

Formulas of the Form

$$k \frac{\pi}{4} = \sum_{i=1}^N m_i \arctan \left(\frac{1}{x_i} \right)$$

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the 1-term formula:

$$\frac{\pi}{4} = \arctan(1) \quad \{ \}$$

2-term formulae:

$$\frac{\pi}{4} = \arctan \left(\frac{1}{2} \right) + \arctan \left(\frac{1}{3} \right) \quad \{ 5 \} \text{ (Euler, 1706)} \quad (2)$$

$$\frac{\pi}{4} = 2 \arctan \left(\frac{1}{2} \right) - \arctan \left(\frac{1}{7} \right) \quad \{ 5 \} \text{ (Herman, 1706)} \quad (3)$$

$$\frac{\pi}{4} = 2 \arctan \left(\frac{1}{3} \right) + \arctan \left(\frac{1}{7} \right) \quad \{ 5 \} \text{ (Hutton, 1776)} \quad (4)$$

$$\frac{\pi}{4} = 4 \arctan \left(\frac{1}{5} \right) - \arctan \left(\frac{1}{239} \right) \quad \{ 13 \} \text{ (Machin, 1706)} \quad (5)$$

3-term formulas:

$$\frac{\pi}{4} = 2 \arctan\left(\frac{1}{2}\right) - \arctan\left(\frac{1}{9}\right) - \arctan\left(\frac{1}{32}\right) \quad (6)$$

$\{5, 41\}$

$$\frac{\pi}{2} = 3 \arctan\left(\frac{1}{2}\right) + 2 \arctan\left(\frac{1}{13}\right) + \arctan\left(\frac{1}{38}\right) \quad (7)$$

$\{5, 17\}$

$$\frac{\pi}{2} = 5 \arctan\left(\frac{1}{3}\right) - 2 \arctan\left(\frac{1}{68}\right) - \arctan\left(\frac{1}{117}\right) \quad (8)$$

$\{5, 37\}$

$$\frac{\pi}{4} = 3 \arctan\left(\frac{1}{4}\right) + \arctan\left(\frac{1}{20}\right) + \arctan\left(\frac{1}{1985}\right) \quad (9)$$

$\{17, 401\}$

$$\frac{\pi}{4} = 5 \arctan\left(\frac{1}{7}\right) + 2 \arctan\left(\frac{1}{23}\right) - 2 \arctan\left(\frac{1}{182}\right) \quad (10)$$

$\{5, 53\}$

$$\frac{\pi}{4} = 5 \arctan\left(\frac{1}{7}\right) + 2 \arctan\left(\frac{1}{26}\right) - 2 \arctan\left(\frac{1}{2057}\right) \quad (11)$$

$\{5, 677\}$

$$\frac{\pi}{4} = 5 \arctan\left(\frac{1}{7}\right) + 2 \arctan\left(\frac{1}{27}\right) + 2 \arctan\left(\frac{1}{1068}\right) \quad (12)$$

$\{5, 73\}$

$$\frac{\pi}{4} = 5 \arctan\left(\frac{1}{7}\right) + 2 \arctan\left(\frac{1}{28}\right) + 2 \arctan\left(\frac{1}{443}\right) \quad (13)$$

$\{5, 157\}$

$$\frac{\pi}{4} = 5 \arctan\left(\frac{1}{7}\right) + 2 \arctan\left(\frac{1}{43}\right) + 2 \arctan\left(\frac{1}{68}\right) \quad (14)$$

$\{5, 37\}$

$$\frac{\pi}{4} = 5 \arctan\left(\frac{1}{7}\right) + 4 \arctan\left(\frac{1}{53}\right) + 2 \arctan\left(\frac{1}{4443}\right) \quad (15)$$

$\{5, 181\}$

$$\frac{\pi}{4} = 6 \arctan\left(\frac{1}{8}\right) + 2 \arctan\left(\frac{1}{57}\right) + \arctan\left(\frac{1}{239}\right) \quad (16)$$

$\{5, 13\}$ (*Størmer, 1896*)

$$\frac{\pi}{4} = 8 \arctan\left(\frac{1}{10}\right) - \arctan\left(\frac{1}{239}\right) - 4 \arctan\left(\frac{1}{515}\right) \quad (17)$$

$\{13, 101\}$ (*Klingstierna*)

$$\frac{\pi}{4} = 12 \arctan\left(\frac{1}{18}\right) + 8 \arctan\left(\frac{1}{57}\right) - 5 \arctan\left(\frac{1}{239}\right) \quad (18)$$

$\{5, 13\}$ (*Gauss*)

4-term formulas:

$$\frac{\pi}{4} = 3 \arctan\left(\frac{1}{7}\right) + 4 \arctan\left(\frac{1}{13}\right) + 2 \arctan\left(\frac{1}{55}\right) + 2 \arctan\left(\frac{1}{123}\right) \quad (19)$$

$\{5, 17, 89\}$

$$2 \frac{\pi}{4} = 11 \arctan\left(\frac{1}{7}\right) + 4 \arctan\left(\frac{1}{443}\right) + \arctan\left(\frac{1}{1393}\right) + 2 \arctan\left(\frac{1}{11018}\right) \quad (20)$$

$\{5, 157, 197\}$

$$5 \frac{\pi}{4} = 44 \arctan\left(\frac{1}{11}\right) - 9 \arctan\left(\frac{1}{239}\right) - 16 \arctan\left(\frac{1}{682}\right) - 12 \arctan\left(\frac{1}{12943}\right) \quad (21)$$

$\{5, 13, 61\}$

$$\frac{\pi}{4} = 12 \arctan\left(\frac{1}{15}\right) - \arctan\left(\frac{1}{239}\right) - 4 \arctan\left(\frac{1}{580}\right) - 4 \arctan\left(\frac{1}{1710}\right) \quad (22)$$

$\{13, 113, 229\}$

$$\frac{\pi}{4} = 10 \arctan\left(\frac{1}{17}\right) + 2 \arctan\left(\frac{1}{38}\right) + 7 \arctan\left(\frac{1}{41}\right) - 4 \arctan\left(\frac{1}{157}\right) \quad (23)$$

$\{5, 17, 29\}$

$$\frac{\pi}{4} = 12 \arctan\left(\frac{1}{17}\right) + 8 \arctan\left(\frac{1}{57}\right) - 5 \arctan\left(\frac{1}{239}\right) - 12 \arctan\left(\frac{1}{307}\right) \quad (24)$$

$\{5, 13, 29\}$

$$\frac{\pi}{4} = 12 \arctan\left(\frac{1}{18}\right) + 8 \arctan\left(\frac{1}{73}\right) + 3 \arctan\left(\frac{1}{239}\right) - 8 \arctan\left(\frac{1}{2943}\right) \quad (25)$$

$\{5, 13, 41\}$

$$\frac{\pi}{4} = 12 \arctan\left(\frac{1}{18}\right) + 3 \arctan\left(\frac{1}{70}\right) + 5 \arctan\left(\frac{1}{99}\right) + 8 \arctan\left(\frac{1}{307}\right) \quad (26)$$

$\{5, 13, 29\}$ (Bennet)

$$\frac{\pi}{4} = 12 \arctan\left(\frac{1}{18}\right) + 8 \arctan\left(\frac{1}{99}\right) + 3 \arctan\left(\frac{1}{239}\right) + 8 \arctan\left(\frac{1}{307}\right) \quad (27)$$

$\{5, 13, 29\}$ (D.H. Lehmer, 1938)

$$\frac{\pi}{4} = 12 \arctan\left(\frac{1}{18}\right) + 8 \arctan\left(\frac{1}{109}\right) + 11 \arctan\left(\frac{1}{239}\right) - 8 \arctan\left(\frac{1}{6826318}\right) \quad (28)$$

$\{5, 13, 457\}$

$$3 \frac{\pi}{4} = 44 \arctan\left(\frac{1}{18}\right) - 23 \arctan\left(\frac{1}{239}\right) + 8 \arctan\left(\frac{1}{682}\right) - 16 \arctan\left(\frac{1}{12943}\right) \quad (29)$$

$\{5, 13, 61\}$

$$\frac{\pi}{4} = 16 \arctan\left(\frac{1}{21}\right) + 3 \arctan\left(\frac{1}{239}\right) + 4 \arctan\left(\frac{1}{421}\right) + 4 \arctan\left(\frac{1}{1985}\right) \quad (30)$$

$\{13, 17, 401\}$

$$\frac{\pi}{4} = 10 \arctan\left(\frac{1}{22}\right) + 7 \arctan\left(\frac{1}{41}\right) + 12 \arctan\left(\frac{1}{75}\right) + 2 \arctan\left(\frac{1}{4193}\right) \quad (31)$$

$\{5, 29, 97\}$ (Størmer, 1896)

$$\frac{\pi}{4} = 17 \arctan\left(\frac{1}{23}\right) + 8 \arctan\left(\frac{1}{182}\right) + 10 \arctan\left(\frac{1}{5118}\right) + 5 \arctan\left(\frac{1}{6072}\right) \quad (32)$$

$\{5, 53, 373\}$

$$\frac{\pi}{4} = 17 \arctan\left(\frac{1}{26}\right) + 5 \arctan\left(\frac{1}{38}\right) + 3 \arctan\left(\frac{1}{2057}\right) - 10 \arctan\left(\frac{1}{8827}\right) \quad (33)$$

$\{5, 17, 677\}$

$$\frac{\pi}{4} = 12 \arctan\left(\frac{1}{28}\right) + 20 \arctan\left(\frac{1}{57}\right) - 5 \arctan\left(\frac{1}{239}\right) + 12 \arctan\left(\frac{1}{443}\right) \quad (34)$$

$$\{5, 13, 157\}$$

$$\frac{\pi}{4} = 22 \arctan\left(\frac{1}{28}\right) + 2 \arctan\left(\frac{1}{443}\right) - 5 \arctan\left(\frac{1}{1393}\right) - 10 \arctan\left(\frac{1}{11018}\right) \quad (35)$$

$\{5, 157, 197\}$ (*Escott*)

$$\frac{\pi}{4} = 24 \arctan\left(\frac{1}{29}\right) - 4 \arctan\left(\frac{1}{57}\right) + 7 \arctan\left(\frac{1}{239}\right) - 12 \arctan\left(\frac{1}{12238}\right) \quad (36)$$

$\{5, 13, 421\}$

$$\frac{\pi}{4} = 12 \arctan\left(\frac{1}{38}\right) + 20 \arctan\left(\frac{1}{57}\right) + 7 \arctan\left(\frac{1}{239}\right) + 24 \arctan\left(\frac{1}{268}\right) \quad (37)$$

$\{5, 13, 17\}$ (*Gauss*)

$$\frac{\pi}{4} = 12 \arctan\left(\frac{1}{49}\right) + 32 \arctan\left(\frac{1}{57}\right) - 5 \arctan\left(\frac{1}{239}\right) + 12 \arctan\left(\frac{1}{110443}\right) \quad (38)$$

$\{5, 13, 1201\}$

$$\frac{\pi}{4} = 24 \arctan\left(\frac{1}{53}\right) + 20 \arctan\left(\frac{1}{57}\right) - 5 \arctan\left(\frac{1}{239}\right) + 12 \arctan\left(\frac{1}{4443}\right) \quad (39)$$

$\{5, 13, 281\}$

$$\frac{\pi}{4} = 20 \arctan\left(\frac{1}{57}\right) + 24 \arctan\left(\frac{1}{68}\right) + 12 \arctan\left(\frac{1}{117}\right) - 5 \arctan\left(\frac{1}{239}\right) \quad (40)$$

$\{5, 13, 37\}$

$$\frac{\pi}{4} = 44 \arctan\left(\frac{1}{57}\right) + 7 \arctan\left(\frac{1}{239}\right) - 12 \arctan\left(\frac{1}{682}\right) + 24 \arctan\left(\frac{1}{12943}\right) \quad (41)$$

$\{5, 13, 61\}$ (*Størmer, 1896*)

5-term formulas:

$$\begin{aligned} \frac{\pi}{2} = & 11 \arctan\left(\frac{1}{7}\right) + 2 \arctan\left(\frac{1}{99}\right) + \arctan\left(\frac{1}{1393}\right) + \\ & -2 \arctan\left(\frac{1}{182}\right) - 2 \arctan\left(\frac{1}{1264557}\right) \\ & \{5, 13, 29, 53, 197\} \end{aligned} \quad (42)$$

$$\begin{aligned} \frac{\pi}{4} = & 22 \arctan\left(\frac{1}{29}\right) + 12 \arctan\left(\frac{1}{392}\right) + 2 \arctan\left(\frac{1}{1068}\right) - \\ & -5 \arctan\left(\frac{1}{1393}\right) - 10 \arctan\left(\frac{1}{5502}\right) \\ & \{5, 73, 197, 421\} \end{aligned} \quad (43)$$

$$\begin{aligned} \frac{\pi}{4} = & 22 \arctan\left(\frac{1}{32}\right) + 17 \arctan\left(\frac{1}{182}\right) + 7 \arctan\left(\frac{1}{818}\right) - \\ & -5 \arctan\left(\frac{1}{1303}\right) - 12 \arctan\left(\frac{1}{3014557}\right) \\ & \{5, 41, 53, 101\} \end{aligned} \quad (44)$$

$$\begin{aligned} \frac{\pi}{4} = & 68 \arctan\left(\frac{1}{99}\right) - 12 \arctan\left(\frac{1}{117}\right) + 39 \arctan\left(\frac{1}{239}\right) + \\ & +20 \arctan\left(\frac{1}{307}\right) - 24 \arctan\left(\frac{1}{882}\right) \\ & \{5, 13, 29, 37\} \end{aligned} \quad (45)$$

$$\begin{aligned} \frac{\pi}{4} = & 56 \arctan\left(\frac{1}{99}\right) + 27 \arctan\left(\frac{1}{239}\right) + 32 \arctan\left(\frac{1}{307}\right) + \\ & +12 \arctan\left(\frac{1}{4193}\right) - 12 \arctan\left(\frac{1}{39307}\right) \\ & \{5, 13, 29, 97\} \end{aligned} \quad (46)$$

$$\begin{aligned} \frac{\pi}{4} = & 44 \arctan\left(\frac{1}{109}\right) + 95 \arctan\left(\frac{1}{239}\right) - 12 \arctan\left(\frac{1}{682}\right) + \\ & +24 \arctan\left(\frac{1}{12943}\right) - 44 \arctan\left(\frac{1}{6826318}\right) \\ & \{5, 13, 61, 457\} \end{aligned} \quad (47)$$

$$\begin{aligned} \frac{\pi}{4} = & 88 \arctan\left(\frac{1}{111}\right) + 7 \arctan\left(\frac{1}{239}\right) - 44 \arctan\left(\frac{1}{515}\right) + \\ & +32 \arctan\left(\frac{1}{682}\right) + 24 \arctan\left(\frac{1}{12943}\right) \\ & \{5, 13, 61, 101\} \end{aligned} \quad (48)$$

$$\begin{aligned} \frac{\pi}{4} = & 88 \arctan\left(\frac{1}{172}\right) + 51 \arctan\left(\frac{1}{239}\right) + 32 \arctan\left(\frac{1}{682}\right) + \\ & +44 \arctan\left(\frac{1}{5357}\right) + 68 \arctan\left(\frac{1}{12943}\right) \\ & \{5, 13, 61, 97\} \text{ (Størmer, 1896)} \end{aligned} \quad (49)$$

$$\begin{aligned} \frac{\pi}{4} = & 88 \arctan\left(\frac{1}{192}\right) + 39 \arctan\left(\frac{1}{239}\right) + 100 \arctan\left(\frac{1}{515}\right) - \\ & -32 \arctan\left(\frac{1}{1068}\right) - 56 \arctan\left(\frac{1}{173932}\right) \\ & \{5, 13, 73, 101\} \end{aligned} \quad (50)$$

6-term formulas:

$$134 \frac{\pi}{4} = 227 \arctan\left(\frac{1}{2}\right) - 13 \arctan\left(\frac{1}{682}\right) + 12 \arctan\left(\frac{1}{2059}\right) + \\ + 24 \arctan\left(\frac{1}{2943}\right) - 6 \arctan\left(\frac{1}{5357}\right) + 20 \arctan\left(\frac{1}{12943}\right) \\ \{5, 13, 41, 61, 97\} \quad (51)$$

$$3 \frac{\pi}{4} = 200 \arctan\left(\frac{1}{99}\right) + 81 \arctan\left(\frac{1}{239}\right) + 4 \arctan\left(\frac{1}{4193}\right) - \\ - 32 \arctan\left(\frac{1}{12238}\right) - 4 \arctan\left(\frac{1}{39307}\right) - 64 \arctan\left(\frac{1}{58911}\right) \\ \{5, 13, 29, 97, 421\} \quad (52)$$

$$\frac{\pi}{4} = 100 \arctan\left(\frac{1}{216}\right) + 71 \arctan\left(\frac{1}{265}\right) + 24 \arctan\left(\frac{1}{1068}\right) + \\ + 39 \arctan\left(\frac{1}{2436}\right) + 44 \arctan\left(\frac{1}{5507}\right) + 56 \arctan\left(\frac{1}{6962}\right) \\ \{5, 13, 37, 73, 97\} \quad (53)$$

$$\frac{\pi}{4} = 7 \arctan\left(\frac{1}{239}\right) + 176 \arctan\left(\frac{1}{293}\right) + 44 \arctan\left(\frac{1}{616}\right) + \\ + 32 \arctan\left(\frac{1}{682}\right) + 132 \arctan\left(\frac{1}{3141}\right) - 64 \arctan\left(\frac{1}{12943}\right) \\ \{5, 13, 17, 61, 101\} \quad (54)$$

$$\frac{\pi}{4} = 27 \arctan\left(\frac{1}{239}\right) + 200 \arctan\left(\frac{1}{307}\right) + 68 \arctan\left(\frac{1}{4193}\right) + \\ + 56 \arctan\left(\frac{1}{12238}\right) - 68 \arctan\left(\frac{1}{39307}\right) + 112 \arctan\left(\frac{1}{58911}\right) \\ \{5, 13, 29, 97, 421\} \quad (55)$$

$$\frac{\pi}{4} = 95 \arctan\left(\frac{1}{239}\right) + 44 \arctan\left(\frac{1}{515}\right) + 32 \arctan\left(\frac{1}{682}\right) + \\ + 176 \arctan\left(\frac{1}{782}\right) + 88 \arctan\left(\frac{1}{4030}\right) + 112 \arctan\left(\frac{1}{12943}\right) \\ \{5, 13, 61, 101, 401\} \quad (56)$$

$$\frac{\pi}{4} = 183 \arctan\left(\frac{1}{255}\right) + 44 \arctan\left(\frac{1}{515}\right) - 56 \arctan\left(\frac{1}{682}\right) + \\ + 88 \arctan\left(\frac{1}{1303}\right) + 7 \arctan\left(\frac{1}{2943}\right) - 71 \arctan\left(\frac{1}{12943}\right) \\ \{5, 13, 41, 61, 101\} \quad (57)$$

$$\frac{\pi}{4} = 127 \arctan\left(\frac{1}{265}\right) + 100 \arctan\left(\frac{1}{515}\right) + 88 \arctan\left(\frac{1}{697}\right) - \\ - 32 \arctan\left(\frac{1}{1068}\right) + 39 \arctan\left(\frac{1}{2436}\right) - 56 \arctan\left(\frac{1}{173932}\right) \\ \{5, 13, 37, 73, 101\} \quad (58)$$

$$\frac{\pi}{4} = 68 \arctan\left(\frac{1}{268}\right) + 115 \arctan\left(\frac{1}{307}\right) + 32 \arctan\left(\frac{1}{447}\right) + \\ + 95 \arctan\left(\frac{1}{1143}\right) + 51 \arctan\left(\frac{1}{34208}\right) + 39 \arctan\left(\frac{1}{44179}\right) \\ \{5, 13, 17, 29, 53\} \quad (59)$$

$$\frac{\pi}{4} = 183 \arctan\left(\frac{1}{268}\right) + 32 \arctan\left(\frac{1}{682}\right) + 95 \arctan\left(\frac{1}{1568}\right) + \\ + 44 \arctan\left(\frac{1}{4662}\right) - 166 \arctan\left(\frac{1}{12943}\right) - 51 \arctan\left(\frac{1}{32807}\right) \\ \{5, 13, 17, 61, 89\} \quad (60)$$

$$\frac{\pi}{4} = 337 \arctan\left(\frac{1}{307}\right) - 193 \arctan\left(\frac{1}{463}\right) + 151 \arctan\left(\frac{1}{4193}\right) + \\ + 305 \arctan\left(\frac{1}{4246}\right) - 122 \arctan\left(\frac{1}{39307}\right) - 83 \arctan\left(\frac{1}{390112}\right) \quad (61)$$

$$\{5, 13, 17, 29, 97\}$$

$$\begin{aligned} \frac{\pi}{4} = & 100 \arctan\left(\frac{1}{319}\right) + 127 \arctan\left(\frac{1}{378}\right) + 71 \arctan\left(\frac{1}{557}\right) - \\ & -15 \arctan\left(\frac{1}{1068}\right) + 66 \arctan\left(\frac{1}{2943}\right) + 44 \arctan\left(\frac{1}{478707}\right) \end{aligned} \quad (62)$$

$\{5, 13, 17, 41, 73\}$

$$\begin{aligned} \frac{\pi}{4} = & 322 \arctan\left(\frac{1}{577}\right) + 76 \arctan\left(\frac{1}{682}\right) + 139 \arctan\left(\frac{1}{1393}\right) + \\ & +156 \arctan\left(\frac{1}{12943}\right) + 132 \arctan\left(\frac{1}{32807}\right) + 44 \arctan\left(\frac{1}{1049433}\right) \end{aligned} \quad (63)$$

$\{5, 13, 61, 89, 197\}$

7-term formulas:

$$\begin{aligned}
62 \frac{\pi}{4} = & 537 \arctan\left(\frac{1}{11}\right) + 43 \arctan\left(\frac{1}{5357}\right) - 52 \arctan\left(\frac{1}{5507}\right) + \\
& + 6 \arctan\left(\frac{1}{12943}\right) + 138 \arctan\left(\frac{1}{17923}\right) + 112 \arctan\left(\frac{1}{32807}\right) \\
& + 26 \arctan\left(\frac{1}{157318}\right) \\
& \{5, 13, 37, 61, 89, 97\}
\end{aligned} \tag{64}$$

$$\begin{aligned}
\frac{\pi}{4} = & 44 \arctan\left(\frac{1}{515}\right) + 95 \arctan\left(\frac{1}{538}\right) + 127 \arctan\left(\frac{1}{682}\right) + \\
& + 176 \arctan\left(\frac{1}{782}\right) + 95 \arctan\left(\frac{1}{1068}\right) + 88 \arctan\left(\frac{1}{4030}\right) + \\
& + 17 \arctan\left(\frac{1}{12943}\right) \\
& \{5, 13, 61, 73, 101, 401\}
\end{aligned} \tag{65}$$

$$\begin{aligned}
\frac{\pi}{4} = & 366 \arctan\left(\frac{1}{577}\right) - 12 \arctan\left(\frac{1}{682}\right) + 44 \arctan\left(\frac{1}{1023}\right) + \\
& + 183 \arctan\left(\frac{1}{1393}\right) - 44 \arctan\left(\frac{1}{5832}\right) + 24 \arctan\left(\frac{1}{12943}\right) - \\
& - 88 \arctan\left(\frac{1}{6826318}\right) \\
& \{5, 13, 61, 229, 197, 457\}
\end{aligned} \tag{66}$$

$$\begin{aligned}
\frac{\pi}{4} = & 215 \arctan\left(\frac{1}{682}\right) + 827 \arctan\left(\frac{1}{1568}\right) - 139 \arctan\left(\frac{1}{4662}\right) - \\
& - 349 \arctan\left(\frac{1}{12943}\right) + 132 \arctan\left(\frac{1}{32807}\right) - 366 \arctan\left(\frac{1}{83270}\right) - \\
& - 183 \arctan\left(\frac{1}{1493208}\right) \\
& \{5, 13, 17, 61, 89, 233\}
\end{aligned} \tag{67}$$

$$\begin{aligned}
\frac{\pi}{4} = & 537 \arctan\left(\frac{1}{746}\right) + 44 \arctan\left(\frac{1}{5357}\right) + 259 \arctan\left(\frac{1}{5507}\right) + \\
& + 156 \arctan\left(\frac{1}{12943}\right) - 171 \arctan\left(\frac{1}{17923}\right) + 227 \arctan\left(\frac{1}{32807}\right) \\
& + 139 \arctan\left(\frac{1}{157318}\right) \\
& \{5, 13, 37, 61, 89, 97\}
\end{aligned} \tag{68}$$

$$\begin{aligned}
\frac{\pi}{4} = & 1074 \arctan\left(\frac{1}{1568}\right) + 657 \arctan\left(\frac{1}{4662}\right) + 183 \arctan\left(\frac{1}{5357}\right) - \\
& - 779 \arctan\left(\frac{1}{12943}\right) - 32 \arctan\left(\frac{1}{17923}\right) - 449 \arctan\left(\frac{1}{32807}\right) + \\
& + 398 \arctan\left(\frac{1}{390112}\right) \\
& \{5, 13, 17, 61, 89, 97\}
\end{aligned} \tag{69}$$

$$\begin{aligned}
\frac{\pi}{4} = & 1587 \arctan\left(\frac{1}{2852}\right) + 295 \arctan\left(\frac{1}{4193}\right) + 593 \arctan\left(\frac{1}{4246}\right) + \\
& + 359 \arctan\left(\frac{1}{39307}\right) + 481 \arctan\left(\frac{1}{55603}\right) + 625 \arctan\left(\frac{1}{211050}\right) - \\
& - 708 \arctan\left(\frac{1}{390112}\right) \\
& \{5, 13, 17, 29, 97, 433\}
\end{aligned} \tag{70}$$

$primes := \text{subset of } 5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, 109, 113$

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) \quad \{13\} \quad (71)$$

$$\frac{\pi}{4} = 12 \arctan\left(\frac{1}{18}\right) + 8 \arctan\left(\frac{1}{57}\right) - 5 \arctan\left(\frac{1}{239}\right) \quad \{5, 13\} \quad (72)$$

$$\frac{\pi}{4} = 44 \arctan\left(\frac{1}{57}\right) + 7 \arctan\left(\frac{1}{239}\right) - 12 \arctan\left(\frac{1}{682}\right) + 24 \arctan\left(\frac{1}{12943}\right) \quad \{5, 13, 61\} \quad (73)$$

$$\begin{aligned} \frac{\pi}{4} = & 88 \arctan\left(\frac{1}{192}\right) + 39 \arctan\left(\frac{1}{239}\right) + 100 \arctan\left(\frac{1}{515}\right) \\ & - 32 \arctan\left(\frac{1}{1068}\right) - 56 \arctan\left(\frac{1}{173932}\right) \end{aligned} \quad \{5, 13, 73, 101\} \quad (74)$$

$$\begin{aligned} \frac{\pi}{4} = & 100 \arctan\left(\frac{1}{319}\right) + 127 \arctan\left(\frac{1}{378}\right) + 71 \arctan\left(\frac{1}{557}\right) \\ & - 15 \arctan\left(\frac{1}{1068}\right) + 66 \arctan\left(\frac{1}{2943}\right) + 44 \arctan\left(\frac{1}{478707}\right) \end{aligned} \quad \{5, 13, 17, 41, 73\} \quad \text{cf. formula 63} \quad (75)$$

$$\begin{aligned} \frac{\pi}{4} = & 1074 \arctan\left(\frac{1}{1568}\right) + 657 \arctan\left(\frac{1}{4662}\right) + 183 \arctan\left(\frac{1}{5357}\right) \\ & - 779 \arctan\left(\frac{1}{12943}\right) - 32 \arctan\left(\frac{1}{17923}\right) - 449 \arctan\left(\frac{1}{32807}\right) \\ & + 398 \arctan\left(\frac{1}{390112}\right) \end{aligned} \quad \{5, 13, 17, 61, 89, 97\} \quad \text{cf. formula 70} \quad (76)$$

$$\begin{aligned} \frac{\pi}{4} = & 1074 \arctan\left(\frac{1}{4246}\right) + 1257 \arctan\left(\frac{1}{5357}\right) + 1731 \arctan\left(\frac{1}{6107}\right) \\ & + 295 \arctan\left(\frac{1}{12943}\right) + 625 \arctan\left(\frac{1}{19703}\right) - 481 \arctan\left(\frac{1}{32807}\right) \\ & - 1042 \arctan\left(\frac{1}{39307}\right) + 398 \arctan\left(\frac{1}{390112}\right) \end{aligned} \quad \{primes\} \quad (77)$$

$$\begin{aligned} \frac{\pi}{4} = & 7162 \arctan\left(\frac{1}{12943}\right) + 3796 \arctan\left(\frac{1}{32807}\right) + 2558 \arctan\left(\frac{1}{34208}\right) \\ & + 2729 \arctan\left(\frac{1}{44179}\right) - 708 \arctan\left(\frac{1}{51387}\right) + 2192 \arctan\left(\frac{1}{114669}\right) \\ & - 2805 \arctan\left(\frac{1}{157318}\right) - 3696 \arctan\left(\frac{1}{485298}\right) - 2407 \arctan\left(\frac{1}{24208144}\right) \end{aligned} \quad \{primes\} \quad (78)$$

$$\begin{aligned} \frac{\pi}{4} = & 2805 \arctan\left(\frac{1}{5257}\right) - 398 \arctan\left(\frac{1}{9466}\right) + 1950 \arctan\left(\frac{1}{12943}\right) \\ & + 1850 \arctan\left(\frac{1}{34208}\right) + 2021 \arctan\left(\frac{1}{44179}\right) + 2097 \arctan\left(\frac{1}{85353}\right) \\ & + 1484 \arctan\left(\frac{1}{114669}\right) + 1389 \arctan\left(\frac{1}{330182}\right) + 808 \arctan\left(\frac{1}{485298}\right) \end{aligned} \quad \{5, 13, 17, 29, 37, 41, 53, 61\} \quad (Gauss) \quad (79)$$

$$\begin{aligned} \frac{\pi}{4} = & 50539 \arctan\left(\frac{1}{51387}\right) + 1555 \arctan\left(\frac{1}{114669}\right) \\ & - 6601 \arctan\left(\frac{1}{157318}\right) - 20678 \arctan\left(\frac{1}{390112}\right) - 5617 \arctan\left(\frac{1}{485298}\right) \\ & - 64126 \arctan\left(\frac{1}{617427}\right) + 10958 \arctan\left(\frac{1}{1984933}\right) - 30569 \arctan\left(\frac{1}{3449051}\right) \\ & + 23407 \arctan\left(\frac{1}{22709274}\right) + 25433 \arctan\left(\frac{1}{24208144}\right) \end{aligned} \quad \{primes\} \quad (80)$$

$$\begin{aligned}
\frac{\pi}{4} = & 36462 \arctan\left(\frac{1}{390112}\right) + 135908 \arctan\left(\frac{1}{485298}\right) \\
& + 274509 \arctan\left(\frac{1}{683982}\right) - 39581 \arctan\left(\frac{1}{1984933}\right) \\
& + 178477 \arctan\left(\frac{1}{2478328}\right) - 114569 \arctan\left(\frac{1}{3449051}\right) \\
& - 146571 \arctan\left(\frac{1}{18975991}\right) + 61914 \arctan\left(\frac{1}{22709274}\right) \\
& - 69044 \arctan\left(\frac{1}{24208144}\right) - 89431 \arctan\left(\frac{1}{201229582}\right) \\
& - 43938 \arctan\left(\frac{1}{2189376182}\right) \\
& \{5, 13, 17, 29, 37, 53, 61, 89, 97, 101\}
\end{aligned} \tag{81}$$

$$\begin{aligned}
\frac{\pi}{4} = & 446879 \arctan\left(\frac{1}{683982}\right) + 172370 \arctan\left(\frac{1}{1635786}\right) \\
& - 193720 \arctan\left(\frac{1}{1984933}\right) + 369078 \arctan\left(\frac{1}{2478328}\right) \\
& + 18231 \arctan\left(\frac{1}{3014557}\right) + 21339 \arctan\left(\frac{1}{3449051}\right) \\
& - 154139 \arctan\left(\frac{1}{6225244}\right) - 110109 \arctan\left(\frac{1}{18975991}\right) \\
& + 80145 \arctan\left(\frac{1}{22709274}\right) - 223183 \arctan\left(\frac{1}{24208144}\right) \\
& - 107662 \arctan\left(\frac{1}{201229582}\right) - 216308 \arctan\left(\frac{1}{2189376182}\right) \\
& \{primes\}
\end{aligned} \tag{82}$$

$$\begin{aligned}
\frac{\pi}{4} = & 872408 \arctan\left(\frac{1}{1984933}\right) + 619249 \arctan\left(\frac{1}{2298668}\right) \\
& + 369078 \arctan\left(\frac{1}{2478328}\right) + 18231 \arctan\left(\frac{1}{3014557}\right) \\
& - 1217159 \arctan\left(\frac{1}{5033696}\right) + 911989 \arctan\left(\frac{1}{6225244}\right) \\
& + 783649 \arctan\left(\frac{1}{18975991}\right) - 70886 \arctan\left(\frac{1}{22709274}\right) \\
& - 374214 \arctan\left(\frac{1}{24208144}\right) - 1044789 \arctan\left(\frac{1}{168623905}\right) \\
& + 339217 \arctan\left(\frac{1}{201229582}\right) - 446879 \arctan\left(\frac{1}{284862638}\right) \\
& + 402941 \arctan\left(\frac{1}{2189376182}\right) \\
& \{primes\}
\end{aligned} \tag{83}$$