

113. Let the original times (in hours after noon) be $X_1, \ldots, X_n$, which we're assuming are Unif[0, 1]. Their pdf is 1 and cdf is x, so the pdf of the kth order statistic is $g_k(y) = \dfrac{n!}{(k-1)!(n-k)!} y^{k-1}(1-y)^{n-k}$, which we can recognize as the Beta(k, $n - k + 1$) distribution. The expected value, from the Beta formulas, is then $E(Y_k) = \dfrac{k}{k+n-k+1} = \dfrac{k}{n+1}$. The expected ordered arrival times are evenly spaced throughout the hour, $1/(n+1)$ hours apart. (For example, if $n = 5$, the arrival times are 1/6, …, 5/6 hours past noon, aka 12:10pm, 12:20pm, …, 12:50pm.)

115. The pdf of the underlying population distribution is $f(x) = \theta x^{\theta-1}$. The pdf of Y_i is

$$g_i(y) = \frac{n!}{(i-1)!(n-i)!}[y^\theta]^{i-1}[1-y^\theta]^{n-i}[\theta y^{\theta-1}] = \frac{n!\theta}{(i-1)!(n-i)!} y^{i\theta-1}[1-y^\theta]^{n-i}.$$ Thus,

$$E(Y_i) = \int_0^1 y g_i(y)dy = \frac{n!\theta}{(i-1)!(n-i)!}\int_0^1 y^{i\theta}[1-y^\theta]^{n-i}dy$$ = [via the substitution $u = y^\theta$]

$$\frac{n!\theta}{(i-1)!(n-i)!}\int_0^1 u^i[1-u]^{n-i}\frac{u^{1/\theta-1}}{\theta}du = \frac{n!}{(i-1)!(n-i)!}\int_0^1 u^{i+1/\theta-1}(1-u)^{n-i}du.$$

The integral is the "kernel" of the Beta($i + 1/\theta$, $n - i + 1$) distribution, and so this entire expression equals $\dfrac{n!}{(i-1)!(n-i)!}\dfrac{\Gamma(i+1/\theta)\Gamma(n-i+1)}{\Gamma(n+1/\theta+1)} = \dfrac{n!\Gamma(i+1/\theta)}{(i-1)!\Gamma(n+1/\theta+1)}$. Similarly, $E(Y_i^2) = \dfrac{n!\Gamma(i+2/\theta)}{(i-1)!\Gamma(n+2/\theta+1)}$,

from which $V(Y_i) = \dfrac{n!\Gamma(i+2/\theta)}{(i-1)!\Gamma(n+2/\theta+1)} - \left[\dfrac{n!\Gamma(i+1/\theta)}{(i-1)!\Gamma(n+1/\theta+1)}\right]^2$.

117. $f(x) = 3/x^4$ for $x > 1 \Rightarrow F(x) = \int_1^x 3/y^4\,dy = 1 - x^{-3}$ for $x > 1$.

a. P(at least one claim > \$5000) = 1 – P(all 3 claims are ≤ \$5000) = $1 - P(X_1 \le 5 \cap X_2 \le 5 \cap X_3 \le 5) = 1 - F(5)\cdot F(5)\cdot F(5)$ by independence $= 1 - (1 - 5^{-3})^3 = .0238$.

b. The pdf of Y_3, the largest claim, is $g_3(y) = 3f(y)[F(y)]^{3-1} = 3(3y^{-4})[1-y^{-3}]^2 = 9(y^{-4} - 2y^{-7} + y^{-10})$ for $y > 1$. Hence, $E(Y_3) = \int_1^\infty y\cdot 9(y^{-4} - 2y^{-7} + y^{-10})dy = 9\int_1^\infty (y^{-3} - 2y^{-6} + y^{-9})dy$ = 2.025, or \$2,025.

119. As suggested in the section, divide the number line into five intervals: $(-\infty, y_i]$, $(y_i, y_i + \Delta_1]$, $(y_i + \Delta_1, y_j]$, $(y_j, y_j + \Delta_2]$, and $(y_j + \Delta_2, \infty)$. For a rv X having cdf F, the probability X falls into these five intervals are $p_1 = P(X \le y_i) = F(y_i)$, $p_2 = F(y_i + \Delta_1) - F(y_i) \approx f(y_i)\Delta_1$, $p_3 = F(y_j) - F(y_i + \Delta_1)$, $p_4 = F(y_j + \Delta_2) - F(y_j) \approx f(y_j)\Delta_2$, and $p_5 = P(X > y_j + \Delta_2) = 1 - F(y_j + \Delta_2)$.

Now consider a random sample of size n from this distribution. Let Y_i and Y_j denote the ith and jth smallest values (order statistics) with $i < j$. It is unlikely that more than one X will fall in the 2nd interval or the 4th interval, since they are very small (widths Δ_1 and Δ_2). So, the event that Y_i falls in the 2nd interval and Y_j falls in the 4th interval is approximately the probability that: $i - 1$ of the X's falls in the 1st interval; one X (the ith smallest) falls in the 2nd interval; $j - i - 1$ fall in the 3rd interval; one X (the jth smallest) falls in the 4th interval; and the largest $n - j$ X's fall in the 5th interval. Apply the multinomial formula:

$$P(y_i < Y_i \le y_i + \Delta_1, y_j < Y_j \le y_j + \Delta_2) \approx \frac{n!}{(i-1)!1!(j-i-1)!1!(n-j)!} p_1^{i-1} p_2^1 p_3^{j-i-1} p_4^1 p_5^{n-j}$$

$$\approx \frac{n!}{(i-1)!(j-i-1)!(n-j)!} [F(y_i)]^{i-1} f(y_i)\Delta_1 [F(y_j) - F(y_i + \Delta_1)]^{j-i-1} f(y_j)\Delta_2 [1 - F(y_j + \Delta_2)]^{n-j}$$

Dividing the left-hand side by $\Delta_1\Delta_2$ and letting $\Delta_1 \to 0$, $\Delta_2 \to 0$ yields the joint pdf $g(y_i, y_j)$. Taking the same action on the right-hand side returns

$\dfrac{n!}{(i-1)!(j-i-1)!(n-j)!} F(y_i)^{i-1} [F(y_j) - F(y_i)]^{j-i-1} [1 - F(y_j)]^{n-j} f(y_i) f(y_j)$, as claimed.

121.

a. Substitute $i = 1$ and $j = n$ to get

$$g_{1,n}(y_1, y_n) = \frac{n!}{(1-1)!(n-1-1)!(n-n)!} [F(y_n) - F(y_1)]^{n-1-1} f(y_1) f(y_n)$$
$$= n(n-1)[F(y_n) - F(y_1)]^{n-2} f(y_1) f(y_n) \quad \text{for } y_1 < y_n$$

b. Let $W = W_2 = Y_n - Y_1$ (drop the subscript "2") and $W_1 = Y_1$. The Jacobian of this transformation is clearly 1. With $Y_1 = W_1$ and $Y_n = Y_1 + W = W_1 + W$, the desired joint pdf is

$f(w_1, w) = n(n-1)[F(w + w_1) - F(w_1)]^{n-2} f(w_1) f(w + w_1) \quad \text{for } w > 0$.

To find the marginal pdf of the range, integrate out the other variable, W_1:

$$f_W(w) = \int n(n-1)[F(w + w_1) - F(w_1)]^{n-2} f(w_1) f(w + w_1) dw_1 \quad .$$

c. For the Uniform[0, 1] distribution, $f(x) = 1$ and $F(x) = x$. The limits $0 \le y_1 < y_n \le 1$ are equivalent to $0 \le w_1 \le w + w_1 \le 1$, from which the dw_1 limits are $0 \le w_1 \le 1 - w$. Hence,

$$f_W(w) = \int_0^{1-w} n(n-1)[(w + w_1) - w_1]^{n-2} \cdot 1 \cdot 1 dw_1 = n(n-1)w^{n-2} \int_0^{1-w} 1 dw_1$$
$$= n(n-1)w^{n-2}(1-w) \quad \text{for } 0 \le w \le 1$$

That is, the sample range has a Beta($n - 1$, 2) distribution.

Supplementary Exercises

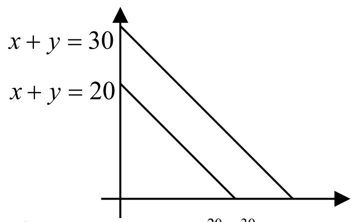
123. Let X and Y be the transmission times, so the joint pdf of X and Y is $f(x, y) = f_X(x) \cdot f_Y(y) = e^{-x}e^{-y} = e^{-(x+y)}$ for $x, y > 0$. Define $T = 2X + Y$ = the total cost to send the two messages. The cdf of T is given by

$F_T(t) = P(T \le t) = P(2X + Y \le t) = P(Y \le t - 2X)$. For $X > t/2$, this probability is zero (since Y can't be negative). Otherwise, for $X \le t/2$,

$P(Y \le t - 2X) = \int_0^{t/2} \int_0^{t-2x} e^{-(x+y)} dy dx = \cdots = 1 - 2e^{-t/2} + e^{-t}$ for $t > 0$. Thus, the pdf of T is

$f_T(t) = F_T'(t) = e^{-t/2} - e^{-t}$ for $t > 0$.

125.



a. $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_0^{20}\int_{20-x}^{30-x} kxydydx + \int_{20}^{30}\int_0^{30-x} kxydydx = \dfrac{81,250}{3}\cdot k \Rightarrow k = \dfrac{3}{81,250}$.

b. $f_X(x) = \begin{cases} \int_{20-x}^{30-x} kxydy = k(250x - 10x^2) & 0 \le x \le 20 \\ \int_0^{30-x} kxydy = k(450x - 30x^2 + \frac{1}{2}x^3) & 20 \le x \le 30 \end{cases}$

and, by symmetry, $f_Y(y)$ is obtained by substituting y for x in $f_X(x)$. Since $f_X(25) > 0$ and $f_Y(25) > 0$ but $f(25, 25) = 0$, $f_X(x)\cdot f_Y(y) \ne f(x, y)$ for all x, y and so X and Y are not independent.

c. $P(X+Y \le 25) = \int_0^{20}\int_{20-x}^{25-x} kxydydx + \int_{20}^{25}\int_0^{25-x} kxydydx = \dfrac{3}{81,250}\cdot\dfrac{230,625}{24} = .355$

d. $E(X+Y) = E(X) + E(Y) = 2E(X) = 2\int_0^{20} x\cdot k\left(250x - 10x^2\right)dx$

$+2\int_{20}^{30} x\cdot k\left(450x - 30x^2 + \frac{1}{2}x^3\right)dx = 2k(351,666.67) = 25.969$ lb.

e. $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\cdot f(x,y)dxdy = \int_0^{20}\int_{20-x}^{30-x} kx^2y^2dydx$

$+\int_{20}^{30}\int_0^{30-x} kx^2y^2dydx = \dfrac{k}{3}\cdot\dfrac{33,250,000}{3} = 136.4103$, so

Cov(X, Y) = 136.4103 – $(12.9845)^2$ = –32.19. Also, $E(X^2) = E(Y^2) = 204.6154$, so

$\sigma_x^2 = \sigma_y^2 = 204.6154 - (12.9845)^2 = 36.0182$ and $\rho = \dfrac{-32.19}{36.0182} = -.894$.

f. $V(X+Y) = V(X) + V(Y) + 2\text{Cov}(X, Y) = 7.66$.

127. $E(X+Y-t)^2 = \int_0^1\int_0^1 (x+y-t)^2\cdot f(x,y)dxdy.$ To find the minimizing value of t, take the derivative with respect to t and equate it to 0:

$0 = \int_0^1\int_0^1 2(x+y-t)(-1)f(x,y) = 0 \Rightarrow \int_0^1\int_0^1 tf(x,y)dxdy = t = \int_0^1\int_0^1 (x+y)\cdot f(x,y)dxdy = E(X+Y)$, so the best prediction is the individual's expected score (= 1.167).

129.

a. First, $E(V) = E(\rho Z_1 + \sqrt{1-\rho^2} Z_2) = \rho E(Z_1) + \sqrt{1-\rho^2} E(Z_2) = \rho(0) + \sqrt{1-\rho^2}(0) = 0.$ Second, since Z_1 and Z_2 are independent,

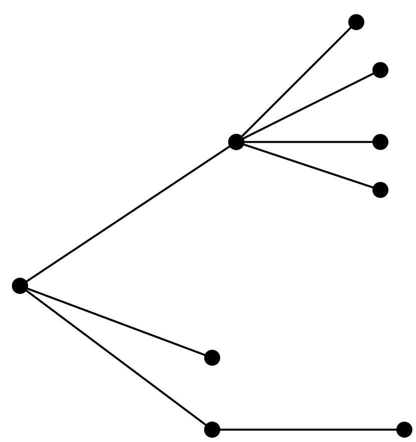
$$V(V) = V(\rho Z_1 + \sqrt{1-\rho^2} Z_2) = \rho^2 V(Z_1) + \left[\sqrt{1-\rho^2}\right]^2 V(Z_2) = \rho^2(1) + (1-\rho^2)(1) = 1.$$

b.
$$\begin{aligned}\text{Cov}(U_1, V) &= \text{Cov}(Z_1, \rho Z_1 + \sqrt{1-\rho^2} Z_2) = \rho \text{Cov}(Z_1, Z_1) + \sqrt{1-\rho^2}\text{Cov}(Z_1, Z_2) \\ &= \rho \text{Var}(Z_1) + \sqrt{1-\rho^2}\text{Cov}(Z_1, Z_2) = \rho(1) + \sqrt{1-\rho^2}(0) = \rho.\end{aligned}$$

c. $\text{Corr}(U, V) = \dfrac{\text{Cov}(U_1, V_2)}{\text{SD}(U)\text{SD}(V_2)} = \dfrac{\rho}{(1)(1)} = \rho.$

131.

a.



b. By the Law of Total Probability, $A = \bigcup_{x=0}^{\infty} A \cap \{X_1 = x\} \Rightarrow P(A) = \sum_{x=0}^{\infty} P(A \cap \{X_1 = x\}) =$

$\sum_{x=0}^{\infty} P(A \mid X_1 = x) P(X_1 = x) = \sum_{x=0}^{\infty} P(A \mid X_1 = x) p(x)$. With x members in generation 1, the process becomes extinct iff these x new, independent branching processes all become extinct. By definition, the extinction probability for each new branch is $P(A) = p^*$, and independence implies $P(A \mid X_1 = x) =$ $(p^*)^x$. Therefore, $p^* = \sum_{x=0}^{\infty} (p^*)^x p(x)$.

c. Check $p^* = 1$: $\sum_{x=0}^{\infty} (1)^x p(x) = \sum_{x=0}^{\infty} p(x) = 1 = p^*$. [We'll drop the * notation from here forward.]

In the first case, we get $p = .3 + .5p + .2p^2$. Solving for p gives $p = 3/2$ and $p = 1$; the smaller value, $p = 1$, is the extinction probability. Why will this model die off with probability 1? Because the expected number of progeny from a single individual is $0(.3)+1(.5)+2(.2) = .9 < 1$.

On the other hand, the second case gives $p = .2 + .5p + .3p^2$, whose solutions are $p = 1$ and $p = 2/3$. The extinction probability is the smaller value, $p = 2/3$. Why does this model have positive probability of eternal survival? Because the expected number of progeny from a single individual is $0(.2)+1(.5)+2(.3) = 1.1 > 1$.

133.

a. Use a sort of inclusion-exclusion principle:
$P(a \le X \le b,\ c \le Y \le d) = P(X \le b,\ Y \le d) - P(X < a,\ Y \le d) - P(X \le b,\ Y < c) + P(X < a,\ Y < c)$.
Then, since these variables are continuous, we may write
$P(a \le X \le b,\ c \le Y \le d) = F(b, d) - F(a, d) - F(b, c) + F(a, c)$.

b. In the discrete case, the strict inequalities in (a) must be re-written as follows:
$P(a \le X \le b,\ c \le Y \le d) = P(X \le b,\ Y \le d) - P(X \le a-1,\ Y \le d) - P(X \le b,\ Y \le c-1) +$
$P(X \le a-1,\ Y \le c-1) = F(b, d) - F(a-1, d) - F(b, c-1) + F(a-1, c-1)$. For the values specified, this becomes $F(10,6) - F(4,6) - F(10,1) + F(4,1)$.

c. Use the cumulative joint cdf table below. At each (x^*, y^*), $F(x^*, y^*)$ is the sum of the probabilities at points (x, y) such that $x \le x^*$ and $y \le y^*$.

	$F(x, y)$	x = 100	250
	200	.50	1
y	100	.30	.50
	0	.20	.25

d. Integrating long-hand and exhausting all possible options for (x,y) pairs, we arrive at the following:
$F(x, y) = .6x^2y + .4xy^3$, $0 \le x,\ y \le 1$; $F(x, y) = 0$, $x \le 0$ or $y \le 0$; $F(x, y) = .6x^2 + .4x$, $0 \le x \le 1$, $y > 1$;
$F(x, y) = .6y + .4y^3$, $x > 1$, $0 \le y \le 1$; and, obviously, $F(x, y) = 1$, $x > 1$, $y > 1$. (Whew!)
Thus, from (a), $P(.25 \le x \le .75,\ .25 \le y \le .75) = F(.75,.75) - F(.25,.75) - F(.75,.25) + F(.25,.25) = \ldots$
$=.23125$. [This only requires the main form of $F(x, y)$; i.e., that for $0 \le x,\ y \le 1$.]

e. Again, we proceed on a case-by case basis. The results are:
$F(x, y) = 6x^2y^2$, $x+ y \le 1$, $0 \le x \le 1$; $0 \le y \le 1$;
$F(x, y) = 3x^4 - 8x^3 + 6x^2 + 3y^4 - 8y^3 + 6y^2 - 1$, $x+ y > 1$, $x \le 1$, $y \le 1$;
$F(x, y) = 0$, $x \le 0$; $F(x, y) = 0$, $y \le 0$;
$F(x, y) = 3x^4 - 8x^3 + 6x^2$, $0 \le x \le 1$, $y > 1$;
$F(x, y) = 3y^4 - 8y^3 + 6y^2$, $0 \le y \le 1$, $x > 1$; and, obviously,
$F(x, y) = 1$, $x > 1$, $y > 1$.

135.

a. For an individual customer, the expected number of packages is $1(.4)+2(.3)+3(.2)+4(.1) = 2$ with a variance of 1 (by direct computation). Given $X=x$, Y is the sum of x independent such customers, so $E(Y|X=x) = x(2) = 2x$ and $V(Y|X=x) = x(1) = x$.

b. By the law of total expectation, $E(Y) = E[E(Y|X)] = E(2X) = 2E(X) = 2(20) = 40$.

c. By the law of total variance, $V(Y) = V(E(Y|X)) + E(V(Y|X)) = V(2X) + E(X) = 4V(X) + E(X) = 4(20) + 20 = 100$. (Recall that the mean and variance of a Poisson rv are equal.)

137. Let $a = 1/1000$ for notational ease. W is the <u>maximum</u> of the two exponential rvs, so its pdf is $f_W(w) = 2F_X(w)f_X(w) = 2(1 - e^{-aw})ae^{-aw} = 2ae^{-aw}(1 - e^{-aw})$. From this, $M_W(t) = E[e^{tW}] = \int_0^\infty e^{tw}2ae^{-aw}(1-e^{-aw})dw = 2a\int_0^\infty e^{-(a-t)w}dw - 2a\int_0^\infty e^{-(2a-t)w}dw = \frac{2a}{a-t} - \frac{2a}{2a-t} = \frac{2a^2}{(a-t)(2a-t)} = \frac{2}{(1-1000t)(2-1000t)}$. From this, $E[W] = M'_W(0) = 1500$ hours.

139. The roll-up procedure is <u>not</u> valid for the 75th percentile unless $\sigma_1 = 0$ and/or $\sigma_2 = 0$, as described below.
Sum of percentiles: $(\mu_1 + z\sigma_1) + (\mu_2 + z\sigma_2) = \mu_1 + \mu_2 + z(\sigma_1 + \sigma_2)$

Percentile of sums: $(\mu_1 + \mu_2) + z\sqrt{\sigma_1^2 + \sigma_2^2}$

These are equal when $z = 0$ (i.e. for the median) or in the unusual case when $\sigma_1 + \sigma_2 = \sqrt{\sigma_1^2 + \sigma_2^2}$, which happens when $\sigma_1 = 0$ and/or $\sigma_2 = 0$.

141.

a. Let $X_1, \ldots, X_{12}$ denote the weights for the business-class passengers and $Y_1, \ldots, Y_{50}$ denote the tourist-class weights. Then T = total weight = $X_1 + \ldots + X_{12} + Y_1 + \ldots + Y_{50} = X + Y$.
$E(X) = 12E(X_1) = 12(30) = 360$; $V(X) = 12\ V(X_1) = 12(36) = 432$.
$E(Y) = 50E(Y_1) = 50(40) = 2000$; $V(Y) = 50\ V(Y_1) = 50(100) = 5000$.
Thus $E(T) = E(X) + E(Y) = 360 + 2000 = 2360$,
and $V(T) = V(X) + V(Y) = 432 + 5000 = 5432 \Rightarrow \text{SD}(T) = 73.7021$.

b. $P(T \le 2500) = \Phi\left(\frac{2500 - 2360}{73.7021}\right) = \Phi(1.90) = .9713$.

143. The student will not be late if $X_1 + X_3 \le X_2$, i.e. if $X_1 - X_2 + X_3 \le 0$. This linear combination has mean –2 and variance 4.25, so $P(X_1 - X_2 + X_3 \le 0) = \Phi\left(\frac{0-(-2)}{\sqrt{4.25}}\right) = \Phi(.97) = .8340$.

145.

a. $V(X_1) = V(W + E_1) = \sigma_W^2 + \sigma_E^2 = V(W + E_2) = V(X_2)$ and
$\text{Cov}(X_1, X_2) = \text{Cov}(W + E_1, W + E_2) = \text{Cov}(W, W) + \text{Cov}(W, E_2) + \text{Cov}(E_1, W) + \text{Cov}(E_1, E_2) =$
$V(W) + 0 + 0 + 0 = \sigma_W^2$.

Thus, $\rho = \frac{\text{Cov}(X_1, X_2)}{\text{SD}(X_1)\text{SD}(X_2)} = \frac{\sigma_W^2}{\sqrt{\sigma_W^2 + \sigma_E^2} \cdot \sqrt{\sigma_W^2 + \sigma_E^2}} = \frac{\sigma_W^2}{\sigma_W^2 + \sigma_E^2}$.

b. $\rho = \frac{1^2}{1^2 + .01^2} = .9999$.

147. $E(Y) \doteq h(\mu_1, \mu_2, \mu_3, \mu_4) = 120\left[\frac{1}{10} + \frac{1}{15} + \frac{1}{20}\right] = 26.$

The partial derivatives of $h(\mu_1, \mu_2, \mu_3, \mu_4)$ with respect to x_1, x_2, x_3, and x_4 are $-\frac{x_4}{x_1^2}$, $-\frac{x_4}{x_2^2}$, $-\frac{x_4}{x_3^2}$, and $\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}$, respectively. Substituting $x_1 = 10$, $x_2 = 15$, $x_3 = 20$, and $x_4 = 120$ gives –1.2, –.5333, –.3000, and .2167, respectively, so $V(Y) = (1)(-1.2)^2 + (1)(-.5333)^2 + (1.5)(-.3000)^2 + (4.0)(.2167)^2 = 2.6783$, and the approximate sd of Y is 1.64.

149.

a. The marginal pdf of X is

$$f_X(x) = \int_0^\infty \frac{1}{2\pi} \frac{e^{-[(\ln x)^2 + (\ln y)^2]/2}}{xy}[1 + \sin(2\pi \ln x)\sin(2\pi \ln y)]dy$$

$$= \frac{1}{2\pi}\frac{e^{-[(\ln x)^2]/2}}{x}\int_0^\infty e^{-[(\ln y)^2]/2}[1 + \sin(2\pi \ln x)\sin(2\pi \ln y)]\frac{dy}{y}$$

$$= \frac{1}{2\pi}\frac{e^{-[(\ln x)^2]/2}}{x}\int_{-\infty}^\infty e^{-u^2/2}[1 + \sin(2\pi \ln x)\sin(2\pi u)]du$$

$$= \frac{1}{2\pi}\frac{e^{-[(\ln x)^2]/2}}{x}\left[\int_{-\infty}^\infty e^{-u^2/2}du + \sin(2\pi \ln x)\int_{-\infty}^\infty e^{-u^2/2}\sin(2\pi u)du\right]$$

The first integral is $\sqrt{2\pi}$, since the integrand is the $N(0, 1)$ pdf without the constant. The second integral is 0, since the integral is an odd function over $(-\infty, \infty)$. Hence, the final answer is

$f_X(x) = \frac{1}{2\pi}\frac{e^{-[(\ln x)^2]/2}}{x}\cdot\frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}x}e^{-[(\ln x)^2]/2}$, the lognormal pdf with $\mu = 0$ and $\sigma = 1$.

By symmetry, this is also the marginal pdf of Y.

b. The conditional distribution of Y given $X = x$ is

$f(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-[(\ln y)^2]/2}}{\sqrt{2\pi}y}[1 + \sin(2\pi \ln x)\sin(2\pi \ln y)]$, from which

$$E(Y^n \mid X = x) = \int_0^\infty y^n \cdot \frac{1}{\sqrt{2\pi}}\frac{e^{-[(\ln y)^2]/2}}{y}[1 + \sin(2\pi \ln x)\sin(2\pi \ln y)]dy$$

$$= \int_0^\infty y^n \cdot \frac{e^{-[(\ln y)^2]/2}}{\sqrt{2\pi}y}dy + \frac{\sin(2\pi \ln x)}{\sqrt{2\pi}}\int_0^\infty y^n \cdot e^{-[(\ln y)^2]/2}\sin(2\pi \ln y)\frac{dy}{y}$$

The first integral is $\int_0^\infty y^n \cdot \frac{e^{-[(\ln y)^2]/2}}{\sqrt{2\pi}y}dy = \int_0^\infty y^n \cdot f_Y(y)dy = E(Y^n)$. So, the goal is now to show that the second integral equals zero. For the second integral, make the suggested substitution $\ln(y) = u + n$, for which $du = dy/y$ and $y = e^{u+n}$:

$\int_0^\infty y^n \cdot e^{-[(\ln y)^2]/2}\sin(2\pi \ln y)\frac{dy}{y} = \int_{-\infty}^\infty (e^{u+n})^n \cdot e^{-(u+n)^2/2}\sin(2\pi(u+n))du = \int_{-\infty}^\infty e^{-u^2/2 + n^2/2}\sin(2\pi u + 2\pi n)du$

$= e^{n^2/2}\int_{-\infty}^\infty e^{-u^2/2}\sin(2\pi u + 2\pi n)du = e^{n^2/2}\int_{-\infty}^\infty e^{-u^2/2}\sin(2\pi u)du$. The second equality comes from expanding the exponents on e; the last equality comes from the basic fact that $\sin(\theta + 2\pi n) = \sin(\theta)$ for any integer n. The integral that remains has an odd integrand (u^2 is even and sine is odd), so the integral on $(-\infty, \infty)$ equals zero. At last, we have that $E(Y^n \mid X = x) = E(Y^n)$ for any positive integer n.

c. Since the pdf is symmetric in X and Y, the same derivation will yield $E(X^n \mid Y = y) = E(X^n)$ for all positive integers n.

d. Despite the fact that the expectation of every polynomial in Y is unaffected by conditioning on X (and vice versa), the two rvs are <u>not</u> independent. From **a**, the marginal pdfs of X and Y are lognormal, from which $f_X(x)\cdot f_Y(y) = \frac{1}{\sqrt{2\pi}x}e^{-[(\ln x)^2]/2}\cdot\frac{1}{\sqrt{2\pi}y}e^{-[(\ln y)^2]/2} = \frac{1}{2\pi xy}e^{-[(\ln x)^2+(\ln y)^2]/2} \neq f(x, y)$. Therefore, by definition X and Y are not independent.