CHAPTER 4

Section 4.1

1.

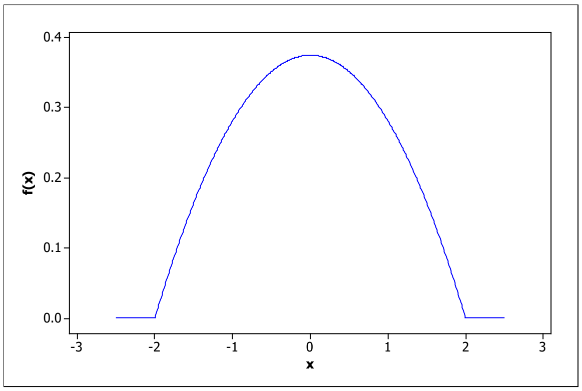
a. $P(X \le 1) = \int_{-\infty}^{1} f(x)dx = \int_0^1 \frac{1}{2}xdx = \frac{1}{4}x^2\Big]_0^1 = .25$.

b. $P(.5 \le X \le 1.5) = \int_{.5}^{1.5} \frac{1}{2}xdx = \frac{1}{4}x^2\Big]_{.5}^{1.5} = .5$.

c. $P(X > 1.5) = \int_{1.5}^{\infty} f(x)dx = \int_{1.5}^{2} \frac{1}{2}xdx = \frac{1}{4}x^2\Big]_{1.5}^{2} = \frac{7}{16} \approx .438$.

3.

a.



b. $P(X > 0) = \int_0^2 .09375(4 - x^2)dx = .09375\left(4x - \frac{x^3}{3}\right)\Big]_0^2 = .5$.

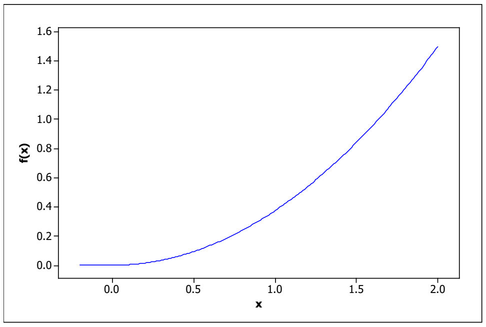
This matches the symmetry of the pdf about $x = 0$.

c. $P(-1 < X < 1) = \int_{-1}^{1} .09375(4 - x^2)dx = .6875$.

d. $P(X < -.5 \text{ or } X > .5) = 1 - P(-.5 \le X \le .5) = 1 - \int_{-.5}^{.5} .09375(4 - x^2)dx = 1 - .3672 = .6328$.

5.

a. $1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^2 kx^2 dx = \left.\frac{kx^3}{3}\right]_0^2 = \frac{8k}{3} \Rightarrow k = \frac{3}{8}$.



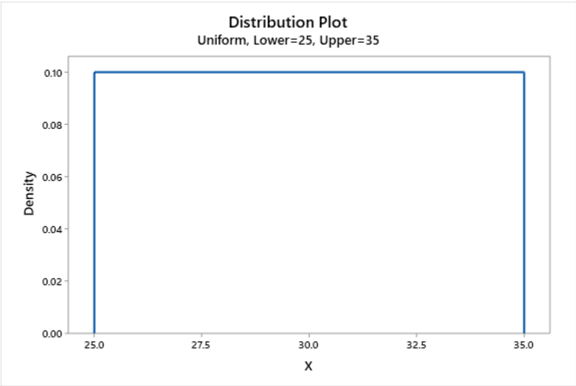
b. $P(0 \le X \le 1) = \int_0^1 \frac{3}{8}x^2 dx = \left.\frac{1}{8}x^3\right]_0^1 = \frac{1}{8} = .125$.

c. $P(1 \le X \le 1.5) = \int_1^{1.5} \frac{3}{8}x^2 dx = \left.\frac{1}{8}x^3\right]_1^{1.5} = \frac{1}{8}\left(\frac{3}{2}\right)^3 - \frac{1}{8}(1)^3 = \frac{19}{64} = .296875$.

d. $P(X \ge 1.5) = \int_{1.5}^{2} \frac{3}{8}x^2 dx = \left.\frac{1}{8}x^3\right]_{1.5}^{2} = \frac{1}{8}(2)^3 - \frac{1}{8}(1.5)^3 = .578125$.

7.

a. $f(x) = \frac{1}{B-A} = \frac{1}{35-25} = \frac{1}{10} = .1$ for $25 \le x \le 35$ and $f(x) = 0$ otherwise.



b. $P(X > 33) = \int_{33}^{35} \frac{1}{10} dx = \frac{35-33}{10} = .2$.

c. The mean is clearly the midpoint of 30 min. $30 \pm 2 = 28$ to 32: $P(28 < X < 32) = \int_{28}^{32} \frac{1}{10} dx = .4$.

d. $P(a \le X \le a+2) = \int_a^{a+2} \frac{1}{10} dx = \frac{(a+2)-a}{10} = \frac{2}{10} = .2$.

9.

a. $P(X \le 40) = \int_{10}^{40} .04e^{-.04(x-10)}dx = .04\int_{0}^{30} e^{-.04u}du$ (after the substitution $u = x - 10$)

$= -e^{-.04u}\Big]_0^{30} = 1 - e^{-.04(30)} \approx .699$.

b. $P(X > 40) = 1 - P(X \le 40) = 1 - .699 = .301$. Since X is continuous, $P(X \ge 40) = P(X > 40) = .301$ as well.

c. $P(40 \le X \le 60) = \int_{40}^{60} .04e^{-.04(x-10)}\,dx = .04\int_{30}^{50} e^{-.04u}\,du = -e^{-.04u}\Big|_{30}^{50} = .166$.

11.

a. $P(X \le 1) = F(1) = \dfrac{1^2}{4} = .25$.

b. $P(.5 \le X \le 1) = F(1) - F(.5) = \dfrac{1^2}{4} - \dfrac{.5^2}{4} = .1875$.

c. $P(X > 1.5) = 1 - P(X \le 1.5) = 1 - F(1.5) = 1 - \dfrac{1.5^2}{4} = .4375$.

d. $.5 = F(\tilde{\mu}) = \dfrac{\tilde{\mu}^2}{4} \Rightarrow \tilde{\mu}^2 = 2 \Rightarrow \tilde{\mu} = \sqrt{2} \approx 1.414$ hours.

e. $f(x) = F'(x) = \dfrac{x}{2}$ for $0 \le x < 2$, and $= 0$ otherwise.

13.

a. $1 = \int_1^\infty \dfrac{k}{x^4}dx = k\int_1^\infty x^{-4}dx = \dfrac{k}{-3}x^{-3}\Big]_1^\infty = 0 - \left(\dfrac{k}{-3}\right)(1)^{-3} = \dfrac{k}{3} \Rightarrow k = 3$.

b. For $x \ge 1$, $F(x) = \int_{-\infty}^{x} f(y)dy = \int_1^x \dfrac{3}{y^4}dy = -y^{-3}\Big|_1^x = -x^{-3} + 1 = 1 - \dfrac{1}{x^3}$. For $x < 1$, $F(x) = 0$ since the distribution begins at 1. Put together, $F(x) = \begin{cases} 0 & x < 1 \\ 1 - \dfrac{1}{x^3} & 1 \le x \end{cases}$.

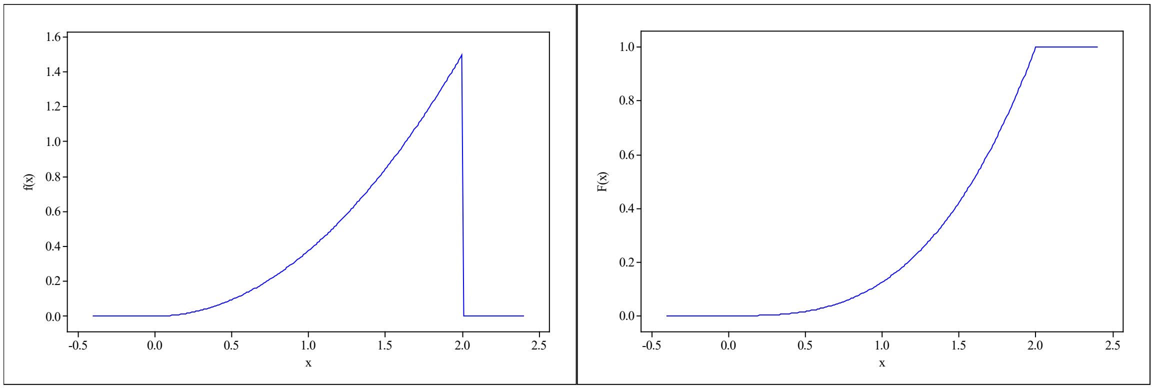
c. $P(X > 2) = 1 - F(2) = 1 - \frac{7}{8} = \frac{1}{8}$ or .125;

$P(2 < X < 3) = F(3) - F(2) = \left(1 - \frac{1}{27}\right) - \left(1 - \frac{1}{8}\right) = .963 - .875 = .088$.

15.

a. Since X is restricted to the interval [0, 2], $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 2$.

For $0 \le x \le 2$, $F(x) = \int_0^x \frac{3}{8} y^2 dy = \frac{1}{8} y^3 \Big]_0^x = \frac{x^3}{8}$. Both graphs appear below.



b. $P(X \le .5) = F(.5) = \frac{(.5)^3}{8} = \frac{1}{64} = .015625.$

c. $P(.25 < X \le .5) = F(.5) - F(.25) = .015625 - .001953125 = .0137.$
Since X is continuous, $P(.25 \le X \le .5) = P(.25 < X \le .5) = .0137.$

d. The 75th percentile is the value of x for which $F(x) = .75$: $\frac{x^3}{8} = .75 \Rightarrow x = 1.817$.

17.

a. Let $F(x)$ denote the cdf of X, so that $F(\eta) = .5$. Also, let η_Y denote the median of Y, so that by definition $P(Y \le \eta_Y) = .5$. Substitute $Y = 1.8X + 32$ into the previous expression:

$$.5 = P(Y \le \eta_Y) = P(1.8X + 32 \le \eta_Y) = P\left(X \le \frac{\eta_Y - 32}{1.8}\right) = F\left(\frac{\eta_Y - 32}{1.8}\right).$$

Looking at the two ends of this expression implies that $\frac{\eta_Y - 32}{1.8}$ is the median of X; i.e., $\eta = \frac{\eta_Y - 32}{1.8}$.

Finally, solving for η_Y gives $\eta_Y = 1.8\eta + 32$.

b. Replace .5 with .9 above and one obtains $\eta_{.9,Y} = 1.8\eta_{.9,X} + 32$.

c. If $Y = aX + b$ with $a > 0$, then the $(100p)$th percentiles of X and Y are related by $\eta_{p,Y} = a\eta_{p,X} + b$.
For $a < 0$, the relationship is slightly more complicated because of the sign reversal:

$$p = P(Y \le \eta_{p,Y}) = P(aX + b \le \eta_{p,Y}) = P(aX \le \eta_{p,Y} - b) = P\left(X \ge \frac{\eta_{p,Y} - b}{a}\right) = 1 - F\left(\frac{\eta_{p,Y} - b}{a}\right) \Rightarrow$$

$F\left(\frac{\eta_{p,Y} - b}{a}\right) = 1 - p \Rightarrow \frac{\eta_{p,Y} - b}{a} = \eta_{1-p,X} \Rightarrow \eta_{p,Y} = a\eta_{1-p,X} + b$. So, for example, the 90th percentile of Y (when $a < 0$) is the linear rescaling of the 10th percentile of X.

Section 4.2

19.

a. $E(X)=\int_{10}^{\infty} x\cdot .04e^{-.04(x-10)}\,dx = .04e^{(-.04)(-10)}\int_{10}^{\infty} x\cdot e^{-.04x}\,dx$. Let $u = x$ and $dv = e^{-.04x}\,dx$, so $du = dx$ and $v = -\frac{1}{.04}e^{-.04x}$. Applying integration by parts, $E(X)=\frac{-.04e^{4}}{.04}xe^{-.04x}\Big|_{10}^{\infty} - .04e^{4}\int_{10}^{\infty}-\frac{1}{.04}e^{-.04x}\,dx$ = $0-(-10)+e^{4}\int_{10}^{\infty}e^{-.04x}\,dx = 10-\frac{e^{.4}}{.04}e^{-.04x}\Big|_{10}^{\infty}$ = 10 – (0 – 25) = 10 + 25 = 35 m³/s.

Similarly, integration by parts twice(!) yields $E(X^2)=1850$, so $V(X)$ = 1850 – 35² = 625 and $\sigma=\sqrt{625}$ = 25 m³/s.

b. $P(\mu-\sigma \le X \le \mu+\sigma) = P(10 \le X \le 60) = \int_{10}^{60} .04e^{-.04(x-10)}\,dx = 1-e^{-2} = .865$.

21.

a. $E(X) = \int_{-\infty}^{\infty} x\cdot f(x)dx = \int_0^1 x\cdot 90x^8(1-x)dx = \int_0^1 (90x^9-90x^{10})dx = 9x^{10}-\frac{90}{11}x^{11}\Big]_0^1 = \frac{9}{11} \approx .8182$ ft³.

Similarly, $E(X^2) = \int_{-\infty}^{\infty} x^2\cdot f(x)dx = \int_0^1 x^2\cdot 90x^8(1-x)dx = \ldots = .6818$, from which $V(X) = .6818 - (.8182)^2 = .0124$ and SD(X) = .11134 ft³.

b. $\mu \pm \sigma$ = (.7068, .9295). Thus, $P(\mu - \sigma \le X \le \mu + \sigma) = F(.9295) - F(.7068)$ = .8465 – .1602 = .6863, and the probability X is <u>more</u> than 1 standard deviation from its mean value equals 1 – .6863 = .3137.

23.

a. To find the (100p)th percentile, set $F(\eta_p) = p$ and solve for η_p: $\frac{\eta_p - A}{B-A} = p \Rightarrow \eta_p = A + (B - A)p$.

b. Set p = .5 to obtain $\tilde{\mu} = A + (B - A)(.5) = .5B + .5A = \frac{A+B}{2}$. This is exactly the same as the mean of X, which is no surprise: since the uniform distribution is symmetric about $\frac{A+B}{2}$, $\mu = \tilde{\mu} = \frac{A+B}{2}$.

c. $E(X^n)=\int_A^B x^n\cdot\frac{1}{B-A}dx = \frac{1}{B-A}\frac{x^{n+1}}{n+1}\Big]_A^B = \frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)}$.

25. $E(\text{area}) = E(\pi R^2) = \int_{-\infty}^{\infty}\pi r^2 f(r)dr = \int_9^{11}\pi r^2\frac{3}{4}\left(1-(10-r)^2\right)dr = \cdots = \frac{501}{5}\pi = 314.79$ m².

27. With X = temperature in °C, the temperature in °F equals $1.8X + 32$, so the mean and standard deviation in °F are $1.8\mu_X + 32 = 1.8(120) + 32 = 248$°F and $|1.8|\sigma_X = 1.8(2) = 3.6$°F. Notice that the additive constant, 32, affects the mean but does <u>not</u> affect the standard deviation.

29. First, $E(X)=\int_0^\infty x\cdot 4e^{-4x}\,dx$. Apply integration by parts with $u=x \to du=dx$ and $dv=4e^{-4x}\,dx \to v=-e^{-4x}$:

$$E(X)=uv-\int v\,du = x\cdot(-e^{-4x})\Big|_0^\infty-\int_0^\infty -e^{-4x}\,dx=(0-0)+\int_0^\infty e^{-4x}\,dx=\frac{e^{-4x}}{-4}\Bigg|_0^\infty=\frac{1}{4}\text{ min.}$$

Similarly, $E(X^2)=\int_0^\infty x^2\cdot 4e^{-4x}\,dx$; with $u=x^2$ and $dv=4e^{-4x}\,dx$,

$$E(X^2)=x^2\cdot(-e^{-4x})\Big|_0^\infty-\int_0^\infty(-e^{-4x})(2x)dx=(0-0)+\int_0^\infty 2xe^{-4x}\,dx=\int_0^\infty 2xe^{-4x}\,dx=\frac{1}{2}\int_0^\infty x\cdot 4e^{-4x}\,dx=$$

$\frac{1}{2}E(X)=\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}$. Hence, $V(X)=\frac{1}{8}-\left(\frac{1}{4}\right)^2=\frac{1}{16}\Rightarrow \text{SD}(X)=\frac{1}{4}$ min.

31.

a. We have $R=h(I)=v/I$, so $h'(I)=-v/I^2$. The first-order approximation to μ_R is $h(\mu_I)=v/\mu_I=v/20$. The first-order approximation to σ_R^2 is $[h'(\mu_I)]^2V(I)=\left[-\frac{v}{\mu_I^2}\right]^2\cdot\sigma_I^2=\frac{v^2}{(20^2)^2}\cdot(.5)^2=\frac{v^2}{640{,}000}$; taking the square root, the first-order approximation to σ_R is $\frac{v}{800}$.

b. From Exercise 25, the exact value of $E[\pi R^2]$ was $100.2\pi\approx 314.79$ m^2. The first-order approximation via the delta method is $h(\mu_R)=h(10)=\pi(10)^2=100\pi\approx 314.16$ m^2. These are quite close.

c. The derivative of $h(R)=\pi R^2$ is $h'(R)=2\pi R$. The delta method approximation to $V[h(R)]$, therefore, is $[h'(\mu_R)]^2\cdot V(R)=[2\pi\mu_R]^2\cdot V(R)=[2\pi(10)]^2\cdot\frac{1}{5}=80\pi^2$. This is very close to the exact variance, given by $14008\pi^2/175\approx 80.046\pi^2$.

33. A linear function that maps 0 to –5 and 1 to 5 is $g(x)=10x-5$. Let $X\sim$ Unif[0, 1], so that from Exercise 32 we know that $M_X(t)=(e^t-1)/t$ for $t\neq 0$. Define $Y=g(X)=10X-5$; applying the mgf rescaling property with $a=10$ and $b=-5$, the mgf of Y is given by

$M_Y(t)=e^{-5t}M_X(10t)=e^{-5t}\cdot\frac{e^{(10t)}-1}{(10t)}=\frac{e^{5t}-e^{-5t}}{10t}$. This is an exact match to the mgf of the Unif[–5, 5] mgf based on Exercise 32. Therefore, by uniqueness of mgfs, Y must follow a Unif[–5, 5] distribution. (Equivalently, the pdf of Y is $f(y)=1/10$ for $-5\le y\le 5$.)

35. $f(x)=.04e^{-.04(x-10)}$ for $x\ge 10$.

a.

$$M_X(t)=\int_{10}^\infty e^{tx}\cdot .04e^{-.04(x-10)}\,dx=.04e^{+.4}\int_{.5}^\infty e^{(t-.04)x}\,dx=.04e^{.4}\frac{e^{(t-.04)x}}{t-.04}\Bigg|_{10}^\infty$$

$$=.04e^{.4}\left[0-\frac{e^{(t-.04)(10)}}{t-.04}\right]=\frac{.04e^{10t}}{.04-t}\qquad\text{for } t<.04$$

The condition $t<.04$ is necessary so that $(t-.04)<0$ and the improper integral converges. To find the mean and variance, re-write the mgf as $M_X(t)=.04e^{10t}(.04-t)^{-1}$ and use the product rule:

$$M_X'(t) = \frac{.4e^{10t}}{.04-t} + \frac{.04e^{10t}}{(.04-t)^2} \Rightarrow E(X) = M_X'(0) = \cdots = 35 \text{ m}^3\text{/s}$$

$$M_X''(t) = \frac{4e^{10t}}{.04-t} + \frac{2(.4e^{10t})}{(.04-t)^2} + \frac{.08e^{10t}}{(.04-t)^3} \Rightarrow E(X^2) = M_X''(0) = \cdots = 1850 \Rightarrow$$

$V(X) = (1850) - (35)^2 = 625$.

b. The mgf of the given pdf is

$M(t) = \int_0^\infty e^{tx} \cdot .04e^{-.04x}\, dx = .04\int_0^\infty e^{(t-.04)x}\, dx = .04\left.\frac{e^{(t-.04)x}}{t-.04}\right|_0^\infty = \frac{.04}{.04-t}$ for $t < .04$. Taking derivatives here (which is much easier than in part **a**!) gives $E(X) = M'(0) = \frac{1}{.04} = 25$, $E(X^2) = M''(0) = \frac{2}{.04^2} = 1250$ and $V(X) = 1250 - (25)^2 = 625$. The hazardous flood rate pdf and this pdf have exactly the same variance, while the mean of the hazardous flood rate pdf is 10 more than the mean of this pdf.

c. If $Y = X - 10$, then $M_Y(t) = e^{-10t}M_X(1t) = e^{-10t} \cdot \frac{.04e^{10t}}{.04-t} = \frac{.04}{.04-t}$, which is exactly the mgf in **b**. By uniqueness of mgfs, we conclude that Y follows the pdf specified in **b**: $f(y) = .04e^{-.04y}$ for $y > 0$.
In other words, the two pdfs represent "shifted" versions of two variables, X and Y. The rv X is on $[10, \infty)$, while $Y = X - 10$ is on $[0, \infty)$. This is consistent with the moments as well: the mean of Y is 10 less than the mean of X, as suggested by $Y = X - 10$, and the two rvs have the same variance because the shift of 10 doesn't affect variance.

37.

a. For $t \le x < \infty$, $x \cdot f(x) \ge t \cdot f(x)$. Thus,
$\int_t^\infty x \cdot f(x)dx \ge \int_t^\infty t \cdot f(x)dx = t \cdot \int_t^\infty f(x)dx = t \cdot P(X > t) = t \cdot [1 - F(t)]$.

b. By definition, $\mu = \int_0^\infty x \cdot f(x)dx = \int_0^t x \cdot f(x)dx + \int_t^\infty x \cdot f(x)dx$, from which it follows that
$\int_t^\infty x \cdot f(x)dx = \mu - \int_0^t x \cdot f(x)dx$.

Now consider the expression $t \cdot [1 - F(t)]$. Since $t > 0$ and $F(t) \le 1$, $t \cdot [1 - F(t)] \ge 0$. Combining that with part **a**, we have $0 \le t \cdot [1 - F(t)] \le \int_t^\infty x \cdot f(x)dx = \mu - \int_0^t x \cdot f(x)dx$.
As $t \to \infty$, the upper bound on $t \cdot [1 - F(t)]$ converges to 0:
$\lim_{t\to\infty}\left[\mu - \int_0^t x \cdot f(x)dx\right] = \mu - \lim_{t\to\infty}\left[\int_0^t x \cdot f(x)dx\right] = \mu - \int_0^\infty x \cdot f(x)dx = \mu - \mu = 0$.
(Those operations rely on the integral converging.)

Therefore, by the squeeze theorem, $\lim_{t\to\infty} t \cdot [1 - F(t)] = 0$ as well.

Section 4.3

39.

a. $P(0 \le Z \le 2.17) = \Phi(2.17) - \Phi(0) = .4850$.

b. $\Phi(1) - \Phi(0) = .3413$.

c. $\Phi(0) - \Phi(-2.50) = .4938$.

d. $\Phi(2.50) - \Phi(-2.50) = .9876$.

e. $\Phi(1.37) = .9147$.

f. $P(-1.75 < Z) + [1 - P(Z < -1.75)] = 1 - \Phi(-1.75) = .9599$.

g. $\Phi(2) - \Phi(-1.50) = .9104$.

h. $\Phi(2.50) - \Phi(1.37) = .0791$.

i. $1 - \Phi(1.50) = .0668$.

j. $P(|Z| \le 2.50) = P(-2.50 \le Z \le 2.50) = \Phi(2.50) - \Phi(-2.50) = .9876$.

41.

a. $\Phi(c) = .9100 \Rightarrow c \approx 1.34$, since .9099 is the entry in the 1.3 row, .04 column.

b. Since the standard normal distribution is symmetric about $z = 0$, the 9^{th} percentile = $-$[the 91^{st} percentile] = -1.34.

c. $\Phi(c) = .7500 \Rightarrow c \approx .675$, since .7486 and .7517 are in the .67 and .68 entries, respectively.

d. Since the standard normal distribution is symmetric about $z = 0$, the 25^{th} percentile = $-$[the 75^{th} percentile] = $-.675$.

e. $\Phi(c) = .06 \Rightarrow c \approx -1.555$, since .0594 and .0606 appear as the -1.56 and -1.55 entries, respectively.

43.

a. $P(X \le 100) = P\left(Z \le \frac{100-80}{10}\right) = P(Z \le 2) = \Phi(2) = .9772$.

b. $P(X \le 80) = P\left(Z \le \frac{80-80}{10}\right) = P(Z \le 0) = \Phi(0) = .5$.

c. $P(65 \le X \le 100) = P\left(\frac{65-80}{10} \le Z \le \frac{100-80}{10}\right) = P(-1.5 \le Z \le 2) = \Phi(2) - \Phi(-1.5) = .9104$.

d. $P(70 \le X) = P\left(Z \ge \frac{70-80}{10}\right) = P(Z \ge -1) = 1 - \Phi(-1) = .8413$.

e. $P(85 \le X \le 95) = P\left(\frac{85-80}{10} \le Z \le \frac{95-80}{10}\right) = P(.5 \le Z \le 1.5) = \Phi(1.5) - \Phi(.5) = .2417$.

f. $P(|X-80| \le 10) = P(-10 \le X - 80 \le 10) = P(70 \le X \le 90) = P\left(\frac{70-80}{10} \le Z \le \frac{90-80}{10}\right) =$
$P(-1 \le Z \le 1) = \Phi(1) - \Phi(-1) = .6826$.

45. $X \sim N(.30, .06)$

a. $P(X > .25) = 1 - \Phi\left(\frac{.25-.30}{.06}\right) = 1 - \Phi(-0.83) = .7967$.

b. $P(X \le .10) = \Phi\left(\frac{.10-.30}{.06}\right) = \Phi(-3.33) = .0004$.

c. We want the 95th percentile: $.95 = \Phi\left(\frac{\eta_{.95}-.30}{.06}\right) \Rightarrow \frac{\eta_{.95}-.30}{.06} = -1.645 \Rightarrow \eta_{.95} = .29013$. So, the highest 5% of concentrations are those greater than .29013 mg/cm^3.

47. Let X denote the diameter of a randomly selected cork made by the first machine, and let Y be defined analogously for the second machine.
$P(2.9 \le X \le 3.1) = P(-1.00 \le Z \le 1.00) = .6826$, while
$P(2.9 \le Y \le 3.1) = P(-7.00 \le Z \le 3.00) = .9987$. So, the second machine wins handily.

49.

a. $P(X < 40) = P\left(Z \le \frac{40-43}{4.5}\right) = P(Z < -0.667) = .2514$.

$P(X > 60) = P\left(Z > \frac{60-43}{4.5}\right) = P(Z > 3.778) \approx 0$.

b. We desire the 25th percentile. Since the 25th percentile of a standard normal distribution is roughly $z = -0.67$, the answer is $43 + (-0.67)(4.5) = 39.985$ ksi.

51. The probability X is within .1 of its mean is given by $P(\mu - .1 \le X \le \mu + .1) =$
$P\left(\frac{(\mu-.1)-\mu}{\sigma} < Z < \frac{(\mu+.1)-\mu}{\sigma}\right) = \Phi\left(\frac{.1}{\sigma}\right) - \Phi\left(-\frac{.1}{\sigma}\right) = 2\Phi\left(\frac{.1}{\sigma}\right) - 1$. If we require this to equal 95%, we find $2\Phi\left(\frac{.1}{\sigma}\right) - 1 = .95 \Rightarrow \Phi\left(\frac{.1}{\sigma}\right) = .975 \Rightarrow \frac{.1}{\sigma} = 1.96$ from the standard normal table. Thus, $\sigma = \frac{.1}{1.96} = .0510$.
Alternatively, use the empirical rule: 95% of all values lie within 2 standard deviations of the mean, so we want $2\sigma = .1$, or $\sigma = .05$. (This is not quite as precise as the first answer.)

53.

a. $P(\mu - 1.5\sigma \le X \le \mu + 1.5\sigma) = P(-1.5 \le Z \le 1.5) = \Phi(1.50) - \Phi(-1.50) = .8664$.

b. $P(X < \mu - 2.5\sigma$ or $X > \mu + 2.5\sigma) = 1 - P(\mu - 2.5\sigma \le X \le \mu + 2.5\sigma)$
$= 1 - P(-2.5 \le Z \le 2.5) = 1 - .9876 = .0124$.

c. $P(\mu - 2\sigma \le X \le \mu - \sigma$ or $\mu + \sigma \le X \le \mu + 2\sigma) = P(\text{within 2 sd's}) - P(\text{within 1 sd}) =$
$P(\mu - 2\sigma \le X \le \mu + 2\sigma) - P(\mu - \sigma \le X \le \mu + \sigma) = .9544 - .6826 = .2718$.

55.

a. $P(67 < X < 75) = P\left(\frac{67-70}{3} < \frac{X-70}{3} < \frac{75-70}{3}\right) = P(-1 < Z < 1.67) = \Phi(1.67) - \Phi(-1) = .9525 -$ $.1587 = .7938$.

b. By the Empirical Rule, c should equal 2 standard deviations. Since $\sigma = 3$, $c = 2(3) = 6$.
We can be a little more precise, as in Exercise 42, and use $c = 1.96(3) = 5.88$.

c. Let Y = the number of acceptable specimens out of 10, so $Y \sim \text{Bin}(10, p)$, where $p = .7938$ from part **a**.
Then $E(Y) = np = 10(.7938) = 7.938$ specimens.

d. Now let Y = the number of specimens out of 10 that have a hardness of less than 73.84, so
$Y \sim \text{Bin}(10, p)$, where
$p = P(X < 73.84) = P\left(Z < \frac{73.84-70}{3}\right) = P(Z < 1.28) = \Phi(1.28) = .8997$. Then
$P(Y \le 8) = \sum_{y=0}^{8} \binom{10}{y} (.8997)^y (.1003)^{10-y} = .2651$.

You can also compute $1 - P(Y = 9, 10)$ and use the binomial formula, or round slightly to $p = .9$ and use the binomial table: $P(Y \le 8) = B(8; 10, .9) = .265$.

57.

a. By symmetry, $P(-1.72 \le Z \le -.55) = P(.55 \le Z \le 1.72) = \Phi(1.72) - \Phi(.55)$.

b. $P(-1.72 \le Z \le .55) = \Phi(.55) - \Phi(-1.72) = \Phi(.55) - [1 - \Phi(1.72)]$.

No, thanks to the symmetry of the z curve about 0.

59. $X \sim N(119, 13.1)$.

a. $P(100 \le X \le 120) = \Phi\left(\frac{120-119}{13.1}\right) - \Phi\left(\frac{100-119}{13.1}\right) \approx \Phi(0.08) - \Phi(-1.45) = .5319 - .0735 = .4584$.

b. The goal is to find the speed, s, so that $P(X > s) = 10\% = .1$ (the fastest 10%). That's equivalent to $P(X \le s) = 1 - .1 = .9$ (the 90th percentile), so $.9 = \Phi\left(\frac{s-119}{13.1}\right) \Rightarrow \frac{s-119}{13.1} \approx 1.28 \Rightarrow s = 119 +$ $1.28(13.1) \approx 135.8$ kph.

c. $P(X > 100) = 1 - \Phi\left(\dfrac{100-119}{13.1}\right) = 1 - \Phi(-1.45) = 1 - .0735 = .9265.$

d. P(at least one is <u>not</u> exceeding 100 kph) = 1 – P(all five are exceeding 100 kph). Using independence and the answer from **c**, this equals 1 – P(first > 100 kph) × … × P(fifth > 100 kph) = 1 – $(.9265)^5$ = .3173.

e. Convert: 70 miles per hour ≈ 112.65 kilometers per hour. Thus $P(X > 70 \text{ mph}) = P(X > 112.65 \text{ kph})$ = $1 - \Phi\left(\dfrac{112.65-119}{13.1}\right) = 1 - \Phi(-.48) = 1 - .3156 = .6844.$

61.

a. $P(20 \le X \le 30) = P(20 - .5 \le X \le 30 + .5) = P(19.5 \le X \le 30.5) = P(-1.1 \le Z \le 1.1) = .7286.$

b. $P(X \le 30) = P(X \le 30.5) = P(Z \le 1.1) = .8643$, while
$P(X < 30) = P(X \le 29.5) = P(Z < .9) = .8159.$

63. Use the normal approximation to the binomial, with a continuity correction. With $p = .10$ and $n = 200$, $\mu = np = 20$, and $\sigma^2 = npq = 18$. So, Bin(200, .10) ≈ $N(20, \sqrt{18})$.

a. $P(X \le 30) = \Phi\left(\dfrac{(30+.5)-20}{\sqrt{18}}\right) = \Phi(2.47) = .9932.$

b. $P(X < 30) = P(X \le 29) = \Phi\left(\dfrac{(29+.5)-20}{\sqrt{18}}\right) = \Phi(2.24) = .9875.$

c. $P(15 \le X \le 25) = P(X \le 25) - P(X \le 14) = \Phi\left(\dfrac{(25+.5)-20}{\sqrt{18}}\right) - \Phi\left(\dfrac{(14+.5)-20}{\sqrt{18}}\right)$
$= \Phi(1.30) - \Phi(-1.30) = .9032 - .0968 = .8064.$

65. We use a normal approximation to the binomial distribution: Let X denote the number of people in the sample of 1000 who <u>can</u> taste the difference, so $X \sim$ Bin(1000, .03). Because $\mu = np = 1000(.03) = 30$ and $\sigma = \sqrt{np(1-p)} = 5.394$, X is approximately $N(30, 5.394)$.

a. Using a continuity correction, $P(X \ge 40) = 1 - P(X \le 39) = 1 - P\left(Z \le \dfrac{39.5-30}{5.394}\right) = 1 - P(Z \le 1.76) =$ $1 - \Phi(1.76) = 1 - .9608 = .0392.$

b. 5% of 1000 is 50, and $P(X \le 50) = P\left(Z \le \dfrac{50.5-30}{5.394}\right) = \Phi(3.80) \approx 1.$

67. As in the previous exercise, $u = (x - \mu)/\sigma \rightarrow du = dx/\sigma$. Here, we'll also need $x = \mu + \sigma u$.

a. $E(X) = \int_{-\infty}^{\infty} x \cdot f(x;\mu,\sigma)dx = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} e^{-[(x-\mu)/\sigma]^2/2} \frac{dx}{\sigma} = \int_{-\infty}^{\infty} \frac{\mu + \sigma u}{\sqrt{2\pi}} e^{-u^2/2} du =$

$\mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du + \sigma \int_{-\infty}^{\infty} \frac{u}{\sqrt{2\pi}} e^{-u^2/2} du$. The first integral is the area under the standard normal pdf, which equals 1. The second integrand is an odd function over a symmetric interval, so that second integral equals 0. Put it all together: $E(X) = \mu(1) + \sigma(0) = \mu$.

b. $V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x;\mu,\sigma)dx = \int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sqrt{2\pi}} e^{-[(x-\mu)/\sigma]^2/2} \frac{dx}{\sigma} = \int_{-\infty}^{\infty} \frac{(\sigma u)^2}{\sqrt{2\pi}} e^{-u^2/2} du =$

$\sigma^2 \int_{-\infty}^{\infty} u^2 \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$. We need to show the integral equals 1; toward this goal, use integration by parts with $u = u$ and $dv = u \cdot \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \rightarrow v = -\frac{1}{\sqrt{2\pi}} e^{-u^2/2}$:

$$\int_{-\infty}^{\infty} u^2 \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = -u \cdot u \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = 0 + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du .$$

Both limits required for the first evaluation are 0 because the exponential term dominates. What remains is, once again, the integral of the standard normal pdf, which is 1.
Therefore, $V(X) = \sigma^2(1) = \sigma^2$.

69.

a. $P(Z \geq 1) \approx .5 \cdot \exp\left(\frac{83 + 351 + 562}{703 + 165}\right) = .1587$, which matches $1 - \Phi(1)$.

b. $P(Z < -3) = P(Z > 3) \approx .5 \cdot \exp\left(\frac{-2362}{399.3333}\right) = .0013$, which matches $\Phi(-3)$.

c. $P(Z > 4) \approx .5 \cdot \exp\left(\frac{-3294}{340.75}\right) = .0000317$, so $P(-4 < Z < 4) = 1 - 2P(Z \geq 4) \approx 1 - 2(.0000317) = .999937$.

d. $P(Z > 5) \approx .5 \cdot \exp\left(\frac{-4392}{305.6}\right) = .00000029$.

Section 4.4

71. For **a** and **b**, we use the properties of the gamma function provided in this section.

a. $\Gamma(6) = 5! = 120$.

b. $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \left(\frac{3}{4}\right)\sqrt{\pi} \approx 1.329$.

c. $G(4; 5) = .371$ from row 4, column 5 of Table A.4.

d. $G(5; 4) = .735$.

e. $G(0; 4) = P(X \le 0$ when $\alpha = 4) = 0$, since the gamma distribution is positive.

73.

a. $\alpha\beta = 20, \alpha\beta^2 = 80 \Rightarrow \beta = 4 \Rightarrow \alpha = 5$.

b. Let $X \sim$ Gamma(5, 4). $P(X \le 24) = G(24/4;5) = G(6;5) = .715$.

c. $P(20 \le X \le 40) = G(40/4;5) - G(20/4;5) = G(10;5) - G(5;5) = .411$.

75.

a. $E(X) = \dfrac{1}{\lambda} = 1$.

b. $\sigma = \dfrac{1}{\lambda} = 1$.

c. Using the exponential cdf, $P(X \le 4) = 1 - e^{-(1)(4)} = 1 - e^{-4} = .982$.

d. Similarly, $P(2 \le X \le 5) = (1 - e^{-(1)(5)}) - (1 - e^{-(1)(2)}) = e^{-2} - e^{-5} = .129$.

77. Note that a mean value of 10 for the exponential distribution implies $\lambda = \dfrac{1}{10} = .1$. Let X denote the survival time of a mouse without treatment.

a. $P(X \ge 8) = 1 - [1 - e^{-(.1)(8)}] = e^{-(.1)(8)} = .4493$. $P(X \le 12) = 1 - e^{-(.1)(12)} = .6988$. Combining these two answers, $P(8 \le X \le 12) = P(X \le 12) - P(X < 8) = .6988 - [1 - .4493] = .1481$.

b. The standard deviation equals the mean, 10 hours. So, $P(X > \mu + 2\sigma) = P(X > 30) = 1 - [1 - e^{-(.1)(30)}] = e^{-(.1)(30)} = .0498$. Similarly, $P(X > \mu + 3\sigma) = P(X > 40) = e^{-(.1)(40)} = .0183$.

79.

a. $\{X \geq t\}$ = {the lifetime of the system is at least t}. Since the components are connected in series, this equals {all 5 lifetimes are at least t} = $A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$.

b. Since the events A_i are assumed to be independent, $P(X \geq t) = P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4) \cdot P(A_5)$. Using the exponential cdf, for any i we have
$P(A_i) = P(\text{component lifetime is} \geq t) = 1 - F(t) = 1 - [1 - e^{-.01t}] = e^{-.01t}$.
Therefore, $P(X \geq t) = (e^{-.01t}) \cdots (e^{-.01t}) = e^{-.05t}$, and $F_X(t) = P(X \leq t) = 1 - e^{-.05t}$.
Taking the derivative, the pdf of X is $f_X(t) = .05e^{-.05t}$ for $t \geq 0$. Thus X also has an exponential distribution, but with parameter $\lambda = .05$.

c. By the same reasoning, $P(X \leq t) = 1 - e^{-n\lambda t}$, so X has an exponential distribution with parameter $n\lambda$.

81.

a. If X is exponential, then $f_X(x) = \lambda e^{-\lambda x}$ for some $\lambda > 0$. Thus $f_W(w) \propto w^2 \cdot \lambda e^{-\lambda w} \propto w^2 e^{-\lambda w} = w^{3-1}e^{-\lambda w}$. This is the kernel of a gamma distribution with $\alpha = 3$ and $\beta = 1/\lambda$.

b. $3 = V(W) = \alpha\beta^2 = 2\beta^2 \Rightarrow \beta = \sqrt{3/2} \approx 1.225$. Then, $\lambda = 1/\beta = \sqrt{2/3} \approx .8165$.

83. Using (4.5), for any positive exponent k we have

$$E(X^k) = \int_0^\infty x^k \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\, dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{k+\alpha-1} e^{-x/\beta}\, dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \cdot \beta^{k+\alpha}\Gamma(k+\alpha)$$
$$= \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)}\beta^k$$

So, $E(X) = \frac{\Gamma(1+\alpha)}{\Gamma(\alpha)}\beta^1 = \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)}\beta = \alpha\beta$; $E(X^2) = \frac{\Gamma(2+\alpha)}{\Gamma(\alpha)}\beta^2 = \frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)}\beta^2 = \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)}$
$= (\alpha+1)\alpha\beta^2$; and $V(X) = (\alpha+1)\alpha\beta^2 - [\alpha\beta]^2 = \alpha\beta^2$.

Section 4.5

85.

a. $P(20 \leq X \leq 40) = F(40;1.18,21.61) - F(20;1.18,21.61) = [1 - e^{-(40/21.61)^{1.18}}] - [1 - e^{-(20/21.61)^{1.18}}] =$ $.874 - .599 = .275$.
$P(X < 20) = F(20;1.18,21.61) = 1 - e^{-(20/21.61)^{1.18}} = .599$.
$P(X > 40) = 1 - F(40;1.18,21.61) = 1 - [1 - e^{-(40/21.61)^{1.18}}] = 1 - .874 = .126$.

b. Software provides $\Gamma\left(1 + \frac{1}{1.18}\right) = .94484$ and $\Gamma\left(1 + \frac{2}{1.18}\right) = 1.53845$, from which
$\mu = (21.61)(.94484) = 20.418$ and $\sigma^2 = (21.61)^2\left\{1.53845 - [.94484]^2\right\} = 301.55$, or $\sigma = 17.365$.

c. Solve $F(x) = .5$: $.5 = 1 - e^{-(x/21.61)^{1.18}} \Rightarrow$
$e^{-(x/21.61)^{1.18}} = .5 \Rightarrow (x/21.61)^{1.18} = -\ln(.5) \Rightarrow x = 21.61(-\ln(.5))^{1/1.18} = 15.84$ microns.

87. Use the substitution $y = \left(\frac{x}{\beta}\right)^{\alpha} = \frac{x^{\alpha}}{\beta^{\alpha}}$. Then $dy = \frac{\alpha x^{\alpha-1}}{\beta^{\alpha}}dx$, and $\mu = \int_0^{\infty} x \cdot \frac{\alpha}{\beta^{\alpha}} x^{\alpha-1} e^{-(x/\beta)^{\alpha}} dx$ = $\int_0^{\infty} (\beta^{\alpha} y)^{1/\alpha} \cdot e^{-y} dy = \beta \int_0^{\infty} y^{1/\alpha} e^{-y} dy = \beta \cdot \Gamma\left(1 + \frac{1}{\alpha}\right)$ by definition of the gamma function.

89.

a. $P(X \le 105) = F(105; 20, 100) = 1 - e^{-(105/100)^{20}} = .9295$.

b. $P(100 \le X \le 105) = F(105; 20, 100) - F(100; 20, 100) = .9295 - .6321 = .2974$.

c. Set $.5 = F(x)$ and solve: $.5 = 1 - e^{-(x/100)^{20}} \Rightarrow -(x/100)^{20} = \ln(.5) \Rightarrow x = 100[-\ln(.5)]^{1/20} = 98.184$ ksi.

91.

a. $E(X) = e^{1.513 + 1.006^2/2} = 7.53$ square microns. $V(X) = e^{2(1.513)+1.006^2}(e^{1.006^2} - 1) = 99.319$, so SD($X$) = 9.966 square microns.

b. $P(X < 10) = \Phi\left(\frac{\ln(10) - 1.513}{1.006}\right) = \Phi(.78) = .7823$.

$P(10 \le X \le 20) = \Phi\left(\frac{\ln(20) - 1.513}{1.006}\right) - \Phi\left(\frac{\ln(10) - 1.513}{1.006}\right) = \Phi(1.47) - \Phi(.78) = .9292 - .7823 = .1469$.

c. $P(X < e^{1.513+1.006^2/2}) = \Phi\left(\frac{\ln(e^{1.513+1.006^2/2}) - 1.513}{1.006}\right) = \Phi\left(\frac{1.513 + 1.006^2/2 - 1.513}{1.006}\right) = \Phi\left(\frac{1.006}{2}\right)$

$= \Phi(.503) \approx .6925$. While the *normal* distribution is symmetric (and so the mean and median are equal), the *lognormal* distribution is right-skewed. As a result, the mean exceeds the median, and the probability X is less than its mean exceeds .5.

93.

a. $E(X) = e^{5+(.01)/2} = e^{5.005} = 149.157$; $V(X) = e^{10+(.01)} \cdot \left(e^{.01} - 1\right) = 223.594$.

b. $P(X > 125) = 1 - P(X \le 125) = = 1 - \Phi\left(\frac{\ln(125) - 5}{.1}\right) = 1 - \Phi(-1.72) = .9573$.

c. $P(110 \le X \le 125) = \Phi(-1.72) - \Phi\left(\frac{\ln(110) - 5}{.1}\right) = .0427 - .0013 = .0414$.

d. $\tilde{\mu} = e^{\mu} = e^{5} = 148.41$.

e. P(any particular one has $X > 125$) = .9573 $\Rightarrow$ expected # = 10(.9573) = 9.573.

f. We want the 5th percentile, which is $e^{5+(-1.645)(.1)} = 125.90$.

95. Since the standard beta distribution lies on (0, 1), the point of symmetry must be ½, so we require that $f\left(\frac{1}{2}-\mu\right)=f\left(\frac{1}{2}+\mu\right)$. Cancelling out the constants, this implies $\left(\frac{1}{2}-\mu\right)^{\alpha-1}\left(\frac{1}{2}+\mu\right)^{\beta-1}=\left(\frac{1}{2}+\mu\right)^{\alpha-1}\left(\frac{1}{2}-\mu\right)^{\beta-1}$, which (by matching exponents on both sides) in turn implies that $\alpha=\beta$.

Alternatively, symmetry about ½ requires μ = ½, so $\dfrac{\alpha}{\alpha+\beta}$ = .5. Solving for α gives $\alpha=\beta$.

97.

a. Notice from the definition of the standard beta pdf that, since a pdf must integrate to 1,

$$1=\int_0^1\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\,dx\Rightarrow\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Using this, $E(X)=\int_0^1 x\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\,dx=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\alpha}(1-x)^{\beta-1}\,dx=$

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)}=\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(\alpha+\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)}=\frac{\alpha}{\alpha+\beta}.$$

b. Similarly, $E[(1-X)^m]=\int_0^1(1-x)^m\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\,dx=$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\alpha-1}(1-x)^{m+\beta-1}\,dx=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha)\Gamma(m+\beta)}{\Gamma(\alpha+m+\beta)}=\frac{\Gamma(\alpha+\beta)\cdot\Gamma(m+\beta)}{\Gamma(\alpha+m+\beta)\Gamma(\beta)}.$$

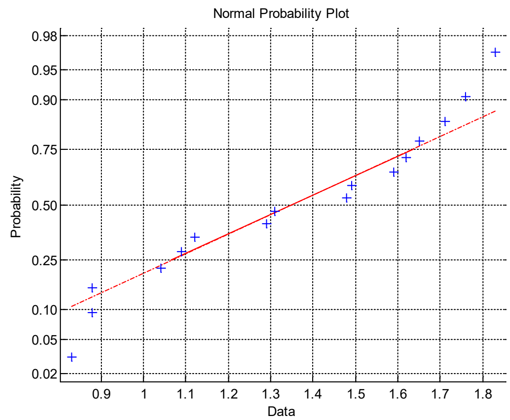
If X represents the proportion of a substance consisting of an ingredient, then $1-X$ represents the proportion <u>not</u> consisting of this ingredient. For m = 1 above,

$$E(1-X)=\frac{\Gamma(\alpha+\beta)\cdot\Gamma(1+\beta)}{\Gamma(\alpha+1+\beta)\Gamma(\beta)}=\frac{\Gamma(\alpha+\beta)\cdot\beta\Gamma(\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)\Gamma(\beta)}=\frac{\beta}{\alpha+\beta}.$$

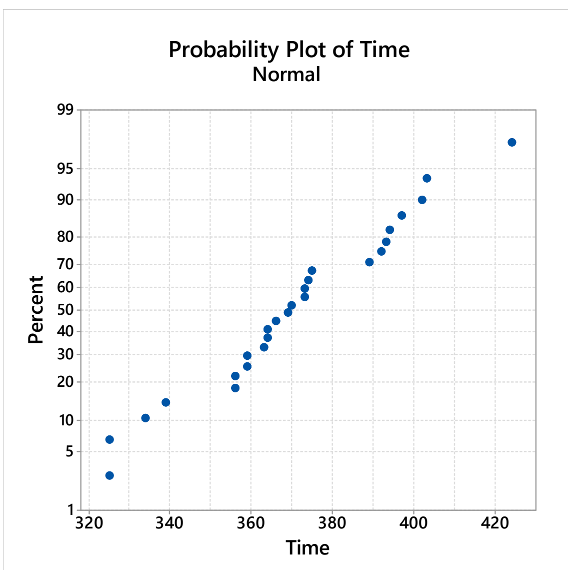
Section 4.6

99. The given probability plot is quite linear, and thus it is quite plausible that the tension distribution is normal.

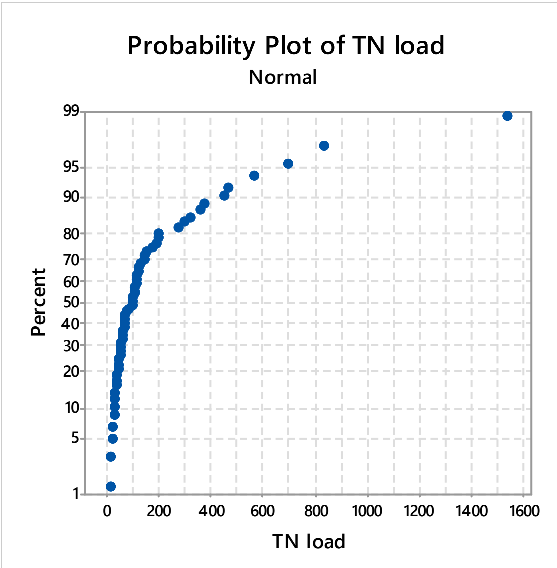
101. The z percentile values are as follows: –1.86, –1.32, –1.01, –0.78, –0.58, –0.40, –0.24,–0.08, 0.08, 0.24, 0.40, 0.58, 0.78, 1.01, 1.30, and 1.86. The accompanying probability plot has some curvature but (arguably) not enough to worry about. It would be reasonable to use estimating methods that assume a normal population distribution.

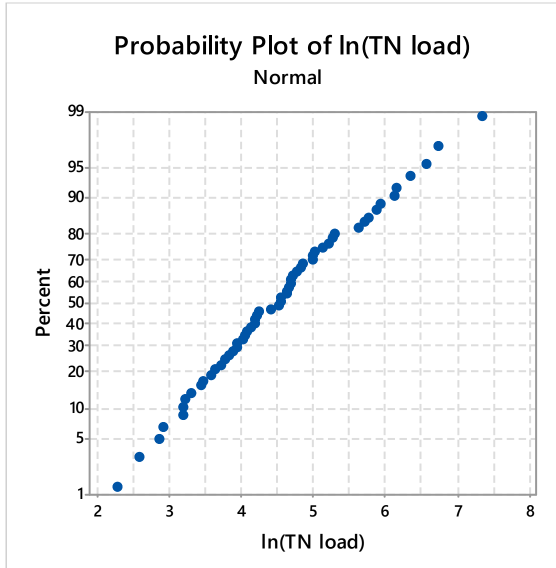


103. The accompanying normal probability plot is fairly straight, suggesting that an assumption of population normality is plausible.

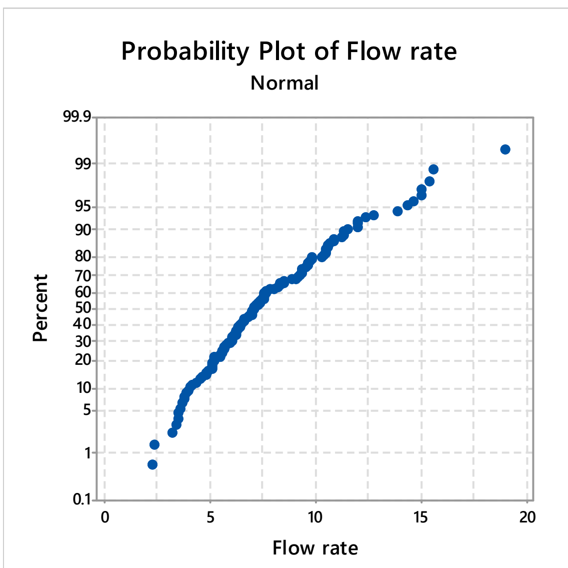


105. To check for plausibility of a lognormal population distribution for this data, take the natural logs and construct a normal probability plot. This plot and a normal probability plot for the original data appear below. Clearly the log transformation gives quite a straight plot, so lognormality is plausible. The curvature in the plot for the original data implies a positively skewed population distribution — like the lognormal distribution.



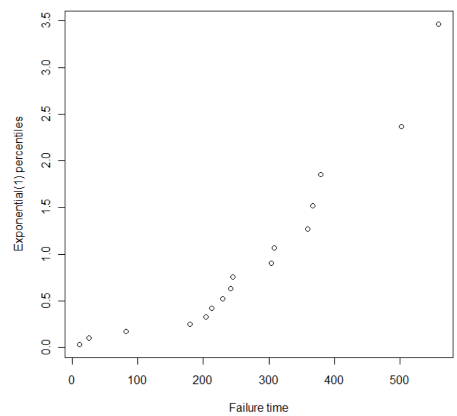


107. The pattern in the normal probability plot is curved downward, consistent with a right-skewed distribution. It is not plausible that shower flow rates have a normal population distribution.



109. The $(100p)^{\text{th}}$ percentile η_p for the exponential distribution with $\lambda = 1$ is given by the formula $\eta_p = -\ln(1-p)$. With $n = 16$, we need η_p for $p = \frac{0.5}{16}, \frac{1.5}{16}, \ldots, \frac{15.5}{16}$. These percentiles are .032, .398, .170, .247, .330, .421, .521, .633, .758, .901, 1.068, 1.269, 1.520, 1.856, 2.367, 3.466.

The accompanying plot of (failure time value, exponential(1) percentile) pairs exhibits substantial curvature, casting doubt on the assumption of an exponential population distribution.



Because λ is a scale parameter (as is σ for the normal family), $\lambda = 1$ can be used to assess the plausibility of the entire exponential family. If we used a different value of λ to find the percentiles, the slope of the graph would change, but not its linearity (or lack thereof).

Section 4.7

111. $y = 1/x \Rightarrow x = 1/y$ and $0 < x < 1 \Rightarrow 0 < 1/y < 1 \Rightarrow y > 1$. Apply the transformation theorem: $f_Y(y) = f_X(1/y)|dx/dy| = f_X(1/y)|-1/y^2| = 2(1/y)(1/y^2) = 2/y^3$ for $y > 1$.
(If you're paying attention you might notice this is just the previous exercise in reverse!)

113. $y = \sqrt{x} \Rightarrow x = y^2$ and $x > 0 \Rightarrow y > 0$. Apply the transformation theorem:

$f_Y(y) = f_X(y^2)|dx/dy| = \frac{1}{2}e^{-y^2/2}\left|2y\right| = ye^{-y^2/2}$ for $y > 0$.

115. $y = \text{area} = x^2 \Rightarrow x = \sqrt{y}$ and $0 < x < 4 \Rightarrow 0 < y < 16$. Apply the transformation theorem:

$f_Y(y) = f_X(\sqrt{y})|dx/dy| = \sqrt{y}/8|1/(2\sqrt{y})| = 1/16$ for $0 < y < 16$. That is, the area Y is uniform on (0,16).

117. $y = \tan(\pi(x-.5)) \Rightarrow x = [\arctan(y)+.5]/\pi$ and $0 < x < 1 \Rightarrow -\pi/2 < \pi(x-.5) < \pi/2 \Rightarrow -\infty < y < \infty$ (since $\tan\theta \to \pm\infty$ as $\theta \to \pm\pi/2$). Also, $X \sim \text{Unif}(0, 1) \Rightarrow f_X(x) = 1$. Apply the transformation theorem:

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = 1\cdot\left|\frac{d}{dy}\left[\frac{\arctan(y)+.5}{\pi}\right]\right| = \left|\frac{1}{\pi}\frac{1}{1+y^2}\right| = \frac{1}{\pi(1+y^2)} \quad \text{for } -\infty < y < \infty.$$

119. Assume the target function $g(x)$ is differentiable and increasing, so that $h(y) = g^{-1}(y)$ is also differentiable and increasing. Apply the transformation theorem:

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)|$$

$$1 = \frac{h(y)}{8} \cdot h'(y)$$

$$8 = h(y)h'(y)$$

Take the antiderivative of both sides to obtain $8y = (1/2)[h(y)]^2$, from which $h(y) = 4\sqrt{y}$. Now reverse the roles of x and y to find the inverse of h, aka g: $x = 4\sqrt{g(x)} \Rightarrow g(x) = x^2/16$.
As a check, apply the transformation theorem with $Y = X^2/16$ and $f_X(x) = x/8$ for $0 \le x \le 4$ and you indeed obtain $Y \sim$ Unif[0, 1].

You might notice that $x^2/16$ is the antiderivative of $x/8$; i.e., $g(x) = F_X(x)$. This is a special case of a more general result: if X is a continuous rv with cdf $F_X(x)$, then $F_X(X) \sim$ Unif[0, 1].

121. The transformation $y = |x|$ is not monotone on [–1, 1], so we must proceed via the cdf method.
For $y > 0$, $F_Y(y) = P(Y \le y) = P(|X| \le y) = P(-y \le X \le y) = \Phi(y) - \Phi(-y) = \Phi(y) - [1 - \Phi(y)] = 2\Phi(y) - 1$.
Thus, $f_Y(y) = d/dy[2\Phi(y) - 1] = 2\Phi'(y) = \dfrac{2}{\sqrt{2\pi}} e^{-y^2/2}$ for $y > 0$. In the last step, we use the fact that the derivative of the standard normal cdf is, of course, the standard normal pdf.

123.

a. By assumption, the probability that you hit the disc centered at the bulls-eye with area x is proportional to x; in particular, this probability is x/[total area of target] $= x/[\pi(1)^2] = x/\pi$.
Therefore, $F_X(x) = P(X \le x) = P$(you hit disc centered at the bulls-eye with area x) $= x/\pi$. From this, $f_X(x) = d/dx[x/\pi] = 1/\pi$ for $0 < x < \pi$. That is, X is uniform on $(0, \pi)$.

b. $x = \pi y^2$ and $0 < x < \pi \Rightarrow 0 < y < 1$. Thus, $f_Y(y) = f_X(\pi y^2)|dx/dy| = 1/\pi\ |2\pi y| = 2y$ for $0 < y < 1$.

Section 4.8

125.

a. $F(x) = x^2/4$. Set $u = F(x)$ and solve for x: $u = x^2/4 \Rightarrow x = 2\sqrt{u}$.

b. The one-line "program" below have been vectorized for speed; i.e., all 10,000 Unif[0, 1] values are generated simultaneously.
In R: `x<-2*sqrt(runif(10000))`

c. One execution of the code gave `mean(x)` = 1.331268 and `sd(x)` = 0.4710592. These are very close to the exact mean and sd of X, which we can obtain through simple polynomial integrals:

$$\mu = \int_0^2 x \cdot \frac{x}{2} dx = \frac{1}{2}\int_0^2 x^2\,dx = \frac{4}{3} = 1.333;\ E(X^2) = \int_0^2 x^2 \cdot \frac{x}{2} dx = 2 \Rightarrow \sigma = \sqrt{2 - \left(\frac{4}{3}\right)^2} = \frac{\sqrt{2}}{3} = 0.4714.$$

127. $f(x)=\frac{1}{8}+\frac{3}{8}x \Rightarrow F(x)=\frac{1}{8}x+\frac{3}{16}x^2$. Set $u = F(x)$ and solve for x: $u = \frac{1}{8}x+\frac{3}{16}x^2 \Rightarrow 3x^2 + 2x - 16u = 0 \Rightarrow$
$x=\frac{-2\pm\sqrt{4-4(3)(-16u)}}{2(3)}=\frac{-2\pm 2\sqrt{1+48u}}{2(3)}=\frac{\sqrt{1+48u}-1}{3}$. (The other root of the quadratic would place x between –2 and 0 rather than 0 and 2.) The code below implements this transformation and returns 10,000 values from the desired pdf.

In R:
```
x<-(sqrt(1+48*runif(10000))-1)/3
```

129.

a. The cdf of this model is $F(x)=1-\left(1-\frac{x}{\tau}\right)^{\theta}$ for $0 < x < \tau$. Set $u = F(x)$ and solve for x:

$u=1-\left(1-\frac{x}{\tau}\right)^{\theta} \Rightarrow x=\tau\cdot\left[1-(1-u)^{1/\theta}\right]$. This transform is implemented in the function below.

```
waittime<-function(n,theta,tau){
u<-runif(n)
x<- tau*(1-(1-u)^(1/theta))
}
```

b. Calling `x<-waittime(10000,4,80)` and `mean(x)` in R returned 15.9188, quite close to 16.

Supplementary Exercises

131. Let Y = the amount paid out. Since the insurance company only pays up to 5 thousand dollars, this means $Y = X$ if $X \le 5$ but $Y = 5$ if $X > 5$. So, the expected payout is
$E(Y)=\int_1^5 x\cdot f(x)dx+\int_5^\infty 5\cdot f(x)dx=\int_1^5 x\cdot\frac{3}{x^4}dx+\int_5^\infty 5\cdot\frac{3}{x^4}dx=\frac{36}{25}+\frac{1}{25}$ = 1.48, or \$1480.

133.

a. Clearly $f(x) \ge 0$. Now check that the function integrates to 1:

$$\int_0^\infty \frac{32}{(x+4)^3}dx=\int_0^\infty 32(x+4)^{-3}\,dx=-\frac{16}{(x+4)^2}\Big|_0^\infty=0--\frac{16}{(0+4)^2}=1.$$

b. For $x \le 0$, $F(x) = 0$. For $x > 0$,

$$F(x)=\int_{-\infty}^x f(y)dy=\int_0^x \frac{32}{(y+4)^3}dy=-\frac{1}{2}\cdot\frac{32}{(y+4)^2}\Bigg]_0^x=1-\frac{16}{(x+4)^2}.$$

c. $P(2 \le X \le 5) = F(5) - F(2) = 1-\frac{16}{81}-\left(1-\frac{16}{36}\right)=.247$.

d. $E(X)=\int_{-\infty}^\infty x\cdot f(x)dx=\int_{-\infty}^\infty x\cdot\frac{32}{(x+4)^3}dx=\int_0^\infty (x+4-4)\cdot\frac{32}{(x+4)^3}dx$

$=\int_0^\infty \frac{32}{(x+4)^2}dx-4\int_0^\infty \frac{32}{(x+4)^3}dx=8-4=4$ years.

e. $E\left(\frac{100}{X+4}\right)=\int_0^\infty \frac{100}{x+4}\cdot\frac{32}{(x+4)^3}dx=3200\int_0^\infty \frac{1}{(x+4)^4}dx=\frac{3200}{(3)(64)}=16.67$.

135.

a. $P(39 < X < 42) = \Phi\left(\frac{42-40}{1.5}\right) - \Phi\left(\frac{39-40}{1.5}\right) = \Phi(1.33) - \Phi(-.67) = .9082 - .2514 = .6568$.

b. We desire the 85th percentile: $\Phi(z) = .85 \Rightarrow z = 1.04$ from the standard normal table, so the 85th percentile of this distribution is $40 + (1.04)(1.5) = 41.56$ V.

c. For a single diode, $P(X > 42) = 1 - P(X \le 42) = 1 - \Phi\left(\frac{42-40}{1.5}\right) = 1 - \Phi(1.33) = .0918$.

Now let D represent the number of diodes (out of four) with voltage exceeding 42. The random variable D is binomial with $n = 4$ and $p = .0918$, so

$P(D \ge 1) = 1 - P(D = 0) = 1 - \binom{4}{0}(.0918)^0(.9082)^4 = 1 - .6803 = .3197$.

137.

a. Let X = the number of defectives in the batch of 250, so $X \sim \text{Bin}(250, .05)$. We can approximate X by a normal distribution, since $np = 12.5 \ge 10$ and $nq = 237.5 \ge 10$. The mean and sd of X are $\mu = np = 12.5$ and $\sigma = 3.446$. Using a continuity correction and realizing 10% of 250 is 25,

$P(X \ge 25) = 1 - P(X < 25) = 1 - P(X \le 24.5) \approx 1 - \Phi\left(\frac{24.5-12.5}{3.446}\right) = 1 - \Phi(3.48) =$

$1 - .9997 = .0003$. (The exact binomial probability, from software, is .00086.)

b. Using the same normal approximation with a continuity correction, $P(X = 10) =$

$P(9.5 \le X \le 10.5) \approx \Phi\left(\frac{10.5-12.5}{3.446}\right) - \Phi\left(\frac{9.5-12.5}{3.446}\right) = \Phi(-.58) - \Phi(-.87) = .2810 - .1922 = .0888$.

(The exact binomial probability is $\binom{250}{10}(.05)^{10}(.95)^{240} = .0963$.)

139.

a. $F(x) = 0$ for $x < 1$ and $F(x) = 1$ for $x > 3$. For $1 \le x \le 3$, $F(x) = \int_1^x \frac{1.5}{y^2}\,dy = 1.5\left(1 - \frac{1}{x}\right)$.

b. $P(X \le 2.5) = F(2.5) = 1.5(1 - .4) = .9$; $P(1.5 \le X \le 2.5) = F(2.5) - F(1.5) = .4$.

c. $E(X) = \int_1^3 x \cdot \frac{1.5}{x^2}\,dx = 1.5\int_1^3 \frac{1}{x}\,dx = 1.5\ln(x)\big]_1^3 = 1.648$ seconds.

d. $E(X^2) = \int_1^3 x^2 \cdot \frac{1.5}{x^2}\,dx = 1.5\int_1^3 dx = 3$, so $V(X) = E(X^2) - [E(X)]^2 = .284$ and $\sigma = .553$ seconds.

e. From the description, $h(x) = 0$ if $1 \le x \le 1.5$; $h(x) = x - 1.5$ if $1.5 \le x \le 2.5$ (one second later), and $h(x) = 1$ if $2.5 \le x \le 3$. Using those terms,

$E[h(X)] = \int_1^3 h(x)\,dx = \int_{1.5}^{2.5} (x - 1.5) \cdot \frac{1.5}{x^2}\,dx + \int_{2.5}^3 1 \cdot \frac{1.5}{x^2}\,dx = .267$ seconds.

141.

a. Since X is exponential, $E(X) = \frac{1}{\lambda} = 1.075$ and $\sigma = \frac{1}{\lambda} = 1.075$.

b. $P(X > 3.0) = 1 - P(X \le 3.0) = 1 - F(3.0) = 1 - [1 - e^{-.93(3.0)}] = .0614$.
$P(1.0 \le X \le 3.0) = F(3.0) - F(1.0) = [1 - e^{-.93(3.0)}] - [1 - e^{-.93(1.0)}] = .333$.

c. The 90th percentile is requested: $.9 = F(\eta_{.9}) = 1 - e^{-.93\eta_{.9}} \Rightarrow \eta_{.9} = \frac{\ln(.1)}{(-.93)} = 2.476$ mm.

143. We have a random variable $T \sim N(\mu, \sigma)$. Let $f(t)$ denote its pdf.

a. The "expected loss" is the expected value of a piecewise-defined function, so we should first write the function out in pieces (two integrals, as seen below). Call this expected loss $Q(a)$, to emphasize we're interested in its behavior as a function of a. We have:

$$Q(a) = E[L(a,T)] = \int_{-\infty}^{a} k(a-t) f(t)\,dt + \int_{a}^{\infty} (t-a) f(t)\,dt$$

$$= ka\int_{-\infty}^{a} f(t)\,dt - k\int_{-\infty}^{a} tf(t)\,dt + \int_{a}^{\infty} tf(t)\,dt - a\int_{a}^{\infty} f(t)\,dt = kaF(a) - k\int_{-\infty}^{a} tf(t)\,dt + \int_{a}^{\infty} tf(t)\,dt - a[1-F(a)]$$

where $F(a)$ denotes the cdf of T. To minimize this expression, take the first derivative with respect to a, using the product rule and the fundamental theorem of calculus where appropriate:

$$\begin{aligned} Q'(a) &= kaF(a) - k\int_{-\infty}^{a} tf(t)\,dt + \int_{a}^{\infty} tf(t)\,dt - a[1-F(a)] \\ &= kF(a) + kaF'(a) - kaf(a) + 0 - af(a) - 1 + F(a) + aF'(a) \\ &= kF(a) + kaf(a) - kaf(a) - af(a) - 1 + F(a) + af(a) \\ &= (k+1)F(a) - 1 \end{aligned}$$

Finally, set this equal to zero, and use the fact that, because T is a normal random variable, $F(a) = \Phi\left(\frac{a-\mu}{\sigma}\right)$:

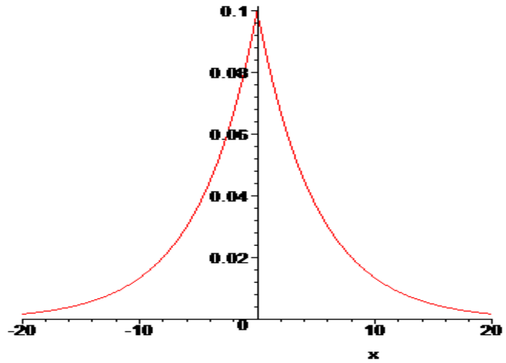
$$(k+1)F(a) - 1 = 0 \Rightarrow (k+1)\Phi\left(\frac{a-\mu}{\sigma}\right) - 1 = 0 \Rightarrow \Phi\left(\frac{a-\mu}{\sigma}\right) = \frac{1}{k+1} \Rightarrow a = \mu + \sigma\cdot\Phi^{-1}\left(\frac{1}{k+1}\right)$$

This is the critical value, a^*, as desired.

b. With the values provided, $a^* = 100{,}000 + 10{,}000\Phi^{-1}\left(\frac{1}{2+1}\right) = 100{,}000 + 10{,}000\Phi^{-1}(0.33) = 100{,}000 +$ $10{,}000(-0.44)$ from the standard normal table $= 100{,}000 - 4{,}400 = \$95{,}600$. The probability of an over-assessment equals $P(95{,}600 > T) = P(T < 96{,}500) = \Phi\left(\frac{95{,}600 - 100{,}000}{10{,}000}\right) = \Phi(-0.44) = .3300$, or 33%. Notice that, in general, the probability of an over-assessment using the optimal value of a is equal to $\frac{1}{k+1}$.

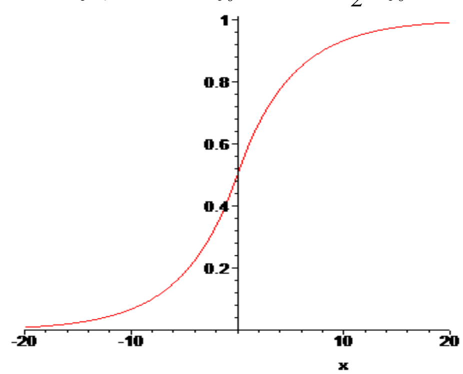
145.

a. $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} .1e^{.2x}dx + \int_{0}^{\infty} .1e^{-.2x}dx = .5 + .5 = 1.$



b. For $x < 0$, $F(x) = \int_{-\infty}^{x} .1e^{.2y}dy = \frac{1}{2}e^{.2x}$.

For $x \geq 0$, $F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{-\infty}^{0} .1e^{-.2y}\,dy + \int_{0}^{x} .1e^{-.2y}dy = \frac{1}{2} + \int_{0}^{x} .1e^{-.2y}dy = 1 - \frac{1}{2}e^{-.2x}$.



c. $P(X < 0) = F(0) = .5$; $P(X < 2) = F(2) = 1 - .5e^{-.4} = .665$; $P(-1 \leq X \leq 2) = F(2) - F(-1) = .256$; and $P(|X| > 2) = 1 - (-2 \leq X \leq 2) = 1 - [F(2) - F(-2)] = .670$.

147.

a. Provided $\alpha > 1$, $1 = \int_{5}^{\infty} \frac{k}{x^\alpha}dx = k \cdot \frac{5^{1-\alpha}}{\alpha - 1} \Rightarrow k = (\alpha - 1)5^{\alpha-1}$.

b. For $x \geq 5$, $F(x) = \int_{5}^{x} \frac{(\alpha-1)5^{\alpha-1}}{y^\alpha}dy = -5^{\alpha-1}\left[x^{1-\alpha} - 5^{1-\alpha}\right] = 1 - \left(\frac{5}{x}\right)^{\alpha-1}$. For $x < 5$, $F(x) = 0$.

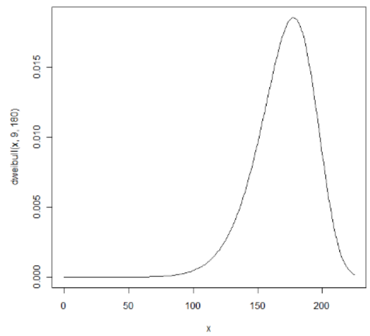
c. Provided $\alpha > 2$, $E(X) = \int_{5}^{\infty} x \cdot \frac{k}{x^\alpha}dx = \int_{5}^{\infty} \frac{(\alpha-1)5^{\alpha-1}}{x^{\alpha-1}}dx = 5\frac{\alpha-1}{\alpha-2}$.

d. Let $Y = \ln(X/5)$. Then $F_Y(y) = P\left(\ln\left(\frac{X}{5}\right) \leq y\right) = P\left(\frac{X}{5} \leq e^y\right) = P\left(X \leq 5e^y\right) = F(5e^y) = 1 - \left(\frac{5}{5e^y}\right)^{\alpha-1} =$

$1 - e^{-(\alpha-1)y}$, the cdf of an exponential rv with parameter $\alpha - 1$.

149.

a. The accompanying Weibull pdf plot was created in R.



b. $P(X > 175) = 1 – F(175; 9, 180) = e^{-(175/180)^9} = .4602$.
$P(150 \le X \le 175) = F(175; 9, 180) – F(150; 9, 180) = .5398 – .1762 = .3636$.

c. From **b**, the probability a specimen is not between 150 and 175 equals 1 – .3636 = .6364. So, P(at least one is between 150 and 175) = 1 – P(neither is between 150 and 175) = 1 – $(.6364)^2$ = .5950.

d. We want the 10th percentile: $.10 = F(x; 9, 180) = 1-e^{-(x/180)^9}$. A small bit of algebra leads us to $x = 180(–\ln(1–.10))^{1/9} = 140.178$. Thus, 10% of all tensile strengths will be less than 140.178 MPa.

151.

a. If we let $\alpha = 2$ and $\beta = \sqrt{2}\sigma$, then we can manipulate $f(v)$ as follows:
$f(v) = \dfrac{v}{\sigma^2}e^{-v^2/2\sigma^2} = \dfrac{2}{2\sigma^2}ve^{-v^2/2\sigma^2} = \dfrac{2}{(\sqrt{2}\sigma)^2}v^{2-1}e^{-\left(v/\sqrt{2}\sigma\right)^2} = \dfrac{\alpha}{\beta^\alpha}v^{\alpha-1}e^{-(v/\beta)^\alpha}$, which is in the Weibull family of distributions.

b. Use the Weibull cdf: $P(V \le 25) = F(25;2,\sqrt{2}\sigma) = 1-e^{-\left(\frac{25}{\sqrt{2}\sigma}\right)^2} = 1-e^{-\frac{625}{800}}$ = 1 – .458 = .542.

153.

a. $f(x) = \lambda e^{-\lambda x}$ and $F(x) = 1-e^{-\lambda x}$, so $r(x) = \dfrac{\lambda e^{-\lambda x}}{1-[1-e^{-\lambda x}]} = \lambda$, a constant. This is consistent with the memoryless property of the exponential distribution.

b. Substituting Expressions (4.9) and (4.10), $r(x) = \dfrac{(\alpha/\beta^\alpha)x^{\alpha-1}e^{-(x/\beta)^\alpha}}{1-[1-e^{-(x/\beta)^\alpha}]} = (\alpha/\beta^\alpha)x^{\alpha-1}$. For $\alpha > 1$, $r(x)$ is an increasing function of x; for $\alpha < 1$, $r(x)$ decreases with x.

c. First, $\ln[1-F(x)] = -\int \alpha\left(1-\dfrac{x}{\beta}\right)dx = -\alpha\left(x-\dfrac{x^2}{2\beta}\right) \Rightarrow F(x) = 1-e^{-\alpha(x-x^2/2\beta)}$ up to $x = \beta$. Then $f(x) = F'(x) = \alpha\left(1-\dfrac{x}{\beta}\right)e^{-\alpha(x-x^2/2\beta)}$ for $0 \le x \le \beta$. Note that this function integrates to less than 1, meaning that some probability has been assigned to $x = \infty$ (a device that "lasts forever").

155. For $y > 0$, $F(y) = P(Y \le y) = P\left(\frac{2X^2}{\beta^2} \le y\right) = P\left(X^2 \le \frac{\beta^2 y}{2}\right) = P\left(X \le \frac{\beta\sqrt{y}}{\sqrt{2}}\right) = F_X(\beta\sqrt{y/2})$

$= 1 - \exp\left[-\left(\frac{\beta\sqrt{y/2}}{\beta}\right)^2\right]$ (that's the Weibull cdf) $= 1 - e^{-y/2}$. Differentiate: $f(y) = (1/2)e^{-y/2}$. We recognize this as an exponential distribution, aka the gamma distribution with parameters 1 and 2.

157. When $X \le q$, gross profits are profit + salvage = $dX + e(q - X)$. But when $X > q$, gross profits are profit – shortage cost = $dq - f \cdot (X - q)$. In any case, there are fixed costs of $c_0 + c_1q$.
If we let Y denote the <u>net</u> profit, then $E[Y]$ =
$\int_0^\infty$ [gross profit] $f_X(x)\,dx - [c_0 + c_1q] = \int_0^q [dx + e(q-x)]f_X(x)dx + \int_q^\infty [dq - f(x-q)]f_X(x)dx - [c_0 + c_1q]$. Expand and simplify:
$E[Y] = (d - e)\int_0^q xf_X(x)dx + eq\,F_X(q) + (dq + fq)[1 - F_X(q)] - f \cdot \int_q^\infty xf_X(x)dx - [c_0 + c_1q]$.

Differentiate, using the Fundamental Theorem of Calculus, and then cancel as much as possible:
$d/dq\ E[Y] = (d - e)qf_X(q) + eF_X(q) + eqf_X(q) + (d + f)[1 - F_X(q)] + (dq + fq)[-f_X(q)] - f \cdot [-qf_X(q)] - [0 + c_1]$
$= eF_X(q) + (d + f)[1 - F_X(q)] - c_1$. Whew!

The optimal value q^* makes the derivative equal zero, so $eF_X(q^*) + (d + f)[1 - F_X(q^*)] - c_1 = 0$, from which we finally get $F_X(q^*) = (d - c_1 + f)/(d - e + f)$. Notice that c_0 is irrelevant to the optimization. For the values provided, $F_X(q^*) = (35 - 15 + 25)/(35 - 5 + 25) = 45/55 = .8182$.