remarks on arithmetical algorithms and the computation of π

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[1]This document is enclosed in the hfloat-package which is online at http://www.jjj.de/hfloat/

Abstract

This is a collection of remarks about some arithmetical algorithms. If you think iterating $x \mapsto \frac{1}{2}(x + \frac{d}{x})$ is the best way to compute $\sqrt{d}$ then have a look.

In addition there is a collection of formulas and iterations for the computation of π. Some of the formulas are highly cryptic and useless. If you think that
$\frac{\pi}{4} = 88 \text{ arctan } \frac{1}{192} + 39 \text{ arctan } \frac{1}{239} + 100 \text{ arctan } \frac{1}{515} - 32 \text{ arctan } \frac{1}{1068} - 56 \text{ arctan } \frac{1}{173932}$
is fun then stare at more formulas there.

Things are treated pretty superficially, always see the references for details. Do not expect mathematics, expect formulas, ideas and algorithms. Some sections are mere formula buckets or enumerations of names or references, don't panic.

Please report errors and typos !

Contents

Chapter 1

Remarks on arithmetical algorithms

1.1 Asymptotics of algorithms

An important feature of an algorithm is the number of operations that must be performed for the completion of a task a certain size N. N should be some reasonable quantity that grows strictly with the size of the task, for high precision computations one will take the length of the numbers counted in decimal digits or bits. For computations with square matrices one may take for N the number of the rows. An operation is typically a (machine word) multiplication plus an addition, one could also simply count machine instructions.

An algorithm is said to have some asymptotics $f(N)$ if it needs proportional $f(N)$ operations for a task of size N.

Examples:

- Addition of an N-digit number needs proportional N operations (here: machine word addition plus some carry operation).
- Ordinary multiplication needs N^2 operations.
- The Fast Fourier Transform (FFT) needs $\sim N \log(N)$ operations (a straightforward implementation of the Fourier Transform, i.e. computing N sums each of length N would be $\sim N^2$).
- Matrix multiplication is $\sim N^3$ (N^2 sums each of N products).

The algorithm with the 'best' asymptotics wins for some, possibly huge, N. For smaller N some other algorithm will be superior. For the exact break-even point the constants omitted elsewhere are of course important.

Example: Let the algorithm `mult1` take $1.0 \cdot N^2$ operations, `mult2` take $8.0 \cdot N \log_2(N)$ operations. Then for $N < 64$ `mult1` is faster and for $N > 64$ `mult2` is faster. Often completely different algorithms are optimal for the same task at different problem sizes.

See [62], [16] and [5].

1.2 Multiplication of large numbers

1.2.1 Fast multiplication via FFT

Ordinary multiplication is $\sim N^2$. Computing the product of two million digit numbers would require $\approx 10^{12}$ operations, taking (in the order of) 1 day on a machine that does 10 million operations per second. But there is a better way:

1.) Note that multiplication of two numbers is essentially a convolution of the sequences of their digits. A convolution $c_k, k = 0...2N-2$ of the two sequences $a_k, b_k, k = 0...N-1$ is defined as

$$c_k \quad := \quad \sum_{i,j=0;i+j=k}^{N-1} a_i b_j. \tag{1.1}$$

A number written (in radix r) as

$$a_P \quad a_{P-1} \quad \ldots \quad a_2 \quad a_1 \quad a_0 \quad . \quad a_{-1} \quad a_{-2} \quad \ldots \quad a_{-p+1} \quad a_{-p}$$

denotes a quantity of

$$\sum_{i=-p}^{P} a_i \cdot r^i = a_P \cdot r^P + a_{P-1} \cdot r^{P-1} + \ldots + a_{-p} \cdot r^{-p}.$$

i.e. digits are coefficients of a polynomial in r. (e.g. for decimal numbers $r = 10$ and $123.4 = 1 \cdot 10^2 + 2 \cdot 10^1 + 3 \cdot 10^0 + 4 \cdot 10^{-1}$). The product of two numbers is almost the polynomial product

$$\sum_{k=0}^{2N-2} c_k r^k := \sum_{i=0}^{N-1} a_i r^i \sum_{j=0}^{N-1} b_j r^j \tag{1.2}$$

The c_k are found by comparing coefficients, one gets $c_k = \sum_{i,j=0;i+j=k}^{N-1} a_i b_j$, apparently a convolution.

As the c_k can be greater (or equal) r, the carry operations have to be appended: go from right to left, replace c_k by $c_k \% r$ and add $(c_k - c_k \% r)/r$ to its left neighbour.

Here is a tiny decimal example:

	.82	$\times$	.34
		32	8
	24	6	
	24	38	8
=.2	^{2} 7	^{3} 8	8

2.) Note that convolution can be done effectively using the Fast Fourier Transform (FFT): Convolution is a simple (elementwise array) multiplication in Fourier space (see e.g. [46]). The FFT itself takes $N \cdot \log N$ operations. Instead of the direct convolution ($\sim N^2$) one does

{ FFT; array multiplication; FFT^{-1}; }

The operation count is dominated by that of the FFT's (the array multiplication is of course $\sim N$), so the whole fast convolution algorithm takes $\sim N \cdot \log N$ operations. The following carry operation is also $\sim N$ and can therefor be neglected when counting operations.

Well, $N \cdot \log N$ is not really the truth: it has to be $N \cdot \log N \cdot \log \log N$. This is because the sums in the convolutions have to be represented as exact integers. The biggest term c_{max} that can possibly occur (when multiplying two N-digit numbers made of all 'nines', i.e. $R-1$) is the central one, it is the sum of N times $(R-1)(R-1)$ or approximately

$$c_{max} \quad \approx \quad N R^2 \tag{1.3}$$

a number with proportional $\log N$ bits. Therefor, working with some fixed radix R one has to do FFTs with $\log N$ bits precision, leading to an operation count of $N \cdot \log N \cdot \log N$. The slightly better $N \cdot \log N \cdot \log\log N$ is obtained by recursive use of FFT multiplies. For realistic applications (where the sums in the convolution all fit into the machine type floating point numbers) it is safe to think of FFT multiplication being proportional $N \cdot \log N$.

Multiplying our million digit numbers will now take only $10^6 \log_2(10^6) \approx 10^6 \cdot 20$ operations, taking (in the order of) 2 seconds on a 10 Mips machine.

See [61].

1.2.2 The Karatsuba algorithm

Split the numbers U and V in two pieces:

$$\begin{aligned} U &= U_0 + U_1 B \\ V &= V_0 + V_1 B \end{aligned} \tag{1.4}$$

Instead of the straightforward multiplication

$$UV = U_0 V_0 + B(U_0 V_1 + V_0 U_1) + B^2 U_1 V_1 \tag{1.5}$$

(4 multiplications with half precision for one multiplication with full precision)

use the relation

$$UV = (1+B) U_0 V_0 + B(U_1 - U_0)(V_0 - V_1) + (B + B^2) U_1 V_1 \tag{1.6}$$

(3 multiplications with half precision for one multiplication with full precision)

recursively.

For Squaring the relation is:

$$U^2 = (1+B) U_0^2 - B(U_1 - U_0)^2 + (B + B^2) U_1^2 \tag{1.7}$$

or

$$U^2 = (1-B) U_0^2 + B(U_1 + U_0)^2 + (-B + B^2) U_1^2 \tag{1.8}$$

The asymptotics of the algorithm is $\sim N^{\log_2(3)} \approx N^{1.585}$.

One can extend the above idea by splitting U and V into more than two pieces each, the resulting algorithm is called Toom Cook algorithm.

See [5], chapter 4.3.3 ('How fast can we multiply ?').

1.2.3 Other methods for multiplication

- Binary algorithm: only shift-, bittest- and add- operations required.
- Systolic arrays.
- Modular multiplication.
- Montgomery multiplication is a method for effective multiplication of two numbers modulo M. With this method the modulus can be replaced by another modulus N (that must be greater than and prime to M). E.g. for M odd one can choose $N = 2^k$. See [6].

1.3 Power computations

See [5] and [6] !

1.4 Division, square root and cube root

1.4.1 Division

The ordinary division algorithm is useless for numbers of extreme precision. Instead one replaces does the division $\frac{a}{b}$ as $a \cdot \frac{1}{b}$. The inverse of b is computed by finding a starting approximation $x_0 \approx \frac{1}{b}$ and then iterating

$$x_{k+1} = x_k + x_k(1 - b\,x_k) \tag{1.9}$$

until the desired presision is reached. The convergence is quadratical (2.order), which means that the number of correct digits is doubled with each step: if $x_k = \frac{1}{b}(1+\epsilon)$ then

$$\begin{aligned} x_{k+1} &= \frac{1}{b}(1+\epsilon) + \frac{1}{b}(1+\epsilon)(1 - b\frac{1}{b}(1+\epsilon)) && (1.10) \\ &= \frac{1}{b}(1-\epsilon^2) && (1.11) \end{aligned}$$

Moreover each step needs only computations with twice the number of digits that were correct at its beginning. Still better: the multiplication $x_k(\ldots)$ needs only to be done with half precision as it computes the 'correcting' digits (which alter only the less significant half of the digits). Thus at each step we have 1.5 multiplications of the 'current' precision. The total work[1] amounts to

$$1.5 \cdot \sum_{n=0}^{n<\infty} \frac{1}{2}^n$$

which is less than 3 full precision multiplications. together with the final multiplication a division costs as much as 4 multiplications. Another nice feature of the algorithm is that it is self-correcting. Cf. figure 1.4.1 for a numerical example.

1.4.2 Square root extraction

is quite similar: first compute $\frac{1}{\sqrt{d}}$ then a final multiply with d gives $\sqrt{d}$. Find a starting approximation $x_0 \approx \frac{1}{\sqrt{b}}$ then iterate

$$x_{k+1} = x_k + x_k \frac{(1 - d\,x_k^2)}{2} \tag{1.25}$$

until the desired precision is reached. Convergence is again 2.order. Similar considerations as above (with squaring considered as expensive as multiplication[2]) yield an operation count of 4 multiplications for $\frac{1}{\sqrt{d}}$ or 5 for $\sqrt{d}$.

Note that this algorithm is considerably better than the usual one where $x_{k+1} := \frac{1}{2}(x_k + \frac{d}{x_k})$ is iterated, as long divisions are involved.

[1] The asymptotics of the multiplication is set to $\sim N$ (instead of $N \log(N)$) for the estimates made here, this gives a realistic picture for large N.

[2] Indeed it costs about $\frac{2}{3}$ of a multiplication.

$$
\begin{aligned}
b &:= 3.1415926 && (1.12)\\
x_0 &= 0.31 \qquad \text{2 digit approximation for } 1/b && (1.13)\\
& \text{now use 4 digits:} && (1.14)\\
b \cdot x_0 &= 3.141 \cdot 0.3100 = 0.9737 && (1.15)\\
y_0 &:= 1.000 - b \cdot x_0 = 0.02629 && (1.16)\\
x_0 \cdot y_0 &= 0.3100 \cdot 0.02629 = 0.0081(49) && (1.17)\\
x_1 &:= x_0 + x_0 \cdot y_0 = 0.3100 + 0.0081 = 0.3181 && (1.18)\\
& \text{now use 8 digits:} && (1.19)\\
b \cdot x_1 &= 3.1415926 \cdot 0.31810000 = 0.9993406 && (1.20)\\
y_1 &:= 1.0000000 - b \cdot x_0 = 0.0006594 && (1.21)\\
x_1 \cdot y_1 &= 0.31810000 \cdot 0.0006594 = 0.0002097(5500) && (1.22)\\
x_2 &:= x_1 + x_1 \cdot y_1 = 0.31810000 + 0.0002097 = 0.31830975 && (1.23)\\
& \text{last step with 8 digits ... homework !} && (1.24)
\end{aligned}
$$

Figure 1.1: Computation of $1/\pi$ to 8 digits by a second order iteration.

1.4.3 Cube root extraction

Use $d^{1/3} = d\,(d^2)^{-1/3}$, i.e. compute the inverse third root of d^2 using the iteration

$$x_{k+1} = x_k + x_k \frac{(1 - d^2\, x_k^3)}{3} \qquad (1.26)$$

finally multiply with d.

1.5 A general procedure for the inverse n-th root

There is a nice general formula that allows to build iterations with arbitrary order of convergence for $d^{-1/a}$ that involve no long division.

One uses the identity

$$
\begin{aligned}
d^{-1/a} &= x\,\left(1 - (1 - x^a\, d)\right)^{-1/a} && (1.27)\\
&= x\,(1 - y)^{-1/a} && (1.28)
\end{aligned}
$$

where $y := (1 - x^a\, d)$.

Taylor expansion gives

$$d^{-1/a} = x \sum_{k=0}^{\infty} (1/a)^{\bar{k}}\, y^k \qquad (1.29)$$

where $z^{\bar{k}} := z(z+1)(z+2)...(z+k-1)$.

$$d^{-1/a} = \qquad (1.30)$$

$$= x\left(1 + \frac{y}{a} + \frac{(1+a)\,y^2}{2\,a^2} + \frac{(1+a)(1+2a)\,y^3}{6\,a^3} + \frac{(1+a)(1+2a)(1+3a)\,y^4}{4!\,a^4} + ...\right) \qquad (1.31)$$

A n-th order iteration for $d^{-1/a}$ is obtained by truncating the above series after the $(n-1)$-th term,

$$\Phi_n(a,x) \quad := \quad x \sum_{k=0}^{n-1} (1/a)^{\bar{k}}\, y^k \tag{1.32}$$

$$x_{k+1} \quad = \quad \Phi_n(a, x_k) \tag{1.33}$$

e.g. second order:

$$\Phi_2(a,x) \quad := \quad x + x \frac{(1 - d x^a)}{a} \tag{1.34}$$

Convergence:

$$\Phi_n(d^{-1/a}(1+\epsilon)) = d^{-1/a}(1 + \epsilon^n + O(\epsilon^{n+1})) \tag{1.35}$$

Examples:

$a = 1$: inverse

$$\frac{1}{d} \quad = \quad x \frac{1}{1-y} \tag{1.36}$$

$$= \quad x \left(1 + y + y^2 + y^3 + y^4 + \dots\right) \tag{1.37}$$

$\Phi_2(1,x)$ was described in the last section.

$a = 2$: inverse square root

$$\frac{1}{\sqrt{d}} \quad = \quad x \frac{1}{\sqrt{1-y}} \tag{1.38}$$

$$= \quad x \left(y + \frac{y}{2} + \frac{3\,y^2}{8} + \frac{5\,y^3}{16} + \frac{35\,y^4}{128} + \dots + \frac{\binom{2k}{k}\, y^k}{4^k} + \dots \right) \tag{1.39}$$

$\Phi_2(2,x)$ was described in the last section.

In hfloat, the second order iterations of this type are used. At the second last step the third order correction is used to assure maximum precision at the last step.

1.6 n-th root by Goldschmidt's algorithm

Set

$$x_0 \quad := \quad d^a \qquad y_0 := d^b \tag{1.40}$$

then iterate:

$$r_k \quad := \quad \frac{a + 1 - y_k}{a} \tag{1.41}$$

$$x_{k+1} \quad := \quad x_k \cdot r_k^b \tag{1.42}$$

$$y_{k+1} \quad := \quad y_k \cdot r_k^a \tag{1.43}$$

until x close enough to

$$x_\infty \quad = \quad d^{\frac{a-b}{a}}. \tag{1.44}$$

This is because

$$\frac{x_0^a}{y_0^b} = d^{a-b} \tag{1.45}$$

and

$$\frac{x_{k+1}^a}{y_{k+1}^b} = \frac{(x_k \cdot r^b)^a}{(y_k \cdot r^a)^b} = \frac{x_k^a}{y_k^b} \tag{1.46}$$

e.g. (with $b = 1$)

$$d^{-1/a} = \prod_{k=0}^{\infty} (2 - E_k) \tag{1.47}$$

where $E_0 := d$ and $E_{k+1} := E_k \left(\frac{a+1-E_k}{a}\right)^a$.

1.7 Trancendental functions & the AGM

1.7.1 The AGM

The AGM (arithmetic geometric mean) plays a central role in the (high precision) computation of logarithms and π.

The $AGM(a, b)$ is defined as the limit of the iteration

$$(a_{i+1}, b_{i+1}) := (\frac{a_i + b_i}{2}, \sqrt{a_i\, b_i}) \tag{1.48}$$

starting with $(a_0, b_0) = (a, b)$. Both of the values converge quadratically to a common limit.

One further defines (cf. [15] p.221)

$$R'(k) := \left[1 - \sum_{n=0}^{\infty} 2^{n-1} c_n^2\right]^{-1} \tag{1.49}$$

where $c_n^2 := a_n^2 - b_n^2$ corresponding to $AGM(1, k)$.

A quartic variant of the AGM ($AGM4$) can be written as

$$(\alpha_{i+1}, \beta_{i+1}) := \left(\frac{\alpha_i + \beta_i}{2}, \left(\frac{\alpha_i\, \beta_i\, (\alpha_i^2 + \beta_i^2)}{2}\right)^{1/4}\right) \tag{1.50}$$

starting with $(\alpha_0, \beta_0) = (\sqrt{a_0}, \sqrt{b_0})$ then $(\alpha_k, \beta_k) = (\sqrt{a_{2k}}, \sqrt{b_{2k}})$.

Then

$$R'(k) = \left[1 - \sum_{n=0}^{\infty} 4^n \left(\alpha_n^4 - \left(\frac{\alpha_n^2 + \beta_n^2}{2}\right)^2\right)\right]^{-1} \tag{1.51}$$

corresponding to $AGM4(1, \sqrt{k})$ (cf. [15] p.17).

1.7.2 log

The (natural) logarithm can be computed using the following relations (cf. [15] p.221)

$$\left|\log(x) - R'(10^{-n}) + R'(10^{-n}\, x)\right| \leq \frac{n}{10^{2(n-1)}} \tag{1.52}$$

that hold for $n \geq 3$ and $x \in]\frac{1}{2}, 1[$.

1.7.3 exp

The exponential function is computed using the log and the iteration that comes from truncating the series

$$\begin{aligned}
\exp(d) &= x \exp\left(d - \log(x)\right) && (1.53)\\
&= x \left(1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \ldots\right) && (1.54)
\end{aligned}$$

where $y := d - \log(x)$.

A n-th oder iteration

$$x_{k+1} = x_k \left(1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \ldots \frac{y^{n-1}}{(n-1)!}\right) \tag{1.55}$$

(where $y := d - \log(x_k)$)

As the computation of one log is expensive one would use a higher (e.g. 8-th) order iteration.

If one had some efficient algorithm for exp one could compute log from exp using

$$\begin{aligned}
\log(d) &= x + \log\left(1 + (d \exp(-x) - 1)\right) && (1.56)\\
&= x + \left(1 + y + \frac{y^2}{2} + \frac{y^3}{3} + \ldots\right) && (1.57)
\end{aligned}$$

where $y := d \exp(-x) - 1$.

1.7.4 sin, cos, tan

For the $\arcsin -, \arccos -, \arctan -$ functions use the complex analogue of the AGM. For the $\sin, \cos, \tan$ use the exp iteration above & think complex.

1.8 Inverting a function

I am aware of two formulas that produce iterations for $x = f^{-1}(d)$ for general $f()$:

1.8.1 Householders formula

$$x_{k+1} \mapsto \Phi_n(x_k) := x_k + (n-1)\,\frac{\left(\frac{1}{f(x_k)}\right)^{(n-2)}}{\left(\frac{1}{f(x_k)}\right)^{(n-1)}} + f(x_k)^{n+1}\,\psi \tag{1.58}$$

(where $n \geq 2$ and ψ is an arbitrary function that is set to zero in what follows, cf. [63])

gives a $n-$th order iteration that converges against x_* so that $f(x_*) = 0$.

For $n = 2$ this is Newton's formula:

$$\Phi_2(x) := x - \frac{f}{f'} \tag{1.59}$$

For $n = 3$ this is Halley's formula:

$$\Phi_3(x) := x - \frac{2ff'}{2f^2 - ff''} \tag{1.60}$$

$n = 3$ gives:

$$\Phi_4(x) \quad := \quad x - \frac{3f(ff'' - 2f'^2)}{6ff'f'' - 6f'^3 - ff'''} \tag{1.61}$$

Second order 1.58 with $f(x) := \frac{1}{x^a} - d$ gives 1.34, but for higher orders one gets iterations that require long divisions.

1.8.2 Schröders formula

$$x_{k+1} \mapsto \Phi_n(x_k) \quad := \quad \sum_{t=0}^{n} (-1)^t \, \frac{f(x_k)^t}{t!} \left(\frac{1}{f'(x_k)} \, \partial \right)^{t-1} \frac{1}{f'(x_k)} - f(x_k)^{n+1} \, \varphi \tag{1.62}$$

(where $n \geq 2$ and φ is an arbitrary function that is set to zero in what follows cf. [14] p.13)

gives a $n-$th order iteration that converges against x_* so that $f(x_*) = 0$.

This is, written out,

$$\Phi_n \quad = \quad x - \frac{f}{1!} \cdot \frac{1}{f'} - \frac{f^2}{2!} \cdot \frac{f''}{f'^3} - \frac{f^3}{3!} \cdot \frac{3f''^2 - f'f'''}{f'^5} - \tag{1.63}$$

$$-\frac{f^4}{4!} \cdot \frac{15f''^3 - 10f'f''f'' + f'^2f''f''''}{f'^7} \tag{1.64}$$

$$-\frac{f^7}{7!} \cdot \frac{105f''^4 - 105f'f''^2f''' + 10f'^2f'''^2 + 15f'^2f'''' - f'^3f'''''}{f'^9} \tag{1.65}$$

$$-\ldots \tag{1.66}$$

The second order iteration is the same as the corresponding iteration from 1.58 while all higher order iterations are different.

“If we denote the general term by

$$-\frac{f^a}{a!} \frac{\chi_a}{f^{2a-1}} \tag{1.67}$$

the numbers χ_a can easily computed by the recurrence

$$\chi_{a+1} \quad = \quad (2a - 1)f''\chi_a - f'\partial\chi_a \tag{1.68}$$

.“ (cited from [14], p.16).

Formula 1.62 with $f(x) := 1/x^a - d$ gives the iteration 1.32 for arbitrary order.

Formula 1.62 with $f(x) := log(x) - d$ gives the iteration 1.7.3.

The file `doc/bucket/iter.mu` is a mupad[3] script to experiment with. It produces the iterations 1.62 and 1.58 for arbitrary $f(x)$. A sample output is in `doc/bucket/itermu.out`.

1.9 Addition of floating point numbers

coding a function `add(a,b,c)` := ‘add a to b and put result into c’

and `sub(a,b,c)` := ‘sub b form a and put result into c’

one must consider that

[3]mupad is a freeware computer algebra system that should be on your computer.

1.) any of a,b,c can be identical

2.) different exponents cause an offset in add/sub

3.) action is also determined by the signs (of a and b)

4.) any combination of precisions can occur

Chapter 2

Remarks on the computation of π

2.1 Arctan formulas

Formulas of the form

$$k\,\frac{\pi}{4} \quad = \quad \sum_{i=1}^{N} m_i \arctan\frac{1}{x_i} \tag{2.1}$$

$(k \in \mathbb{N}, \quad m_i \in \mathbb{Z}, \quad x_i \in \mathbb{N})$
where all x_i^2+1 factor completely into factors in

$$F \quad := \{2, 5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, 109, 113\} \tag{2.2}$$

(exceptions are formulas 2.13 and 2.15).

For some of the formulas the factors are given in curly braces[1].

All these formulas were built in 1992 with a mixture of C-programs and Computer Algebra. For the known formulas the original authors are given.

Formulas 2.7 and 2.8 were used for the 100,000 digit computation of π in 1961, see [33]. Machin's formula (2.6) was often used in π-computations before 1960. The computations used the expansion

$$\arctan\frac{1}{x} \quad = \quad \frac{1}{x} - \frac{1}{3\,x^3} + \frac{1}{5\,x^5} - \frac{1}{7\,x^7} + - ... \tag{2.3}$$

Another expansion is given by

$$\arctan\frac{1}{x} \quad = \quad x\left(\frac{1}{x^2+1} + \frac{2}{3}\,\frac{1}{(x^2+1)^2} + \frac{2\cdot 4}{3\cdot 5}\,\frac{1}{(x^2+1)^3} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\,\frac{1}{(x^2+1)^4} + ...\right) \tag{2.4}$$

$$\frac{\pi}{4} \quad = \quad \arctan\frac{1}{2} + \arctan\frac{1}{3} \tag{2.5}$$

$\{5\}$ *(Euler, 1706)*

$$\frac{\pi}{4} \quad = \quad 4\arctan\frac{1}{5} - \arctan\frac{1}{239} \tag{2.6}$$

$\{13\}$ *(Machin, 1776)*

[1] the 2 is always omitted.

$$\frac{\pi}{4} = 6\arctan\frac{1}{8} + 2\arctan\frac{1}{57} + \arctan\frac{1}{239} \tag{2.7}$$

{5,13} *(Størmer, 1896)*

$$\frac{\pi}{4} = 12\arctan\frac{1}{18} + 8\arctan\frac{1}{57} - 5\arctan\frac{1}{239} \tag{2.8}$$

{5,13} *(Gauss)*

$$\frac{\pi}{4} = 44\arctan\frac{1}{57} + 7\arctan\frac{1}{239} - 12\arctan\frac{1}{682} + 24\arctan\frac{1}{12943} \tag{2.9}$$

{5,13,61} *(Størmer, 1896)*

$$\begin{aligned}\frac{\pi}{4} = {} & 88\arctan\frac{1}{172} + 51\arctan\frac{1}{239} + 32\arctan\frac{1}{682} + \\ & +44\arctan\frac{1}{5357} + 68\arctan\frac{1}{12943}\end{aligned} \tag{2.10}$$

{5,13,61,97} *(Størmer, 1896)*

$$\begin{aligned}\frac{\pi}{4} = {} & 88\arctan\frac{1}{192} + 39\arctan\frac{1}{239} + 100\arctan\frac{1}{515} - \\ & -32\arctan\frac{1}{1068} - 56\arctan\frac{1}{173932}\end{aligned} \tag{2.11}$$

{5,13,73,101}

$$\begin{aligned}\frac{\pi}{4} = {} & 100\arctan\frac{1}{319} + 127\arctan\frac{1}{378} + 71\arctan\frac{1}{557} - \\ & -15\arctan\frac{1}{1068} + 66\arctan\frac{1}{2943} + 44\arctan\frac{1}{478707}\end{aligned} \tag{2.12}$$

{5,13,17,41,73}

$$\begin{aligned}\frac{\pi}{4} = {} & 322\arctan\frac{1}{577} + 76\arctan\frac{1}{682} + 139\arctan\frac{1}{1393} + \\ & +156\arctan\frac{1}{12943} + 132\arctan\frac{1}{32807} + 44\arctan\frac{1}{1049433}\end{aligned} \tag{2.13}$$

{5,13,61,89,197}

$$\begin{aligned}\frac{\pi}{4} = {} & 1074\arctan\frac{1}{1568} + 657\arctan\frac{1}{4662} + 183\arctan\frac{1}{5357} - \\ & -779\arctan\frac{1}{12943} - 32\arctan\frac{1}{17923} - 449\arctan\frac{1}{32807} + \\ & +398\arctan\frac{1}{390112}\end{aligned} \tag{2.14}$$

{5,13,17,61,89,97}

$$\begin{aligned}\frac{\pi}{4} = {} & 1587\arctan\frac{1}{2852} + 295\arctan\frac{1}{4193} + 593\arctan\frac{1}{4246} + \\ & +359\arctan\frac{1}{39307} + 481\arctan\frac{1}{55603} + 625\arctan\frac{1}{211050} - \\ & -708\arctan\frac{1}{390112}\end{aligned} \tag{2.15}$$

{5,13,17,29,97,433}

$$\begin{aligned}\frac{\pi}{4} &= 1074 \arctan\frac{1}{4246} + 1257 \arctan\frac{1}{5357} + 1731 \arctan\frac{1}{6107} + \\ &+295 \arctan\frac{1}{12943} + 625 \arctan\frac{1}{19703} - 481 \arctan\frac{1}{32807} - \\ &-1042 \arctan\frac{1}{39307} + 398 \arctan\frac{1}{390112}\end{aligned} \tag{2.16}$$

{...}

$$\begin{aligned}\frac{\pi}{4} &= 7162 \arctan\frac{1}{12943} + 3796 \arctan\frac{1}{32807} + 2558 \arctan\frac{1}{34208} + \\ &+2729 \arctan\frac{1}{44179} - 708 \arctan\frac{1}{51387} + 2192 \arctan\frac{1}{114669} - \\ &-2805 \arctan\frac{1}{157318} - 3696 \arctan\frac{1}{485298} - 2407 \arctan\frac{1}{24208144}\end{aligned} \tag{2.17}$$

{...}

$$\begin{aligned}\frac{\pi}{4} &= 2805 \arctan\frac{1}{5257} - 398 \arctan\frac{1}{9466} + 1950 \arctan\frac{1}{12943} + \\ &+1850 \arctan\frac{1}{34208} + 2021 \arctan\frac{1}{44179} + 2097 \arctan\frac{1}{85353} + \\ &+1484 \arctan\frac{1}{114669} + 1389 \arctan\frac{1}{330182} + 808 \arctan\frac{1}{485298}\end{aligned} \tag{2.18}$$

{5,13,17,29,37,41,53,61} (Gauss)

$$\begin{aligned}\frac{\pi}{4} &= 50539 \arctan\frac{1}{51387} + 1555 \arctan\frac{1}{114669} \\ &-6601 \arctan\frac{1}{157318} - 20678 \arctan\frac{1}{390112} - 5617 \arctan\frac{1}{485298} \\ &-64126 \arctan\frac{1}{617427} + 10958 \arctan\frac{1}{1984933} - 30569 \arctan\frac{1}{3449051} \\ &+23407 \arctan\frac{1}{22709274} + 25433 \arctan\frac{1}{24208144}\end{aligned} \tag{2.19}$$

{...}

$$\begin{aligned}\frac{\pi}{4} &= 36462 \arctan\frac{1}{390112} + 135908 \arctan\frac{1}{485298} \\ &+274509 \arctan\frac{1}{683982} - 39581 \arctan\frac{1}{1984933} \\ &+178477 \arctan\frac{1}{2478328} - 114569 \arctan\frac{1}{3449051} \\ &-146571 \arctan\frac{1}{18975991} + 61914 \arctan\frac{1}{22709274} \\ &-69044 \arctan\frac{1}{24208144} - 89431 \arctan\frac{1}{201229582} \\ &-43938 \arctan\frac{1}{2189376182}\end{aligned} \tag{2.20}$$

{5,13,17,29,37,53,61,89,97,101}

$$\begin{aligned}\frac{\pi}{4} &= 446879 \arctan\frac{1}{683982} + 172370 \arctan\frac{1}{1635786} \\ &-193720 \arctan\frac{1}{1984933} + 369078 \arctan\frac{1}{2478328} \\ &+18231 \arctan\frac{1}{3014557} + 21339 \arctan\frac{1}{3449051}\end{aligned} \tag{2.21}$$

$$
\begin{aligned}
&-154139 \arctan\frac{1}{6225244} - 110109 \arctan\frac{1}{18975991}\\
&+80145 \arctan\frac{1}{22709274} - 223183 \arctan\frac{1}{24208144}\\
&-107662 \arctan\frac{1}{201229582} - 216308 \arctan\frac{1}{2189376182}\\
&\{...\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\pi}{4} \quad = \quad & 872408 \arctan\frac{1}{1984933} + 619249 \arctan\frac{1}{2298668} \qquad (2.22)\\
&+369078 \arctan\frac{1}{2478328} + 18231 \arctan\frac{1}{3014557}\\
&-1217159 \arctan\frac{1}{5033696} + 911989 \arctan\frac{1}{6225244}\\
&+783649 \arctan\frac{1}{18975991} - 70886 \arctan\frac{1}{22709274}\\
&-374214 \arctan\frac{1}{24208144} - 1044789 \arctan\frac{1}{168623905}\\
&+339217 \arctan\frac{1}{201229582} - 446879 \arctan\frac{1}{284862638}\\
&+402941 \arctan\frac{1}{2189376182}\\
&\{...\}
\end{aligned}
$$

2.2 How to build arctan formulas

For a n-term arctan relation of the form

$$\frac{k\,\pi}{4} \quad = \quad m_1 \arctan\frac{1}{a_1} + m_2 \arctan\frac{1}{a_2} + ... + m_n \arctan\frac{1}{a_n} \qquad (2.23)$$

1. Choose a set $F = \{p_1, ..., p_{n-1}\}$ of primes of the form $4k+1$
2. Find $a_1, ...a_y$ so that a_i^2+1 factor completely in $2 \cup F$,
   i.e. $a_i^2 + 1 = 2^{e_{i,0}} \cdot p_1^{e_{i,1}} \cdot p_2^{e_{i,2}} \cdot ... \cdot p_{n-1}^{e_{i,n-1}} \qquad i = 1, 2, ..., y$
   (the factor 2 for the odd x_i is ignored in what follows).
3. If you can't find more than n a_i, i.e. $y < n$, then restart with another F
4. give the $e_{i,j}$ a minus sign if $(a_i \% p_j) > \frac{p_j}{2}$
5. Find the nullspace of the $(n-1) \times y$ matrix $M_{ij} := \{e_{ij}\}$
6. Find linear combinations of the basis vectors of the nullspace that correspond to nice and nontrivial (i.e. $k \neq 0$) arctan relations.

An example: for a 5-term relation

1. Choose $F = \{5, 13, 61, 101\}$

2. Find $\{2, 3, 5, 7, 8, 18, ..., 57, ..., 111, 239, 515, 682, 12943\}$
   (from which i choose the 5 largest $111, 239, 515, 682, 12943$)

3. $y >= 5$ ok.

4. and

5. $M =$

	5	13	61	101
111	0	0	-1	+1
239	0	+4	0	0
515	0	-1	0	2
682	+3	0	+2	0
12943	-4	-3	+1	0

   (e.g. $239^2 + 1 = 2 \cdot 13^4$ and $239\%13 = 5 < 13/2$)

6. we were lucky:

$$\begin{aligned} \frac{1\,\pi}{4} &= 88 \arctan \frac{1}{111} + 7 \arctan \frac{1}{239} - 44 \arctan \frac{1}{515} + \\ &+32 \arctan \frac{1}{682} + 24 \arctan \frac{1}{12943} \end{aligned} \tag{2.24}$$

Often one ends up with a trivial relation, i.e.

$$0 = m_1 \arctan \frac{1}{a_1} + m_2 \arctan \frac{1}{a_2} + ... + m_n \arctan \frac{1}{a_n} \tag{2.25}$$

To build formulas with terms of the type $\arctan \frac{a}{b}$ use the factors of $a^2 + b^2$.

Open questions:

1. What is the upper bound for the a_i for a certain set of factors F ?

2. Is there an algorithm that (in subexponential running time) finds for a certain set $A = \{a_i\}, i = 1, ..., y$ (for which $a_i^2 + 1$ factor completely in F) the arctan relations also for the subsets of A that contain only a_i for which $a_i^2 + 1$ factor completely in a subset of ?

3. Is there a better algorithm to find the a_i for a certain F than these:

   (a) tree search over all products $\prod p_j^{e_{ij}}$
   (checking for each product if the product minus one is square)

   (b) brute force checking all $a = 2...\infty$ if $a^2 + 1$ factors in F

2.3 Ramanujan type formulas

Some nice explicit 'Ramanujan-type' formulas for $1/\pi$ follow. For more formulas and explanation cf. [15] and [17] but don't see appendix A.

The explicit formulas were made in 1994 using Mathematica (for numerical computation of the quantities in the general formulas and for finding the minimal polynomials) and MapleV (for solving the polynomials and numerical verification of the results).

The ‘huge’ formulas here are given rather for fun than for the computation of pi.
The ‘type X’ are the same as in [17].
It is

$$(z)_k \quad := \quad z^{\bar{k}} := z(z+1)(z+2)...(z+k-1) \tag{2.26}$$

in what follows.

2.3.1 Type 1 $n = 58$

$$\frac{1}{\pi} \quad = \quad \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3} \left(2\sqrt{2}\,(1103 + 26390\,n)\right) \frac{1}{\left(99^2\right)^{2n+1}} \tag{2.27}$$

$$\quad = \quad \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!\,(1103 + n\,26390)}{(n!)^4\,396^{4n}} \tag{2.28}$$

(Ramanujan) about 8 correct digits per term

2.3.2 Type 1 $n = 862$

$$\frac{1}{\pi} \quad = \quad \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3} \frac{A + n\,B}{X^{2n+1}} \tag{2.29}$$

(type 1, $n = 862$)

$$\begin{aligned} A \quad := \quad & \Big[4521962731044058367634998271455136035/4 + \\ & +799377627848523458605912125112563234\sqrt{2} + \\ & +12\Big(17750127552909235203012377369182079345275390781190873870656491261057219 + \\ & +1255123555958829884236839904476079251826408616374387198187634303258534\sqrt{2}\Big)^{1/2}\Big]^{1/2} \end{aligned} \tag{2.30}$$

$$\begin{aligned} B \quad := \quad & \Big[9617761395088953485915444091307636106000 + \\ & +6800784302301588686616253973429782154400\sqrt{2} \\ & +52003425600\sqrt{2}\Big(34204566586722903151731072537516469136640672047198830592963 + \\ & +24186280981018566606552309811255775851849456510216830399522\sqrt{2}\Big)^{1/2}\Big]^{1/2} \end{aligned} \tag{2.31}$$

$$\begin{aligned} X \quad := \quad & 1670141896514232075 + 1180968660568974600\sqrt{2} + \\ & +2736\sqrt{2}\Big(372627201865017746341791564603 + \\ & +263487221293322577155951514850\sqrt{2}\Big)^{1/2} \end{aligned} \tag{2.32}$$

(each term adds 37 digits)

2.3.3 Type 2 $n = 37$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3} \frac{(1123 + 21460\, n)}{4} \frac{1}{882^{2n+1}} \tag{2.33}$$

(Ramanujan)

2.3.4 Type 3a $n = 7...163$

$$\frac{1}{\pi} = \frac{1}{\sqrt{-1728\, J}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^3} \frac{A + n\, B}{J^n} \tag{2.34}$$

$$= \frac{1}{\sqrt{-1728\, J}} \sum_{n=0}^{\infty} \frac{(6n)!}{12^{3n}\, (3n)!\, (n!)^3} \frac{A + n\, B}{J^n} \tag{2.35}$$

n	A	B	J	correct digits per term
7	24	189	-125/64	≤ 1
11	60	616	-512/27	1
19	300	4104	-512	3
27	1116	18216	-64000/9	4
43	9468	195048	-80^3	6
67	122124	3140242	-440^3	8
163	163096908	6541681608	-53360^3	15

The last ($n = 163$) is known as Chudnovsky's formula:

$$\frac{1}{\pi} = \frac{6541681608}{\sqrt{640320}^3} \sum_{k=0}^{\infty} \left(\frac{13591409}{545140134} + k\right) \left(\frac{(6k)!}{(k!)^3\, (3k)!} \frac{(-1)^k}{640320^{3k}}\right) \tag{2.36}$$

$$= \frac{12}{\sqrt{640320}^3} \sum_{k=0}^{\infty} (-1)^k \frac{(6k)!}{(k!)^3\, (3k)!} \frac{13591409 + k\, 545140134}{(640320)^{3k}} \tag{2.37}$$

2.3.5 Type 3c $n = 1555$

$$\frac{1}{\pi} = \frac{1}{\sqrt{-12\, J}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^3} \frac{A + n\, B}{J^n} \tag{2.38}$$

$$= \frac{1}{\sqrt{-12\, J}} \sum_{n=0}^{\infty} \frac{(6n)!}{12^n\, (3n)!\, (n!)^3} \frac{A + n\, B}{J^n} \tag{2.39}$$

(type 3c, $n = 1555$)

$$\begin{aligned} A \;:=\;& 52804190260809999965452185 + 23614751784000701705688800\,\sqrt{5} + \\ & +32\,\sqrt{5}\big(10891728551171178200467436212395209160385656017 + \\ & +4870929086578810225077338534541688721351255040\,\sqrt{5}\big)^{1/2} \end{aligned} \tag{2.40}$$

$$\begin{aligned} B &:= 654159204458052267524145750 + 292548889855077669080467200\sqrt{5} + \\ &\quad +209664\sqrt{3110}\Big(6260208323789001636993322654444020882161 + \\ &\quad +2799650273060444296577206890718825190235\sqrt{5}\Big)^{1/2} \end{aligned} \tag{2.41}$$

$$\begin{aligned} J &:= -\Big[17897749588626020 + 8004116944887336\sqrt{5} + \\ &\quad +108\sqrt{5}\big(10985234579463550323713318473 + \\ &\quad +4912746253692362754607395912\sqrt{5}\big)^{1/2}\Big]^3 \end{aligned} \tag{2.42}$$

(each term adds 50 correct digits)

$$\begin{aligned} 0 &= 91056965337194438815073158624225 + \\ &\quad +9214187265360390391808927003100\,A + \\ &\quad +672035320036821921804675631270\,A^2 - \\ &\quad -21121676104323999861808740\,A^3 + A^4 \end{aligned} \tag{2.43}$$

$$\begin{aligned} 0 &= 17514180018137387326565131389045795600 + \\ &\quad +193756947585743300725193322013380000\,B + \\ &\quad +2063419786805410130433556462222680\,B^2 - \\ &\quad -2616636817832209070096583000\,B^3 + B^4 \end{aligned} \tag{2.44}$$

$$\begin{aligned} 0 &= 27192565672854630400 + \\ &\quad +49698245345181030400\,U + \\ &\quad +22885453089727782720\,U^2 - \\ &\quad -71590998354504080\,U^3 + U^4 \end{aligned} \tag{2.45}$$

$$J = -U^3 \tag{2.46}$$

2.3.6 Type 3b $n = 190$

$$\frac{1}{\pi} = \frac{1}{\sqrt{3J}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^3} \frac{A + n\,B}{J^n} \tag{2.47}$$

(type 3b, $n = 190$)

$$\begin{aligned} A &:= 21242668516504965 + \\ &\quad +15020834958518500\sqrt{2} + \\ &\quad +2\sqrt{5}\Big(45125096427586568251645610141659 + \\ &\quad +31908261685643312902173585434250\sqrt{2}\Big)^{1/2} \end{aligned} \tag{2.48}$$

$$B := 1839779353703421900 + \tag{2.49}$$

$$
\begin{aligned}
& +1300920456890691000\sqrt{2}+ \\
& +24337404\sqrt{10}\Big(1142912476713024496667+ \\
& +808161162586491705750\sqrt{2}\Big)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
J \;:=\; & \Big[71864175655+22725423252\sqrt{10}+ \\
& +2808\sqrt{5}\left(261993316778681+82849561276216\sqrt{10}\right)^{1/2}\Big]^3
\end{aligned} \tag{2.50}
$$

(each term adds 34 digits)

$$
\begin{aligned}
0 \;=\; & 18983882886895192207942622025- \\
& -22331544571858356551863737000\,A+ \\
& +570499890229502944324099\,0\,A^2- \\
& 84970674066019860\,A^3+A^4
\end{aligned} \tag{2.51}
$$

$$
\begin{aligned}
0 \;=\; & 11316047287507303785105891917318400- \\
& -1435640467917904306324392329280\,00\,B+ \\
& +213962358988652911130249985\,60\,B^2- \\
& -7359117414813687600\,B^3+B^4
\end{aligned} \tag{2.52}
$$

$$
\begin{aligned}
0 \;=\; & 1860185517864501025-12623836948343599\,00\,U+ \\
& +44498697145120230\,U^2-287456702620\,U^3+U^4
\end{aligned} \tag{2.53}
$$

$$
J \;=\; -U^3 \tag{2.54}
$$

2.4 How to build Ramanujan type formulas

Here is how to build ‘Ramanujan-type’ formulas like those in section 2.3:

1. Read the definitions of the general formulas.
2. Pick out the necessary definitions from the messy appendix A or from Borwein’s book ([15]).
3. Write a mupad-package that implements all the needed quantities (steal from the mathematica package `src/pi/bucket/piram.m`).
4. For each formula and n do
   (a) Get numeric approximations for the quantities you need (e.g. f_n and J_n), compute 500 digits or so.
   (b) Find the minimal polynomials for those quantities.
   (c) Solve the polynomials & beautify the results.
   (d) Check the symbolic results by comparing them to the quantities they were made of, using a higher precision than before.

For more formulas and explanation cf. [17] and [15].

2.5 Approximations for π

In what follows $\pi(n)$ denotes an $n-$digit approximation for π.

$$\pi(2) = \frac{22}{7} = 3.1428... \tag{2.55}$$
$$\pi(6) = \frac{355}{113} = 3.14159292... \tag{2.56}$$

The last approximation tells us that it is a particularly bad idea to use π as an irrational value e.g. in chaos theoretic programs: it is almost up to floating point (single-) precision a rational number with pretty small denominator. Use $\phi = \frac{\sqrt{5}-1}{2}$ instead, cf. section C.

$$\pi(2) = \sqrt{3} + \sqrt{2} = 3.1462... \tag{2.57}$$

Ramanujan [28] gives in his paper (among many other):

$$\pi(5) = \frac{12}{\sqrt{22}} \log\left(\sqrt{2}+2\right) \tag{2.58}$$
$$\pi(15) = \frac{12}{\sqrt{130}} \log\left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+2\right)\left(\sqrt{13}+3\right)\right) \tag{2.59}$$
$$\pi(16) = \frac{24}{\sqrt{142}} \log\left(\frac{1}{2}\sqrt{7\sqrt{2}+10}+\sqrt{11\sqrt{2}+10}\right) \tag{2.60}$$
$$\pi(18) = \frac{12}{\sqrt{190}} \log\left(\left(\sqrt{10}+3\right)\left(2\sqrt{2}+\sqrt{10}\right)\right) \tag{2.61}$$
$$\pi(22) = \frac{12}{\sqrt{310}} \log\left(\left(\sqrt{2}+2\right)\left(\sqrt{5}+3\right)\left(2\sqrt{10}+\sqrt{20\sqrt{10}+61}+5\right)\right) \tag{2.62}$$
$$\pi(31) = \frac{4}{\sqrt{522}} \log\left(\frac{1}{256}\sqrt{2}\left(\sqrt{29}+5\right)^3\left(11\sqrt{6}+5\sqrt{29}\right)\left(\sqrt{3\sqrt{6}+5}+\sqrt{3}\sqrt{\sqrt{6}+3}\right)^6\right) \tag{2.63}$$

From the definition of the J-function

$$\begin{aligned} J(\tau) &:= 1/q + 744 + 196884\,q + 21493760\,q^2 + 864299970\,q^3 + 20245856256\,q^4 + \\ &+ 333202640600\,q^5 + 4252023300096\,q^6 + 44656994071935\,q^7 + 401490886656000\,q^8 + \\ &+ 3176440229784420\,q^9 + 22567393309593600\,q^{10} + ... \end{aligned} \tag{2.64}$$

(where $q := e^{2\,i\,\pi\,\tau}$) it is possible to give approximations to π for certain values of τ (cf. [11]).
E.g.

$$\pi(17) = \frac{\log\left(5280^3+744\right)}{\sqrt{67}} \tag{2.65}$$
$$\pi(30) = \frac{\log\left(640320^3+744\right)}{\sqrt{163}} \tag{2.66}$$

for $\tau = 67$ and $\tau = 163$, respectively.

As q is close to an integer for these values, one can take more terms from the above series to get (approximating q by $[q]$) better approximations of the same type. With $z := -J(\frac{1+i\sqrt{163}}{2}) + 744 =$

$640320^3 + 744$ one gets

$$\pi(46) = \frac{\log\left(z - 196884/z\right)}{\sqrt{163}} \tag{2.67}$$

$$\pi(60) = \frac{\log\left(z - 196884/z + 21493760/z^2\right)}{\sqrt{163}} \tag{2.68}$$

Using more terms doesn't improve the accuracy anymore because of the approximation made for q.

2.6 Iterations

For general forms of the examples given here see J. & P. Borwein's book [15] and their papers.

For some iterations an operation count (in units of full precision multiplications) is given. Operations different from multiplication are counted as follows:
1 squaring = 2/3 mult.
1 division = 4 mult.
1 inverse sqrt = 4 mult.
1 sqrt = 5 mult.
1 cuberoot = mult.
1 inverse 4th root = mult.
1 4th root = mult.

full prec mult $\rightarrow$ efficiency measure

2.order iteration, cf. `src/pi/pi2nd.cc`:

$$y_0 = \frac{1}{\sqrt{2}} \tag{2.69}$$

$$a_0 = \frac{1}{2} \tag{2.70}$$

$$y_{k+1} = \frac{1-(1-y_k^2)^{1/2}}{1+(1-y_k^2)^{1/2}} \quad \rightarrow 0+ \tag{2.71}$$

$$= \frac{(1-y_k^2)^{-1/2}-1}{(1-y_k^2)^{-1/2}+1} \tag{2.72}$$

$$a_{k+1} = a_k\,(1+y_{k+1})^2 - 2^{k+1}\,y_{k+1} \quad \rightarrow \frac{1}{\pi} \tag{2.73}$$

$$a_k - \pi^{-1} \leq 16 \cdot 2^{k+1}\,e^{-2^{k+1}\,\pi} \tag{2.74}$$

2.72 shows how to save 1 multiplication per step (cf. section 1.4).

Operations per step: 1 inverse sqrt, 1 division, 2 squarings, 1 multiplication.

Quartic (4.order) iterations, cf. `src/pi/pi4th.cc`:

variant $r = 4$:

$$y_0 = \sqrt{2} - 1 \tag{2.75}$$

$$a_0 = 6 - 4\sqrt{2} \tag{2.76}$$

$$y_{k+1} = \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \quad \rightarrow 0+ \tag{2.77}$$

$$= \frac{(1-y_k^4)^{-1/4}-1}{(1-y_k^4)^{-1/4}+1} \tag{2.78}$$

$$a_{k+1} = a_k\,(1+y_{k+1})^4 - 2^{2k+3}\,y_{k+1}\,(1+y_{k+1}+y_{k+1}^2) \quad \to \frac{1}{\pi} \tag{2.79}$$

$$= a_k\,((1+y_{k+1})^2)^2 - 2^{2k+3}\,y_{k+1}\,((1+y_{k+1})^2 - y_{k+1}) \tag{2.80}$$

$$0 < a_k - \pi^{-1} \le 16 \cdot 4^n\,2\,e^{-4^n\,2\,\pi} \tag{2.81}$$

Identities 2.78 and 2.80 show how to save operations.

Operations per step: 1 inverse 4th root, 1 division, 2 squarings, 1 multiplication.

variant $r = 16$:

$$y_0 = \frac{1-2^{-1/4}}{1+2^{-1/4}} \tag{2.82}$$

$$a_0 = \frac{8/\sqrt{2}-2}{(2^{-1/4}+1)^4} \tag{2.83}$$

$$y_{k+1} = \frac{(1-y_k^4)^{-1/4}-1}{(1-y_k^4)^{-1/4}+1} \quad \to 0+ \tag{2.84}$$

$$a_{k+1} = a_k\,(1+y_{k+1})^4 - 2^{2k+4}\,y_{k+1}\,(1+y_{k+1}+y_{k+1}^2) \quad \to \frac{1}{\pi} \tag{2.85}$$

$$0 < a_k - \pi^{-1} \le 16 \cdot 4^n\,4\,e^{-4^n\,4\,\pi} \tag{2.86}$$

Same operation count as last, but this variant gives approximately twice as much precision after the same number of steps.

AGM (2.order) iteration, cf.1.48 :

$$a_{k+1} = \frac{a_k + b_k}{2} \tag{2.87}$$

$$b_{k+1} = \sqrt{a_k\,b_k} \tag{2.88}$$

$$c_k^2 = a_k^2 - b_k^2 \tag{2.89}$$

$$= (a_{k-1} - a_k)^2 \tag{2.90}$$

Operations: 1 multiplication, 1 sqrt, 1 squaring.

AGM variant 1, cf. `src/pi/piagm.cc`:

$$a_0 = 1 \tag{2.91}$$

$$b_0 = \frac{1}{\sqrt{2}} \tag{2.92}$$

$$p_n = \frac{2\,a_{n+1}^2}{1 - \sum_{k=0}^{n} 2^k\,c_k^2} \quad \to \pi \tag{2.93}$$

$$\pi - p_n = \frac{\pi^2\,2^{n+4}\,e^{-\pi\,2^{n+1}}}{AGM^2(a_0, b_0)} \tag{2.94}$$

A 4.order version uses 1.50, cf. also `src/pi/piagm.cc`.

AGM variant 3fast, cf. `src/pi/piagm3.cc`:

$$a_0 = 1 \tag{2.95}$$

$$b_0 = \frac{\sqrt{6}+\sqrt{2}}{4} \tag{2.96}$$

$$p_n = \frac{2\,a_{n+1}^2}{\sqrt{3}\,(1 - \sum_{k=0}^{n} 2^k\,c_k^2) - 1} \quad \to \pi \tag{2.97}$$

$$\pi - p_n < \frac{\sqrt{3}\,\pi^2\,2^{n+4}\,e^{-\sqrt{3}\,\pi\,2^{n+1}}}{AGM^2(a_0, b_0)} \tag{2.98}$$

AGM variant 3slow, cf. `src/pi/piagm3.cc`:

$$\begin{aligned}
a_0 &= 1 && (2.99)\\
b_0 &= \frac{\sqrt{6}-\sqrt{2}}{4} && (2.100)\\
p_n &= \frac{6\,a_{n+1}^2}{\sqrt{3}\,(1-\sum_{k=0}^{n} 2^k\,c_k^2)+1} \quad \to \pi && (2.101)\\
\pi - p_n &< \frac{\frac{1}{\sqrt{3}}\,\pi^2\,2^{n+4}\,e^{-\frac{1}{\sqrt{3}}\,\pi\,2^{n+1}}}{AGM(a_0,b_0)^2} && (2.102)
\end{aligned}$$

Derived AGM iteration (2.order), cf. `src/pi/pideriv.cc`:

$$\begin{aligned}
x_0 &= \sqrt{2} && (2.103)\\
p_0 &= 2+\sqrt{2} && (2.104)\\
y_1 &= 2^{1/4} && (2.105)\\
x_{k+1} &= \frac{1}{2}\left(\sqrt{x_k}+\frac{1}{\sqrt{x_k}}\right) \quad (k \ge 0) \quad \to 1+ && (2.106)\\
y_{k+1} &= \frac{y_k\,\sqrt{x_k}+\frac{1}{\sqrt{x_k}}}{y_k+1} \quad (k \ge 1) \quad \to 1+ && (2.107)\\
p_{k+1} &= p_k\,\frac{x_k+1}{y_k+1} \quad (k \ge 1) \quad \to \pi+ && (2.108)\\
p_k - \pi &= 10^{-2^{k+1}} && (2.109)
\end{aligned}$$

Cubic AGM, from [22], cf. `src/pi/picubagm.cc`:

$$\begin{aligned}
a_0 &= 1 && (2.110)\\
b_0 &= \frac{\sqrt{3}-1}{2} && (2.111)\\
a_{n+1} &= \frac{a_n+2\,b_n}{3} && (2.112)\\
b_{n+1} &= \sqrt[3]{\frac{b_n\,(a_n^2+a_n\,b_n+b_n^2)}{3}} && (2.113)\\
p_n &= \frac{3\,a_n^2}{1-\sum_{k=0}^{n} 3^k\,(a_k^2-a_{k+1}^2)} && (2.114)
\end{aligned}$$

Quintic (5. order) iteration, from the article [18] cf. `src/pi/pi5th.cc`:

$$\begin{aligned}
s_0 &= 5(\sqrt{5}-2) && (2.115)\\
a_0 &= \frac{1}{2} && (2.116)\\
s_{n+1} &= \frac{25}{s_n(z+x/z+1)^2} \quad \to 1 && (2.117)\\
\text{where } x &= \frac{5}{s_n}-1 \quad \to 4 && (2.118)\\
\text{and } y &= (x-1)^2+7 \quad \to 16 && (2.119)
\end{aligned}$$

$$\begin{aligned}
\text{and } z &= \left(\frac{x}{2}\left(y+\sqrt{y^2-4x^3}\right)\right)^{1/5} \quad \to 2 && (2.120)\\
a_{n+1} &= s_n^2 a_n - 5^n \left(\frac{s_n^2-5}{2}+\sqrt{s_n\,(s_n^2-2s_n+5)}\right) \quad \to \frac{1}{\pi} && (2.121)\\
a_n - \frac{1}{\pi} &< 16\cdot 5^n\, e^{-\pi\, 5^n} && (2.122)
\end{aligned}$$

Btw. the 5th order algorithm is the slowest of the above, because compared to the other iterations much more operations are needed for each step.

Cubic (3. order) iteration, from [25], cf. `src/pi/pi3rd.cc`:

$$\begin{aligned}
a_0 &= \frac{1}{3} && (2.123)\\
s_0 &= \frac{\sqrt{3}-1}{2} && (2.124)\\
r_{k+1} &= \frac{3}{1+2\,(1-s_k^3)^{1/3}} && (2.125)\\
s_{k+1} &= \frac{r_{k+1}-1}{2} && (2.126)\\
a_{k+1} &= r_{k+1}^2\, a_k - 3^k\,(r_{k+1}^2-1) \quad \to \frac{1}{\pi} && (2.127)
\end{aligned}$$

Nonic (9. order) iteration, from [25], cf. `src/pi/pi9th.cc`:

$$\begin{aligned}
a_0 &= \frac{1}{3} && (2.128)\\
r_0 &= \frac{\sqrt{3}-1}{2} && (2.129)\\
s_0 &= (1-r_0^3)^{1/3} && (2.130)\\
t &= 1+2\,r_k && (2.131)\\
u &= \left(9\,r_k\,(1+r_k+r_k^2)\right)^{1/3} && (2.132)\\
v &= t^2+t\,u+u^2 && (2.133)\\
m &= \frac{27\,(1+s_k+s_k^2)}{v} && (2.134)\\
a_{k+1} &= m\,a_k + 3^{2\,k-1}\,(1-m) \quad \to \frac{1}{\pi} && (2.135)\\
s_{k+1} &= \frac{(1-r_k)^3}{(t+2\,u)\,v} && (2.136)\\
r_{k+1} &= (1-s_k^3)^{1/3} && (2.137)
\end{aligned}$$

2.7 How to build iterations

See Borwein's book [15] and their papers (in particular [49]), enjoy !

E.g. learn that the general form of the quartic iterations (2.75 and 2.82) is[2]

$$y_0 = \sqrt{\lambda^*(r)} \qquad (2.138)$$

[2] [15], p.170f

$$\begin{aligned}
a_0 &= \alpha(r) \\
y_{k+1} &= \frac{(1-y_k^4)^{-1/4}-1}{(1-y_k^4)^{-1/4}+1} \quad \to 0+ \\
a_{k+1} &= a_k\,(1+y_{k+1})^4 - 2^{2k+2}\sqrt{r}\,y_k\,(1+y_{k+1}+y_{k+1}^2) \quad \to \frac{1}{\pi} \\
0 &< a_k - \pi^{-1} \le 16\cdot 4^n\,\sqrt{r}\,e^{-4^n\,\sqrt{r}\,\pi}
\end{aligned} \tag{2.139}$$

2.8 Geometric iterations for π

Let r_k and R_k be the radii of a circles that are inscribed and circumscribed, respectively, to a regular polygon with 2^k sides and circumference 2. Then $2\,\pi\,r_k < 2 < 2\,\pi\,R_k$ and the relations

$$r_2 = \frac{1}{4} \tag{2.140}$$

$$R_2 = \frac{1}{\sqrt{8}} \tag{2.141}$$

$$r_{k+1} = \frac{R_k + r_k}{2} \tag{2.142}$$

$$R_{k+1} = \sqrt{R_k\,r_{k+1}} \tag{2.143}$$

allow to compute better and better approximations to π. This is called Cusanus' method, it was discovered around 1450, cf. [60] pp.155-156. Note the different subscripts on the right hand side of the last equation: if they were equal the above iteration would compute the $AGM(r_2, R_2)$.

Archimedes used the circumferences of regular polygons with $3\cdot 2^k$ that are inscribed (s_k) and circumscribed (t_k) the unit circle.

$$t_0 = 2\sqrt{3} \tag{2.144}$$

$$s_0 = 3 \tag{2.145}$$

$$t_{k+1} = \frac{2\,t_k\,s_k}{t_k + s_k} \tag{2.146}$$

$$s_{k+1} = \sqrt{t_{k+1}\,s_k} \tag{2.147}$$

This is also called the Borchard-Pfaff algorithm. Again, if the subscipts on the right hand side of the last equation were the same then one would compute the arithmetic-harmonic mean $AHM(a_0, b_0)$ that also has the quadratic convergence property of the AGM. One can identify equation 2.146 as

$$\tan\frac{\phi}{2} = \frac{2\,\tan\phi\,\sin\phi}{\tan\phi + \sin\phi} \tag{2.148}$$

and equation 2.147 as

$$\sin\frac{\phi}{2} = \sqrt{\tan\frac{\phi}{2}\,\sin\phi} \tag{2.149}$$

The quantities in the above algorithms can be identified with (values of) trigonometrical functions and their half-argument relations.

It is easy to contruct one-valued iterations of similar kind. Consider $4\cdot 2^k$-sided regular polygons circumscribed around the unit circle. The cicumference is used as an approximation for that of the circle: $2\,\pi \approx 8\cdot 2^k\,x_k$ where x_k is the length of one side of the $4\cdot 2^k$-gons. One has

$$x_k = \tan\frac{x_0}{2^k} \tag{2.150}$$

Start with the quadrat ($x_0 = 1$) and use

$$\tan\frac{\phi}{2} = \frac{\tan\phi}{1+\sqrt{1+\tan^2\phi}} \tag{2.151}$$

i.e. iterate

$$x_{k+1} = \frac{x_k}{1+\sqrt{1+x_k^2}} \tag{2.152}$$

This can be rewritten in many ways, this one is quite elegant: use

$$\arctan\frac{1}{x} = 2\ \arctan\frac{1}{x+\sqrt{x^2+1}} \tag{2.153}$$

i.e. iterate

$$\phi_{k+1} = \phi_k + \sqrt{\phi_k^2+1} \tag{2.154}$$

The approximation made in these algorithms is always pof the type $\sin\phi \approx \phi$ or $\tan\phi \approx \phi$.
Note that all these iterations converge only linear.
Cf. [3] and [66].

2.9 Products for π

Wallis product

$$\begin{aligned} \frac{\pi}{2} &= \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{8\cdot 8}{7\cdot 9}\cdot\ldots \\ &= \prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1} \end{aligned} \tag{2.155}$$

From

$$\sin 2\,x = 2\,\sin x\,\cos x \tag{2.156}$$

or equivalent

$$\frac{\sin 2\,x}{2\,x} = \frac{\sin x}{x}\,\cos x \tag{2.157}$$

follows by repeated substitution

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\cos\frac{x}{2^n} \tag{2.158}$$

using

$$\cos\frac{x}{2} = \sqrt{\frac{1}{2}+\frac{\cos x}{2}} \tag{2.159}$$

$$= \frac{\sqrt{2+2\,\cos x}}{2} \tag{2.160}$$

and $x = \frac{\pi}{2}$ one gets

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{1}{2}}}} \cdot \ldots \tag{2.161}$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \ldots \tag{2.162}$$

From

$$\frac{\tan x}{x} = \prod_{n=1}^{\infty} \frac{1}{1 - \tan \frac{x}{2^n}} \tag{2.163}$$

and

$$\cos x = \frac{\sin x}{x} \Big/ \frac{\tan x}{x} \tag{2.164}$$

and formula 2.158 follows

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \tan^2 \frac{x}{2^n}\right)^{2^n - 1} \tag{2.165}$$

$$\prod_{k=0}^{\infty} \begin{bmatrix} \frac{2\,(k-\frac{1}{2})\,(k+2)}{27\,(k+\frac{2}{3})\,(k+\frac{4}{3})} & 10 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \pi + 6 \\ 0 & 1 \end{bmatrix} \tag{2.166}$$

R.W.Gosper gave

$$\prod_{k=0}^{\infty} \begin{bmatrix} \frac{(k-\frac{5}{2})\,(k+\frac{3}{2})\,(k+3)\,(k+\frac{7}{2})}{64\,(k+\frac{3}{4})\,(k+\frac{5}{4})\,(k+\frac{9}{4})\,(k+\frac{11}{4})} & 48\,(k + \frac{41}{21}) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 15\,\pi + 32 \\ 0 & 1 \end{bmatrix} \tag{2.167}$$

and

$$\prod_{k=0}^{\infty} \begin{bmatrix} \frac{(k-\frac{19}{6})\,(k-\frac{13}{6})\,(k+1)\,(k+\frac{19}{6})\,(k+\frac{25}{6})}{64\,(k-\frac{7}{12})\,(k-\frac{1}{12})\,(k+\frac{3}{2})\,(k+\frac{31}{12})\,(k+\frac{37}{12})} & (k - \frac{23}{42})\,(k + \frac{1}{3})\,(k + \frac{11}{6}) \\ 0 & 1 \end{bmatrix} \tag{2.168}$$

$$= \begin{bmatrix} \frac{6175}{2268}\,\pi - \frac{325}{378} & \cdot \\ \cdot & \cdot \end{bmatrix}$$

Gosper gives two matrix products for $\arctan(x)$ (and thereby for π): Define

$$K(k,n) := \begin{bmatrix} \frac{k\,x^2}{(n+k)(x^2+1)} & \frac{x}{x^2+1} \\ 0 & 1 \end{bmatrix} \tag{2.169}$$

and

$$N(k,n) := \begin{bmatrix} -\frac{n\,x^2}{n+k} & x \\ 0 & 1 \end{bmatrix} \tag{2.170}$$

then

$$\arctan(x) = \text{upper-right}\left(K(1,\frac{1}{2}) \cdot K(2,\frac{1}{2}) \cdot K(3,\frac{1}{2}) \cdot K(4,\frac{1}{2}) \cdot \ldots\right) \tag{2.171}$$

$$= \text{upper-right}\left(N(1,\frac{1}{2}) \cdot N(1,\frac{3}{2}) \cdot N(1,\frac{5}{2}) \cdot N(1,\frac{7}{2}) \cdot \ldots\right) \tag{2.172}$$

2.10 Continued fractions for π

2.10.1 The simple continued fraction for π

$$\pi = 3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}}} \tag{2.173}$$

there is no pattern in the occuring numbers, in figure 2.1 (page 35) the first 500 terms are given.

2.10.2 other continued fractions for π

$$\frac{4}{\pi} = 1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cfrac{9^2}{2+\cfrac{11^2}{2+\ldots}}}}}} \tag{2.174}$$

(Brouncker, 1658)

$$\frac{4}{\pi} = 1+\cfrac{1^2}{3+\cfrac{2^2}{5+\cfrac{3^2}{7+\cfrac{4^2}{9+\cfrac{5^2}{11+\ldots}}}}} \tag{2.175}$$

(cf. [59])

$$\frac{6}{\pi^2-6} = 1+\cfrac{1^2}{1+\cfrac{1\cdot 2}{1+\cfrac{2^2}{1+\cfrac{2\cdot 3}{1+\cfrac{3^2}{1+\cfrac{3\cdot 4}{1+\cfrac{4^2}{1+\ldots}}}}}}} \tag{2.176}$$

$$\frac{12}{\pi^2} = 1+\cfrac{1^4}{3+\cfrac{2^4}{5+\cfrac{3^4}{7+\cfrac{4^4}{9+\cfrac{5^4}{+\ldots}}}}} \tag{2.177}$$

$r_k/\pi^k = 1 + 1^k/(3 + 2^k/(...))$

2.11 Series for π

$$\frac{\pi}{4} = \arctan 1 \tag{2.178}$$
$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} + -... \tag{2.179}$$

(Gregory, 1671). The error of the truncated series is about one half of the first neglected term.

Applying the Euler transform (cf. [53] p.253-255) to 2.178 gives the series

$$\frac{\pi}{2} = 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + ... \tag{2.180}$$
$$= 1 + \frac{1}{3}\left(1 + \frac{2}{5}\left(1 + \frac{3}{7}\,(1 + ...)\right)\right) \tag{2.181}$$

which is used in the spigot algorithm, cf. [12].

In [1] the following acceleration of the arctan-series is given:

$$\arctan x = \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{2k+1} + (-1)^n \frac{1}{2} x^{2n-1} \sum_{k=0}^{\infty} \left(\frac{2x^2}{x^2+1}\right)^{k+1} \frac{k!}{((2n+1))_{k+1}} \tag{2.182}$$

where $((2n+1))_{k+1}$ denotes $(2n+1)(2n+3)...(2n+2k+1)$. The error of the truncated series is less than $|1+x|$ times the first neglected term.

For $x = 1$ (formula 2.178) and $n = 500$ this is

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - +... - \frac{1}{999} + \tag{2.183}$$
$$+ \frac{1}{2}\left(\frac{1}{1001} + \frac{1!}{1001 \cdot 1003} + \frac{2!}{1001 \cdot 1003 \cdot 1005} + \frac{3!}{1001 \cdot 1003 \cdot 1005 \cdot 1007} + ...\right)$$

$$\frac{\pi}{6} = \arcsin \frac{1}{2} \tag{2.184}$$
$$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{3 \cdot 2^3}\right) + \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{5 \cdot 2^5}\right) + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\left(\frac{1}{7 \cdot 2^7}\right) + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}\left(\frac{1}{9 \cdot 2^9}\right) + ... \tag{2.185}$$

(Newton, 1665)

$$\frac{\pi}{4} = \frac{3}{4} + \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{4 \cdot 5 \cdot 6} + \frac{1}{6 \cdot 7 \cdot 8} - \frac{1}{8 \cdot 9 \cdot 10} + \frac{1}{10 \cdot 11 \cdot 12} - +... \tag{2.186}$$

$$\frac{4}{\pi} = 1 + \frac{1}{4} + \left(\frac{1 \cdot 1}{2 \cdot 4}\right)^2 + \left(\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\right)^3 + \left(\frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}\right)^4 + ... \tag{2.187}$$

(Gauss)

$$\frac{4}{\pi} = 1 + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \frac{1}{2}\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 + ... \tag{2.188}$$

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 - 13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + - \ldots \tag{2.189}$$

(Ramanujan, cf. p.7 in [67])

$$\pi = 3\sqrt{3}\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!\,n} \tag{2.190}$$

$$\pi = 3\sqrt{3}\sum_{n=0}^{\infty}\frac{1}{(3n+1)\,(3n+2)} \tag{2.191}$$

Using the identity

$$\pi = \sum_{i=0}^{\infty}\frac{1}{16^i}\left(\frac{4}{8i+1}-\frac{2}{8i+4}-\frac{1}{8i+5}-\frac{1}{8i+6}\right) \tag{2.192}$$

it is possible to compute some hexadecimal digits of π *without* computing any of the preceding digits. See the article [42] for the algorithm.

A similar series is

$$\pi = \sum_{i=0}^{\infty}\frac{(-1)^i}{4^i}\left(\frac{2}{4i+1}+\frac{2}{4i+2}+\frac{1}{4i+3}\right) \tag{2.193}$$

$$\pi = 3 + \frac{1}{60}\left(8 + \frac{2\cdot 3}{7\cdot 8\cdot 3}\left(13 + \frac{3\cdot 5}{10\cdot 11\cdot 3}\left(18 + \frac{4\cdot 7}{13\cdot 14\cdot 3}\,(23 + \ldots)\right)\right)\right) \tag{2.194}$$

(Gosper, cf. [**?**])

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sum_{k=1}^{n}k^2} = 6\,(\pi - 3) \tag{2.195}$$

(from [15], p.101)

F.Bellard gives

$$740025\,\pi + 20379280 = \sum_{n=1}^{\infty}\frac{3\,P(n)}{\binom{7\,n}{2\,n}\,2^{n-1}} \tag{2.196}$$

where

$$\begin{aligned} P(n) \; := \; & -885673181\,n^5 + 3125347237\,n^4 - 2942969225\,n^3 \\ & +1031962795\,n^2 - 196882274\,n + 10996648 \end{aligned} \tag{2.197}$$

$$\pi = \log\frac{1}{x} - 2\,x^4 - 13\,x^8 - \frac{368}{3}\,x^{12} - \ldots \tag{2.198}$$
$$\text{where} \qquad x = \frac{1}{2}\,\frac{2^{1/4}-1}{2^{1/4}+1}$$

2.12 Miscellaneous formulas for π

$$e^{\sqrt{-1}\,\pi} = -1 \tag{2.199}$$

(Euler)

In [2] integrals of the form

$$I_{n,m} = \int_0^1 \frac{x^m\,(1-x)^n}{(1+x^2)} \tag{2.200}$$

examples are

$$I_{4,4} = \int_0^1 \frac{x^4\,(1-x)^4}{(1+x^2)} = \frac{22}{7} - \pi \tag{2.201}$$

$$I_{2,4} = \pi - \frac{47}{15} \tag{2.202}$$

$$I_{6,12} = 16\,\pi - \frac{153966181}{3063060} \tag{2.203}$$

$$I_{32,32} = 16384\,\pi - \frac{31694514838868667276634759966 4}{61576400213688659766216 75} \tag{2.204}$$

The fractions on the rhs. are, as shown in the paper, approximations for π, the last gives 23 correct digits.

$$\pi = \frac{426880\,\sqrt{10005}}{13591409\,\left({}_3F_2(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;A) - B\,{}_3F_2(\frac{7}{6},\frac{3}{2},\frac{11}{6};2,2;A)\right)} \tag{2.205}$$

where

$$A := -\frac{1}{151931373056000} \tag{2.206}$$

$$\tag{2.207}$$

$$B := \frac{30285563}{1651969144908540723200} \tag{2.208}$$

(Chudnovskys), this is from [45].

The

From the Poisson summation formula (cf. [46]) follows:

$$\sum_{x=-\infty}^{+\infty} \left(\frac{\sin x}{x}\right) = \pi \tag{2.209}$$

$$\sum_{x=-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^2 = \pi \tag{2.210}$$

$$\sum_{x=-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^3 = \frac{4}{3}\,\pi \tag{2.211}$$

$$\sum_{x=-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^4 = \frac{2}{3}\,\pi \tag{2.212}$$

(cf. appendix **??**).

2.13 A bit recursion for $1/\pi$

$$
\begin{aligned}
a_0 &:= \tan(1) && (2.213)\\
a_{k+1} &:= \frac{2\,a_k}{1-a_k^2} && (2.214)\\
b(x) &:= \begin{cases} 1 & \text{if} \quad x<0 \\ 0 & \text{else} \end{cases} && (2.215)
\end{aligned}
$$

then

$$\sum_{k=0}^{\infty} \frac{b(a_k)}{2^{k+1}} = \frac{1}{\pi} \qquad (2.216)$$

$$\sum_{k=0}^{\infty} \frac{b(a_k)}{2^{k+1}} = \frac{\arctan(a_0)}{\pi} \qquad (2.217)$$

See and the paper [51].

2.14 A self correcting iteration for π

Use

$$\sin\frac{\pi}{2} = 0 \qquad (2.218)$$

therefor the iteration defined by

$$
\begin{aligned}
x_0 &= \frac{\pi}{2} + \epsilon_0 && (2.219)\\
x_{k+1} &= x_k + \sin x_k && (2.220)
\end{aligned}
$$

converges towards π.

Convergence is of third order: if $x_k = \frac{\pi}{2} + \epsilon_k$ then

$$\epsilon_{k+1} \approx \frac{\epsilon_k^3}{6} \qquad (2.221)$$

Of course, this is not an 'efficient' iteration as the computation of a sine function is required. Similar iterations exist for cos() and tan(), see [57].

3	7	15	1	292	1	1	1	2	1
3	1	14	2	1	1	2	2	2	2
1	84	2	1	1	15	3	13	1	4
2	6	6	99	1	2	2	6	3	5
1	1	6	8	1	7	1	2	3	7
1	2	1	1	12	1	1	1	3	1
1	8	1	1	2	1	6	1	1	5
2	2	3	1	2	4	4	16	1	161
45	1	22	1	2	2	1	4	1	2
24	1	2	1	3	1	2	1	1	10
2	5	4	1	2	2	8	1	5	2
2	26	1	4	1	1	8	2	42	2
1	7	3	3	1	1	7	2	4	9
7	2	3	1	57	1	18	1	9	19
1	2	18	1	3	7	30	1	1	1
3	3	3	1	2	8	1	1	2	1
15	1	2	13	1	2	1	4	1	12
1	1	3	3	28	1	10	3	2	20
1	1	1	1	4	1	1	1	5	3
2	1	6	1	4	1	120	2	1	1
3	1	23	1	15	1	3	7	1	16
1	2	1	21	2	1	1	2	9	1
6	4	127	14	5	1	3	13	7	9
1	1	1	1	1	5	4	1	1	3
1	1	29	3	1	1	2	2	1	3
1	1	1	3	1	1	10	3	1	3
1	2	1	12	1	4	1	1	1	1
7	1	1	2	1	11	3	1	7	1
4	1	48	16	1	4	5	2	1	1
4	3	1	2	3	1	2	2	1	2
5	20	1	1	5	4	1	436	8	1
2	2	1	1	1	1	1	5	1	2
1	3	6	11	4	3	1	1	1	2
5	4	6	9	1	5	1	5	15	1
11	24	4	4	5	2	1	4	1	6
1	1	1	4	3	2	2	1	1	2
1	58	5	1	2	1	2	1	1	2
2	7	1	15	1	4	8	1	1	4
2	1	1	1	3	1	1	1	2	1
1	1	1	1	9	1	4	3	15	1
2	1	13	1	1	1	3	24	1	2
4	10	5	12	3	3	21	1	2	1
34	1	1	1	4	15	1	4	44	1
4	20776	1	1	1	1	1	1	1	23
1	7	2	1	94	55	1	1	2	1
1	3	1	1	32	5	1	14	1	1
1	1	1	3	50	2	16	5	1	2
1	4	6	3	1	3	3	1	2	2
2	5	2	2	2	28	1	1	13	1
5	43	1	4	3	5	3	1	4	1

Figure 2.1: The first 500 terms of the simple continued fraction of π. The third term (15) corresponds to the approximation $\pi \approx \frac{22}{7}$, the fifth term (292) corresponds to $\pi \approx \frac{355}{113}$.

Appendix A

How to build Ramanujan type formulas

NOTE: this section may be detrimental to your health, do not read it.

$$\left(\frac{2\,K}{\pi}(k)\right)^2 = m(k)\,F(\Phi(k)) \tag{A.1}$$

$$= \frac{1}{1+k^2}\;{}_3F_2\left(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;\left(\frac{g^{12}+g^{-12}}{2}\right)^{-2}\right) \tag{A.2}$$

$$= \frac{1}{k'^2-k^2}\;{}_3F_2\left(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;-\left(\frac{G^{12}-G^{-12}}{2}\right)^{-2}\right) \tag{A.3}$$

$$= \frac{1}{\sqrt{1-k^2\,k'^2}}\;{}_3F_2\left(\frac{1}{6},\frac{5}{6},\frac{1}{2};1,1;J^{-1}\right) \tag{A.4}$$

1. Series in x_N $N \geq 3$:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n\left(\frac{2}{4}\right)_n\left(\frac{3}{4}\right)_n}{n!^3}\,d_n(N)\,x_N^{2n+1} \tag{A.5}$$

$$x_N := \left(\frac{g_N^{12}+g_N^{-12}}{2}\right)^{-1} \tag{A.6}$$

$$= \frac{4\,k(N)\,k'^2(N)}{\left(1+k^2(N)\right)^2} \tag{A.7}$$

$$d_n := \left(\frac{\alpha(N)\,x_N^{-1}}{1+k^2(N)} - \frac{\sqrt{N}}{4}\,g_N^{-12}\right) + n\,\sqrt{N}\,\left(\frac{g_N^{12}-g_N^{-12}}{2}\right) \tag{A.8}$$

2. Series in y_N $N \geq 4$:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n\,\frac{\left(\frac{1}{4}\right)_n\left(\frac{2}{4}\right)_n\left(\frac{3}{4}\right)_n}{n!^3}\,e_n(N)\,y_N^{2n+1} \tag{A.9}$$

$$y_N := \left(\frac{G_N^{12}-G_N^{-12}}{2}\right)^{-1} \tag{A.10}$$

$$= \frac{4\,k(N)\,k'(N)}{1-\left(2\,k(N)\,k'(N)\right)^2} \tag{A.11}$$

$$e_n := \left(\frac{\alpha(N)\, y_N^{-1}}{k'^2(N) - k^2(N)} + \frac{\sqrt{N}}{2}\, k^2(N)\, G_N^{12} \right) + n\,\sqrt{N}\, \left(\frac{G_N^{12} + G_N^{-12}}{2} \right) \tag{A.12}$$

3. Series in J_N^{-1} $N \geq 2$:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^3} f_n(N) \left(J_N^{-1/2}\right)^{2n+1} \tag{A.13}$$

$$J_N^{-1} := \frac{27\, G_N^{24}}{\left(4\, G_N^{24} - 1\right)^3} \tag{A.14}$$

$$= \frac{27\, g_N^{24}}{\left(4\, g_N^{24} + 1\right)^3} \tag{A.15}$$

$$f_n := \frac{1}{3\sqrt{3}} \left(\sqrt{N}\, \sqrt{1 - G_N^{-24}} + 2\, \left(\alpha(N) - \sqrt{N}\, k^2(N)\right)\, \left(4G_N^{24} - 1\right) \right) + \tag{A.16}$$

$$+ n\,\sqrt{N}\, \frac{2}{3\sqrt{3}} \left(\left(8G_N^{24} + 1\right)\, \sqrt{1 - G_N^{-24}} \right) \tag{A.17}$$

Elliptic integral of the first kind:

$$K(k) := \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) = \frac{\pi}{2} \sum_{i=0}^{\infty} \left(\frac{(2\,i-1)!!}{2^i\, i!} \right)^2 k^{2i} \tag{A.18}$$

$$= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2\,t^2)}} = \int_0^{\pi/2} \frac{d\Theta}{\sqrt{1 - k^2\, sin^2(\Theta)}} \tag{A.19}$$

$$= \frac{\pi}{2\, AGM(1, k')} \tag{A.20}$$

$$= \frac{\pi}{2} \Theta_3^2(q) \qquad q = e^{-\pi K'(k)/K(k)} \tag{A.21}$$

Elliptic integral of the second kind:

$$E(k) := \frac{\pi}{2}\, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right) = \frac{\pi}{2} \left(1 - \sum_{i=0}^{\infty} \left(\frac{(2\,i-1)!!}{2^i\, i!} \right)^2 \frac{k^{2i}}{2i-1} \right) \tag{A.22}$$

$$= \int_0^1 \frac{\sqrt{1-k^2\,t^2}}{\sqrt{1-t^2}} dt = \int_0^{\pi/2} \sqrt{1 - k^2\, sin^2(\Theta)}\, d\Theta \tag{A.23}$$

Derivatives:

$$\frac{dK}{dk} = \frac{E - k'^2\, K}{k\, k'} \tag{A.24}$$

$$\frac{dE}{dk} = \frac{E - K}{k} \tag{A.25}$$

Differential equation for K and E:

$$0 = \left(k^3 - k\right) \frac{d^2y}{dk^2} + \left(3\, k^2 - 1\right) \frac{dy}{dk} + k\, y \tag{A.26}$$

(also satisfied by $AGM(1, k)^{-1}$ and $AGM(1 + k, 1 - k)^{-1}$)

$$k' := \sqrt{1 - k^2} \tag{A.27}$$

$$K'(k) := K(k') \tag{A.28}$$

$$E'(k) := E(k') \tag{A.29}$$

$$\Theta_2(q) := \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \tag{A.30}$$

$$\Theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \tag{A.31}$$

$$\Theta_4(q) := \sum_{n=-\infty}^{\infty} (-1)^n \, q^{n^2} \tag{A.32}$$

$$k(q) := k = \frac{\Theta_2^2(q)}{\Theta_3^2(q)} \tag{A.33}$$

$$k'(q) := k' = \frac{\Theta_4^2(q)}{\Theta_3^2(q)} \tag{A.34}$$

$$q = e^{-\frac{K'(k)}{K(k)}} \tag{A.35}$$

$$E = k'^2 \, K + k \, k'^2 \, \frac{dK}{dk} \tag{A.36}$$

$$E' \, K + K' \, E - K' \, K = \frac{\pi}{2} \tag{A.37}$$

singular value function:

$$k(N): \qquad \frac{K'}{K}(k(N)) = \sqrt{N} \qquad k(N) := \left(\frac{\Theta_2(q)}{\Theta_3(q)}\right)^2, \text{ where } q := e^{-\pi\sqrt{N}} \tag{A.38}$$

$$k(0) = \infty, \quad k(1) = \frac{1}{\sqrt{2}}, \quad k(\infty) = 0 \qquad k(N) \quad k(n)\text{algebraic for } N \text{ rational} \tag{A.39}$$

$$l(N): \qquad \frac{K'}{K}(l(N)) = \frac{L'}{L} = \frac{1}{\sqrt{N}} \qquad l(N) := \left(\frac{\Theta_2(r)}{\Theta_3(r)}\right)^2, \text{ with} \tag{A.40}$$

$$r := e^{-\pi/\sqrt{N}} = q^{1/N} \tag{A.41}$$

$$\frac{K'}{K} = N\frac{L'}{L} \tag{A.42}$$

$$u := k^{1/4} \qquad v := l^{1/4} \tag{A.43}$$

singular value function of the second kind:

$$\alpha(N) := \frac{E'}{K} - \frac{\pi}{4\,K^2} \quad (\text{where } k := k(N)) \tag{A.44}$$

$$= \sqrt{N}\,\frac{E'}{K'} - \frac{N\,\pi}{4\,K'^2} = \frac{\pi}{4\,K^2} - \sqrt{N}\left(\frac{E}{K} - 1\right) \tag{A.45}$$

$$= \frac{\pi^{-1} - \sqrt{N}\,4\,q\,\frac{\dot{\Theta}_4}{\Theta_4}}{\Theta_3^4} \quad (\text{where } q = e^{-\pi\sqrt{N}}) \tag{A.46}$$

$$\alpha(1) = \frac{1}{2}, \quad \alpha(\infty) = \frac{1}{\pi} \tag{A.47}$$

recursions:

$$\alpha(4N) = \frac{4\,\alpha(N) - 2\,\sqrt{N}\,k^2(N)}{(1+k'(N))^2} \tag{A.48}$$

$$= \left(1+y^2\right)\alpha(4N) - 2\sqrt{N}y \tag{A.49}$$

where

$$y := k(4N) = \frac{1-k'(N)}{1+k'(N)} \tag{A.50}$$

$$\alpha(16N) = (1+y)^4\alpha(N) - 4\sqrt{N}y(1+y+y^2) \tag{A.51}$$

$$\text{where} \qquad y := k(16N) = \left(\frac{1-\sqrt[4]{1-k^2(n)}}{1+\sqrt[4]{1-k^2(n)}}\right)^2 \tag{A.52}$$

$$\alpha(N^{-1}) = \frac{\sqrt{N}-\alpha(N)}{N} \tag{A.53}$$

G,g:

$$G := (2k\,k')^{-1/12}, \quad g := \left(\frac{2k}{k'^2}\right)^{-1/12} \tag{A.54}$$

recursions:

$$9 = \left(1+2\sqrt{2}\frac{G_{9n}^3}{G_n^9}\right)\left(1+2\sqrt{2}\,\frac{G_n^3}{G_{9n}^9}\right) \tag{A.55}$$

$$9 = \left(1+2\sqrt{2}\frac{g_{9n}^3}{g_n^9}\right)\left(1-2\sqrt{2}\,\frac{g_n^3}{g_{9n}^9}\right) \tag{A.56}$$

$$G_{81n}^3 = G_{9n}\,\frac{\sqrt{2}G_{9n}+G_n^3}{\sqrt{2}G_n^3-G_{9n}} \tag{A.57}$$

$$g_{81n}^3 = -g_{9n}\,\frac{\sqrt{2}g_{9n}+g_n^3}{\sqrt{2}g_n^3-g_{9n}} \tag{A.58}$$

$$J := \frac{\left(4\,G^{24}-1\right)^3}{27\,G^{24}} \tag{A.59}$$

$$= \frac{\left(4\,g^{24}+1\right)^3}{27\,g^{24}} \tag{A.60}$$

$$= \frac{4}{27}\,\frac{\left(1-k^2(N)+k^4(N)\right)^3}{k^4(N)\,\left(1-k^2(N)\right)^2} \tag{A.61}$$

$$= \frac{4}{27}\,\frac{\left(1-k^2(N)k'^2(N)\right)^3}{k^4(N)k'^4(N)} \tag{A.62}$$

$$j := 1728\,J \tag{A.63}$$

multiplier:

$$M_N(l,k) := \frac{\Theta_3^2(q)}{\Theta_3^2(q^{1/N})} = \frac{K}{L} \tag{A.64}$$

$$\frac{1}{\pi} = \sqrt{N}\,k\,k'^2\,\frac{4K\dot{K}}{\pi^2} + \left(\alpha(N)-\sqrt{N}\,k^2\right)\,\frac{4K^2}{\pi^2} \tag{A.65}$$

$$\frac{1}{K} = \sqrt{N}\, k\, k'^2\, \frac{4\dot{K}}{\pi} + \left(\alpha(N) - \sqrt{N}\, k^2\right) \frac{4K}{\pi} \tag{A.66}$$

$$\Theta_1(z,q) := \Theta_1(z,t) := 2q^{1/4}\sin z - 2q^{9/4}\sin 3z + 2q^{25/4}\sin 5z - 2q^{49/4}\sin 7z + \ldots \tag{A.67}$$
$$\Theta_2(z,q) := \Theta_2(z,t) := 2q^{1/4}\cos z + 2q^{9/4}\cos 3z + 2q^{25/4}\cos 5z + 2q^{49/4}\cos 7z + \ldots \tag{A.68}$$
$$\Theta_3(z,q) := \Theta_3(z,t) := 1 + 2q\cos 2z + 2q^4\cos 4z + 2q^9\cos 6z + 2q^{16}\cos 8z + \ldots \tag{A.69}$$
$$\Theta_4(z,q) := \Theta_4(z,t) := 1 - 2q\cos 2z + 2q^4\cos 4z - 2q^9\cos 6z + 2q^{16}\cos 8z - \ldots \tag{A.70}$$
$$\text{where} \quad q := e^{\pi i t} \tag{A.71}$$

$$\Theta_1(z,q) := \Theta_1(z,t) := \sum_{n\in Z} (-1)^n q^{((2n+1)/2)^2} \sin{(2n+1)z} \tag{A.72}$$
$$\Theta_2(z,q) := \Theta_2(z,t) := \sum_{n\in Z} q^{((2n+1)/2)^2} \cos{(2n+1)z} \tag{A.73}$$
$$\Theta_3(z,q) := \Theta_3(z,t) := \sum_{n\in Z} q^{(2n)^2} \cos{(2n)z} \tag{A.74}$$
$$\Theta_4(z,q) := \Theta_4(z,t) := \sum_{n\in Z} (-1)^n q^{(2n)^2} \cos{(2n)z} \tag{A.75}$$

$$\Theta_1(z) = -\Theta_2(z+\frac{\pi}{2}) = -iM\Theta_3(z+\frac{\pi}{2}+\frac{\pi t}{2}) = -iM\Theta_4(z+\frac{\pi t}{2}) \tag{A.76}$$
$$\Theta_2(z) = M\Theta_3(z+\frac{\pi t}{2}) = M\Theta_4(z+\frac{\pi}{2}+\frac{\pi t}{2}) = \Theta_1(z+\frac{\pi}{2}) \tag{A.77}$$
$$\Theta_3(z) = \Theta_4(z+\frac{\pi}{2}) = M\Theta_1(z+\frac{\pi}{2}+\frac{\pi t}{2}) = M\Theta_2(z+\frac{\pi t}{2}) \tag{A.78}$$
$$\Theta_4(z) = -iM\Theta_1(z+\frac{\pi t}{2}) = iM\Theta_2(z+\frac{\pi}{2}+\frac{\pi t}{2}) = \Theta_3(z+\frac{\pi}{2}) \tag{A.79}$$
$$\text{where} \quad M := q^{1/4}e^{iz} \tag{A.80}$$

$$\Theta_1(z) = \Theta_1(z+\pi) = -qe^{2iz}\Theta_1(z+\pi t) \tag{A.81}$$
$$\Theta_2(z) = \Theta_2(z+\pi) = +qe^{2iz}\Theta_2(z+\pi t) \tag{A.82}$$
$$\Theta_3(z) = \Theta_3(z+\pi) = +qe^{2iz}\Theta_3(z+\pi t) \tag{A.83}$$
$$\Theta_4(z) = \Theta_4(z+\pi) = -qe^{2iz}\Theta_4(z+\pi t) \tag{A.84}$$

$$\frac{\partial^2\Theta(z,t)}{\partial z^2} = \frac{4i}{\pi}\frac{\partial\Theta(z,t)}{\partial t} \tag{A.85}$$
$$\Theta_3^2 = 1 + 4\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}} \tag{A.86}$$

Let

$$T(x,q) := \sum_{-\infty}^{\infty} x^n q^{n^2} \tag{A.87}$$

then:

$$T(x,q^a)T(y,q^b) = \sum_{m,n} x^m y^n q^{am^2+bn^2} \tag{A.88}$$

$$= \sum_{k=0}^{a+b-1} y^k q^{bk^2} T(xyq^{2bk}, q^{a+b}) T(y^a x^{-b} q^{2abk}, q^{ab(a+b)}) \tag{A.89}$$

$$Q_0 := \prod_{n=1}^{\infty} 1 - q^{2n} \tag{A.90}$$

$$Q_1 := \prod_{n=1}^{\infty} 1 + q^{2n} \tag{A.91}$$

$$Q_2 := \prod_{n=1}^{\infty} 1 + q^{2n-1} \tag{A.92}$$

$$Q_3 := \prod_{n=1}^{\infty} 1 - q^{2n-1} \tag{A.93}$$

$$Q_0 Q_1 = Q_0(q^2) \tag{A.94}$$

$$Q_0 Q_3 = Q_0(q^{1/2}) \tag{A.95}$$

$$Q_2 Q_3 = Q_3(q^2) \tag{A.96}$$

$$Q_1 Q_2 = Q_1(q^{1/2}) \tag{A.97}$$

$$\sum_{n=-\infty}^{\infty} (\pm 1)^n q^{kn^2+ln} = \prod_{n=0}^{\infty} (1 \pm q^{2kn+k-l})(1 \pm q^{2kn+k+l})(1 - q^{2kn+2k}) \tag{A.98}$$

$$e.g. :1 = Q_1 Q_2 Q_3 \tag{A.99}$$

$$\Theta_3(q) = Q_0 Q_2^2 \qquad \Theta_4(q) = Q_0 Q_3^2 \qquad \Theta_2(q) = 2q^{1/4} Q_0 Q_1^2 \tag{A.100}$$

(A.101)

$k = 3/2$ and $l = 1/2$ gives: (A.102)

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{-\infty}^{\infty} (-1)^n q^{(3n+1)n/2} \tag{A.103}$$

triple-product identity:

$$\prod_{n=1}^{\infty} (1 - q^{2n})(1 + xq^{2n-1})(1 + x^{-1} q^{2n-1}) = \sum_{n=-\infty}^{\infty} x^n q^{n^2} \tag{A.104}$$

quintuple-product identity:

$$\prod_{n=1}^{\infty} (1 - q^n)(1 - xq^n)(1 - x^{-1} q^{n-1})(1 - x^2 q^{2n-1})(1 - x^{-2} q^{2n-1}) \tag{A.105}$$

$$= \sum_{n=-\infty}^{\infty} (x^{3n} - x^{-3n-1}) q^{n(3n+1)/2} \tag{A.106}$$

equivalent:

$$\prod_{n=1}^{\infty} (1 - q^{2n})(1 - xq^{2n-1})(1 - x^{-1} q^{2n-1})(1 - x^2 q^{4n-4})(1 - x^{-2} q^{4n-4}) \tag{A.107}$$

$$= \sum_{n=-\infty}^{\infty} q^{3n^2-2n}[(x^{3n}+x^{-3n})-(x^{3n-2}+x^{-(3n-2)})] \tag{A.108}$$

Appendix B

More arctan formulas

In what follows let $\mathbf{A}\frac{1}{x}$ denote $\arctan\frac{1}{x}$.

$$\arctan\frac{1}{x} = \arctan\frac{1}{x+d} + \arctan\frac{1}{x+\frac{x^2+1}{d}} \tag{B.1}$$

$$\arctan\frac{1}{x} + \arctan\frac{1}{y} = \arctan\frac{1}{\frac{x\,y-1}{x+y}} \tag{B.2}$$

$$\arctan\frac{1}{z} = \arctan\frac{1}{x_1} + \arctan\frac{1}{x_2} + ... + \arctan\frac{1}{x_n} \tag{B.3}$$

$$\Longleftrightarrow \quad z = \frac{Re(\prod_{k=1}^{n}(x_k+i))}{Im(\prod_{k=1}^{n}(x_k+i))} \tag{B.4}$$

$$\Longleftrightarrow \quad (z^2+1) = \frac{\prod_{k=1}^{n}(x_k^2+1)}{\left(Im(\prod_{k=1}^{n}(x_k+i))\right)^2} \tag{B.5}$$

$$\arctan\frac{1}{x} = \arctan\frac{1}{y} + \arctan\frac{1}{z} \tag{B.6}$$

$$\Longleftrightarrow \quad (x^2+1) = \frac{(y^2+1)(z^2+1)}{(y+z)^2} = (y-x)(z-x) \tag{B.7}$$

$$\frac{k\,\pi}{4} = \arctan\frac{b_1}{a_1} + \arctan\frac{b_2}{a_2} + ... + \arctan\frac{b_n}{a_n} \tag{B.8}$$

$$\Longleftrightarrow \quad (a_1+i\,b_1)(a_2+i\,b_2)...(a_n+i\,b_n)(1-i)^k \in \mathbb{R} \tag{B.9}$$

$$\frac{k\,\pi}{4} = \arctan\frac{1}{a_1} + \arctan\frac{1}{a_2} + ... + \arctan\frac{1}{a_n} \tag{B.10}$$

$$\Longleftrightarrow \quad (a_1+i)(a_2+i)...(a_n+i)(1-i)^k \in \mathbb{R} \tag{B.11}$$

$$\frac{k\,\pi}{4} = m_1\arctan\frac{1}{a_1} + m_2\arctan\frac{1}{a_2} + ... + m_n\arctan\frac{1}{a_n} \tag{B.12}$$

$$\Longleftrightarrow \quad (a_1+i)^{m_1}(a_2+i)^{m_2}\ldots(a_n+i)^{m_n}(1-i)^k \in \mathbb{R} \tag{B.13}$$

Let

$$a > b, \qquad gcd(a,b) = 1, \quad a\,\alpha + b\,\beta = 1 \tag{B.14}$$

then

$$\mathbf{A}\frac{1}{a}b = \mathbf{A}\frac{1}{-\beta}\alpha + \arctan\frac{1}{a\,\beta - b\,\alpha} \tag{B.15}$$

because if

$$r := a\,\beta - b\,\alpha \tag{B.16}$$

$$\arctan\frac{1}{z} := \arctan\frac{1}{a/b} + \arctan\frac{1}{r} \tag{B.17}$$

then

$$z = \frac{(a/b)\,r - 1}{(a/b) + r} \tag{B.18}$$

$$= \frac{a(\alpha\,b - \beta\,a) - b}{a + b(\alpha\,b - \beta\,a)} \tag{B.19}$$

$$= \frac{a(\alpha\,b - \beta\,a) - b(1)}{a(1) + b(\alpha\,b - \beta\,a)} \tag{B.20}$$

$$= \frac{a(\alpha\,b - \beta\,a) - b(\alpha\,a + \beta\,b)}{a(\alpha\,a + \beta\,b) + b(\alpha\,b - \beta\,a)} \tag{B.21}$$

$$= \frac{-a^2\,\beta - b^2\,\beta}{\alpha\,a^2 + \alpha\,b^2} \tag{B.22}$$

$$= \frac{-\beta}{\alpha} \tag{B.23}$$

$$\mathbf{A}\frac{1}{x} = \mathbf{A}\frac{1}{x+1} + \mathbf{A}\frac{1}{x^2+x+1} \tag{B.24}$$

$$\frac{\pi}{4} = \mathbf{A}\frac{1}{1} = \sum_{x=1}^{\infty}\mathbf{A}\frac{1}{x^2+x+1} = \tag{B.25}$$

$$= \mathbf{A}\frac{1}{3} + \mathbf{A}\frac{1}{7} + \mathbf{A}\frac{1}{13} + \mathbf{A}\frac{1}{21} + \mathbf{A}\frac{1}{31} + \mathbf{A}\frac{1}{43} + \mathbf{A}\frac{1}{57} + ... \tag{B.26}$$

$$\mathbf{A}\frac{1}{2x+1} = \mathbf{A}\frac{1}{2x+3} + \mathbf{A}\frac{1}{2x^2+4x+2} \tag{B.27}$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{2n^2} \tag{B.28}$$

$$= \mathbf{A}\frac{1}{2} + \mathbf{A}\frac{1}{8} + \mathbf{A}\frac{1}{18} + \mathbf{A}\frac{1}{50} + \mathbf{A}\frac{1}{98} + \mathbf{A}\frac{1}{128} + \dots \tag{B.29}$$

define

$$jt(a_0, a_1, m_1, m_2, 0) := a0 \tag{B.30}$$

$$jt(a_0, a_1, m_1, m_2, 1) := a1 \tag{B.31}$$

$$jt(a_0, a_1, m_1, m_2, n) := m_1\,jt(a_0, a_1, m_1, m_2, n-1) + m_2\,jt(a_0, a_1, m_1, m_2, n-2) \tag{B.32}$$

then

$$jt(a_0, a_1, m_1, m_2, n) = C_1\, x_1^n + C_2\, x_2^n \quad \text{where:} \tag{B.33}$$
$$x_{1,2}^2 = m_1\, x_{1,2} + m_2, \quad C_1 + C_2 = a_0, \quad C_1\, x_1 + C_2\, x_2 = a_1 \tag{B.34}$$

$$\frac{\pi}{8} = \sum_{n=1}^{\infty} (-1)^n \mathbf{A} \frac{1}{jt(0,1,2,1,2n)} = \mathbf{A}\frac{1}{2} - \mathbf{A}\frac{1}{12} + \mathbf{A}\frac{1}{70} - \mathbf{A}\frac{1}{408} + \mathbf{A}\frac{1}{2378} - + \ldots \tag{B.35}$$
$$\frac{\pi}{8} = \sum_{n=1}^{\infty} \mathbf{A} \frac{1}{jt(1,1,2,1,2n)} = \tag{B.36}$$
$$= \sum_{n=1}^{\infty} (-1)^n\, \mathbf{A} \frac{1}{(1-\sqrt{2})^{2n} + (1+\sqrt{2})^{2n}} \tag{B.37}$$
$$= \mathbf{A}\frac{1}{3} + \mathbf{A}\frac{1}{17} + \mathbf{A}\frac{1}{99} + \mathbf{A}\frac{1}{577} + \mathbf{A}\frac{1}{3363} + \ldots \tag{B.38}$$

$$f(n) := jt(0,1,2,1,2n)\, jt(0,1,2,1,2n+1)^2 + jt(1,1,2,1,2n+1) \tag{B.39}$$
$$\text{eg.} \quad f(0) = 2 \cdot 5^2 + 7 = 57, \quad f(1) = 12 \cdot 29^2 + 41 = 10133, \quad \ldots \tag{B.40}$$
$$\frac{\pi}{8} = \sum_{n=1}^{\infty} 3^n\, \mathbf{A} \frac{1}{jt(0,1,2,1,2n+1)} - \sum_{n=1}^{\infty} 3^n\, \mathbf{A} \frac{1}{f(n)} \tag{B.41}$$
$$= \mathbf{A}\frac{1}{5} + 3\left(\mathbf{A}\frac{1}{29} + 3\left(\mathbf{A}\frac{1}{169} + 3\left(\mathbf{A}\frac{1}{5741} + \ldots\right)\right)\right) - \tag{B.42}$$
$$- \left(\mathbf{A}\frac{1}{57} + 3\left(\mathbf{A}\frac{1}{10133} + 3\left(\mathbf{A}\frac{1}{1999509} + \ldots\right)\right)\right)$$
$$\mathbf{A}\frac{1}{P_n} = 4\,\mathbf{A}\frac{1}{Q_{n+2}} + \mathbf{A}\frac{1}{P_{n+4}} - 2\,\mathbf{A}\frac{1}{q(Q_{n+2})} (\text{n odd}) \tag{B.43}$$
$$\frac{\pi}{4} = \mathbf{A}\frac{1}{P_1} = 4\left(\mathbf{A}\frac{1}{5} + \mathbf{A}\frac{1}{169} + \mathbf{A}\frac{1}{5741} + \ldots\right) - \tag{B.44}$$
$$- 2\left(\mathbf{A}\frac{1}{q(5)} + \mathbf{A}\frac{1}{q(169)} + \mathbf{A}\frac{1}{q(5741)} + \ldots\right)$$

$$\text{with } g(n) := jt(0,1,2,1,n+3)\, jt(0,1,2,1,n+2)^2 + jt(1,1,2,1,n+2) \tag{B.45}$$
$$\mathbf{A}\frac{1}{jt(1,1,2,1,n)} = 5\,\mathbf{A}\frac{1}{jt(1,1,2,1,n+2)} + 2\,\mathbf{A}\frac{1}{jt(1,1,2,1,n+3)} \tag{B.46}$$
$$- \mathbf{A}\frac{1}{q(jt(0,1,2,1,n+2))} + 2\,\mathbf{A}\frac{1}{g(n)} \quad (\text{n odd})$$
$$\frac{\pi}{4} = 2\,\mathbf{A}\frac{1}{3} + \mathbf{A}\frac{1}{7} = 2\,\mathbf{A}\frac{1}{3} + 2\,\mathbf{A}\frac{1}{17} + \mathbf{A}\frac{1}{41} = \tag{B.47}$$
$$= \left(12\,\mathbf{A}\frac{1}{17} + 5\,\mathbf{A}\frac{1}{41}\right) - 2\,\mathbf{A}\frac{1}{70} - 4\,\mathbf{A}\frac{1}{307} \tag{B.48}$$
$$= \left(70\,\mathbf{A}\frac{1}{99} + 29\,\mathbf{A}\frac{1}{239}\right) - 2\,\mathbf{A}\frac{1}{70} - 4\,\mathbf{A}\frac{1}{307} - 12\,\mathbf{A}\frac{1}{12238} - 24\,\mathbf{A}\frac{1}{58911} \tag{B.49}$$
$$= \left(408\,\mathbf{A}\frac{1}{577} + 169\,\mathbf{A}\frac{1}{1393}\right) - \tag{B.50}$$
$$- (2\,\mathbf{A}\frac{1}{70} + 4\,\mathbf{A}\frac{1}{307} + 12\,\mathbf{A}\frac{1}{12238} + 24\,\mathbf{A}\frac{1}{58911} + 70\,\mathbf{A}\frac{1}{\ldots} + 140\,\mathbf{A}\frac{1}{\ldots})$$

$$\begin{aligned} \frac{\pi}{4} &= 2\,(\sqrt{2}-1) - \sum_{n=1}^{\infty} \left(jt(0,1,2,1,2n)\,\mathbf{A}\frac{1}{q(jt(0,1,2,1,2n+1))} + \right. \\ &+ \left. 2\,jt(0,1,2,1,2n)\,\mathbf{A}\frac{1}{g(2n+1)} \right) \end{aligned} \quad \text{(B.51)}$$

$$\frac{\pi}{4} = 3\sum_{n=1}^{\infty}\mathbf{A}\frac{1}{jt(1,1,3,1,2n)} - \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{jt(1,1,3,1,2n)} \quad \text{(B.52)}$$

$$= 3\left(\mathbf{A}\frac{1}{4} + \mathbf{A}\frac{1}{43} + \mathbf{A}\frac{1}{469} + \mathbf{A}\frac{1}{5116} + \ldots\right) \quad \text{(B.53)}$$

$$- \left(\mathbf{A}\frac{1}{q(4)} + \mathbf{A}\frac{1}{q(43)} + \mathbf{A}\frac{1}{q(469)} + \mathbf{A}\frac{1}{q(5116)} + \ldots\right)$$

$$= 3\left(\mathbf{A}\frac{1}{4} + \mathbf{A}\frac{1}{43} + \mathbf{A}\frac{1}{469} + \mathbf{A}\frac{1}{5116} + \ldots\right) \quad \text{(B.54)}$$

$$- \left(\mathbf{A}\frac{1}{38} + \mathbf{A}\frac{1}{39818} + \mathbf{A}\frac{1}{51581558} + \ldots\right)$$

$$q(n) := \frac{3n+n^3}{2} \quad \text{(B.55)}$$

$$F_n := jt(0,1,1,1,n) \quad \text{(B.56)}$$

$$= Fibonacci(n) = 1,1,2,3,5,8,13,21,34,55,... \quad \text{(B.57)}$$

$$\mathbf{A}\frac{1}{x} = 2\,\mathbf{A}\frac{1}{2x} - \mathbf{A}\frac{1}{3x+4x^3} \quad \text{(B.58)}$$

$$\frac{\pi}{4} = 1 - \sum_{n=0}^{\infty} 2^n\,\mathbf{A}\frac{1}{3\,2^n + 4\,2^{3n}} = 1 - \sum_{n=0}^{\infty} 2^n\,\mathbf{A}\frac{1}{q(2^{n+1})} \quad \text{(B.59)}$$

$$= 1 - \left(\mathbf{A}\frac{1}{7} + 2\,\mathbf{A}\frac{1}{38} + 4\,\mathbf{A}\frac{1}{268} + 8\,\mathbf{A}\frac{1}{2072} + 16\,\mathbf{A}\frac{1}{16432} + \ldots\right) \quad \text{(B.60)}$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{F_{2n+1}} \quad \text{(B.61)}$$

$$= \mathbf{A}\frac{1}{2} + \mathbf{A}\frac{1}{5} + \mathbf{A}\frac{1}{13} + \mathbf{A}\frac{1}{34} + \mathbf{A}\frac{1}{89} + \mathbf{A}\frac{1}{233} + \ldots$$

$$\mathbf{A}\frac{1}{F_{2n}} = 3\,\mathbf{A}\frac{1}{F_{2n+2}} - \mathbf{A}\frac{1}{F_{2n+4}} - \mathbf{A}\frac{1}{q(F_{2n+2})} \Rightarrow \quad \text{(B.62)}$$

$$\frac{\pi}{4} = \frac{3\sqrt{5}-5}{2} - \sum_{n=1}^{\infty} F_{2n}\,\mathbf{A}\frac{1}{q(F_{2n+2})} \quad \text{(B.63)}$$

$$= \frac{3\sqrt{5}-5}{2} - \left(1\,\mathbf{A}\frac{1}{18} + 3\,\mathbf{A}\frac{1}{268} + 8\,\mathbf{A}\frac{1}{4662} + 21\,\mathbf{A}\frac{1}{83270} + \ldots\right) \quad \text{(B.64)}$$

$$\mathbf{A}\frac{1}{x+\sqrt{x^2-4}} = \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{x\,jt(0,1,x,-1,n)^2} \quad \text{(B.65)}$$

for $x = 2$:

$$\mathbf{A}\frac{1}{2+\sqrt{2^2-4}} = \mathbf{A}\frac{1}{1} \quad \text{(B.66)}$$

$$= \frac{\pi}{4} = \sum_{n=1}^{\infty} \mathbf{A}\frac{1}{2\,n^2} \tag{B.67}$$

for $x = 4$:

$$\mathbf{A}\frac{1}{4+\sqrt{4^2-4}} = \mathbf{A}\frac{1}{2+\sqrt{3}} \tag{B.68}$$

$$= \frac{\pi}{12} = \sum_{n=1}^{\infty} \mathbf{A}\frac{1}{4\,jt(0,1,4,-1,n)^2} \tag{B.69}$$

$$= \mathbf{A}\frac{1}{4} + \mathbf{A}\frac{1}{64} + \mathbf{A}\frac{1}{900} + \mathbf{A}\frac{1}{12544} + \mathbf{A}\frac{1}{174724} + \mathbf{A}\frac{1}{2433600} + \ldots \tag{B.70}$$

$$\mathbf{A}\frac{1}{2x^2} = 2\,\mathbf{A}\frac{1}{4x^2-2x+1} - \mathbf{A}\frac{1}{4x^2+2x+1} \Rightarrow \tag{B.71}$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\left[\mathbf{A}\frac{1}{4n^2-2n+1} + \mathbf{A}\frac{1}{4n^2+2n+1}\right] \tag{B.72}$$

$$\mathbf{A}\frac{1}{2x^2} = 2\,\mathbf{A}\frac{1}{4x^2+2x+1} - \mathbf{A}\frac{1}{2x^3+x} + \mathbf{A}\frac{1}{4x^3+3x} \tag{B.73}$$

$$\mathbf{A}\frac{1}{2x^2} = 2\,\mathbf{A}\frac{1}{4x^2-2x+1} + \mathbf{A}\frac{1}{2x^3+x} - \mathbf{A}\frac{1}{4x^3+3x} \Rightarrow \tag{B.74}$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\left[\mathbf{A}\frac{1}{4n^2+2n+1} + \mathbf{A}\frac{1}{2n^3+n} - \mathbf{A}\frac{1}{4n^3+3n}\right] \tag{B.75}$$

$$\frac{\pi}{6} = \sum_{n=0}^{\infty} \mathbf{A}\frac{1}{2\,jt(1,3,4,-1,n)^2} \tag{B.76}$$

$$= \mathbf{A}\frac{1}{2} + \mathbf{A}\frac{1}{18} + \mathbf{A}\frac{1}{242} + \mathbf{A}\frac{1}{3362} + \mathbf{A}\frac{1}{46818} + \mathbf{A}\frac{1}{652082} + \ldots \tag{B.77}$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \mathbf{A}\frac{1}{2n^2} = \text{(by changing the order of summation)} \tag{B.78}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)2^{2n+1}}\zeta(4n+2) \tag{B.79}$$

Appendix C

Continued fractions

Set

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ldots}}}} \tag{C.1}$$

For $k > 0$ let $\frac{p_k}{q_k}$ be the value of the above fraction if a_{k+1} is set to zero (set $\frac{p_{-1}}{q_{-1}} := \frac{1}{0}$ and $\frac{p_0}{q_0} := \frac{b_0}{1}$). Then

$$p_k = b_k\, p_{k-1} + a_k\, p_{k-2} \tag{C.2}$$

$$q_k = b_k\, q_{k-1} + a_k\, q_{k-2} \tag{C.3}$$

(Simple continued fractions are those with $a_k = 1\ \forall k$).

Pseudo code for a procedure that computes the $p_k, q_k \quad k = -1 \ldots n$ of a continued fraction :

```
procedure ratios_from_contfrac(a[0..n], b[0..n], n, p[-1..n], q[-1..n])
{
    p[-1] := 1
    q[-1] := 0

    p[0] := b[0]
    q[0] := 1

    for k:=1 to n
    {
        p[k] := b[k] * p[k-1] + a[k] * p[k-2]
        q[k] := b[k] * q[k-1] + a[k] * q[k-2]
    }
}
```

Pseudo code for a procedure that fills the first `n` terms of the simple continued fraction of (the floating point number) `x` into the array `cf[]`:

```
procedure continued_fraction(x, n, cf[0..n-1])
{
    for k:=0 to n-1
```

```
    {
        xi := floor(x)
        cf[k] := xi
        x := 1/(x-xi)
    }
}
```

Pseudo code for a function that computes the numerical value of a number `x` from (the leading `n` terms of) its simple continued fraction representation:

```
function number_from_contfrac(cf[0..n-1], n)
{
    x := cf[n-1]

    for k:=n-2 to 0 step -1
    {
        x := 1/x+cf[k]
    }

    return x
}
```

(cf. [58], [59], [7], [8]).

Appendix D

A modulo multiplication trick

The following trick allows easy multiplication modulo some modulus that has more bits than a halfword. It uses the fact that integer multiplication gets the least significant bits of the result whereas float multiplication gets most significant bits of the result.

The C-code given here assumes that you have 64 bit integer types `sint64` (signed) and `uint64` (unsigned) and a floating point type with 64 bit mantissa, `float64` (typically `long double`).

```
sint64  modulus;
float64 invmod=(float64)1.0/modulus;

sint64 mul(const sint64 a, const sint64 b)
{
    sint64 quot;
    sint64 rem;

    quot = (sint64)((float64)a*(float64)b*invmodulus+(float64)0.5);

    uint64 ab = (uint64)a*(uint64)b;
    uint64 mq = (uint64)modulus*(uint64)quot;

    rem = (sint64)(ab-mq);


    if ( rem<0 )  // correction
    {
        rem += modulus;
        quot--;
    }

    // quot == (a*b)/modulus
    return rem;  // (a*b)%modulus
}
```

The code works if $0 <=$ `a, b` $<$ modulus $< 2^{63}$. Note that for fixed modulus the division for the inverse modulus `invmod` needs only be done once, so the routine avoids any division.

I found this trick in a documentation file (the file 'PROJECTS' in version 2.02) of the gmp package ([55]) where it is ascribed to Peter Montgomery.

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