Solutions Manual

a supplement to

Real Analysis: Foundations
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Contents

0 Using the Manual

In my opinion, the most effective way of learning mathematics is by "doing it". Accordingly, I urge the student not to read the solutions in advance, but rather to make a concerted effort to find a solution to the problem in the exercise before consulting the Solution Manual to verify correctness. If the student's solution differs from the one given in the Manual, a comparison might reveal an unjustified assumption that had been made by the student or a misapplication of a theorem. Meanwhile, the instructor can use the solutions to create balanced assignments and research projects.

Solutions in the Manual are labeled in the same way as exercises in the book. For instance, item **1.10** in the Manual is a solution to the problem in Exercise 1.10.

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1 Rational Numbers

1.1. Evidently, $(mp)n = m(np)$ for all $m, n, p \in \mathbf{Z}$. Hence, $(mp, np) \sim (m, n)$. We need $p \neq 0$ to make sure that (mp, np) is a fraction.

1.2. Suppose that $\frac{n}{1} = \frac{m}{1}$. Then $(n, 1) \sim (m, 1)$, that is, $n \cdot 1 = m \cdot 1$. Hence, $n = m$, so φ is one-to-one. Furthermore (cf. (1,1) on p. 4 in the book),

$$\varphi(m+n) = \frac{m+n}{1} = \frac{m}{1} + \frac{n}{1} = \varphi(m) + \varphi(n),$$

and

$$\varphi(m \cdot n) = \frac{m \cdot n}{1} = \frac{m}{1} \cdot \frac{n}{1} = \varphi(m) \cdot \varphi(n),$$

for all $m, n \in \mathbf{Z}$.

1.3. Straightforward verification of the properties defining a field.

1.4. Suppose that $a + b = 0$ in $\mathbf{F}$. Then, $-a + a + b = -a$. By Property **A4**, $b = -a$. Hence, $-a$ is a unique additive inverse of a.

For $a \neq 0$, let b be an element of $\mathbf{F}$ such that $a \cdot b = 1$. By Property **M3**, we have

$$a^{-1} = 1 \cdot a^{-1} = a \cdot b \cdot a^{-1} = b \cdot a \cdot a^{-1} = b \cdot 1 = b.$$

Hence, a^{-1} is a unique multiplicative inverse of a.

1.5. By Property **D**, $0 + 0 = 0$ implies $a \cdot 0 + a \cdot 0 = a \cdot 0$. Hence,

$$a \cdot 0 = -a \cdot 0 + a \cdot 0 + a \cdot 0 = -a \cdot 0 + a \cdot 0 = 0.$$

1.6. (a) By Property **M3**, $1 \cdot 1 = 1$. By Property **M4** and Exercise 1.4, $1^{-1} = 1$.
(b) By Property **M1**, $a^{-1} \cdot b^{-1} \cdot a \cdot b = a^{-1} \cdot a \cdot b^{-1} \cdot b = 1$. Hence, $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$.
(c) If $c \cdot b = a$, then $c = c \cdot b \cdot b^{-1} = a \cdot b^{-1}$. If $c = a \cdot b^{-1}$, then $c \cdot b = a \cdot b^{-1} \cdot b = a$.
(d) By part (b) above,

$$\frac{a \cdot c}{b \cdot d} = (a \cdot c) \cdot (b \cdot d)^{-1} = a \cdot c \cdot b^{-1} \cdot d^{-1} = a \cdot b^{-1} \cdot c \cdot d^{-1} = \frac{a}{b} \cdot \frac{c}{d}.$$

(e) By part (d), $\frac{a \cdot c}{b \cdot c} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a}{b}$.

(f) We have

$$\begin{aligned}\frac{a \cdot d + c \cdot b}{b \cdot d} &= (a \cdot d + c \cdot b)(b \cdot d)^{-1} = (a \cdot d + c \cdot b)(b^{-1} \cdot d^{-1}) \\ &= a \cdot d \cdot b^{-1} \cdot d^{-1} + c \cdot b \cdot b^{-1} \cdot d^{-1} \\ &= a \cdot b^{-1} + c \cdot d^{-1} = \frac{a}{b} + \frac{c}{d}.\end{aligned}$$

(g) If $\dfrac{a}{b} = \dfrac{c}{d}$, then $(a \cdot b^{-1}) \cdot b \cdot d = (c \cdot d^{-1}) \cdot b \cdot d$, so $a \cdot d = b \cdot c$. If $a \cdot d = b \cdot c$, then $(a \cdot d) \cdot b^{-1} \cdot d^{-1} = (b \cdot c) \cdot b^{-1} \cdot d^{-1}$, so $\dfrac{a}{b} = a \cdot b^{-1} = c \cdot d^{-1} = \dfrac{c}{d}$.

(h) Because, clearly, $(b^{-1})^{-1} = b$,

$$\left(\frac{a}{b}\right)^{-1} = (a \cdot b^{-1})^{-1} = a^{-1} \cdot b = \frac{b}{a}.$$

(i) ***Correction.*** The right hand side of the equality must be $\dfrac{a \cdot d}{b \cdot c}$. It is easy to verify that $\left(\dfrac{c}{d}\right)^{-1} = \dfrac{d}{c}$. Therefore, by part (d) above, $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{a \cdot d}{b \cdot c}$.

1.7. Note that $\mathbf{Z}[x]$ is an integral domain (cf. Exercise A.14)) and repeat the steps in the proof of Theorem 1.1.

1.8. Repeat the steps in the proof of Theorem 1.4.

1.9. Suppose that $1 < 0$ and let $a \neq 0$ be a positive element of $\mathbf{F}$. Then $a \cdot a^{-1} = 1 < 0$. By part (b) of Definition 1.7, $a = a \cdot a^{-1} \cdot a < a \cdot 0 = 0$, a contradiction.

1.10. ***Correction.*** The function τ in the proof of Theorem 1.8 is defined only for nonnegative integers, not for all $n \in \mathbf{Z}$. For negative integers, we set $\tau(-n) = -\tau(n)$ for $n \in \mathbf{N}$. Thus Exercise 1.10 is, in fact, a part of the definition of the function τ.

1.11. By Definition 1.7(b), $ab^{-1} < cd^{-1}$ is equivalent to $ad < cb$, because $b > 0$ and $d > 0$.

1.12. By mathematical induction. The case $n = 3$ follows immediately from the transitivity property of the linear order $<$ (cf. Definition A.7). By the same property, the case of $n = k + 1$ follows from the case of $n = k$.

1.13. First, suppose that $x > 0$. By the Archimedean Property, there is $n \in \mathbf{N}$ such that $x < n$, so the set $\{k \in \mathbf{N} : x < k\}$ is nonempty. By the Well-Ordering Principle (cf. Theorem A.6), there is $m \in \mathbf{N}$ such that $m - 1 \leq x < m$. For a non-positive x, note that the claim is trivial if $x \in \mathbf{Z} \setminus \mathbf{N}$. If $x \notin \mathbf{Z} \setminus \mathbf{N}$, apply the previous argument to $-x$.

1.14. It is shown in Exercise 1.8 that $\mathbf{Z}(x)$ is a field. An order on $\mathbf{Z}(x)$ is defined in Example 1.3. To show that $\mathbf{Z}(x)$ endowed with this order is an ordered field, repeat the steps in the proofs of Theorems 1.6 and 1.7.

1.15. Suppose to the contrary that $D = p^2/q^2$ for some $p, q \in \mathbf{N}$. We may assume that q is the least natural number satisfying this equation (cf. the proof of Theorem 1.10.). Because D is not a perfect square, there is $t \in \mathbf{N}$ such that

$$t^2 < D < (t+1)^2.$$

(Why?) It follows that

$$tq < p < (t+1)q.$$

Let $q' = p - tq$ and $p' = Dq - tp$. It is easy to verify that $0 < q' < q$ and $p' > 0$. We have

$$\begin{aligned} p'^2 - Dq'^2 &= (Dq - tp)^2 - D(p - tq)^2 \\ &= D^2q^2 - 2Dqtp + t^2p^2 - Dp^2 + 2Dptq - Dt^2q^2 \\ &= (D^2q^2 - Dt^2q^2) - (Dp^2 - t^2p^2) = Dq^2(D - t^2) - p^2(D - t^2) \\ &= (Dq^2 - p^2)(D - t^2) = 0, \end{aligned}$$

because $D = p^2/q^2$. This contradicts the minimality of q.

1.16. Let $a = x - y$, $b = y - z$, so $x - z = a + b$. We need to show that $|a+b| \leq |a| + |b|$. By Exercise 1.17(e), $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Hence, $-(|a| + |b|) \leq a + b \leq |a| + |b|$. The desired result follows from Exercise 1.17(f).

1.17. We prove only part (h) using the triangular inequality from the proof of Theorem 1.12 and other parts of this exercise. We have

$$|a| = |(a - b) + b| \leq |a - b| + |b|.$$

Therefore, $|a| - |b| \leq |a - b|$. On the other hand,

$$|b| = |a + (b - a)| \leq |a| + |b - a| = |a| + |a - b|.$$

Hence, $|a| - |b| \geq -|a - b|$. The result follows from part (e).

1.18. If $a \leq b \leq c$, then $|a - b| = b - a$ and $|b - c| = c - b$. Hence,

$$|a - b| + |b - c| = (b - a) + (c - b) = c - a = |a - c|.$$

In the opposite direction, the proof is by contradiction. Suppose that $b < a$, so also $b < c$. Then

$$|a - b| + |b - c| = (a - b) + (c - b) = a + c - 2b > a + c - 2a = c - a = |a - c|,$$

a contradiction. Similarly, we obtain a contradiction by assuming that $b > c$. Therefore, $a \leq b \leq c$.

1.19. Without loss of generality, we may assume that $x > y$. We have

$$b - a = y - a + x - y + b - x \geq x - y,$$

because $y - a \geq 0$ and $b - x \geq 0$. Hence the result.

1.20. ***Correction.*** The inequality $|a_k| > \varepsilon$ in Exercise 1.20 must be replaced with $|a_k - a| > \varepsilon$.

To form the negation of the statement in Definition 1.12, we must replace the quantifier "for every positive ε" by "there exists $\varepsilon > 0$", the quantifier "there exists" by "for every", and the quantifier "for all" by "for some".

1.21. Standard Calculus exercises.

1.22. Inasmuch as (a_n) is Cauchy, for $\varepsilon > 0$ there is $N \in \mathbf{N}$ such that

$$|a_m - a_n| < \varepsilon, \qquad \text{for all } m, n > N.$$

By Exercise 1.17(h),

$$\big||a_m| - |a_n|\big| \leq |a_m - a_n| < \varepsilon, \qquad \text{for all } m, n > N.$$

Therefore, $(|a_n|)$ is Cauchy.

1.23. Apply Theorem 1.20 for a constant sequence $(b_n) = (k, k, \ldots)$.

1.24. First note that the sequence (a_n) is bounded below by a positive rational number. Indeed, because $\lim a_n^2 = p > 0$, there is only a finite number of terms of (a_n^2) that are smaller than $p/2$. Hence, (a_n^2) is bounded below by a positive rational number (cf. Exercise 1.25). It follows that there is a positive $r \in \mathbf{Q}$ such that $a_n > r$ for all $n \in \mathbf{N}$. By Theorem 1.16, (a_n^2) is Cauchy, that is, for $\varepsilon > 0$ there is $N \in \mathbf{N}$ such that $|a_m^2 - a_n^2| < 2r \cdot \varepsilon$ for all $m, n > N$. We have

$$|a_m - a_n| = \frac{|a_m^2 - a_n^2|}{a_m + a_n} < \frac{2r \cdot \varepsilon}{2r} = \varepsilon, \qquad \text{for all } m, n > N.$$

Therefore, (a_n) is a Cauchy sequence.

1.25. The proof is by induction on the cardinality of the set A. The claim is trivial if A consists of one or two elements. Suppose that it holds for every n-element set and let A be a set containing $n + 1$ elements. For an element x of A, let $A' = A \setminus \{x\}$ and $a = \min A'$, $b = \max A'$. If $x < a$, we set $\min A = x$, $\max A = b$, if $x > b$, we set $\max A = x$, $\min A = a$. Finally, if $a \leq x \leq b$, we set $\min A = a$, $\max A = b$. It is not difficult to verify that in all these cases, $\min A \leq c \leq \max A$ for every $c \in A$.

1.26. Recall that a tail of the sequence (a_n) is a subsequence $(a_m, a_{m+1}, \ldots)$, $m \in \mathbf{N}$. For $\varepsilon > 0$ there is $N \in \mathbf{N}$ such that $|a_n - a| < \varepsilon$ for all $n > N$. It follows that $|a_n - a| < \varepsilon$ for all $n > \max\{m, N\}$. Hence, the subsequence converges to the same limit.

1.27. We define the mapping f recursively. By the Well-Ordering Principle (cf. Theorem A.6), the set S has a least element. We define $f(1)$ to be this element. Suppose the values $f(1), f(2), \ldots, f(n)$ are defined and satisfy the required inequalities. By the same Principle, the nonempty set $S \setminus \{f(1), \ldots, f(n)\}$ has a least element. We define $f(n+1)$ to be this element. Clearly, thus defined function f satisfies the required inequalities.

1.28. Straightforward algebra.

2 Real Numbers

2.1. Suppose that $b \neq b'$ are two suprema of a nonempty subset of an ordered field. By Definition 2.1, $b < b'$ and $b' < b$, which contradicts the Trichotomy Property of the relation $<$ (cf. Theorem A.14). A similar argument proves uniqueness of the infimum.

2.2. Let E be a subset of $\mathbf{F}$ bounded below and E' the set of all lower bounds of E. Because $\mathbf{F}$ is Dedekind complete, it has a supremum, $c = \sup E'$. By Definition 2.1, $c = \inf E$.

2.3. $q - p = (r - p) + (s - r) + (q - s) > 0 + (s - r) + 0 = s - r$.

2.4. Suppose that $a_n \leq a_{n+1}$, for all $n \in \mathbf{N}$. For a given n, we prove that $a_n \leq a_{n+k}$, $k \in \mathbf{N}$, by induction on k. The case $k = 1$ is trivial. Suppose that $a_n \leq a_{n+k-1}$. By transitivity, the inequalities $a_n \leq a_{n+k-1}$ and $a_{n+k-1} \leq a_{n+k}$ imply $a_n \leq a_{n+k}$. A similar argument proves that $b_n \geq b_m$ for $n < m$.

2.5. By the Archimedean Property (cf. Definition 1.8), for $\varepsilon > 0$ there is $N \in \mathbf{N}$ such that $N\varepsilon > 1$. Hence, $1/n < \varepsilon$ for all $n > N$, that is $1/n \to 0$.

In the other direction, suppose that $1/n \to 0$ in $\mathbf{F}$. For positive x and y in $\mathbf{F}$ there is $N \in \mathbf{N}$ such that $1/n < x/y$ for all $n > N$. Hence, $(N+1)x > y$, so the Archimedean Property holds.

2.6. Every $b \in B$ is an upper bound of A. Hence, $\sup A \leq b$ for every $b \in B$, so $\sup A$ is a lower bound of B. It follows that $\sup A \leq \inf B$.

2.7. Let x and y be upper bounds of the sets A and B, respectively. Then $\max\{x, y\}$ is, clearly, an upper bound of $A \cup B$, so this union is bounded.

Clearly, $\sup(A \cup B) \geq \sup A$ and $\sup(A \cup B) \geq \sup B$. Therefore,

$$\sup(A \cup B) \geq \max\{\sup A, \sup B\}.$$

We may assume that $\sup A \leq \sup B$. For every $x \in A$, $x \leq \sup A \leq \sup B$ and for every $x \in B$, $x \leq \sup B$. Hence, for every $x \in A \cup B$,

$$x \leq \sup B = \max\{\sup A, \sup B\},$$

so $\sup(A \cup B) \leq \max\{\sup A, \sup B\}$. The result follows.

2.8. Let $x = \sup A$, $y = \sup B$. For $a \in A$ and $b \in B$, we have $a + b \leq x + y$, so $x + y$ is an upper bound of $A + B$. By Theorem 2.1, for $\varepsilon > 0$ there are $a \in A$, $b \in B$ such that $a < x - \varepsilon/2$ and $b < y - \varepsilon/2$, so $a + b < x + y - \varepsilon$. By the same theorem, $x + y$ is the supremum of $A + B$.

2.9. We prove that $\sup(cE) = c \inf E$ for $c < 0$. The other cases are treated similarly.

For $a \in E$, $\inf E \le a$. Hence $c \inf E \ge ca$ for all $a \in E$. Thus, $c \inf E$ is an upper bound of cE.

Let x be an upper bound of cE, that is, $x \ge ca$ for all $a \in E$. It follows that $x/c \le a$ for all $a \in E$, so $x/c \le \inf E$. Then $x \ge c \inf E$, that is, $c \inf E$ is the supremum of cE.

2.10. Clearly, $\inf E' \le \sup E'$. Because $E' \subseteq E$, $\sup E$ is an upper bound of E'. It follows that $\sup E' \le \sup E$. A similar argument shows that $\inf E \le \inf E'$.

2.11. (Necessity.) Part (a) follows from Theorem 2.1 with $\varepsilon = 1/n$. Part (b) is trivial.

(Sufficiency.) First, we show that b is an upper bound of E. Suppose to the contrary that there is $x \in E$ such that $x > b$ and let n be a positive integer such that $1/n < x - b$. Then $b + 1/n < x$ which contradicts part (b). Thus b is an upper bound of E.

For $\varepsilon > 0$ let n be a positive integer such that $1/n < \varepsilon$. Since $b - 1/n > b - \varepsilon$, by part (a), $b - \varepsilon$ is not an upper bound of E. By Theorem 2.1, $b = \sup E$.

2.12. Suppose that $\sqrt{2} + \sqrt{3}$ is a rational number. Then the number

$$\sqrt{2} - \sqrt{3} = \frac{-1}{\sqrt{2} + \sqrt{3}}$$

is rational, as well as the number

$$\frac{(\sqrt{2} + \sqrt{3}) + (\sqrt{2} - \sqrt{3})}{2} = \sqrt{2},$$

contradicting Theorem 1.10. Hence, $\sqrt{2} + \sqrt{3}$ is an irrational number.

2.13. ***Correction.*** It is assumed that $b \neq 0$.

Straightforward proofs in all four cases are by contradiction.

2.14. Clearly, $\lim(a_n - a_n) = 0$, so the relation $\sim$ is reflexive.

Also, $\lim(a_n - b_n) = 0$ implies $\lim(b_n - a_n) = 0$, so the relation $\sim$ is symmetric.

To prove transitivity of $\sim$, suppose that $\lim(a_n - b_n) = 0$ and $\lim(b_n - c_n) = 0$. Then

$$\lim(a_n - c_n) = \lim[(a_n - b_n) + (b_n - c_n)] = \lim(a_n - b_n) + \lim(b_n - c_n) = 0.$$

Hence the result.

2.15. By definition (cf. (2.1) on page 39 in the book), $(a_n) \in [(a)]$ if and only if $\lim(a_n - a) = 0$. Clearly, the latter condition is equivalent to $\lim a_n = a$.

2.16. Let $a = (x + y)/2$ and $\varepsilon = (y - x)/2$. By Definition 2.6, there is $r \in E$ such that

$$\left| r - \frac{x + y}{2} \right| < \frac{y - x}{2}.$$

Elementary algebra shows that the displayed inequality is equivalent to the chain inequality $x < r < y$.

2.17. By Lemma 2.12, $(\widetilde{a}_n)$ is a Cauchy sequence in $\tilde{\mathbf{F}}$, and, by Theorem 2.8, $\widetilde{a}_n \to [(a_m)]$. Hence there is $N \in \mathbf{N}$ such that $\big|[(a_m)] - \widetilde{a}_n\big| < \varepsilon$ for all $n > N$. We obtain the desired result by setting $a = a_{N+1}$.

2.18. (a) Because $a < b$ if and only if $\varphi(a) < \varphi(b)$ for all $a, b \in \mathbf{F}$, the mapping φ is one-to-one.

(b) We have $\varphi(1) = \varphi(1 + 0) = \varphi(1) + \varphi(0)$. Hence, $\varphi(0) = 0$. By part (a), $\varphi(1) \neq 0$. We have $\varphi(1) = \varphi(1 \cdot 1) = \varphi(1) \cdot \varphi(1)$. Therefore, $\varphi(1) = 1$.

(c) Straightforward verification of the properties defining an ordered field.

2.19 (Necessity.) Suppose that $\varphi(a_n) \to 0$ in $\mathbf{G}$. For $\varepsilon > 0$ in $\mathbf{F}$, $\varphi(\varepsilon) > 0$ in $\mathbf{G}$. There is $N \in \mathbf{N}$ such that

$$-\varphi(\varepsilon) < \varphi(a_n) < \varphi(\varepsilon), \qquad \text{for all } n > N.$$

Because φ is an embedding,

$$-\varepsilon < a_n < \varepsilon, \qquad \text{for all } n > N.$$

Hence, $a_n \to 0$ in $\mathbf{F}$.

(Sufficiency.) Let $a_n \to 0$ in $\mathbf{F}$ and $\varepsilon > 0$ in $\mathbf{G}$. By the density property, there is $\delta > 0$ in $\mathbf{F}$ such that $0 < \varphi(\delta) < \varepsilon$. Because $a_n \to 0$, there is $N \in \mathbf{N}$ such that

$$-\delta < a_n < \delta, \qquad \text{for all } n > N.$$

Then

$$-\varepsilon < \varphi(-\delta) < \varphi(a_n) < \varphi(\delta) < \varepsilon, \qquad \text{for all } n > N.$$

It follows that $\varphi(a_n) \to 0$ in $\mathbf{G}$.

2.20. If $\mathbf{F}$ is not an Archimedean field, then there are positive elements x and y in $\mathbf{F}$ such that $nx \leq y$ for all $n \in \mathbf{N}$, that is, $\underbrace{x + \cdots + x}_{n} \leq y$ for all $n \in \mathbf{N}$. Then $\underbrace{\beta(x) + \cdots + \beta(x)}_{n} \leq \beta(y)$. Note that $\beta(x) > 0$ and $\beta(y) > 0$. It follows that $\widetilde{\mathbf{F}}$ is not Archimedean.

2.21. Let $\varphi : \mathbf{F} \to \mathbf{G}$ be an isomorphism and x, y positive elements in $\mathbf{G}$. Because $\mathbf{F}$ is Archimedean, there is $n \in \mathbf{N}$ such that $n\varphi^{-1}(x) > \varphi^{-1}(y)$. Since φ is an isomorphism, $nx > y$ (cf. solution 2.20). Hence, $\mathbf{G}$ is Archimedean.

2.22. Let (a_n) be a Cauchy sequence in $\mathbf{Q}$. For eny $\varepsilon > 0$ in $\mathbf{R}$ there is $\varepsilon' \in \mathbf{Q}$ such that $0 < \varepsilon' < \varepsilon$ (cf. Exercise 2.16). Because (a_n) is Cauchy in $\mathbf{Q}$, there is $N \in \mathbf{N}$ such that

$$|a_m - a_n| < \varepsilon' < \varepsilon, \qquad \text{for all } m, n > N.$$

Hence, (a_n) is Cauchy in $\mathbf{R}$.

Clearly, every Cauchy sequence of rational numbers in $\mathbf{R}$ is also Cauchy in the field $\mathbf{Q}$.

2.23. Hint. Use the approach from Exercise 2.22.

2.24. (Necessity.) Theorem 2.14.

(Sufficiency.) Suppose that $\mathbf{Q}$ is dense is $\mathbf{F}$ and let x and y be positive elements of $\mathbf{F}$. We my assume that $x < y$. Let p and q be positive rational numbers such that $p < x$ and $q > y$ (cf. Exercise 2.16). Because $\mathbf{Q}$ is Archimedean, there is $n \in \mathbf{N}$ such that $np > q$. We have $nx > np > q > y$. Hence, $\mathbf{F}$ is Archimedean.

2.25. Clearly, both sets $\mathbf{F}$ and $\mathbf{G}$ are closed under operations of addition and multiplication. Verifying ordered field properties is a tedious but straightforward exercise.

Suppose that $\varphi : \mathbf{F} \to \mathbf{G}$ is an isomorphism. It is not difficult to see that φ is the identity map on $\mathbf{Q}$ (which is an ordered subfield of both $\mathbf{F}$ and $\mathbf{G}$). We have $(\sqrt{2})^2 = 2$ in $\mathbf{F}$. Because φ is an isomorphism, we must have $(\varphi(\sqrt{2}))^2 = \varphi(2) = 2$. Let $\varphi(\sqrt{2}) = a + b\sqrt{3}$. If $ab \neq 0$, then

$$(a + b\sqrt{3})^2 = a^2 + 2\sqrt{3}\,ab + 3b^2 = 2,$$

which implies

$$\sqrt{3} = \frac{2 - a^2 - 3b^2}{2ab},$$

a contradiction, because the number on the right hand side is rational.

If $a = 0$, then $b^2 = 2/3$, which is not possible, because $\sqrt{2/3}$ is not a rational number (prove it!). If $b = 0$, then $a^2 = 2$, a contradiction.

It follows that $\mathbf{F}$ and $\mathbf{G}$ are not isomorphic.

2.26. It suffices to show that every open interval (a, b) in $\mathbf{R}$ contains an irrational number (cf. Definition 2.6). Because $\mathbf{Q}$ is dense in $\mathbf{R}$ (cf. Theorem 2.14), there is a rational number p such that $a < p < b$. By the Archimedean Property, there is $n \in \mathbf{N}$ such that $n > \dfrac{\sqrt{2}}{b - p}$, which is equivalent to $p + \dfrac{1}{n}\sqrt{2} < b$. Clearly, $p + \sqrt{2}/n$ is an irrational number that belongs to (a, b).

2.27. Let (A, B) be a cut of $\mathbf{F}$. Suppose that $c < c'$ are two cut points for (A, B). The point $d = (c + c')/2$ satisfies $c < d < c'$. Because c and c' are cut points, d must belong to both sets A and B which is impossible.

2.28. Let c be a cut point for a cut (A, B). Clearly, c is an upper bound of the set A (cf. Definition 2.11). Any other upper bound of A must be in the set B. Hence, c is the least upper bound of A. Similarly, c is the greatest lower bound of the set B.

2.29. Suppose there is $a \in A$ such that $a > c$. Then, for $\varepsilon = a - c$ the interval $(c - \varepsilon, c + \varepsilon)$ does not contain points from B a contradiction. Hence, $a \leq c$ for all $a \in A$. Similarly, $c \leq b$ for all $b \in B$.

2.30 Let (A, B) be a gap in $\mathbf{F}$ and $x \in \mathbf{F}$. Clearly, $(A + x, B + x)$ is a cut. Suppose that c is a cut point for $(A + x, B + x)$. Then $c - x$ is a cut point for (A, B), a contradiction. Hence, $(A + x, B + x)$ is a gap for every $x \in \mathbf{F}$.

2.31. By the Cut property, $A = \{x < 0 \text{ or } x^2 \leq 2\}$ and $B = \mathbf{R} \setminus A$ form a cut with the unique cut point that is a solution of $x^2 = 2$.

3 Continuous Functions

3.1. (Necessity.) Let x be a point in an open set G and (a,b) a bounded open interval containing x. Then, for $\varepsilon = \min\{x-a, b-x\}$, $x \in (x-\varepsilon, x+\varepsilon)$. A similar argument works for an unbounded open interval containing x.

(Sufficiency.) Clearly, every ε-neighborhood of x is its neighborhood.

3.2. If $px+q = px'+q$, then $x = x'$ because $p \neq 0$. Thus, f is one-to-one. For $y \in \mathbf{F}$ we have $f((1/p)(y-q)) = y$. Hence, f is onto. Because $p > 0$, we have $px+q < px'+q$ for $x < x'$. Therefore, f preserves order on $\mathbf{F}$.

Let $(x-\varepsilon, x+\varepsilon) \subseteq G$. Then $(f(x)-p\,\varepsilon, f(x)+p\,\varepsilon) \subseteq f(G)$ (Hint: f preserves order). By Theorem 3.1, $f(G)$ is an open set.

3.3. We consider only the case of $I = (a,b]$. The other cases are treated similarly.

Let x be a point in I and (c,d) a bounded open interval containing x. (Clearly, an unbounded neighborhood of x contains a bounded one.) It can be directly verified that the point $p = (1/2)(\max\{a,c\} + x)$ belongs to $I \cap (c,d)$ and $p \neq x$. Hence, x is a limit point of I.

3.4. Let (a_n) be a convergent sequence of points in F and $a = \lim a_n$. Suppose that $a \notin F$. Inasmuch as $a_n \to a$, every neighborhood of a contains a point from (a_n). Hence, a is a limit point of F, a contradiction.

3.5. We have $-\varepsilon < x - y < \varepsilon$. Thus $y + \varepsilon > x$ and $y - \varepsilon < x$, implying that $y - \varepsilon > x - 2\varepsilon$ and $y + \varepsilon < x + 2\varepsilon$. Hence the result.

3.6. The sequence $(1/n)$ is a sequence of points in the set $E = \{1/n : n \in \mathbf{N}\}$ that converges to 0 which is not an element of E. Any subsequence of $(1/n)$ also converges to $0 \notin E$. Thus, E is not a compact subset of $\mathbf{R}$.

3.7. The set of natural numbers $\mathbf{N}$ is not a compact subset of the set $\mathbf{R}$. Indeed, the sequence (n) in $\mathbf{N}$ does not have a convergent subsequence.

Let $\{U_n\}_{n\in\mathbf{N}}$, where $U_n = \left(n - \frac{1}{n}, n + \frac{1}{n}\right)$, be an open covering of $\mathbf{N}$. For every $\varepsilon > 0$ there is $m > 2$ such that $1/m < \varepsilon$. Suppose that $(m-\varepsilon, m+\varepsilon) \subseteq U_n$ for some n. Then $m \in U_n$, that is, $n - \frac{1}{n} < m < n + \frac{1}{n}$, which implies that $n = m$. Because $1/m < \varepsilon$, $(m-\varepsilon, m+\varepsilon) \not\subseteq U_m$. It follows that the family $\{U_n\}_{n\in\mathbf{N}}$ is the desired example.

3.8. Clearly, there is $x \in \{x_n : n \in \mathbf{N}\}$ for which the set M of indexes n such that $x_n = x$ is infinite. By using the Well-Ordering Principle (cf. Theorem A.6), we can enumerate the set M in the increasing order, $n_1 < n_2 < \cdots < n_k < \cdots$. The constant subsequence (x_{n_k}) of (x_n) converges to x.

3.9. (a) Let (A,B) be a gap in $\mathbf{F}$, $a_1 \in A$, $b_1 \in B$, and (a_n), (b_n) sequences defined as in the proof of Theorem 2.4 on pages 34–35 in the book for $E = A$.

We want to show that the set $S = \{a_n : n \in \mathbf{N}\}$, the range of (a_n), is closed and bounded, but not compact.

Clearly, S is a bounded subset of A. In the next paragraph, we show that it is closed.

Let c be a limit point of S. Suppose that there is $a_m > c$. Then $a_n > c$ for all $n \geq m$, because (a_n) is an increasing sequence. The neighborhood $(2c - a_m, a_m)$ of c contains only finite terms of (a_n), contradicting our assumption that c is a limit point of S. It follows that $a_n \leq c$ for all $n \in \mathbf{N}$. Suppose that $a > c$ for some $a \in A$. We have $b_n - a_n > a - c$ for all $n \in \mathbf{N}$, contradicting Lemma 2.1, because the field $\mathbf{F}$ is Archimedean. Hence, $c \geq a$ for all $a \in A$. Suppose that $b < c$ for some $b \in B$. Because c is a limit point of S, there is $a_n \in (b, c+1)$, a contradiction, since $a_n \in A$. It follows that $a \leq c \leq b$ for all $a \in A$ and $b \in B$, that is, c is a cut point for (A, B), contradicting the assumption that (A, B) is a gap. Thus, S has no limit points. Therefore it is closed.

To show that S is not compact, suppose to the contrary that the sequence (a_n) contains a convergent subsequence and let d be the limit of this subsequence. Inasmuch as (a_n) is increasing, it also converges to the same limit d, which then must be a limit point of S (prove it!). It follows that S is not compact.

(b) Clearly, $\mathbf{N}$ has no limit points in $\mathbf{F}$ and therefore closed. Because $\mathbf{F}$ is not Archimedean, there is a positive element $y \in \mathbf{F}$ such that $n < y$ for all $n \in \mathbf{N}$ (cf. Definition 1.8). Hence, $\mathbf{N}$ is bounded. It is also clear that the sequence (n) does not have convergent subsequences. Therefore, $\mathbf{N}$ is not a compact subset of $\mathbf{F}$.

3.10. (Necessity.) By definition, a point $x \in \mathbf{R}$ is the limit of a subsequence of a sequence (x_n) if every neighborhood of x contains a tail of the subsequence. It follows that every neighborhood of x contains infinitely many terms of (x_n).

(Sufficiency.) Suppose that every neighborhood of $x \in \mathbf{R}$ contains infinitely many terms of (x_n). Then the neighborhood $(x-1, x+1)$ contains a term x_{n_1} of (x_n), the neighborhood $(x - 1/2, x + 1/2)$ contains a term x_{n_2} different from x_{n_1}, and so on. This way we construct a subsequence (x_{n_k}) of (x_n) such that $x_{n_m} \in (x - 1/k, x + 1/k)$ for all $m \geq k$. For every $\varepsilon > 0$ there is $k \in \mathbf{N}$ such that $(x - \varepsilon, x + \varepsilon) \subseteq (x - 1/k, x + 1/k)$, that is, $|x_{n_m} - x| < \varepsilon$ for all $m \geq k$. It follows that $x_{n_k} \to x$ in $\mathbf{R}$.

3.11. Let A and B be compact sets and (a_n) a sequence of points in $A \cup B$. The sequence (a_n) must have a subsequence (a_{n_k}) in at least one of the sets A and B, say, in A. Because A is compact, (a_{n_k}) has a convergent subsequence. This subsequence is also a convergent subsequence of points in the set $A \cup B$. Hence, $A \cup B$ is a compact set.

3.12. Let $\mathcal{K} = \{K_i\}_{i \in J}$ be a family of compact sets and $K = \bigcap \mathcal{K}$. We may assume that $K \neq \varnothing$, because the empty set is vacuously compact. A sequence of points in K is a sequence in each of the sets in $\mathcal{K}$ and therefore contains a convergent subsequence that converges to a point in K. Hence, K is a compact set.

3.13. Let K be a compact set and E a closed subset of K. Every sequence (a_n) of points in E is also a sequence in K and therefore has a convergent subsequence, say, $a_{n_k} \to a \in K$. Because E is closed, $a \in E$. Hence, E is compact.

3.14. Let a be an isolated point of E and $(a-\delta, a+\delta)$ a neighborhood of a that does not contain points of E that are different from a. For any $x \in E$, $|x-a| < \delta$ implies $x = a$. Therefore, for any $\varepsilon > 0$, $|f(x) - f(a)| < \varepsilon$. Hence f is continuous at a.

3.15. Hint. The condition $|x-a| < \delta$ and $x \in E$ is equivalent to $x \in E \cap U_\delta$, and the condition $|f(x) - f(a)| < \varepsilon$ is equivalent to $f(x) \in U_\varepsilon$.

3.16. We assume that functions f and g are continuous at $a \in E$.

(a) For $\varepsilon > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|x-a| < \delta_1,\ x \in E \qquad \Rightarrow \qquad |f(x) - f(a)| < \varepsilon/2$$

and

$$|x-a| < \delta_2,\ x \in E \qquad \Rightarrow \qquad |g(x) - g(a)| < \varepsilon/2.$$

Let $\delta = \{\delta_1, \delta_2\}$. Then, for $|x-a| < \delta$, $x \in E$, we have

$$|f(x) + g(x) - f(a) - g(a)| \leq |f(x) - f(a) + |g(x) - g(a)| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

that is, thus the function $f + g$ is continuous at a.

(b) Because, clearly, a constant function is continuous, the result follows from part (c) below.

(c) It is clear that a function that is continuous at $x = a$ is bounded in some neighborhood U_δ of a. Let $A > 0$ be an upper bound of both functions $|f|$ and $|g|$. For $\varepsilon > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that $\delta_1 < \delta$, $\delta_2 < \delta$ and

$$|x-a| < \delta_1,\ x \in E \qquad \Rightarrow \qquad |f(x) - f(a)| < \varepsilon/A$$

and

$$|x-a| < \delta_2,\ x \in E \qquad \Rightarrow \qquad |g(x) - g(a)| < \varepsilon/A.$$

Let $\delta = \{\delta_1, \delta_2\}$. Then, for $|x-a| < \delta$, $x \in E$, we have

$$\begin{aligned}
|f(x)g(x) - f(a)g(a)| &= |f(x)(g(x) - g(a)) + g(a)(f(x) - f(a))| \\
&\leq |f(x)||g(x) - g(a)| + |g(a)||f(x) - f(a)| \\
&< A \cdot \frac{\varepsilon}{A} + A \cdot \frac{\varepsilon}{A} = \varepsilon.
\end{aligned}$$

Hence, the function fg is continuous at $x = a$.

(d) By the result of part (c), it suffices to consider the case of $1/g$ where $g(x) \neq 0$ on E.

It is clear that a function that is continuous at $x = a$ is bounded in some neighborhood U_δ of a. Let $A > 0$ be a lower bound of the function $|g|$. For $\varepsilon > 0$ there is $\delta_1 > 0$ such that $\delta_1 < \delta$, and

$$|x - a| < \delta_1,\ x \in E \qquad \Rightarrow \qquad |g(x) - g(a)| < A^2\varepsilon.$$

Then, for $|x - a| < \delta_1$ and $x \in E$, we have

$$\left|\frac{1}{g(x)} - \frac{1}{g(a)}\right| = \frac{|g(a) - g(x)|}{|g(x)g(a)|} < \frac{A^2\varepsilon}{A^2} = \varepsilon.$$

Hence, $1/g$ is continuous at a.

(e) Hint. By Exercise 1.17(h), $\big||f(x)| - |f(a)|\big| \leq |f(x) - f(a)|$.

3.17. ***Correction.*** The domain of the composition $g \circ f$ is E.

For $a \in E$ let $b = f(a) \in E'$. Let W be an ε-neighborhood of $g(b)$. By Theorem 3.14, there is a δ-neighborhood V of $b = f(a)$ such that for every $y \in E' \cap V$, $g(y) \in W$. By the same theorem, there is a γ-neighborhood U of a such that for every $x \in E \cap U$, $f(x) \in V$. Because $f(E) \subseteq E'$, it follows that if $x \in E \cap U$, then $f(x) \in E' \cap V$, so $(g \circ f)(x) = g(f(x)) \in W$. By Theorem 3.14, the function $f \circ g$ is continuous at $x = a$.

3.18. Follows immediately from Exercise 3.16.

3.19. Suppose that there is $c \in (a, b)$ that does not belong to E. The sets $(-\infty, c) \cap E$ and $(c, \infty) \cap E$ are disjoint, nonempty relatively open subsets of E whose union is E. This is a contradiction, because E is assumed to be connected.

3.20. By Exercise 3.19, $[a, b] \subseteq E$. The result follows from Theorem 3.18.

3.21. (a) For $\varepsilon = (1/2)f(a) > 0$ there is $\delta > 0$ such that

$$|f(x) - f(a)| < (1/2)f(a) \qquad \text{for } x \in (a - \delta, a + \delta).$$

Hence, $f(x) > (1/2)f(a) > 0$ on $(a - \delta, a + \delta)$.

(b) By part (a), the set $B = \{x \in \mathbf{F} : f(x) > 0\}$. Similarly, the set $C = \{x \in \mathbf{F} : f(x) < 0\}$ is open. Therefore, the set A which is the complement of the open set $B \cup C$ is closed.

3.22. Clearly, we need only to verify the continuity of f at $x = b$. Let $\varepsilon > 0$. Because f is continuous on $[a, b]$, there is $\delta_1 > 0$ such that

$$|f(x) - f(b)| < \varepsilon, \qquad \text{if } x \in [a, b] \cap (b - \delta_1, b].$$

Similarly, there is $\delta_2 > 0$ such that

$$|f(x) - f(b)| < \varepsilon, \qquad \text{if } x \in [b, c] \cap [b + \delta_2, b).$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$|f(x) - f(b)| < \varepsilon, \qquad \text{if } x \in [a, c] \cap (b - \delta, b + \delta).$$

Hence the result.

3.23. The proof of Theorem 3.24 is done for intervals $I = [a, b)$. This proof does not assume that b is a finite element of $\mathbf{F}$. Thus we can apply the result of Theorem 3.24 to (clearly, strictly increasing) function x^n on $I = [0, \infty)$.

3.24. A tedious but straightforward repetition of steps in the proof given in the book.

3.25. Let $I = (a, b)$. For every $x \in I$ the end points of the open interval $I_x = ((a + x)/2, (x + b)/2)$ belong to I. For $I = (a, \infty)$, the end points of the open interval $I_x = ((a + x)/2, x + 1)$ belong to I. For $I = (-\infty, b)$, the end points of the open interval $I_x = (x - 1, (x + b)/2)$ belong to I. Clearly, in all three cases, $I = \bigcup_{x \in I} I_x$.

3.26. ***Correction.*** Must be "...a strictly increasing function f ...". The image of a constant function is a singleton which is not an interval according to Definition 3.1. However, a singleton is often considered as a "degenerate" closed interval.

We give a proof for a bounded interval $I = [a, b)$.

(Necessity.) By the Intermediate Value Theorem (Theorem 3.18) and monotonicity of f, for every $c \in (a, b)$, $f([a, c]) = [f(a), f(c)]$. Since $I = \bigcup_{c \in (a,b)} [a, c]$, the image $f(I)$ is the union of the intervals $[f(a), f(c)]$ taken for all $c \in (a, b)$. If the set $A = \{f(c)\}_{c \in (a,b)}$ is bounded above, we have $f(I) = [f(a), \sup A)$. If A is unbounded above, we have $f(I) = [f(a), \infty)$.

(Sufficiency.) We show that f is continuous at $c \in (a, b)$. Let $\varepsilon > 0$. We may assume that $(f(c) - \varepsilon, f(c) + \varepsilon) \subseteq f(I)$. Inasmuch as f is a strictly increasing function, there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$f(c - \delta_1) = f(c) - \varepsilon \qquad \text{and} \qquad f(c + \delta_2) = f(c) + \varepsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, by monotonicity of f, for every $x \in (c - \delta, c + \delta)$ we have $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$. Hence, f is continuous at c. The case $c = a$ is treated similarly.

3.27. Straightforward verification of the definition of continuity.

3.28. Hint. Consider $x_n = 1/2n$, $y_n = 1/n$. Clearly, $|x_n - y_n| = 1/2n \to 0$. However, $|f(x_n) - f(y_n)| = n$ diverges.

4 Differentiation

4.1. It is easy to verify that the limit of a constant function $g(x) = k$, $x \in \mathbf{F}$ is k. The result follows from part (d).

4.2. Hint. Note that $||f(x)|| = |f(x)|$. Then apply Definition 4.1.

4.3. The two statements

$$|f(x) - L| < \varepsilon \qquad \text{if} \qquad |x - c| < \delta$$

and

$$|f(x + c) - L| < \varepsilon \qquad \text{if} \qquad |x| < \delta$$

are equivalent. Hint: Use substitutions $x = y + c$ and $x = y - c$ to prove it.

4.4. Hint. Use the results of Theorem 4.2 and Exercise 3.14.

4.5. (Necessity.) By Lemma 4.4, there is an unbounded strictly increasing sequence (a_n) with positive terms. Then for every $\varepsilon > 0$ there is $N \in \mathbf{N}$ such that $\frac{1}{\varepsilon} < a_n$ for all $n > N$, or, equivalently, $\frac{1}{a_n} < \varepsilon$ for all $n > N$. It follows that the sequence $(1/a_n)$ converges to zero.

(Sufficiency.) Suppose to the contrary that a field $\mathbf{F}$ is not countably cofinal and contains a convergent sequence which is not eventually constant. The desired contradiction follows from Lemma 4.5.

4.6. (a) We have

$$\begin{aligned}\frac{(f+g)(x) - (f+g)(c)}{x - c} &= \frac{(f(x) - f(c)) + (g(x) - g(c))}{x - c} \\ &= \frac{f(x) - f(c)}{x - c} + \frac{g(x) - g(c)}{x - c}\end{aligned}$$

By Theorem 4.5(a), the right-hand side converges to $f'(c) + g'(c)$.

(b) The result follows from part (c) for a constant function $f(x) = k$, $x \in I$.

4.7. (a) By induction. The case $n = 1$ is trivial (we assume that $x^0 = 1$ for all real x). Suppose that $(x^n)' = nx^{n-1}$ holds for some $n \in \mathbf{N}$. By Theorem 4.8(c),

$$(x^{n+1})' = (x^n x)' = (x^n)'x + x^n(x)' = nx^{n-1}x + x^n = (n+1)x^n.$$

(b) Let $y = x^n$ and $x = y^{1/n}$. By Theorem 4.10,

$$(y^{1/n})' = \frac{1}{(x^n)'} = \frac{1}{nx^{n-1}} = \frac{1}{ny^{(n-1)/n}} = \frac{1}{n}y^{\frac{1}{n}-1}.$$

(c) Let $f(x) = x^m$, $g(x) = x^{1/n}$, and $h(x) = (g \circ f)(x) = (x^m)^{1/n} = x^{m/n}$. By Theorem 4.9 (the Chain Rule),

$$h'(x) = g'(f(x))f'(x) = \frac{1}{n}(x^m)^{\frac{1}{n}-1} \cdot mx^{m-1} = \frac{m}{n}x^{(m/n)-1}.$$

4.8. By the Chain Rule (Theorem 4.9),

$$\begin{aligned}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}\right)' &= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\cdot\left(x+\sqrt{x+\sqrt{x}}\right)' \\ &= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\cdot\left(1+\frac{1}{2\sqrt{x+\sqrt{x}}}\cdot\left(x+\sqrt{x}\right)'\right) \\ &= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\cdot\left(1+\frac{1}{2\sqrt{x+\sqrt{x}}}\cdot\left(1+\frac{1}{2\sqrt{x}}\right)\right).\end{aligned}$$

The above expression can be simplified as

$$\frac{1+2\sqrt{x}+4\sqrt{x}\sqrt{x+\sqrt{x}}}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}$$

4.9. False. Consider $f(x)=|x|$ and $g(x)=-|x|$.

4.10. (Necessity.) By Theorem 4.17, $(f(x)-g(x))'=f'(x)-g'(x)=0$ implies $f(x)-g(x)=C$, a constant function.

(Sufficiency.) We have $f'(x)=g'(x)+(C)'=g'(x)$.

4.11. Hint. A function $f:(a,b)\to\mathbf{F}$ is increasing if and only if the function $-f$ is decreasing.

4.12. First, we prove the inequality for $x\geq 0$. Let $f(x)=(1+x)^r$. By Exercise 4.7(c) and Theorem 4.9, $f'(x)=r(1+x)^{r-1}$. By the Mean Value Theorem (cf. Theorem 4.13) applied to the interval $[0,x]$, there is $c\in(0,x)$ such that

$$f(x)-f(0)=f'(c)(x-0),$$

that is,

$$(1+x)^r-1=r(1+c)^{r-1}x\geq rx,$$

because $c>0$ and $r\geq 1$. The Bernoulli inequality trivially holds for $x=0$.

For $-1\leq x\leq 0$ the proof follows similar steps by applying the Mean Value Theorem to the interval $[x,0]$.

4.13. (Necessity.) Let $P=(a,f(a))$, $Q=(b,f(b))$, and $R=(x,f(x))$, where $a\leq x\leq b$ (cf. Fig. 4.1). It is easy to verify that $x=\lambda a+(1-\lambda)b$, where $\lambda=\dfrac{b-x}{b-a}$. Clearly, $0\leq\lambda\leq 1$. Let R' be a point on the chord PQ with the abscissa x. Elementary geometry (similar triangles) shows that

$$R'=(x,\lambda f(a)+(1-\lambda)f(b)).$$

Inasmuch as the function f is convex, we have

$$f(x) \leq \lambda f(a) + (1-\lambda) f(b).$$

Hence, the point R is below the point R' or coincides with it.

(Sufficiency.) In the above notations, the condition that R lies below R' (or coincides with it) is equivalent to the last displayed equation. The result follows.

4.14. For $\lambda = (x_2 - x)/(x_2 - x_1)$ we have $1 - \lambda = (x - x_1)/(x_2 - x_1)$ (cf. page 101 in the book). Then,

$$\lambda x_1 + (1-\lambda) x_2 = \frac{x_2 - x}{x_2 - x_1} x_1 + \frac{x - x_1}{x_2 - x_1} x_2 = x.$$

Suppose that for $\lambda \neq \lambda'$,

$$\lambda x_1 + (1-\lambda) x_2 = \lambda' x_1 + (1-\lambda') x_2.$$

It follows that $x_1 = x_2$, a contradiction.

4.15. Straightforward algebra.

4.16. Assume, without loss of generality, that $\max\{x_1, \ldots, x_n\} = x_1$. Then

$$\begin{aligned} x = \lambda_1 x_1 + \cdots + \lambda_n x_n &\leq \lambda_1 x_1 + \cdots + \lambda_n x_1 \\ &= (\lambda_1 + \cdots + \lambda_n) x_1 = x_1 = \max\{x_1, \ldots, x_n\}. \end{aligned}$$

A similar argument proves the other inequality.

4.17. (Necessity.) We prove by induction that the inequality

$$f\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k f(x_k),$$

holds for a convex function f.

The base case, $n = 1$, holds trivially. Suppose that this inequality holds for some $m \geq 1$ and let

$$x_1, \ldots, x_m, x_{m+1} \in (a, b), \quad 0 \leq \lambda_k \leq 1 \; (1 \leq k \leq m+1), \quad \sum_{k=1}^{m+1} \lambda_k = 1.$$

We may assume that $\lambda_{m+1} \neq 1$. (Otherwise, the induction step is trivial.) For $\lambda = \sum_{k=1}^{m} \lambda_k$, we have $\lambda_{m+1} = 1 - \lambda$, and then, by (4.2) on page 101 and the induction hypothesis,

$$\begin{aligned}
f\biggl(\sum_{k=1}^{m+1}\lambda_k x_k\biggr) &= f\biggl(\lambda\sum_{k=1}^{m}\frac{\lambda_k}{\lambda}x_k+(1-\lambda)x_{m+1}\biggr)\\
&\le \lambda f\biggl(\sum_{k=1}^{m}\frac{\lambda_k}{\lambda}x_k\biggr)+\lambda_{m+1}f(x_{m+1})\\
&\le \lambda\sum_{k=1}^{m}\frac{\lambda_k}{\lambda}f(x_k)+\lambda_{m+1}f(x_{m+1})\\
&= \sum_{k=1}^{m+1}\lambda_k f(x_k).
\end{aligned}$$

The result follows by induction.

(Sufficiency.) For $n=2$ we have the inequality (4.2) on page 101.

4.18. We have $f''(x)=(p-1)px^{p-2}\ge 0$ on (a,b). By Theorem 4.22, f is a convex function.

4.19. We may assume that both numbers

$$u=\biggl(\sum_{k=1}^{n}|x_k|^p\biggr)^{1/p}\quad\text{and}\quad v=\biggl(\sum_{k=1}^{n}|y_k|^p\biggr)^{1/p}$$

are positive. By the triangle inequality, we have

$$\begin{aligned}
|x_k+y_k|^p\le(|x_k|+|y_k|)^p &= \biggl(u\,\frac{|x_k|}{u}+v\,\frac{|y_k|}{v}\biggr)^p\\
&= \biggl[(u+v)\biggl(\frac{u}{u+v}\frac{|x_k|}{u}+\frac{v}{u+v}\frac{|y_k|}{v}\biggr)\biggr]^p\\
&= (u+v)^p\biggl(\frac{u}{u+v}\frac{|x_k|}{u}+\frac{v}{u+v}\frac{|y_k|}{v}\biggr)^p.
\end{aligned}$$

Because

$$\frac{u}{u+v}+\frac{v}{u+v}=1$$

and the power function x^p is convex for $p\ge 1$ (cf. Exercise 4.18), we have

$$\biggl(\frac{u}{u+v}\frac{|x_k|}{u}+\frac{v}{u+v}\frac{|y_k|}{v}\biggr)^p\le\frac{u}{u+v}\frac{|x_k|^p}{u^p}+\frac{v}{u+v}\frac{|y_k|^p}{v^p}.$$

Hence,

$$|x_k+y_k|^p\le(u+v)^p\biggl(\frac{u}{u+v}\frac{|x_k|^p}{u^p}+\frac{v}{u+v}\frac{|y_k|^p}{v^p}\biggr).$$

By summing up both sides of the above inequality, we obtain

$$\begin{aligned}
\sum_{k=1}^{n} |x_k + y_k|^p &\leq (u+v)^p \left(\frac{u}{u+v} \frac{\sum_{k=1}^{n} |x_k|^p}{u^p} + \frac{v}{u+v} \frac{\sum_{k=1}^{n} |y_k|^p}{v^p} \right) \\
&= (u+v)^p \left(\frac{u}{u+v} + \frac{v}{u+v} \right) \\
&= (u+v)^p = \left[\left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p \right)^{1/p} \right]^p,
\end{aligned}$$

which is equivalent to Minkowski's inequality.

5 Integration

5.1. The proof is by induction. The base step, $n = 2$, is trivial. Suppose that we can enumerate any n-element subset of an ordered field $\mathbf{F}$, and let A be an $(n+1)$-element subset of $\mathbf{F}$. We denote the maximum element of A (cf. Exercise 5.2) by a_{n+1}. Let $a_1 < a_2 < \cdots < a_n$ be an enumeration of the n-element set $A \setminus \{a_{n+1}\}$. Clearly, $\{a_1, \ldots, a_n, a_{n+1}\}$ is the desired enumeration of the set A.

5.2. The proof is by induction on the cardinality n of a nonempty subset of an ordered field. The base step, $n = 1$, is trivial. Suppose that every subset of cardinality less than n has a maximum element and let A be a subset of cardinality n. For $a \in A$, let $A' = \{x \in A : a < x\}$. If $A' = \varnothing$, then a is the maximum element of A, so suppose that $A' \neq \varnothing$. By the induction hypothesis, A' has a maximum element, say, y. Then y is the maximum element of A. Indeed, suppose to the contrary that there is $z \in A$ that is greater than y. Clearly, $z \notin A'$, so $z \leq a$. On the other hand, $z > y > a$, a contradiction.

5.3. Let $[x_{k-1}, x_k]$ be a subinterval of the partition P. Points of Q that belong to this subinterval form a partition of it. Hence subintervals of Q are subintervals of subintervals of P. Clearly, $\|Q\| \leq \|P\|$.

5.4. (a) We have $|x| < 1$ for $x \in \mathcal{I}(\mathbf{F})$. Hence, $x \in \mathcal{F}(\mathbf{F})$.

(b) For $x \in \mathbf{F}$, either there is $n \in \mathbf{N}$ such that $|x| < n$, or $|x| > n$ for all $n \in \mathbf{N}$. Therefore, $\mathbf{F} \subseteq \mathcal{F}(\mathbf{F}) \cup \mathcal{L}(\mathbf{F})$. Obviously, $\mathcal{F}(\mathbf{F}) \cup \mathcal{L}(\mathbf{F}) \subseteq \mathbf{F}$.

(c) For $x \in \mathcal{F}(\mathbf{F}) \cap \mathcal{L}(\mathbf{F})$, there is $n \in \mathbf{N}$ such that $|x| < n$ and for all $m \in \mathbf{N}$ we have $|x| > m$. It follows that $n < |x| < n$, a contradiction. Hence, $\mathcal{F}(\mathbf{F}) \cap \mathcal{L}(\mathbf{F}) = \varnothing$.

5.5. ***Correction.*** The set $\mathcal{I}(\mathbf{F})$ must be $\mathcal{F}(\mathbf{F})$, although the claim holds also for $\mathcal{I}(\mathbf{F})$.

Let $u, v \in \mathcal{F}(\mathbf{F})$, so there are $n_1, n_2 \in \mathbf{N}$ such that $|u| < n_1$ and $|v| < n_2$. Then $|u + v| \leq |u| + |v| < \max\{n_1, n_2\}$. Hence, $\mathcal{F}(\mathbf{F})$ is closed under addition.

Let $u, v \in \mathcal{L}(\mathbf{F}) \cap \mathbf{F}^+$. Clearly, $u + v \in \mathbf{F}^+$. Then

$$|u + v| = u + v > |u| > n, \qquad \text{for all } n \in \mathbf{N},$$

so $u + v \in \mathcal{L}(\mathbf{F})$. Therefore, $\mathcal{L}(\mathbf{F}) \cap \mathbf{F}^+$ is closed under addition.

Let $x \in B = \mathcal{L}(\mathbf{F}) \cap \mathbf{F}^+$ and $y \in \mathbf{F} \setminus B$. There are two possible cases.

1) $y \in \mathcal{L}(\mathbf{F}) \cap \mathbf{F}^-$. Here $\mathbf{F}^-$ is the set of negative elements of $\mathbf{F}$. Then $-y \in B$, so $x - y = x + (-y) \in \mathcal{L}(\mathbf{F}) \cap \mathbf{F}^+$, because $\mathcal{L}(\mathbf{F}) \cap \mathbf{F}^+$ is closed under addition. Hence, $x - y$ is an infinitely large element.

2) $y \in \mathcal{F}(\mathbf{F})$. Suppose that $z = x - y \in \mathcal{F}(\mathbf{F})$. Then $x = z + y \in \mathcal{F}(\mathbf{F})$, because $\mathcal{F}(\mathbf{F})$ is closed under addition. This contradicts our assumption that $x \in B$. It follows that $x - y$ is an infinitely large element.

5.6. Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$ and $c_i \in [x_{i-1}, x_i]$ for $i = 1, \dots, n$. We have

$$S_P = \sum_{i=1}^{n} f(c_i)\triangle_i = k\sum_{i-1}^{n}(x_i - x_{i-1}) = k(b-a).$$

By Definition 5.3, $\int_a^b f = k(b-a)$.

5.7. Hint. $|kf(x)| = |k| \cdot |f(x)|$ and $|f(x)+g(x)| \le |f(x)| + |g(x)|$ for $x \in [a, b]$.

5.8. ***Correction.*** In Definition 5.4, $U_P = \sum_{k=1}^{n} M_k\triangle_k$, not $\sum_{k=1}^{n} m_k\triangle_k$.

We prove the case of U_P. The proof for the other case is similar. For $\varepsilon > 0$ and every $k = 1, \dots, n$ there is $c_k \in [x_{k-1}, x_k]$ such that

$$f(c_k) > M_k - \frac{\varepsilon}{b-a}$$

(cf. Theorem 2.1). Hence,

$$S_P = \sum_{k=1}^{n} f(c_k)\triangle_k > \sum_{k=1}^{n}\big(M_k - \varepsilon/(b-a)\big)\triangle_k = \sum_{k=1}^{n} M_k \triangle_k - \varepsilon = U_P - \varepsilon.$$

By Theorem 2.1, the desired result follows.

5.9. Let f be an increasing function on $[a, b]$. We may assume that $f(b) > f(a)$. (Otherwise, f is constant (cf. Exercise 5.6.) For $\varepsilon > 0$, let $\delta = \varepsilon/(f(b) - f(a))$. Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$ such that $\triangle_k < \delta$ for all $k = 1, \dots, n$. Inasmuch as f is an increasing function, we have $m_k = f(_{k-1})$ and $M_k = f(x_k)$ for all k. Therefore,

$$\begin{aligned} U_P - L_P &= \sum_{k=1}^{n}(f(x_k) - f(x_{k-1}))\triangle_k \\ &< \delta\sum_{k=1}^{n}(f(x_k) - f(x_{k-1})) = \delta(f(b) - f(a)) = \varepsilon. \end{aligned}$$

(The second sum is telescopic.) The function f is Riemann integrable by Theorem 5.3.

5.10. Hint. Apply Theorem 5.7 to the function $f(x) - g(x)$ and use Theorem 5.6.

5.11. ***Correction.*** Theorem 5.9 is proved in the text.

5.12. Follows from Exercise 5.3.

5.13. Let P and Q be step partitions for functions s and t, respectively, and $P \cup Q$ the common refinement of P and Q. It is clear that $P \cup Q$ is a step refinement for both functions s and t. The functions s and t are constant over interiors of subintervals of $P \cup Q$. The statements (a)–(d) of Theorem 5.10 obviously hold for constant functions.

5.14. Let f be a step function on $[a, b]$. Then, trivially, $f \in \mathcal{U}_f$ and $f \in \mathcal{L}_f$ (cf. definitions on page 122 in the book). Hence, the Riemann integral $\int_a^b f = J$ is the Darboux integral of f (cf. Definition 5.8).

5.15. ***Correction.*** The proof of the statement of the exercise is not as simple as it is claimed in the book. A thorough proof is presented below.

(Necessity.) Let f be a Darboux integrable function on $[a, b] \subseteq \mathbf{R}$. The sets $A = \{\int_a^b g : g \in \mathcal{L}_f\}$ and $B = \{\int_a^b h : h \in \mathcal{U}_f\}$ are bounded above and below, respectively. Let $J' = \sup A$ and $J = \inf B$. Clearly, $J' \leq J$. In fact, $J' = J$. Indeed, suppose to the contrary that $J' < J$. By Definition 5.7 with $\varepsilon = J - J'$, we have $\int_a^b h - \int_a^b g < J - J'$ for some $g \in \mathcal{L}_f$ and $h \in \mathcal{U}_f$. Hence, $\int_a^b h < J - (J' - \int_a^b g) \leq J$, which contradicts the definition of J. It follows that $\int_a^b g \leq J \leq \int_a^b h$ for all $g \in \mathcal{L}_f$ and $h \in \mathcal{U}_f$. Hence, J is the Darboux integral of the function f. Thus $I[a, b]$ is contained in $D[a, b]$. By Theorem 5.12, $D[a, b] = I[a, b]$, that is, the class of Darboux integrable functions coincides with the class of functions having a Darboux integral.

Now, we show that the number J defined in the above paragraph is the Riemann integral of f. For a given $\varepsilon > 0$, let $g \in \mathcal{L}_f$ and $h \in \mathcal{U}_f$ be step functions such that

$$\int_a^b g > J - \frac{1}{2}\,\varepsilon \qquad \text{and} \qquad \int_a^b h < J + \frac{1}{2}\,\varepsilon$$

(recall that $J = \sup A = \inf B$). Inasmuch as g and h are Riemann integrable functions (cf. Theorem 5.11), there is a $\delta > 0$ such that for every partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of points $c_k \in [x_{k-1}, x_k]$, $k = 1, \ldots, n$, we have (cf. (5.1) in Definition 5.3)

$$\sum_{k=1}^{n} g(c_k)\triangle_k > \int_a^b g - \frac{1}{2}\,\varepsilon \qquad \text{and} \qquad \sum_{k=1}^{n} h(c_k)\triangle_k < \int_a^b h + \frac{1}{2}\,\varepsilon.$$

It follows that

$$J - \varepsilon < \sum_{k=1}^{n} g(c_k)\triangle_k \leq \sum_{k=1}^{n} f(c_k)\triangle_k \leq \sum_{k=1}^{n} h(c_k)\triangle_k < J + \varepsilon,$$

that is J is the Riemann integral of the function f. Thus, f is Riemann integrable.

(Sufficiency.) Let f be a Riemann integrable function, $J = \int_a^b f$, $\varepsilon > 0$, and $P = \{x_0, \ldots, x_n\}$ a partition of $[a, b]$ such that

$$J - \frac{1}{3}\,\varepsilon < \sum_{k=1}^{n} f(c_k)\triangle_k < J + \frac{1}{3}\,\varepsilon$$

for every choice of $(c_1, \ldots, c_k)$ (cf. Definition 5.3). For every $1 \le k \le n$, we define

$$g_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} \quad \text{and} \quad h_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

We have

$$\sum_{k=1}^{n} h_k \triangle_k \le J + \frac{1}{3}\,\varepsilon.$$

Indeed, suppose to the contrary that $\sum_{k=1}^{n} h_k \triangle_k > J + \frac{1}{3}\,\varepsilon$ and let

$$\gamma = \frac{1}{n}\Big[\sum_{k=1}^{n} h_k \triangle_k - \Big(J + \frac{1}{3}\,\varepsilon\Big)\Big] > 0.$$

For every $k = 1, \ldots, b$, choose $c_k \in [x_{k-1}, x_k]$ such that $f(c_k) > h_k - \gamma/\triangle_k$. Then

$$\begin{aligned}\sum_{k=1}^{n} f(c_k)\triangle_k > \sum_{k=1}^{n}(h_k \triangle_k - \gamma) &= \sum_{k=1}^{n} h_k \triangle_k - \Big[\sum_{k=1}^{n} h_k \triangle_k - \Big(J + \frac{1}{3}\,\varepsilon\Big)\Big] \\ &= J + \frac{1}{3}\,\varepsilon,\end{aligned}$$

a contradiction. Similarly, we obtain

$$\sum_{k=1}^{n} g_h \triangle_k \ge J - \frac{1}{3}\,\varepsilon.$$

Let step functions g and h be defined by

$$g(x) = \begin{cases} g_k, & \text{for } x \in (x_{k-1}, x_k), \\ f(x_k), & \text{for } x = x_k, \end{cases} \qquad k = 1, \ldots, n,$$

and

$$h(x) = \begin{cases} h_k, & \text{for } x \in (x_{k-1}, x_k), \\ f(x_k), & \text{for } x = x_k, \end{cases} \qquad k = 1, \ldots, n,$$

respectively. Clearly, $g \in \mathcal{L}_f$ and $h \in \mathcal{U}_f$. We have

$$\int_a^b h - \int_a^b g = \sum_{k=1}^{n} h_k \triangle_k - \sum_{k=1}^{n} g_k \triangle_k \le \frac{2}{3}\,\varepsilon < \varepsilon.$$

Hence, f is Darboux integrable.

5.16. Hint. $x^- = -(-x)^+$.

5.17. (a) Let $x_1 < x_2$. If $0 \le x_1 < x_2$, then $x_1^+ = x_1 < x_2 = x_2^+$. If $x_1 < 0$, then $x_1^+ = 0 \le x_2^+$. A similar argument shows that $x^- = x - x^+$ is an increasing function.

(b) If $x \geq 0$, then $x^+ - x^- = x$. Otherwise, $x^+ - x^- = -x$.

5.18. (Necessity.) Let $f : [a,b] \to \mathbf{F}$ be a Darboux integrable function and $\varepsilon > 0$. There are step functions $g \in \mathcal{L}_f$ and $h \in \mathcal{U}_f$ such that

$$\int_a^b h - \int_a^b g < \varepsilon.$$

Let f', g', h' and f'', g'', h'' be restrictions of the functions f, g, h to the intervals $[a,c]$ and $[c,b]$, respectively. Clearly, g', h', g'', and h'' are step functions. By Theorems 5.9 and 5.11,

$$\int_a^b g = \int_a^c g' + \int_c^b g'' \qquad \text{and} \qquad \int_a^b h = \int_a^c h' + \int_c^b h''.$$

We have

$$\left[\int_a^c h' - \int_a^c g'\right] + \left[\int_c^b h'' - \int_c^b g''\right] = \int_a^b h - \int_a^b g < \varepsilon.$$

Because the two terms in brackets are nonnegative, it follows that

$$\int_a^c h' - \int_a^c g' < \varepsilon \qquad \text{and} \qquad \int_c^b h'' - \int_c^b g'' < \varepsilon,$$

that is, the functions f' and f'' are Darboux integrable.

(Sufficiency.) Let f' and f'' be the same functions as above and assume that they are Darboux integrable. Then for $\varepsilon > 0$ there are step functions

$$g' \in \mathcal{L}_{f'}, \quad h' \in \mathcal{U}_{f'}, \quad g'' \in \mathcal{L}_{f''}, \quad h'' \in \mathcal{U}_{f''}$$

such that

$$\int_a^c h' - \int_a^c g' < \frac{\varepsilon}{2} \qquad \text{and} \qquad \int_a^c h'' - \int_a^c g'' < \frac{\varepsilon}{2}.$$

We define

$$g(x) = \begin{cases} g'(x), & \text{if } x \in [a,c], \\ g''(x), & \text{if } x \in (c,b], \end{cases} \qquad \text{and} \qquad h(x) = \begin{cases} h'(x), & \text{if } x \in [a,c], \\ h''(x), & \text{if } x \in (c,b]. \end{cases}$$

It is easy to verify that g and h are step functions on $[a,b]$ and $g \in \mathcal{L}_f$, $h \in \mathcal{U}_f$. Because

$$\int_a^c g' + \int_c^b g'' = \int_a^b g \qquad \text{and} \qquad \int_a^c h' + \int_c^b h'' = \int_a^b h,$$

we have

$$\int_a^b h - \int_a^b g = \left[\int_a^c h' - \int_a^c g'\right] + \left[\int_c^b h'' - \int_c^b g''\right] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, f is a Darboux integrable function on $[a,b]$.

6 Infinite Series

6.1. Hint. For $n \geq m > 1$, $a_m + \cdots + a_n = s_n - s_{m-1}$.

6.2. Apply Theorem 1.20 to partial sums $\sum_{k=1}^{n}(a_k + b_k)$ and $\sum_{k=1}^{n}(ca_k)$.

6.3. We have

$$\frac{1}{n+1} + \frac{1}{n+2} \cdots + \frac{1}{2n} > \underbrace{\frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}}_{n \text{ terms}} = \frac{1}{2}.$$

6.4. (1) The base case, $n = 1$, trivially holds. Suppose that $(1+x)^n \geq 1 + nx$ for some $n \in \mathbf{N}$. Then

$$(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = 1+(n+1)x+nx^2 \geq 1+(n+1)x.$$

(2) We may assume that $r \neq 0$. Let $x = 1/|r| - 1 > 0$, so $|r| = 1/(1+x)$. We have

$$0 < |r|^n = \frac{1}{(1+x)^n} \leq \frac{1}{1+nx} < \frac{1}{nx}.$$

By Exercise 1.21(a), $1/(nx) \to 0$. By the Squeeze Theorem (cf. Theorem 1.14), $r^n \to 0$.

6.5. Hint. By Exercise 2.5, $1/n \to 0$ in every Archimedean field. Hence, for $|r| < 1$, $r^n \to 0$ (cf. solution to Exercise 6.4(2)).

6.6. Hint. $|x| < 1/n$ for all $n \in \mathbf{N}$ if and only if $1/|x| > n$ for all $n \in \mathbf{N}$.

6.7. Let (x_n) be a Cauchy sequence in an ordered field $\mathbf{F}$ that is not countably cofinal. Because (x_n) is Cauchy, $a_n = (x_{n+1} - x_n) \to 0$. By Lemma 4.5, the sequence (a_n) is eventually zero, so the sequence (x_n) is eventually constant and therefore converges. Hence, $\mathbf{F}$ is Cauchy complete.

6.8. We have

$$s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n} > s_{2n}$$

and

$$s_{2n+1} = s_{2n-1} - a_{2n} + a_{2n+1} < s_{2n-1}.$$

Because the series converges to S, $s_{2n} < S < s_{2m-1}$ for all $m, n \in \mathbf{N}$.

6.9. Suppose that the sequence $s_n = n$ converges to S. Then (for $\varepsilon = 1$) there is $N \in \mathbf{N}$ such that $n - 1 < S < n + 1$ for all $n > N$. In particular, $S < N + 2$ and $S > N + 2$, a contradiction.

6.10. We use the following simple fact: A sequence $(a_1, a_2, \ldots)$ converges if and only if the subsequence $(a_k, a_{k+1}, \ldots)$ (a tail of (a_n)) converges (cf. Theorem 6.1).

Let $a_{n_1}, a_{n_2}, \ldots, a_{n_m}$ where $n_1 < n_2 < \cdots < n_m$ be the negative terms of a convergent series (a_n) and $\sigma = 2|a_{n_1}| + 2|a_{n_2}| + \cdots + 2|a_{n_m}|$. We denote by (s_n) and (s'_n) the sequences of partial sums of series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} |a_k|$, respectively. It is not difficult to verify that $s'_n = s_n + \sigma$ if $n > n_m$. Because (s_n) is a convergent sequence, so is the subsequence $(s_{n_m+1}, s_{n_m+2}, \ldots)$. It follows that the subsequence $(s'_{n_m+1}, s'_{n_m+2}, \ldots)$ converges, which implies that (s'_n) converges, that is, the series $\sum_{k=1}^{\infty} a_k$ converges absolutely.

6.11. Multiply the right hand side of the identity in (6.5) by 2^n to obtain

$$2^{n-m} - (2^{n-m-1} + \cdots + 1) = 2^{n-m} - (2^{n-m} - 1) = 1.$$

6.12. Because x is an irrational number, there are infinitely many zero and nonzero terms in the series $\sum_{k=1}^{\infty} c_k/2^k$. Let $c_{k_m}/2^{k_m}$ be the last zero term in the m'th maximal set of consecutive zeros. Clearly, the subsequence S_{k_m} of the sequence of partial sums converges to x. This sequence is the same as the corresponding subsequence of the partial sums in the second series. Because the second sequence converges in $\mathbf{R}$, it converges to x.

6.13. We have

$$\lim \frac{\dfrac{|x|^{n+1}}{(n+1)!}}{\dfrac{|x|^n}{n!}} = \lim \frac{|x|}{n+1} = 0.$$

The series converges by the Ratio Test.

6.14. Hint. Use the Ratio Test as in Exercise 6.13.

7 Appendix A: Natural Numbers and Integers

A.1. By Axiom **P1**, $1 \notin s(N)$, and by Axiom **P2**, s is a one-to-one function. To prove that s is a bijection we need to show that $s(N) = N \setminus \{1\}$.

Let $M = s(N) \cup \{1\}$. Clearly, $1 \in M$. If $a \in M$, then $a \in N$ because $M \subseteq N$. Therefore, $s(a) \in s(N) \subseteq M$. By Axiom **P3**, $M = N$, that is, $s(N) \cup \{1\} = N$. Because $1 \notin s(N)$ we have $s(N) = N \setminus \{1\}$, as required.

By definition, a set A is infinite if there is a bijection from A onto a proper subset of A (cf. Section A.4 in Ovchinnikov (2015)).

A.2. Let $M = \{a \in N : s(a) \neq a\}$. By Axiom **P1**, $1 \in M$. By Axiom **P2**, $a \neq b$ implies $s(a) \neq s(b)$ for all $a, b \in N$. Therefore, $s(a) \neq a$ implies $s(s(a)) \neq s(a)$ for all $a \in N$. Hence, $s(a) \in M$ for every $a \in M$. By the Axiom of Induction (Axiom **P3**), $M = N$, that is, $s(a) \neq a$ for all $a \in N$.

A.3. Fix $a \in N$ and define $M = \{b \in N : a + b \neq b\}$. We have $1 \in M$ because, by (A.1) and Axiom **P1**,

$$a + 1 = s(a) \neq 1.$$

Suppose $b \in M$, that is, $a + b \neq b$. Then we have

$$a + s(b) = s(a + b) \neq s(b),$$

by (A.2) and Axiom **P2**. Hence, $b \in M$ implies $s(b) \in M$. By the Axiom of Induction, $M = N$. Because a is an arbitrary element of N, we conclude that $a + b \neq b$ for all $a, b \in N$.

A.4. (a) Suppose that $a \neq b$ are two least elements of a nonempty subset of N. Then $a \leq b$ and $b \leq a$, contradicting the Trichotomy Property (cf. Theorem A.5(i)).

(b) Let $a \neq 1$ be an element of N. Because s is a bijection from N onto $N \setminus \{1\}$ (cf. Exercise A.1), a is a successor of some $b \in N$, that is $a = b + 1$ (cf. (A.1) in Theorem A.1). It follows (cf. Definition A.2) that $a > 1$. Hence, 1 is the least element of N.

Suppose that b is the greatest element of N. Clearly, $b < b + 1$, a contradiction.

A.5. Because $(m', n') \in [m, n]$ and $(p', q') \in [p, q]$, we have $(m', n') \sim (m, n)$ and $(p', q') \sim (p, q)$, that is,

$$m' + n = m + n' \qquad \text{and} \qquad p' + q = p + q'.$$

By adding these equalities and applying the Associative and Commutative Laws of Addition for the natural numbers, we obtain

$$(m' + p') + (n + q) = (m + p) + (n' + q').$$

It follows that $(m' + p', n' + q') \sim (m + p, n + q)$.

We proved that $[m'+p', n'+q'] = [m+p, n+q]$.

A.6. We need to prove that

$$[m'q'+n'p', n'q'+m'p'] = [mq+np, nq+mp].$$

For the proof, we need to show that

$$(m'q'+n'p', n'q'+m'p') \sim (mq+np, nq+mp),$$

that is,

$$m'q'+n'p'+nq+mp = mq+np+n'q'+m'p'. \tag{7.1}$$

Inasmuch as $(m', n') \sim (m, n)$ and $(p', q') \sim (p, q)$, we have

$$m'+n = m+n' \tag{7.2}$$

and

$$p'+q = p+q'. \tag{7.3}$$

From these equalities, we obtain

$$\begin{array}{rll} mq+n'q = m'q+nq, & & \text{by multiplying Eq. (7.2) by } q, \\ m'p+np = mp+n'p, & & \text{by multiplying Eq. (7.2) by } p, \\ n'p+n'q' = n'p'+n'q, & & \text{by multiplying Eq. (7.3) by } n', \\ m'p'+m'q = m'p+m'q', & & \text{by multiplying Eq. (7.3) by } m'. \end{array}$$

By adding the four equations displayed above, we have

$$\begin{aligned} &mq+n'q+m'p+np+n'p+n'q'+m'p'+m'q \\ &\qquad = m'q+nq+mp+n'p+n'p'+n'q+m'p+m'q'. \end{aligned}$$

By applying the Cancellation Law for addition of natural numbers, we obtain the equality in (7.1).

A.7. (Existence.) Let a be an element of R. By Property **R3**, there is $c \in R$ such that $a+c = a$. For any $b \in R$, again by property **R3**, there is $d \in R$ such that $a+d = b$. By applying the Commutative and Associative Laws of Addition, we obtain

$$b+c = (a+d)+c = (a+c)+d = a+d = b.$$

Because b is an arbitrary element of R, the element c is a zero element.

(Uniqueness.) Let c' be another zero element of R. Then

$$c = c+c' = c'+c = c',$$

by the definition of a zero element.

A.8. (Existence.) Let $a \in R$. By property **R3**, there is an element b of R such that $a + b = 0$. Hence, b is an additive inverse of a.

(Uniqueness.) Suppose that for some $a \in R$ there are two additive inverses b and c of a, that is,

$$a + b = 0 \qquad \text{and} \qquad a + c = 0.$$

By the definition of 0 and the Commutative and Associative Laws of Addition,

$$b = b + 0 = b + (a + c) = c + (a + b) = c + 0 = c.$$

A.9. We consider only the case (d). Verifications in other cases are straightforward.

R1. Trivial.

R3. Let $X = A \triangle B$. Then $A \triangle X = A \triangle (A \triangle B) = (A \triangle A) \triangle B = B$ (cf. Property **R2**).

R4. Simple property of the intersection operation.

A simple way of verifying properties **R2** and **R5** is to use characteristic functions of subsets. A characteristic function of a subset A of the set X is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, two subsets are equal if and only if their characteristic function are equal. It is not difficult to show that

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B, \qquad \chi_{A\cap B} = \chi_A \cdot \chi_B, \qquad \chi_{A\setminus B} = \chi_A - \chi_A \cdot \chi_B,$$

and

$$\chi_{A\triangle B} = \chi_A + \chi_B - 2\chi_A \cdot \chi_B.$$

Below, we verify **R5** and **R2**.

R5. We have

$$\chi_{A\cap(B\triangle C)} = \chi_A(\chi_B + \chi_C - 2\chi_B\chi_C) = \chi_A\chi_B + \chi_A\chi_C - 2\chi_A\chi_B\chi_C$$

and

$$\chi_{(A\cap B)\triangle(A\cap C)} = \chi_{A\cap B} + \chi_{A\cap C} - 2\chi_{A\cap B}\chi_{A\cap C} = \chi_A\chi_B + \chi_A\chi_C - 2\chi_A\chi_B\chi_C$$

(clearly, $\chi_A\chi_A = \chi_A$).

R2. It is not difficult to verify that

$$\chi_{(A\triangle B)\triangle C} = \chi_A + \chi_B + \chi_C - 2\chi_A\chi_B - 2\chi_A\chi_C - 2\chi_B\chi_C - 4\chi_A\chi_B\chi_C = \chi_{A\triangle(B\triangle C)}$$

A.10. It suffices to add $-c$ to both sides of the equation $a + c = b + c$.

A.11. Suppose that there are two elements of R such that

$$a \cdot 1 = 1 \cdot a = a \qquad \text{and} \qquad a \cdot 1' = 1' \cdot a = a,$$

for all $a \in R$. Then $1' = 1' \cdot 1 = 1$, and the result follows.

A.12. We have

$$ab - ac = a(b - c) = 0,$$

because $ab = ac$. Inasmuch as R is an integral domain and $a \neq 0$, we must have $b - c = 0$, that is, $b = c$.

A.13. ***Correction.*** Exercise A.13 should refer to Example A.1, not to Definition A.10.

It is convenient to represent elements of $R[x]$ by series $f = \sum_{k=0}^{\infty} a_k x^k$, where $a_k = 0$ for all but finitely many values of k.

Then, for $f = \sum_{k=0}^{\infty} a_k x^k$ and $g = \sum_{k=0}^{\infty} b_k x^k$, we have

$$f + g = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

and

$$f \cdot g = \sum_{k=0}^{\infty} c_k x^k, \qquad \text{where } c_k = \sum_{p+q=k} a_p b_q.$$

We verify the properties defining a ring. **R1** and **R2** hold because they hold in R.

R3. Clearly, $\sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} (b_k - a_k) x^k = \sum_{k=0}^{\infty} b_k x^k$.

R4. We have

$$\sum_{p+q=k} a_p \left(\sum_{r+s=q} b_r c_s \right) = \sum_{p+r+s=k} a_p b_r c_s$$

and

$$\sum_{q+s=k} \left(\sum_{p+r=q} a_p b_r \right) c_s = \sum_{p+r+s=k} a_p b_r c_s.$$

R5. We have

$$\sum_{p+q=k} a_p (b_q + c_q) = \sum_{p+q=k} a_p b_q + \sum_{p+q=k} a_p c_q.$$

Suppose that R is an integral domain. If $f \neq 0$ and $g \neq 0$, let a_p and b_q be nonzero coefficients with greatest indexes p and q in f and g, respectively. Then $a_p b_q \neq 0$ is the coefficient of the term with x^{p+q} in $f \cdot g$. Hence, $f \cdot g \neq 0$ in $R[x]$.

A.14. ***Correction.*** The relation $<$ defined on $\mathbf{Z}[x]$ in Example A.1 is not a linear order. Indeed, two polynomials $f = x + 1$ and $g = x + 2$ are distinct,

$f \neq g$, but incomparable, that is, $f \not< g$ and $g \not< f$. Therefore, $<$ does not satisfy the Trichotomy Property (cf. Definition A.7). There are two equivalent ways to correct the definition.

1) Let $f = \sum_{k=0}^{\infty} a_k x^k$ and $g = \sum_{k=0}^{\infty} b_k x^k$ (cf. item **A.13** above) be distinct elements of $\mathbf{Z}[x]$. We define

$$f < g \qquad \text{if and only if} \qquad a_p < b_p,$$

where p is the maximum index k such that $a_k \neq b_k$

2) We say that a polynomial $f = \sum_{k=0}^{\infty} a_k x^k$ is positive, $f > 0$, and negative, $f < 0$, if the last nonzero term in the sequence $(a_0, a_1, \ldots, a_n, \ldots)$ is positive or, respectively, negative. Now we define $f < g$ if $g - f > 0$ and proceed with showing that $\mathbf{Z}[x]$ is an ordered integral domain (cf. Exercise A.13).

We use the second definition to prove that $<$ is a linear ordering on $\mathbf{Z}[x]$. Clearly, every nonzero polynomial is either positive or negative (but not both). Therefore, for $f \neq g$ either $g - f > 0$, so $f < g$, or $g - f < 0$, so $g < f$. Hence the Trichotomy Property is satisfied. Furthermore, suppose that $f < g$ and $g < h$. Then, $g - f > 0$ and $h - g > 0$. Clearly, the sum of two positive polynomials is positive. Hence, $h - f = (h - g) + (g - f) > 0$, that is, $f < h$. Therefore the Transitivity Property is satisfied.

By Exercise A.13, $\mathbf{Z}[x]$ is an integral domain. Below we show that conditions (i) and (ii) of Definition A.10 are satisfied for the order $<$.

(i) Clearly,

$$g - f > 0 \qquad \text{if and only if} \qquad (g + h) - (f + h) > 0.$$

Hence, $f < g$ if and only if $f + h < g + h$ for all $f, g, h \in \mathbf{Z}[x]$.

(ii) Note that the product of two positive polynomials is positive and the product of a positive polynomial by a negative one is negative. Hence, $g - f > 0$ implies $g \cdot h - f \cdot h = (g - f) \cdot h > 0$, if $h > 0$. In the opposite direction, suppose that $h > 0$ and $(g - f) \cdot h > 0$. Clearly, $g - f \neq 0$ and $g - f \not< 0$. Hence, $g - f > 0$, that is, $f < g$.