# CHAPTER 2

## Section 2.1

**1.**

**a.** $A$ but not $B = A \cap B'$

**b.** at least one of $A$ and $B = A \cup B$

**c.** exactly one hired = $A$ and not $B$, or $B$ and not $A = (A \cap B') \cup (B \cap A')$

**3.**

**a.** $\mathcal{S}$ = {1324, 1342, 1423, 1432, 2314, 2341, 2413, 2431, 3124, 3142, 4123, 4132, 3214, 3241, 4213, 4231}

**b.** Event $A$ contains the outcomes where 1 is first in the list:
$A$ = {1324, 1342, 1423, 1432}.

**c.** Event $B$ contains the outcomes where 2 is first or second:
$B$ = {2314, 2341, 2413, 2431, 3214, 3241, 4213, 4231}.

**d.** The event $A \cup B$ contains the outcomes in $A$ or $B$ or both:
$A \cup B$ = {1324, 1342, 1423, 1432, 2314, 2341, 2413, 2431, 3214, 3241, 4213, 4231}.
$A \cap B = \varnothing$, since 1 and 2 can't both get into the championship game.
$A' = \mathcal{S} - A$ = {2314, 2341, 2413, 2431, 3124, 3142, 4123, 4132, 3214, 3241, 4213, 4231}.

**5.**

**a.** $A = \{SSF, SFS, FSS\}$.

**b.** $B = \{SSS, SSF, SFS, FSS\}$.

**c.** For event $C$ to occur, the system must have component 1 working ($S$ in the first position), then at least one of the other two components must work (at least one $S$ in the second and third positions): $C = \{SSS, SSF, SFS\}$.

**d.** $C' = \{SFF, FSS, FSF, FFS, FFF\}$.
$A \cup C = \{SSS, SSF, SFS, FSS\}$.
$A \cap C = \{SSF, SFS\}$.
$B \cup C = \{SSS, SSF, SFS, FSS\}$. Notice that $B$ contains $C$, so $B \cup C = B$.
$B \cap C = \{SSS\ SSF, SFS\}$. Since $B$ contains $C$, $B \cap C = C$.

**7.**

**a.** The $3^3 = 27$ possible outcomes are numbered below for later reference.

| Outcome Number | Outcome | Outcome Number | Outcome |
|---|---|---|---|
| 1 | 111 | 15 | 223 |
| 2 | 112 | 16 | 231 |
| 3 | 113 | 17 | 232 |
| 4 | 121 | 18 | 233 |
| 5 | 122 | 19 | 311 |
| 6 | 123 | 20 | 312 |
| 7 | 131 | 21 | 313 |
| 8 | 132 | 22 | 321 |
| 9 | 133 | 23 | 322 |
| 10 | 211 | 24 | 323 |
| 11 | 212 | 25 | 331 |
| 12 | 213 | 26 | 332 |
| 13 | 221 | 27 | 333 |
| 14 | 222 | | |

**b.** Outcome numbers 1, 14, 27 above.

**c.** Outcome numbers 6, 8, 12, 16, 20, 22 above.

**d.** Outcome numbers 1, 3, 7, 9, 19, 21, 25, 27 above.

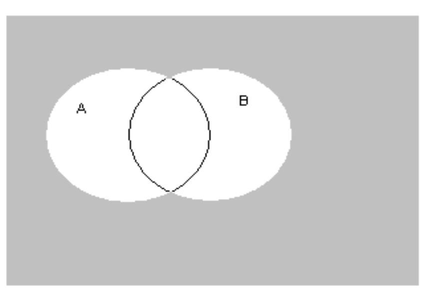
**9.**

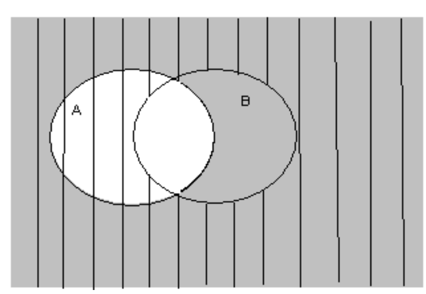
**a.** $\mathcal{S}$ = {*BBBAAAA, BBABAAA, BBAABAA, BBAAABA, BBAAAAB, BABBAAA, BABABAA, BABAABA, BABAAAB, BAABBAA, BAABABA, BAABAAB, BAAABBA, BAAABAB, BAAAABB, ABBBAAA, ABBABAA, ABBAABA, ABBAAAB, ABABBAA, ABABABA, ABABAAB, ABAABBA, ABAABAB, ABAAABB, AABBBAA, AABBABA, AABBAAB, AABABBA, AABABAB, AABAABB, AAABBBA, AAABBAB, AAABABB, AAAABBB*}.

**b.** *AAAABBB, AAABABB, AAABBAB, AABAABB, AABABAB*.

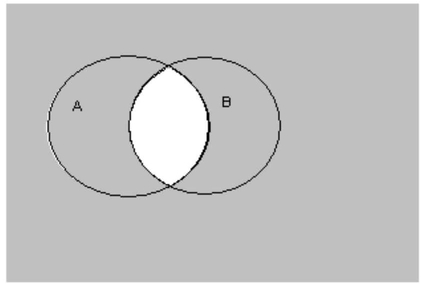
**11.**

**a.** In the diagram on the left, the shaded area is $(A \cup B)'$. On the right, the shaded area is $A'$, the striped area is $B'$, and the intersection $A' \cap B'$ occurs where there is both shading <u>and</u> stripes. These two diagrams display the same area.



**b.** In the diagram below, the shaded area represents $(A \cap B)'$. Using the right-hand diagram from (a), the union of $A'$ and $B'$ is represented by the areas that have either shading or stripes (or both). Both of the diagrams display the same area.




## Section 2.2

**13.**

**a.** .07.

**b.** .15 + .10 + .05 = .30.

**c.** Let $A$ = the selected individual owns shares in a stock fund. Then $P(A)$ = .18 + .25 = .43. The desired probability, that a selected customer does not shares in a stock fund, equals $P(A') = 1 - P(A) = 1 - .43 = .57$. This could also be calculated by adding the probabilities for all the funds that are not stocks.

**15.**

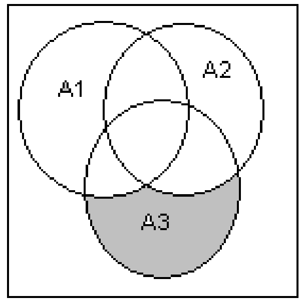
**a.** $A_1 \cup A_2$ = "awarded either #1 or #2 (or both)": from the addition rule,
$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = .22 + .25 - .11 = .36$.

**b.** $A_1' \cap A_2'$ = "awarded neither #1 or #2": using the hint and part (a),
$P(A_1' \cap A_2') = P((A_1 \cup A_2)') = 1 - P(A_1 \cup A_2) = 1 - .36 = .64$.

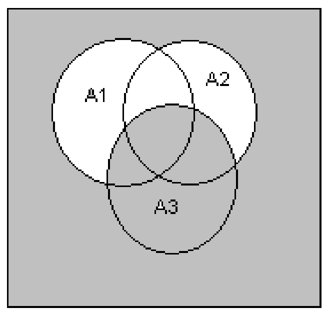
**c.** $A_1 \cup A_2 \cup A_3$ = "awarded at least one of these three projects": using the addition rule for 3 events,
$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) =$
$.22 + .25 + .28 - .11 - .05 - .07 + .01 = .53$.

**d.** $A_1' \cap A_2' \cap A_3'$ = "awarded none of the three projects":
$P(A_1' \cap A_2' \cap A_3') = 1 - P(\text{awarded at least one}) = 1 - .53 = .47$.

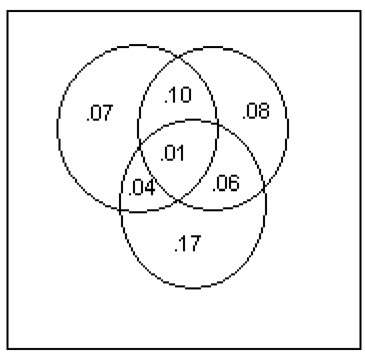
**e.** $A_1' \cap A_2' \cap A_3$ = "awarded #3 but neither #1 nor #2": from a Venn diagram,
$P(A_1' \cap A_2' \cap A_3) = P(A_3) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) =$
$.28 - .05 - .07 + .01 = .17$. The last term addresses the "double counting" of the two subtractions.




**f.** $(A_1' \cap A_2') \cup A_3$ = "awarded neither of #1 and #2, or awarded #3": from a Venn diagram,
$P((A_1' \cap A_2') \cup A_3) = P(\text{none awarded}) + P(A_3) = .47$ (from **d**) $+ .28 = 75$.




Alternatively, answers to **a-f** can be obtained from probabilities on the accompanying Venn diagram:



**17.**

**a.** Let $E$ be the event that at most one purchases an electric dryer. Then $E'$ is the event that at least two purchase electric dryers, and $P(E') = 1 - P(E) = 1 - .428 = .572$.

**b.** Let $A$ be the event that all five purchase gas, and let $B$ be the event that all five purchase electric. All other possible outcomes are those in which at least one of each type of clothes dryer is purchased. Thus, the desired probability is $1 - [P(A) - P(B)] = 1 - [.116 + .005] = .879$.

**19.**

**a.** The probabilities do not add to 1 because there are other software packages besides SPSS and SAS for which requests could be made.

**b.** $P(A') = 1 - P(A) = 1 - .30 = .70$.

**c.** Since $A$ and $B$ are mutually exclusive events, $P(A \cup B) = P(A) + P(B) = .30 + .50 = .80$.

**d.** By deMorgan's law, $P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B) = 1 - .80 = .20$.
In this example, deMorgan's law says the event "neither $A$ nor $B$" is the complement of the event "either $A$ or $B$." (That's true regardless of whether they're mutually exclusive.)

**21.** Let $A$ be that the selected joint was found defective by inspector $A$, so $P(A) = \frac{724}{10{,}000}$. Let $B$ be analogous for inspector $B$, so $P(B) = \frac{751}{10{,}000}$. The event "at least one of the inspectors judged a joint to be defective is $A \cup B$, so $P(A \cup B) = \frac{1159}{10{,}000}$.

**a.** By deMorgan's law, $P(\text{neither } A \text{ nor } B) = P(A' \cap B') = 1 - \mathrm{P(A \cup B)} = 1 - \frac{1159}{10{,}000} = \frac{8841}{10{,}000} = .8841$.

**b.** The desired event is $B \cap A'$. From a Venn diagram, we see that $P(B \cap A') = P(B) - P(A \cap B)$. From the addition rule, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ gives $P(A \cap B) = .0724 + .0751 - .1159 = .0316$. Finally, $P(B \cap A') = P(B) - P(A \cap B) = .0751 - .0316 = .0435$.

**23.** In what follows, the first letter refers to the auto deductible and the second letter refers to the homeowner's deductible.

**a.** $P(MH) = .10$.

**b.** $P(\text{low auto deductible}) = P(\{LN, LL, LM, LH\}) = .04 + .06 + .05 + .03 = .18$. Following a similar pattern, $P(\text{low homeowner's deductible}) = .06 + .10 + .03 = .19$.

**c.** $P(\text{same deductible for both}) = P(\{LL, MM, HH\}) = .06 + .20 + .15 = .41$.

**d.** $P(\text{deductibles are different}) = 1 - P(\text{same deductible for both}) = 1 - .41 = .59$.

**e.** $P(\text{at least one low deductible}) = P(\{LN, LL, LM, LH, ML, HL\}) = .04 + .06 + .05 + .03 + .10 + .03 = .31$.

**f.** $P(\text{neither deductible is low}) = 1 - P(\text{at least one low deductible}) = 1 - .31 = .69$.

**25.** Assume that the computers are numbered 1-6 as described and that computers 1 and 2 are the two laptops. There are 15 possible outcomes: (1,2) (1,3) (1,4) (1,5) (1,6) (2,3) (2,4) (2,5) (2,6) (3,4) (3,5) (3,6) (4,5) (4,6) and (5,6).

**a.** $P$(both are laptops) = $P$({(1,2)}) = $\frac{1}{15}$ =.067.

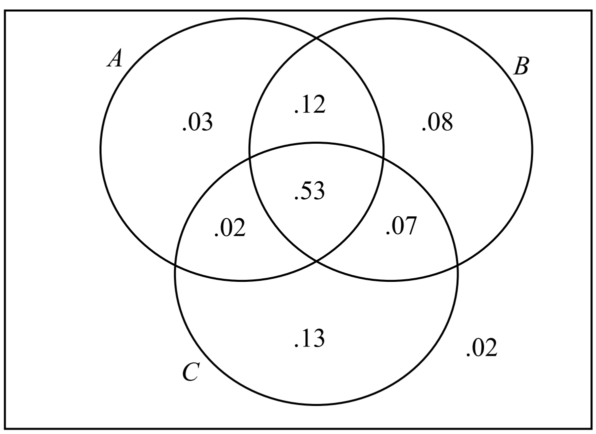
**b.** $P$(both are desktops) = $P$({(3,4) (3,5) (3,6) (4,5) (4,6) (5,6)}) = $\frac{6}{15}$ = .40.

**c.** $P$(at least one desktop) = 1 – $P$(no desktops) = 1 – $P$(both are laptops) = 1 – .067 = .933.

**d.** $P$(at least one of each type) = 1 – $P$(both are the same) =
1 – [$P$(both are laptops) + $P$(both are desktops)] = 1 – [.067 + .40] = .533.

**27.** By rearranging the addition rule, $P(A \cap B) = P(A) + P(B) - P(A \cup B) = .70 + .80 - .85 = .65$. By the same method, $P(A \cap C) = .70 + .75 - .90 = .55$ and $P(B \cap C) = .80 + .75 - .95 = .60$. Finally, rearranging the addition rule for 3 events gives
$P(A \cap B \cap C) = P(A \cup B \cup C) - P(A) - P(B) - P(C) + P(A \cap B) + P(A \cap C) + P(B \cap C) = .98 - .70 - .80 - .85 + .65 + .55 + .60 = .53$.

These probabilities are reflected in the Venn diagram below.




**a.** $P(A \cup B \cup C) = .98$, as given.

**b.** $P$(none selected) = 1 – $P$(at least one selected) = 1 – $P(A \cup B \cup C)$ = 1 – .98 = .02.

**c.** From the Venn diagram, $P$(only automatic transmission selected) = .03.

**d.** From the Venn diagram, $P$(exactly one of the three) = .03 + .08 + .13 = .24.

**29.** Recall there are 27 equally likely outcomes.

**a.** $P$(all the same station) = $P$((1,1,1) or (2,2,2) or (3,3,3)) = $\frac{3}{27} = \frac{1}{9}$.

**b.** $P$(at most 2 are assigned to the same station) = 1 – $P$(all 3 are the same) = 1 – $\frac{1}{9} = \frac{8}{9}$.

**c.** $P$(all different stations) = $P$((1,2,3) or (1,3,2) or (2,1,3) or (2,3,1) or (3,1,2) or (3,2,1)) = $\frac{6}{27} = \frac{2}{9}$.

## Section 2.3

**31.**

**a.** Since offices are distinct, order matters, and ${}_5P_2 = (5)(4) = 20$ (5 choices for president, 4 remain for vice president)

**b.** ${}_5P_3 = (5)(4)(3) = 60$

**c.** ${}_5C_2 = \binom{5}{2} = \frac{5!}{2!3!} = 10$ (No ordering is implied in the choice.)

**33.**

**a.** Use the Fundamental Counting Principle: (9)(27) = 243.

**b.** By the same reasoning, there are (9)(27)(15) = 3645 such sequences, so such a policy could be carried out for 3645 successive nights, or approximately 10 years, without repeating exactly the same program.

**35.** The first four songs must be non-Beatles song and the fifth a Beatles song. The total number of possible five-song sequences, assuming no repeats, is ${}_{100}P_5 = (100)(99)(98)(97)(96)$. The number of such sequences meeting our requirements (starting with 90 non-Beatles songs and 10 Beatles songs) is (90)(89)(88)(87)(10) or ${}_{90}P_4 \times 10$. The probability is ${}_{90}P_4 \times 10 / {}_{100}P_5 = .0679$.

**37.**

**a.** Since order doesn't matter, the number of possible rosters is $\binom{16}{6} = 8008$.

**b.** The number of ways to select 2 women from among 5 is $\binom{5}{2} = 10$, and the number of ways to select 4 men from among 11 is $\binom{11}{4} = 330$. By the Fundamental Counting Principle, the total number of (2-woman, 4-man) teams is (10)(330) = 3300.

**c.** Using the same idea as in part **b**, the count is $3300 + \binom{5}{3}\binom{11}{3} + \binom{5}{4}\binom{11}{2} + \binom{5}{5}\binom{11}{1} = 5236$.

**d.** $P$(exactly 2 women) = $\frac{3300}{8008} = .4121$; $P$(at least 2 women) = $\frac{5236}{8008} = .6538$.

**39.** There are $\binom{5}{2}$ = 10 possible ways to select the positions for *B*'s votes: *BBAAA, BABAA, BAABA, BAAAB, ABBAA, ABABA, ABAAB, AABBA, AABAB*, and *AAABB*. Only the last two have *A* ahead of *B* throughout the vote count. Since the outcomes are equally likely, the desired probability is 2/10 = .20.

**41.** For each of the 5 specific catalysts, there are (3)(4) = 12 pairings of temperature and pressure. Imagine we separate the 60 possible runs into those 5 sets of 12. The number of ways to select exactly one run from each of these 5 sets of 12 is $\binom{12}{1}^5 = 12^5$.

Since there are $\binom{60}{5}$ ways to select the 5 runs overall, the desired probability is $\dfrac{\binom{12}{1}^5}{\binom{60}{5}} = \dfrac{12^5}{\binom{60}{5}} = .0456$.

**43.**

**a.** We want to choose all of the 5 cordless, and 5 of the 10 others, to be among the first 10 serviced, so the desired probability is $\dfrac{\binom{5}{5}\binom{10}{5}}{\binom{15}{10}} = \dfrac{252}{3003} = .0839$.

**b.** Isolating one group, say the cordless phones, we want the other two groups (cellular and corded) represented in the last 5 serviced. The number of ways to choose all 5 cordless phones and 5 of the other phones in the first 10 selections is $\binom{5}{5}\binom{10}{5} = \binom{10}{5}$. However, we don't want two types to be eliminated in the first 10 selections, so we must subtract out the ways that either (all cordless and all cellular) or (all cordless and all corded) are selected among the first 10, which is $\binom{5}{5}\binom{5}{5} + \binom{5}{5}\binom{5}{5} = 2$. So, the number of ways to have only cellular and corded phones represented in the last five selections is $\binom{10}{5} - 2$. We have three types of phones, so the total number of ways to have exactly two types left over is $3 \cdot \left[\binom{10}{5} - 2\right]$, and the probability is $\dfrac{3 \cdot \left[\binom{10}{5} - 2\right]}{\binom{15}{5}} = \dfrac{3(250)}{3003} = .2498$.

**c.** We want to choose 2 of the 5 cordless, 2 of the 5 cellular, and 2 of the corded phones:
$\dfrac{\binom{5}{2}\binom{5}{2}\binom{5}{2}}{\binom{15}{6}} = \dfrac{1000}{5005} = .1998$.

**45.** Label the seats 1 2 3 4 5 6. The probability Jim and Paula sit in the two seats to the far left is $P(\text{J\&P in 1\&2}) = \dfrac{2\times1\times4\times3\times2\times1}{6\times5\times4\times3\times2\times1} = \dfrac{1}{15}$.

Similarly, $P$(J&P next to each other) = $P$(J&P in 1&2) + … + $P$(J&P in 5&6) = $5\times\dfrac{1}{15}=\dfrac{1}{3}$.

Third, $P$(at least one H next to his W) = 1 – $P$(no H next to his W), and we count the number of ways of no H sits next to his W as follows:

\# of orderings with a H-W pair in seats #1 and 3 and no H next to his W = $6^* \times 4 \times 1^* \times 2^{\#} \times 1 \times 1 = 48$

*= pair, # =can't put the mate of seat #2 here or else a H-W pair would be in #5 and 6

\# of orderings without a H-W pair in seats #1 and 3, and no H next to his W = $6 \times 4 \times 2^{\#} \times 2 \times 2 \times 1 = 192$

\#= can't be mate of person in seat #1 or #2

So, the number of seating arrangements with no H next to W = 48 + 192 = 240, and $P$(no H next to his W) = $= \dfrac{240}{6\times5\times4\times3\times2\times1} = \dfrac{1}{3}$. Therefore, $P$(at least one H next to his W) = $1 - \dfrac{1}{3} = \dfrac{2}{3}$.

**47.** $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!} = \dfrac{n!}{(n-k)!k!} = \dbinom{n}{n-k}$

The number of subsets of size $k$ equals the number of subsets of size $n - k$, because to each subset of size $k$ there corresponds exactly one subset of size $n - k$: the $n - k$ objects not in the subset of size $k$. The combinations formula counts the number of ways to split $n$ objects into two subsets: one of size $k$, and one of size $n - k$.

## Section 2.4

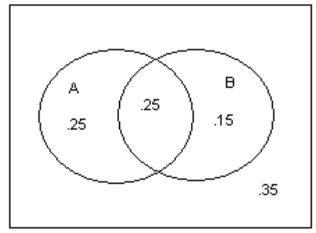
**49.**

**a.** $P(A)$ = .106 + .141 + .200 = .447, $P(C)$ =.215 + .200 + .065 + .020 = .500, and $P(A \cap C)$ = .200.

**b.** $P(A|C) = \dfrac{P(A\cap C)}{P(C)} = \dfrac{.200}{.500} = .400$. If we know that the individual came from ethnic group 3, the probability that he has Type A blood is .40. $P(C|A) = \dfrac{P(A\cap C)}{P(A)} = \dfrac{.200}{.447} = .447$. If a person has Type A blood, the probability that he is from ethnic group 3 is .447.

**c.** Define $D$ = "ethnic group 1 selected." We are asked for $P(D|B')$. From the table, $P(D\cap B')$ = .082 + .106 + .004 = .192 and $P(B')$ = 1 – $P(B)$ = 1 – [.008 + .018 + .065] = .909. So, the desired probability is $P(D|B') = \dfrac{P(D\cap B')}{P(B')} = \dfrac{.192}{.909} = .211$.

**51.** Refer to the Venn Diagram.




**a.** $P(B|A) = \dfrac{P(A\cap B)}{P(A)} = \dfrac{.25}{.50} = .50$ .

**b.** $P(B'|A) = \dfrac{P(A\cap B')}{P(A)} = \dfrac{.25}{.50} = .50$ .

**c.** $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{.25}{.40} = .625$ .

**d.** $P(A'|B) = \dfrac{P(A'\cap B)}{P(B)} = \dfrac{.15}{.40} = .375$ .

**e.** $P(A|A\cup B) = \dfrac{P(A\cap(A\cup B))}{P(A\cup B)} = \dfrac{P(A)}{P(A\cup B)} = \dfrac{.50}{.65} = .7692$ . It should be clear from the Venn diagram that $A\cap(A\cup B) = A$.

**53.** The box has four 40Ws, five 60W (so nine non-75W), and six 75W bulbs. Let $B$ = {at least one selected is 75 W}. Then $P(B) = 1 - P$(neither is 75W) $= 1 - \binom{9}{2}\binom{6}{0}/\binom{15}{2} = 1 - 36/105 = 23/35$. [You can also use the multiplication rule: $P$(neither is 75W) = (9/15)(8/14) = 12/35.] Let $A$ = {both are 75W}. Since $A$ is a subset of $B$, $P(A\cap B) = P(A) = \binom{9}{0}\binom{6}{2}/\binom{15}{2} = 15/105 = 1/7$. Then, by definition,

$$P(A\mid B) = \frac{P(A\cap B)}{P(B)} = \frac{1/7}{23/35} = \frac{5}{23} = .2174.$$

Next, Let $D$ = {at least one is not 75 W}. Notice that $D = A'$, so $P(D) = 1 - 1/7 = 6/7$. Finally, let $C$ = {both bulbs have the same rating}. The event $C\cap D$ is the event {two 40W or two 60W}, whose probability is

$P(C\cap D) = \binom{4}{2}/\binom{15}{2} + \binom{5}{2}/\binom{15}{2} = 16/105$. Thus $P(C\mid D) = \dfrac{16/105}{6/7} = \dfrac{8}{45} = .1778$.

**55.**

**a.** If a red ball is drawn from the first box, the composition of the second box becomes eight red and three green. Use the multiplication rule:

$P$(R from 1st $\cap$ R from 2nd ) = $P$(R from 1st ) × $P$(R from 2nd|R from 1st) = $\dfrac{6}{10}\times\dfrac{8}{11} = .436$ .

**b.** $P$(same numbers as originally) = $P$(both selected balls are the same color) = $P$(both R) + $P$(both G) = $\dfrac{6}{10}\times\dfrac{8}{11}+\dfrac{4}{10}\times\dfrac{4}{11} = .581$.

**57.** $P(B|A) = \dfrac{P(A\cap B)}{P(A)} = \dfrac{P(B)}{P(A)}$ (since $B$ is contained in $A$, $A\cap B = B$)

$$= \frac{.05}{.60} = .0833$$

**59.** Since the intersection is contained in the union, $[A \cup B \cup C] \cap [A \cap B \cap C] = A \cap B \cap C$.

**a.** $P(A \cap B \cap C \mid A \cup B \cup C) = \dfrac{P([A \cup B \cup C] \cap [A \cap B \cap C])}{P(A \cup B \cup C)} = \dfrac{P(A \cap B \cap C)}{P(A \cup B \cup C)} = \dfrac{.05}{.49} = .102.$

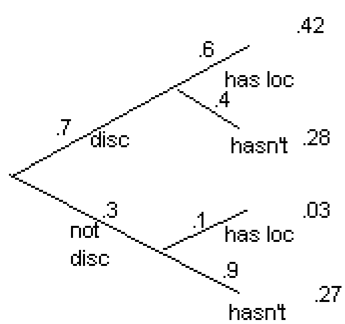
**b.** If she reads every section, then she automatically reads "at least one" section! The correct probability is 1. More formally, $P(A \cup B \cup C \mid A \cap B \cap C) = \dfrac{P(A \cap B \cap C)}{P(A \cap B \cap C)} = \dfrac{.05}{.05} = 1$.

**61.** $P(A \mid B) + P(A' \mid B) = \dfrac{P(A \cap B)}{P(B)} + \dfrac{P(A' \cap B)}{P(B)} = \dfrac{P(A \cap B) + P(A' \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

**63.** $P(A \cup B \mid C) = \dfrac{P[(A \cup B) \cap C)}{P(C)} = \dfrac{P[(A \cap C) \cup (B \cap C)]}{P(C)} = \dfrac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)} =$

$P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$

**65.** The tree diagram below shows the probability for the four disjoint options; e.g., *P*(the flight is discovered and has a locator) = *P*(discovered)*P*(locator | discovered) = (.7)(.6) = .42.




**a.** $P(\text{not discovered} \mid \text{has locator}) = \dfrac{P(\text{not discovered} \cap \text{has locator})}{P(\text{has locator})} = \dfrac{.03}{.03 + .42} = .067$.

**b.** $P(\text{discovered} \mid \text{no locator}) = \dfrac{P(\text{discovered} \cap \text{no locator})}{P(\text{no locator})} = \dfrac{.28}{.55} = .509$.

**67.** First, use the definition of conditional probability and the associative property of intersection:

$$P(A \cap B \mid C) = \frac{P((A \cap B) \cap C)}{P(C)} = \frac{P(A \cap (B \cap C))}{P(C)}$$

Second, use the Multiplication Rule to re-write the numerator:

$$\frac{P(A \cap (B \cap C))}{P(C)} = \frac{P(B \cap C)P(A \mid B \cap C)}{P(C)}$$

Finally, by definition, the ratio $\frac{P(B \cap C)}{P(C)}$ equals $P(B \mid C)$.

Substitution gives $P(A \cap B \mid C) = P(B \mid C) \cdot P(A \mid B \cap C)$, QED.

**69.** First, partition the sample space into statisticians with both life and major medical insurance, just life insurance, just major medical insurance, and neither. We know that *P*(both) = .20; subtracting them out, *P*(life only) = *P*(life) – *P*(both) = .75 – .20 = .55; similarly, *P*(medical only) = *P*(medical) – *P*(both) = .45 – .20 = .25.

**a.** Apply the Law of Total Probability:

*P*(renew) = *P*(life only)*P*(renew | life only) + *P*(medical only)*P*(renew | medical only) + *P*(both)*P*(renew | both)

= (.55)(.70) + (.25)(.80) + (.20)(.90) = .765.

**b.** Apply Bayes' Rule: *P*(both | renew) = $\frac{P(\text{both})P(\text{renew} \mid \text{both})}{P(\text{renew})} = \frac{(.20)(.90)}{.765}$ = .2353.

**71.** Let's see how we can implement the hint. If she's flying airline #1, the chance of 2 late flights is (30%)(10%) = 3%; the two flights being "unaffected" by each other means we can multiply their probabilities. Similarly, the chance of 0 late flights on airline #1 is (70%)(90%) = 63%. Since percents add to 100%, the chance of exactly 1 late flight on airline #1 is 100% – (3% + 63%) = 34%. A similar approach works for the other two airlines: the probability of exactly 1 late flight on airline #2 is 35%, and the chance of exactly 1 late flight on airline #3 is 45%.

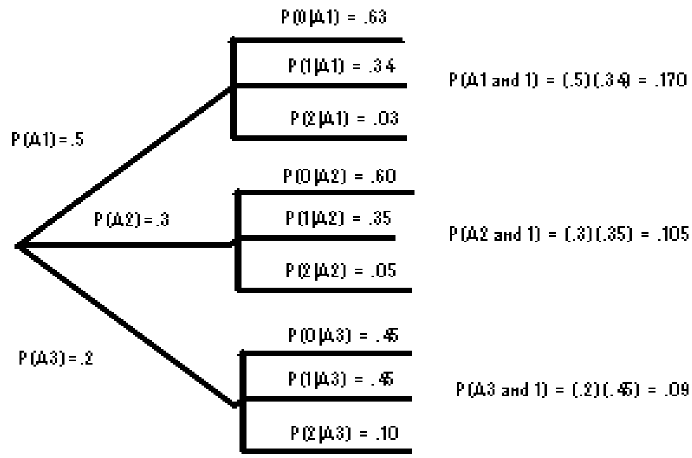
The initial ("prior") probabilities for the three airlines are $P(A_1)$ = 50%, $P(A_2)$ = 30%, and $P(A_3)$ = 20%. Given that she had exactly 1 late flight (call that event *B*), the conditional ("posterior") probabilities of the three airlines can be calculated using Bayes' Rule:

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)} = \frac{(.5)(.34)}{(.5)(.34) + (.3)(.35) + (.2)(.45)} = \frac{.170}{.365} = .4657;$$

$$P(A_2 \mid B) = \frac{P(A_2)P(B \mid A_2)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)} = \frac{(.3)(.35)}{.365} = .2877; \text{ and}$$

$$P(A_3 \mid B) = \frac{P(A_3)P(B \mid A_3)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)} = \frac{(.2)(.45)}{.365} = .2466.$$

Notice that, except for rounding error, these three posterior probabilities add to 1. The tree diagram below shows these probabilities.




**73.**

**a.** Since dad is *BB* and mom is *Bb*, their offspring II-2 will have genotype (*BB*) × (*Bb*) = *BB, Bb, BB,* or *Bb*, each with probability 1/4. Combining idential genotypes gives *BB* or *Bb*, with probability 1/2 each.

**b.** The likelihoods of hamster III1 having genotype *BB, Bb/bB,* or *bb* depends on the genotype of her mother (hamster II2). Applying the law of total probability, the fact that hamster II1 is *Bb*, and the rules of genetic recombination described in the problem,

$P(\text{III1} = BB) = P(\text{II2} = BB)P(\text{III1} = BB \mid \text{II2} = BB) + P(\text{II2} = Bb)P(\text{III1} = BB \mid \text{II2} = Bb)$
$= (1/2)(1/2) + (1/2)(1/4) = 3/8$

$P(\text{III1} = bb) = P(\text{II2} = BB)P(\text{III1} = bb \mid \text{II2} = BB) + P(\text{II2} = Bb)P(\text{III1} = bb \mid \text{II2} = Bb)$
$= (1/2)(0) + (1/2)(1/4) = 1/8$

And, thus, $P(\text{III1} = Bb/bB) = 1 - [3/8 + 1/8] = 1/2$. Finally, the conditional probability that III1is *Bb*, given that she has a black coat (i.e., is not *bb*), equals

$$P(\text{III1 is } Bb \mid \text{III1 is black}) = \frac{P(\text{III1} = Bb \cap \text{III1 is black})}{P(\text{III1 is black})} = \frac{P(\text{III1} = Bb)}{P(\text{III1 is black})} = \frac{1/2}{1-1/8} = \frac{4}{7}.$$

**c.** Apply Bayes' Rule:

$$P(\text{II2} = BB \mid \text{III1} = BB) = \frac{P(\text{II2} = BB)P(\text{III1} = BB \mid \text{II2} = BB)}{P(\text{III1} = BB)} = \frac{(1/2)(1/2)}{(3/8)} = \frac{2}{3}.$$

## Section 2.5

**75.**

**a.** Since the events are independent, then $A'$ and $B'$ are independent, too. (See the paragraph below Equation (2.7).) Thus, $P(B'|A') = P(B') = 1 - .7 = .3$.

**b.** Using the addition rule, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = .4 + .7 - (.4)(.7) = .82$. Since $A$ and $B$ are independent, we are permitted to write $P(A \cap B) = P(A)P(B) = (.4)(.7)$.

**c.** $$P(A \cap B' \mid A \cup B) = \frac{P((A \cap B') \cap (A \cup B))}{P(A \cup B)} = \frac{P(A \cap B')}{P(A \cup B)} = \frac{P(A)P(B')}{P(A \cup B)} = \frac{(.4)(1-.7)}{.82} = \frac{.12}{.82} = .146.$$

**77.** From a Venn diagram, $P(B) = P(A' \cap B) + P(A \cap B) = P(B) \Rightarrow P(A' \cap B) = P(B) - P(A \cap B)$.
If $A$ and $B$ are independent, then $P(A' \cap B) = P(B) - P(A)P(B) = [1 - P(A)]P(B) = P(A')P(B)$.
Thus, $A'$ and $B$ are independent.
Alternatively, $P(A' \mid B) = \dfrac{P(A' \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)} = \dfrac{P(B) - P(A)P(B)}{P(B)} = 1 - P(A) = P(A')$.

**79.** Let $E_i$ denote the event that an error was made in grading the $i$th question. We have $P(E_i) = .1$ for each $i$, and so $P(E_i') = .9$. Using independence, the probability that no errors are made is $P(E_1' \cap \cdots \cap E_{10}') = P(E_1') \cdots P(E_{10}') = (.9)\ldots(.9) = (.9)^{10} = .3487$. The probability that at least one error is made is the complementary probability: $P(\text{at least one error}) = P(E_1 \cup \ldots \cup E_{10}) = 1 - P(E_1' \cap \cdots \cap E_{10}') = 1 - .3487 = .6513$.
Replacing 10 with $n$ and .1 with $p$, the probability no errors are made is $(1 - p)^n$, and the probability that at least one error is made is $1 - (1 - p)^n$.

**81.** $P(\text{at least one opens}) = 1 - P(\text{none open}) = 1 - (.05)^5 = .99999969$.
$P(\text{at least one fails to open}) = 1 - P(\text{all open}) = 1 - (.95)^5 = .2262$.

**83.** Let $A_i$ denote the event that component #$i$ works ($i = 1, 2, 3, 4$). Based on the design of the system, the event "the system works" is $(A_1 \cup A_2) \cup (A_3 \cap A_4)$. We'll eventually need $P(A_1 \cup A_2)$, so work that out first: $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = (.9) + (.9) - (.9)(.9) = .99$. The third term uses independence of events. Also, $P(A_3 \cap A_4) = (.9)(.9) = .81$, again using independence.

Now use the addition rule and independence for the system:

$$\begin{aligned} P((A_1 \cup A_2) \cup (A_3 \cap A_4)) &= P(A_1 \cup A_2) + P(A_3 \cap A_4) - P((A_1 \cup A_2) \cap (A_3 \cap A_4)) \\ &= P(A_1 \cup A_2) + P(A_3 \cap A_4) - P(A_1 \cup A_2) \times P(A_3 \cap A_4) \\ &= (.99) + (.81) - (.99)(.81) = .9981 \end{aligned}$$

(You could also use deMorgan's law in a couple of places.)

**85.** $A = \{(3,1)(3,2)(3,3)(3,4)(3,5)(3,6)\} \Rightarrow P(A) = \frac{6}{36} = \frac{1}{6}$; $B = \{(1,4)(2,4)(3,4)(4,4)(5,4)(6,4)\} \Rightarrow P(B) = \frac{1}{6}$; and $C = \{(1,6)(2,5)(3,4)(4,3)(5,2)(6,1)\} \Rightarrow P(C) = \frac{1}{6}$.

**a.** $A \cap B = \{(3,4)\} \Rightarrow P(A \cap B) = \frac{1}{36} = P(A)P(B)$; $A \cap C = \{(3,4)\} \Rightarrow P(A \cap C) = \frac{1}{36} = P(A)P(C)$; and $B \cap C = \{(3,4)\} \Rightarrow P(B \cap C) = \frac{1}{36} = P(B)P(C)$. Therefore, these three events are pairwise independent.

**b.** However, $A \cap B \cap C = \{(3,4)\} \Rightarrow P(A \cap B \cap C) = \frac{1}{36}$, while $P(A)P(B)P(C) = \ = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$, so $P(A \cap B \cap C) \neq P(A)P(B)P(C)$ and these three events are not mutually independent.

**87.**

**a.** Let $D_1$ = detection on 1st fixation, $D_2$ = detection on 2nd fixation.
$P$(detection in at most 2 fixations) = $P(D_1) + P(D_1' \cap D_2)$ ; since the fixations are independent,
$P(D_1) + P(D_1' \cap D_2) = P(D_1) + P(D_1')\, P(D_2) = p + (1-p)p = p(2-p)$.

**b.** Define $D_1, D_2, \ldots, D_n$ as in **a**. Then $P$(at most $n$ fixations) =
$P(D_1) + P(D_1' \cap D_2) + P(D_1' \cap D_2' \cap D_3) + \ldots + P(D_1' \cap D_2' \cap \cdots \cap D_{n-1}' \cap D_n) =$
$p + (1-p)p + (1-p)^2 p + \ldots + (1-p)^{n-1}p = p[1 + (1-p) + (1-p)^2 + \ldots + (1-p)^{n-1}] =$
$p \cdot \dfrac{1-(1-p)^n}{1-(1-p)} = 1-(1-p)^n$ .
Alternatively, $P$(at most $n$ fixations) = 1 – $P$(at least $n$+1 fixations are required) =
1 – $P$(no detection in 1st n fixations) = $1 - P(D_1' \cap D_2' \cap \cdots \cap D_n') = 1 - (1-p)^n$.

**c.** $P$(no detection in 3 fixations) = $(1-p)^3$.

**d.** $P$(passes inspection) = $P(\{\text{not flawed}\} \cup \{\text{flawed and passes}\})$
= $P$(not flawed) + $P$(flawed and passes)
= .9 + $P$(flawed) $P$(passes | flawed) = $.9 + (.1)(1-p)^3$.

**e.** Borrowing from **d**, $P(\text{flawed} \mid \text{passed}) = \dfrac{P(\text{flawed} \cap \text{passed})}{P(\text{passed})} = \dfrac{.1(1-p)^3}{.9+.1(1-p)^3}$ . For $p$ = .5,

$P(\text{flawed} \mid \text{passed}) = \dfrac{.1(1-.5)^3}{.9+.1(1-.5)^3} = .0137$ .

**89.** Use the series and parallel computations illustrated previously. The probability the 1–2 subsystem works is .9 + .9 – (.9)(.9) = .99. The probability a series pair works is $(.9)^2 = .81$, so the probability that the 3–4–5–6 subsystem works is .81 + .81 – (.81)(.81) = .9639. Therefore,
$P$(system works) = $P$(1–2 works ∩ 3–4–5–6 works ∩ 7 works)
= $P$(1–2 works) · $P$(3–4–5–6 works) · $P$(7 works)
= (.99)(.9639)(.9) = .8588
The subsystem in Figure 2.15(a) works with probability .927. If it were connected in parallel to this subsystem, $P$(system works) = .8588 + .927 – (.8588)(.927) = .9897.

**91.** The question asks for $P$(<u>exactly</u> one tag lost | at <u>most</u> one tag lost) = $P((C_1 \cap C_2') \cup (C_1' \cap C_2) \mid (C_1 \cap C_2)')$ .
Since the first event is contained in (a subset of) the second event, this equals $\dfrac{P((C_1 \cap C_2') \cup (C_1' \cap C_2))}{P((C_1 \cap C_2)')} =$

$\dfrac{P(C_1 \cap C_2') + P(C_1' \cap C_2)}{1 - P(C_1 \cap C_2)} = \dfrac{P(C_1)P(C_2') + P(C_1')P(C_2)}{1 - P(C_1)P(C_2)}$ by independence =

$\dfrac{p(1-p) + (1-p)p}{1-p^2} = \dfrac{2p(1-p)}{1-p^2} = \dfrac{2p}{1+p}$ .

## Section 2.6

**93.**

**a.** Let $A$ = exactly one of $B_1$ or $B_2$ occurs = $(B_1 \cap B_2') \cup (B_2 \cap B_1')$. The R code below has been modified from Example 1.40 to count how often, out of 10,000 independent runs event $A$ occurs.

```
A<-0
for(i in 1:10000){
      u1<-runif(1); u2<-runif(1)
      if((u1<.6 && u2>=.7)||
         (u1>=.6 && u2<.7)){
            A<-A+1
      }
}
```

Executing the code returned `A=4588`, so $\hat{P}(A) = \frac{4588}{10{,}000} = .4588$.

The exact probability is $P(A) = P(B_1 \cap B_2') + P(B_2 \cap B_1') = P(B_1) - P(B_1 \cap B_2) + P(B_2) - P(B_1 \cap B_2) = P(B_1) + P(B_2) - 2P(B_1 \cap B_2) = P(B_1) + P(B_2) - 2P(B_1)P(B_2) = .6 + .7 - 2(.6)(.7) = .46$.

Note: The code `(u1<.6 && u2>=.7)||(u1>=.6 && u2<.7)` can be replaced by a single "exclusive or" command: `xor(u1<.6,u2<.7)`.

**b.** The estimated standard error of $\hat{P}(A)$ is $\sqrt{\frac{\hat{P}(A)[1-\hat{P}(A)]}{n}} = \sqrt{\frac{(.4588)(1-.4588)}{10{,}000}} \approx .00498$.

**95.** In the code below, seven random numbers are generated, one for each of the seven components. The sequence of and/or conjunctions matches the series and parallel ties in the system design.

```
A<-0
for(i in 1:10000){
      u<-runif(7)
      if((u[1]<.9 | u[2]<.9) &
        ((u[3]<.8 & u[4]<.8) |
        (u[5]<.8 & u[6]<.8)) &
        u[7]<.95){
            A<-A+1
      }
}
```

Executing the code gave `A=8159`, so $\hat{P}(A) = \frac{8159}{10{,}000} = .8159$.

**97.** The program below is written as a function, meaning it can receive inputs and generate outputs. The program take two inputs: $n$ = the number of games to be simulated and $p$ = the probability a contestant makes a correct guess. The program outputs the estimated probability $\hat{P}$ of winning the game Now or Then.

```
nowthen<-function(n,p){
win<-0
for(i in 1:n){
    u=runif(6);
    x=(u<p);
    if(x[1]+x[2]+x[3]==3 ||
       x[2]+x[3]+x[4]==3 ||
       x[3]+x[4]+x[5]==3 ||
       x[4]+x[5]+x[6]==3 ||
       x[5]+x[6]+x[1]==3 ||
       x[6]+x[1]+x[2]==3){
        win=win+1;
    }
}
return(win/n)
}
```

The above code is executed at the command line to create the function in R. After this, you may call this function at the command line.

(1) Typing `nowthen(10000,.5)` at the command line gave .3993.
(2) Typing `nowthen(10000,.8)` at the command line gave .8763.

**99.** Modify the program from the previous exercise, as illustrated below. Of interest is whether the difference between the largest and smallest entries of the vector `dollar` is at least 5.

```
A<-0
for(i in 1:10000){
    u<-runif(25)
    flips<-(u<.4)-(u>=.4)
    dollar<-cumsum(flips)
    if(max(dollar)-min(dollar)>=5){
        A<-A+1
    }
}
```

Executing the code above gave `A=9189`, so $\hat{P}$ = .9189.

**101.** Divide the 40 questions into the four types. For the first type (two choices), the probability of correctly guessing the right answer is 1/2. Similarly, the probability of correctly guessing a three-choice question correctly is 1/3, and so on. In the programs below, four vectors contain random numbers for the four types of questions; the binary vectors `(u<1/2)`, `(v<1/3)`, and so on code right and wrong guesses with 1s and 0s, respectively. Thus, `right` represents the total number of correct guesses out of 40. A student gets at least half of the questions correct if that total is at least 20.

```
A<-0
for(i in 1:10000){
    u<-runif(10); v<-runif(13)
    w<-runif(13); x<-runif(5)
    right<-sum(u<1/2)+sum(v<1/3)
             +sum(w<1/4)+sum(x<1/5)
    if(right>=20){
        A<-A+1
    }
}
```

Executing the code once gave `A=227`, so $\hat{P}$ = .0227.

**103.**

**a.** In the program below, `test` is the vector [1 2 3 … 12]. A random permutation is generated and then compared to `test`. If <u>any</u> of the 12 numbers are in the right place, `match` will equal 1; otherwise, `match` equals 0 and we have a derangement. The scalar `D` counts the number of derangements in 10,000 simulations.

```
D<-0
test<-1:12
for(i in 1:10000){
    permutation<-sample(test,12)
    match<-any(permutation==test)
    if(match==0){
        D<-D+1
    }
}
```

**b.** One execution of the code in **a** gave `D=3670`, so $\hat{P}(D)$ = .3670.

**c.** We know there are 12! possible permutations of the numbers 1 through 12. According to **b**, we estimate that 36.70% of them are derangements. This suggests that the estimated <u>number</u> of derangements of the numbers 1 through 12 is .3670(12!) = .3670(479,001,600) = 175,793,587.
(In fact, it is known that the exact number of such derangements is 176,214,841.)

**105.** The program below keeps a simultaneous record of whether the player wins the game and whether the game ends within 10 coin flips. These counts are stored in `win` and `ten`, respectively. The while loop insures that game play continues until the player has \$0 or \$100.

```
win<-0; ten<-0
for(i in 1:10000){
    money<-20; numflips<-0
    while(money>0 && money<100){
        numflips<-numflips+1
        change<-sample(c(-10,10),1)
        money<-money+change
    }
    if(money==100){
        win<-win+1
    }
    if(numflips<=10){
        ten<-ten+1
    }
}
```

**a.** One execution gave `win=2080`, so $\hat{P}$ (player wins) = .2080. (In fact, it can be shown using more sophisticated methods that the exact probability of winning in this scenario is .2, corresponding to the player starting with \$20 of a potential \$100 stake and the coin being fair.)

**b.** One execution gave `ten=5581`, so $\hat{P}$ (game ends within 10 coin flips) = .5581.

**107.**

**a.** Code appears below. One execution gave `A=5224`, so $\hat{P}$ (at least one ⚅ in four rolls) = .5224. Using independence, it can be shown that the exact probability is $1 – (5/6)^4 = .5177$.

```
A<-0
for(i in 1:10000){
    rolls<-sample(1:6,4,TRUE)
    numsixes<-sum(rolls==6)
    if(numsixes>=1){
        A<-A+1
    }
}
```

**b.** Code appears below. One execution gave `A=4935`, so $\hat{P}$ (at least one ⚅⚅ in 24 rolls) = .4935. Using independence, it can be shown that the exact probability is $1 – (35/36)^{24} = .4914$.

```
A<-0
for(i in 1:10000){
    die1<-sample(1:6,24,TRUE)
    die2<-sample(1:6,24,TRUE)
    numdblsixes<-
        sum((die1==6)&(die2==6))
    if(numdblsixes>=1){
        A<-A+1
    }
}
```

In particular, the probability in **a** is greater than 1/2, while the probability in **b** is less than 1/2. So, you should be willing to wager even money on seeing at least one ⚅ in 4 rolls of one die, but not on seeing at least one ⚅⚅ in 24 rolls of two dice.

**109.** Let 1 represent a vote for candidate A and –1 a vote for candidate B. A randomization of the 12 A's and 8 B's can be achieved by sampling <u>without</u> replacement from a vector [1 … 1 –1 … –1] with 12 1's and 8 –1's. To keep track of how far ahead candidate A stands as each vote is counted, employ the `cumsum` command. As long as A is ahead, the cumulative total will be positive; if A and B are ever tied, the cumulative sum will be 0; and a negative cumulative sum indicates that B has taken the lead. (Of course, the final cumulative sum will always be 4, signaling A's victory.)

```
A<-0
for(i in 1:10000){
    die1<-sample(1:6,24,TRUE)
    die2<-sample(1:6,24,TRUE)
    numdblsixes<-
        sum((die1==6)&(die2==6))
    if(numdblsixes>=1){
        A<-A+1
    }
}
```

One execution of the code above returned `A=2013`, so $\hat{P}$ (candidate A leads throughout the count) = .2013.

**111.**

**a.** In the code below, the criterion $x^2 + y^2 \leq 1$ determines whether $(x, y)$ lies in the unit quarter-disk.

```
A<-0
for(i in 1:10000){
    x=runif(1); y=runif(1)
    if(x^2+y^2<=1){
        A<-A+1
    }
}
```

**b.** Since $P(A) = \pi/4$, it follows that $\pi = 4P(A) \approx 4\hat{P}(A)$. One run of the above program returned `A=7837`, which implies that $\hat{P}(A) = .7837$ and $\pi \approx 4(.7837) = 3.1348$.

(While this may seem like a silly application, since we know how to determine π to arbitrarily many decimal places, the <u>idea</u> behind it is critical to lots of modern applications. The technique presented here is a special case of the method called *Monte Carlo integration*.)

## Supplementary Exercises

**113.**

**a.** $\binom{24}{4} = 10{,}626$.

**b.** Order matters here (being selected as VP isn't the same as Treasurer): ${}_{24}P_4 = (24)(23)(22)(21) = 255{,}024$.

**c.** There are $\binom{24}{2} = 276$ ways to choose co-chairs, then 22 choices for secretary and 21 for treasurer. Apply the Fundamental Counting Principle: (276)(22)(21) = 127,512 (exactly half of **b**).

**115.**

**a.** $P(\text{line 1}) = \dfrac{500}{1500} = .333$;

$P(\text{crack}) = \dfrac{.50(500) + .44(400) + .40(600)}{1500} = \dfrac{666}{1500} = .444$.

**b.** This is one of the percentages provided: $P(\text{blemish} \mid \text{line 1}) = .15$.

**c.** $P(\text{surface defect}) = \dfrac{.10(500) + .08(400) + .15(600)}{1500} = \dfrac{172}{1500}$;

$P(\text{line 1} \cap \text{surface defect}) = \dfrac{.10(500)}{1500} = \dfrac{50}{1500}$;

so, $P(\text{line 1} \mid \text{surface defect}) = \dfrac{50/1500}{172/1500} = \dfrac{50}{172} = .291$.

**117.** Apply the addition rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow .626 = P(A) + P(B) - .144$. Apply independence: $P(A \cap B) = P(A)P(B) = .144$.
So, $P(A) + P(B) = .770$ and $P(A)P(B) = .144$.
Let $x = P(A)$ and $y = P(B)$. Using the first equation, $y = .77 - x$, and substituting this into the second equation yields $x(.77 - x) = .144$ or $x^2 - .77x + .144 = 0$. Use the quadratic formula to solve:
$x = \dfrac{.77 \pm \sqrt{(-.77)^2 - (4)(1)(.144)}}{2(1)} = \dfrac{.77 \pm .13}{2} = .32$ or .45. Since $x = P(A)$ is assumed to be the larger probability, $x = P(A) = .45$ and $y = P(B) = .32$.

**119.**

**a.** There are 5! = 120 possible orderings, so $P(\text{BCDEF}) = \frac{1}{120} = .0833$.

**b.** The number of orderings in which F is third equals 4×3×1*×2×1 = 24 (*because F must be here), so $P(\text{F is third}) = \frac{24}{120} = .2$. Or more simply, since the five friends are ordered completely at random, there is a 1-in-5 chance F is specifically in position three.

**c.** Similarly, $P(\text{F last}) = \frac{4\times3\times2\times1\times1}{120} = .2$.

**121.** When three experiments are performed, there are 3 different ways in which detection can occur on exactly 2 of the experiments: (i) #1 and #2 and not #3; (ii) #1 and not #2 and #3; and (iii) not #1 and #2 and #3. If the impurity is present, the probability of exactly 2 detections in three (independent) experiments is (.8)(.8)(.2) + (.8)(.2)(.8) + (.2)(.8)(.8) = .384. If the impurity is absent, the analogous probability is 3(.1)(.1)(.9) = .027. Thus, applying Bayes' theorem, *P*(impurity is present | detected in exactly 2 out of 3) =
$\dfrac{P(\text{detected in exactly } 2 \cap \text{present})}{P(\text{detected in exactly } 2)} = \dfrac{(.384)(.4)}{(.384)(.4)+(.027)(.6)} = .905.$

**123.**

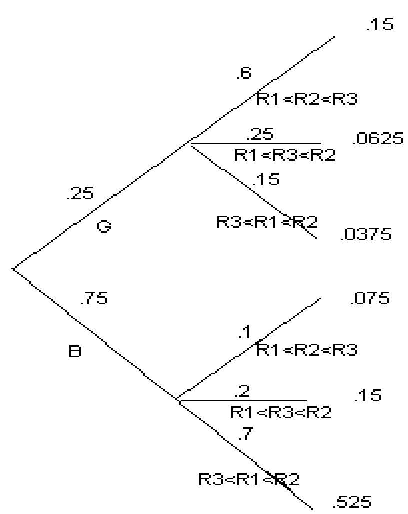
**a.** First, *P*(both +) = *P*(carrier $\cap$ both +) + *P*(not a carrier $\cap$ both +) =
*P*(carrier)*P*(both + | carrier) + *P*(not a carrier)*P*(both + | not a carrier). Assuming independence of the tests, this equals $(.01)(.90)^2 + (.99)(.05)^2 = .010575$.
Similarly, *P*(both –) = $(.01)(.10)^2 + (.99)(.95)^2 = .893575$.
Therefore, *P*(tests agree) = .010575 + .893575 = .90415.

**b.** From the first part of **a**, *P*(carrier | both +) = $\dfrac{P(\text{carrier} \cap \text{both } +)}{P(\text{both } +)} = \dfrac{(.01)(.90)^2}{.010575} = .766.$

**125.** $P(E_1 \cap L) = P(E_1)P(L \mid E_1) = (.40)(.02) = .008.$

**127.** Let *B* denote the event that a component needs rework. By the law of total probability,
$P(B) = \sum P(A_i)P(B \mid A_i) = (.50)(.05) + (.30)(.08) + (.20)(.10) = .069.$
Thus, $P(A_1 \mid B) = \dfrac{(.50)(.05)}{.069} = .362$, $P(A_2 \mid B) = \dfrac{(.30)(.08)}{.069} = .348$, and $P(A_3 \mid B) = .290$.

**129.** See the accompanying tree diagram.




**a.** $P(G \mid R_1 < R_2 < R_3) = \dfrac{.15}{.15+.075} = .67$ while $P(B \mid R_1 < R_2 < R_3) = .33$, so classify the specimen as granite. Equivalently, $P(G \mid R_1 < R_2 < R_3) = .67 > ½$ so granite is more likely.

**b.** $P(G \mid R_1 < R_3 < R_2) = \dfrac{.0625}{.2125} = .2941 < ½$, so classify the specimen as basalt.

$P(G \mid R_3 < R_1 < R_2) = \dfrac{.0375}{.5625} = .0667 < ½$, so classify the specimen as basalt.

**c.** $P$(erroneous classification) = $P$($B$ classified as G) + $P$(G classified as $B$) =
$P(B)P$(classified as G | $B$) + $P(G)P$(classified as $B$ | $G$) =
$(.75)P(R_1 < R_2 < R_3 \mid B) + (.25)P(R_1 < R_3 < R_2 \text{ or } R_3 < R_1 < R_2 \mid G)$ =
(.75)(.10) + (.25)(.25 + .15) = .175.

**d.** For what values of $p$ will $P(G \mid R_1 < R_2 < R_3)$, $P(G \mid R_1 < R_3 < R_2)$, and $P(G \mid R_3 < R_1 < R_2)$ all exceed ½? Replacing .25 and .75 with $p$ and $1 - p$ in the tree diagram,

$P(G \mid R_1 < R_2 < R_3) = \dfrac{.6p}{.6p + .1(1-p)} = \dfrac{.6p}{.1 + .5p} > .5 \text{ iff } p > \dfrac{1}{7}$;

$P(G \mid R_1 < R_3 < R_2) = \dfrac{.25p}{.25p + .2(1-p)} > .5 \text{ iff } p > \dfrac{4}{9}$;

$P(G \mid R_3 < R_1 < R_2) = \dfrac{.15p}{.15p + .7(1-p)} > .5 \text{ iff } p > \dfrac{14}{17}$ (most restrictive). Therefore, one would always classify a rock as granite iff $p > \dfrac{14}{17}$.

**131.**

**a.** There are 4! = 24 possible ways the calculators could be randomly allocated back to the four friends. Since only one of those 24 possibilities results in everyone getting her own calculator back, the chance this randomly occurs is $\dfrac{1}{24}$.

**b.** Our goal is to find $P(A \cup B \cup C \cup D)$. We'll need all of the following probabilities:

$P(A)$ = $P$(Allison gets her calculator back) = 1/4. This is intuitively obvious; you can also see it by writing out the 24 orderings in which the friends could get calculators (ABCD, ABDC, …, DCBA) and observe that 6 of the 24 have A in the first position. So, $P(A)$ = 6/24 = 1/4. By the same reasoning, $P(B) = P(C) = P(D)$ = 1/4.

$P(A \cap B)$ = $P$(Allison and Beth get their calculators back) = 1/12. This can be computed by considering all 24 orderings and noticing that two — ABCD and ABDC — have A and B in the correct positions. Or, you can use the multiplication rule: $P(A \cap B) = P(A)P(B \mid A)$ = (1/4)(1/3) = 1/12. All other pairwise intersection probabilities are also 1/12.

$P(A \cap B \cap C)$ = $P$(Allison and Beth and Carol get their calculators back) = 1/24, since this can only occur if all four friends get their own calculators back. So, all three-wise intersections have probability 1/24, as does $P(A \cap B \cap C \cap D)$, which was part **a**.

Finally, put all the parts together, using a general inclusion-exclusion rule for unions:

$$
\begin{aligned}
P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D) \\
&- P(A \cap B) - P(A \cap C) - \cdots - P(C \cap D) \\
&+ P(A \cap B \cap C) + \cdots + P(B \cap C \cap D) \\
&- P(A \cap B \cap C \cap D) \\
&= 4 \cdot \frac{1}{4} - 6 \cdot \frac{1}{12} + 4 \cdot \frac{1}{24} - \frac{1}{24} \\
&= 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} = \frac{15}{24} = .625
\end{aligned}
$$

**c.** The final answer in **b** has the form $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}$. Generalizing to $n$ friends, the probability at least one will get her own calculator back is $\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^{n-1}\frac{1}{n!}$.

When $n$ is large, we can relate this to the power series for $e^x$ evaluated at $x = -1$:

$$
\begin{aligned}
e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \Rightarrow \\
e^{-1} &= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots = 1 - \left[\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots\right] \Rightarrow \\
1 - e^{-1} &= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots
\end{aligned}
$$

So, for large $n$, $P$(at least one friend gets her own calculator back) $\approx 1 - e^{-1} = .632$. Contrary to intuition, the chance of this event does not converge to 1 (because "someone is bound to get hers back") or to 0 (because "there are just too many possible arrangements"). Rather, in a large group, there's about a 63.2% chance someone will get her own item back (a match), and about a 36.8% chance that nobody will get her own item back (no match).

**133.** Note: $s = 0$ means that the very first candidate interviewed is hired. Each entry below is the candidate hired for the given policy and outcome.

| **Outcome** | $s = 0$ | $s = 1$ | $s = 2$ | $s = 3$ | **Outcome** | $s = 0$ | $s = 1$ | $s = 2$ | $s = 3$ |
|---|---|---|---|---|---|---|---|---|---|
| 1234 | 1 | 4 | 4 | 4 | 3124 | 3 | 1 | 4 | 4 |
| 1243 | 1 | 3 | 3 | 3 | 3142 | 3 | 1 | 4 | 2 |
| 1324 | 1 | 4 | 4 | 4 | 3214 | 3 | 2 | 1 | 4 |
| 1342 | 1 | 2 | 2 | 2 | 3241 | 3 | 2 | 1 | 1 |
| 1423 | 1 | 3 | 3 | 3 | 3412 | 3 | 1 | 1 | 2 |
| 1432 | 1 | 2 | 2 | 2 | 3421 | 3 | 2 | 2 | 1 |
| 2134 | 2 | 1 | 4 | 4 | 4123 | 4 | 1 | 3 | 3 |
| 2143 | 2 | 1 | 3 | 3 | 4132 | 4 | 1 | 2 | 2 |
| 2314 | 2 | 1 | 1 | 4 | 4213 | 4 | 2 | 1 | 3 |
| 2341 | 2 | 1 | 1 | 1 | 4231 | 4 | 2 | 1 | 1 |
| 2413 | 2 | 1 | 1 | 3 | 4312 | 4 | 3 | 1 | 2 |
| 2431 | 2 | 1 | 1 | 1 | 4321 | 4 | 3 | 2 | 1 |

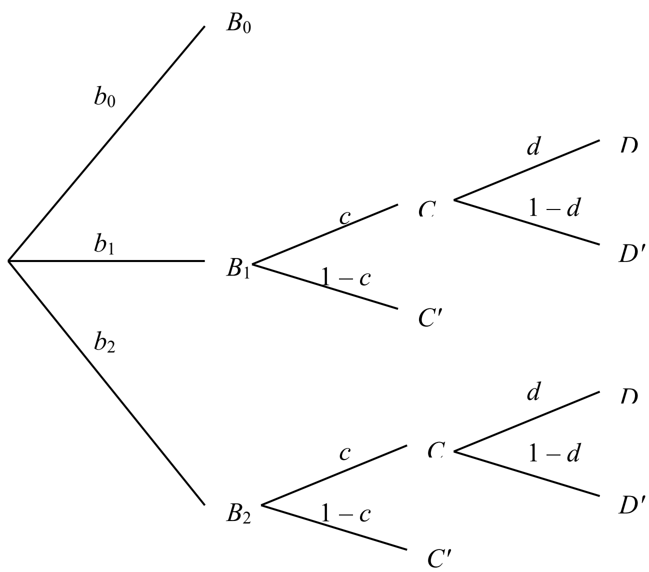
From the table, we derive the following probability distribution based on $s$:

| $s$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P$(hire #1) | $\frac{6}{24}$ | $\frac{11}{24}$ | $\frac{10}{24}$ | $\frac{6}{24}$ |

Therefore $s = 1$ is the best policy.

**135.** $P(A_1) = P(\text{draw slip 1 or 4}) = ½$; $P(A_2) = P(\text{draw slip 2 or 4}) = ½$;
$P(A_3) = P(\text{draw slip 3 or 4}) = ½$; $P(A_1 \cap A_2) = P(\text{draw slip 4}) = ¼$;
$P(A_2 \cap A_3) = P(\text{draw slip 4}) = ¼$; $P(A_1 \cap A_3) = P(\text{draw slip 4}) = ¼$.
Hence, $P(A_1 \cap A_2) = ¼ = P(A_1)P(A_2)$, $P(A_2 \cap A_3) = ¼ = P(A_2)P(A_3)$, and
$P(A_1 \cap A_3) = ¼ = P(A_1)P(A_3)$, thus there exists pairwise independence. However,
$P(A_1 \cap A_2 \cap A_3) = P(\text{draw slip 4}) = ¼ \neq ⅛ = P(A_1)P(A_2)P(A_3)$, so the events are not mutually independent.

**137.** A tree diagram for this problem is given below.




**a.** Using the probabilities provided,

$$P(B_0 \mid D') = \frac{P(B_0 \cap D')}{P(D')} = \frac{b_0(1)(1)}{1 - P(D)} = \frac{b_0}{1 - [b_1cd + b_2cd]} = \frac{b_0}{1 - (b_1 + b_2)cd}$$

Similarly, $P(B_i \mid D') = \dfrac{b_i c(1-d) + b_i(1-c)(1)}{P(D')} = \dfrac{b_i(1-cd)}{1-(b_1+b_2)cd}$ for $i = 1,2$. It's straightforward to show these sum to 1, using $b_0 + b_1 + b_2 = 1$.

**b.** With the numbers provided, $P(B_0 \mid D') = .7117$, $P(B_1 \mid D') = .0577$, $P(B_2 \mid D') = .2306$.

**139.**

**a.** $A$ attracts $B \Rightarrow P(B \mid A) > P(B) \Rightarrow 1 - P(B \mid A) < 1 - P(B)$, because multiplying by –1 reverses the direction of the inequality $\Rightarrow P(B' \mid A) < P(B') \Rightarrow$ by definition, $A$ repels $B'$. In other words, if the occurrence of $A$ makes $B$ more likely, then it must make $B'$ less likely. Notice this is really an iff statement; i.e., all of the implication arrows can be reversed.

**b.** This one is much trickier, since the complementation idea in **a** can't be applied here (i.e., to the conditional event $A$). One approach is as follows, which uses the fact that $P(B) - P(B \cap A) = P(B \cap A')$:

$A$ attracts $B \Rightarrow P(B \mid A) > P(B) \Rightarrow \dfrac{P(A \cap B)}{P(A)} > P(B) \Rightarrow P(A \cap B) > P(A)P(B) \Rightarrow$

$P(B) - P(A \cap B) < P(B) - P(A)P(B)$ because multiplying by –1 is order-reversing $\Rightarrow$

$P(B \cap A') < P(B)[1 - P(A)] = P(B)P(A') \Rightarrow \dfrac{P(B \cap A')}{P(A')} < P(B) \Rightarrow P(B \mid A') < P(B) \Rightarrow$

by definition, $A'$ repels $B$. (Whew!) Notice again this is really an iff statement.

**c.** Apply the simplest version of Bayes' rule:

$A$ attracts $B \Leftrightarrow P(B \mid A) > P(B) \Leftrightarrow \dfrac{P(B)P(A \mid B)}{P(A)} > P(B) \Leftrightarrow \dfrac{P(A \mid B)}{P(A)} > 1 \Leftrightarrow P(A \mid B) > P(A) \Leftrightarrow$

by definition, $B$ attracts $A$.