

To assist in the interpretation of magnetic data a Landau formulation is provided for ferromagnetic, antiferromagnetic and mixed magnetic systems. The functional dependence of the expansion coefficients is derived using a high temperature series expansion of the partition function as a function of the magnetic moment and temperature. The validity and range of applicability of this expansion is discussed.

In subsect. 1.5.5.3.2.2. the Landau free energy formulation is used for modelling magnetisation data of a ferromagnetic system. As a means of relating the field and temperature dependence of the magnetic moment to the coefficients of the free energy expansion of the Landau theory, Arrott plots are introduced in subsect. 1.5.5.3.2.3.

In subsects. 1.5.5.3.2.4 and 1.5.5.3.2.5, the analysis of magnetic data using Arrott plots is extended to systems with more complex forms of ferromagnetic or more complicated magnetic order. Arrott plots are shown to be a valuable means of analysis even for these cases. The physics and the resulting Arrott plots are illustrated by way of example using simple model systems.

#### 1.5.5.3.2.2 Mean field description of magnetic phase transition and Landau form of the free energy

In this section a derivation is given for the form of the free energy of a ferromagnetic system as a function of the magnetic order parameter. Within the Landau theory of magnetic phase transitions the free energy is considered as a power series expansion in terms of the magnetic order parameter, and explicit expressions are derived for the coupling coefficients. For the magnetically ordered state and alternatively also in the limit of high temperature, these coupling coefficients are evaluated as a function of temperature and the quantum numbers which characterise the magnetic moment.

The Hamiltonian describing the short range interactions between magnetic moments is taken to be of the form

$$H = - \sum_{i,j} J_{i,j} \mathbf{M}_i \mathbf{M}_j - \sum_i \mathbf{M}_i \mathbf{B}_0 \quad (3)$$

where the magnetic moment  $\mathbf{M}_i$  at lattice site  $i$  interacts with the magnetic moment at lattice site  $j$  and with the external magnetic field  $\mu_0 \mathbf{H} = \mathbf{B}_0$ . For simplicity the magnetic moment is taken to be of fixed magnitude.

In order to analyse the above Hamiltonian a mean field approximation is employed. First the Hamiltonian is written as the sum of two contributions:

$$H = H_0 + H' \quad (4)$$

with

$$H_0 = -(B_0 + b_0) \sum_i M_i^z \quad (5)$$

Here  $H_0$  is the sum of single atom Hamiltonians. It consists of the sum of magnetic moments interacting with an effective magnetic field of magnitude  $(B_0 + b_0)$ . The internal field  $b_0$  is added again in  $H'$ . The idea is to choose  $b_0$  in such a way as to make  $H'$  as small as possible. The question of the magnitude and the optimal choice of  $b_0$  will be discussed below.

Reformulating the Hamiltonian in (3) in the form of (5) is a simple re-writing with no approximations involved. In order to progress with the analysis of the properties of a system described by the Hamiltonian (5), a mean field approximation is employed combined with the assumption that the  $H'$  contribution is effectively small. The mean field approximation then consists of replacing all operators in  $H'$  by their expectation values which are evaluated using  $H_0$ . With

$\langle M_x \rangle_0 = \langle M_y \rangle_0 = 0$ ,  $\langle M_z \rangle_0 \neq 0$  and assuming translational invariance of the magnetic moments one obtains

$$\langle H' \rangle_0 = Nb_0 \langle M_z \rangle_0 - NJ_M \langle M_z \rangle_0^2 \quad (6)$$

Here  $J_M$  is the  $k = 0$  component of the Fourier transformed coupling constants  $J_{ij}$  in (3) and

$$\langle M_z \rangle_0 = \frac{1}{Z_0} \sum_i M_z(\varepsilon_i) \cdot e^{-\beta \varepsilon_i} \quad (7)$$

The summation runs over all energy levels and the partition function  $Z_0$  is given by

$$Z_0 = \sum_i e^{-\beta \varepsilon_i} \quad (8)$$

Note that the partition function  $Z_0$  is the partition function of a single magnetic moment and that  $Z_0$  has been obtained using the eigenstates of the magnetic moment as defined by  $H_0$ . Thus only  $H_0$  has been used for the evaluation of the expectation values of the magnetic moment.

For a system with  $N$  magnetic moments and with the above approximation one obtains an upper bound of the true free energy [90F1] as:

$$F_{\text{true}} \leq \Phi = -\frac{N}{\beta} \ln(Z_0) + \langle H' \rangle_0 \quad (9)$$

The upper bound is minimised by choosing the internal field  $b_0$  appropriately. The best possible internal field which minimises  $\Phi/N$  necessitates that

$$\frac{\partial \Phi}{\partial b_0} = -\frac{1}{\beta Z_0} \frac{\partial Z_0}{\partial b_0} + \langle M_z \rangle_0 + (b_0 - 2J_M \langle M_z \rangle_0) \frac{\partial \langle M_z \rangle_0}{\partial (b_0)} = 0 \quad (10)$$

$\underbrace{\frac{1}{\beta Z_0} \frac{\partial Z_0}{\partial b_0}}_{=-\langle M_z \rangle_0}$

With the choice of

$$b_0 = 2J_M \cdot \langle M_z \rangle_0 \quad (11)$$

$\Phi$  is minimised. This internal field is identical to the result obtained when employing a mean field approximation directly to the Hamiltonian. The mean field approximation is obtained by using the operator identity  $\hat{O}_i \cdot \hat{O}_j = \hat{O}_i \cdot \langle \hat{O}_j \rangle + \langle \hat{O}_i \rangle \cdot \hat{O}_j - \langle \hat{O}_i \rangle \cdot \langle \hat{O}_j \rangle + (\hat{O}_i - \langle \hat{O}_i \rangle) \cdot (\hat{O}_j - \langle \hat{O}_j \rangle)$  and neglecting the fluctuation part  $(\hat{O}_i - \langle \hat{O}_i \rangle) \cdot (\hat{O}_j - \langle \hat{O}_j \rangle)$ . The remaining part of the free energy

$$F_0 = -\frac{N}{\beta} \ln(Z_0) \quad (12)$$

is made up of the sum of single atom terms in an effective magnetic field. This contribution is evaluated next.

The energy of the magnetic moment  $M_z = \frac{g_J \mu_B}{\hbar} \langle \hat{J}_z \rangle$  (which is associated to an angular momentum operator  $\hat{J}_z$  and its thermal average  $\langle \hat{J}_z \rangle$ ) in an effective field  $(B_0)_{\text{eff}} = B_0 + b_0$  is determined by the Hamiltonian  $H_0 = \frac{g_J \mu_B}{\hbar} \hat{J}_z (B_0)_{\text{eff}}$ . With the projections of  $\hat{J}_z$  denoted by  $\hbar m$ , the energy of the magnetic moment is given by

$$\mathcal{E}(m) = g_J \mu_B m \cdot (B_0)_{\text{eff}} \quad (13)$$

With the energy eigenvalues given by (13) the partition function (8) is now obtained as

$$Z_0 = \sum_{m=-J}^J e^{-\beta \epsilon(m)} \quad (14)$$

The knowledge of the partition function suffices for the modelling of the thermodynamic behaviour of the system of non-interacting magnetic moments in an effective magnetic field which, within a mean field approximation employed here, contains the interaction between the magnetic moments.

In order to obtain the free energy  $F$  as a function of the magnetic order parameter, first an expression is derived for the free energy which is valid in the limit of high temperatures, i.e. small values of  $\beta = 1/k_B T$ . For  $\beta \epsilon(m) \ll 1$  the partition function can be expanded into a power series in  $\beta$ :

$$Z_0 = \sum_{m=-J}^J 1 - \beta \sum_{m=-J}^J [g_J \mu_B m (B_0)_{\text{eff}}] + \frac{\beta^2}{2} \sum_{m=-J}^J [g_J \mu_B m (B_0)_{\text{eff}}]^2 - \frac{\beta^3}{6} \sum_{m=-J}^J [g_J \mu_B m (B_0)_{\text{eff}}]^3 + \frac{\beta^4}{24} \sum_{m=-J}^J [g_J \mu_B m (B_0)_{\text{eff}}]^4 + \dots \quad (15)$$

Using

$$\begin{aligned} \sum_{m=-J}^J 1 &= (2J+1) & \sum_{m=-J}^J m^{2n+1} &= 0, n \text{ integer} \\ \sum_{m=-J}^J m^2 &= \frac{J(J+1)}{3} (2J+1) & \sum_{m=-J}^J m^4 &= \frac{J(J+1)}{15} (2J+1)(3J^2+3J-1) \end{aligned} \quad (16)$$

the expansion of the partition function takes the form

$$\begin{aligned} \frac{Z_0}{(2J+1)} &= 1 + \frac{\beta^2}{6} J(J+1) \cdot g_J^2 \mu_B^2 ((B_0)_{\text{eff}})^2 \\ &+ \beta^4 J(J+1)(2J-1)(3J^2+3J-1) \frac{g_J^4 \mu_B^4}{360} ((B_0)_{\text{eff}})^3 + \dots \end{aligned} \quad (17)$$

From the above high temperature expansion of the partition function an approximate form of the free energy  $F(M)$  is obtained, from which in turn the average magnetic moment per magnetic atom is calculated using

$$M = -\frac{1}{N} \frac{\partial F_0(M)}{\partial (B_0)_{\text{eff}}} = \frac{1}{\beta Z_0} \frac{\partial Z_0}{\partial (B_0)_{\text{eff}}} \quad (18)$$

This results in

$$M = \beta(J+1) \frac{g_J^2 \mu_B^2}{3} (B_0)_{\text{eff}} - \frac{\beta^3}{90} J(J+1)(2J^2+2J+1) g_J^4 \mu_B^4 ((B_0)_{\text{eff}})^3 + \dots \quad (19)$$

The above formula yields the dependence of the induced magnetic moment in the paramagnetic state on the magnetic field and including terms up to 4th order in the applied magnetic field. A comparison with

$$M = \chi_M (B_0)_{\text{eff}} + \chi_M^{(3)} ((B_0)_{\text{eff}})^3 \quad (20)$$

identifies the magnetic susceptibilities as

$$\chi_M = \frac{J(J+1)}{3k_B T} g_J^2 \mu_B^2 \quad \chi_M^{(3)} = -\frac{J(J+1)(2J^2+2J+1)}{15k_B^3 T^3} g_J^4 \mu_B^4 \quad (21)$$

This formulation as given above allows the free energy to be obtained as a function of the (effective) magnetic field. However, it is more instructive to write  $F_0$  as a function of the order parameter  $M$ . Using general symmetry arguments [90F1, 87T1] the free energy, normalised to one magnetic atom and as a function of the magnetic moment, is obtained as

$$\frac{F_0}{N} = \frac{\tilde{F}_0}{N} + \frac{1}{2}AM^2 + \frac{1}{4}CM^4 - M(B_0)_{\text{eff}} \quad (22)$$

Here  $A$  and  $C$  are (temperature dependent) expansion coefficients.  $\tilde{F}_0$  contains all contributions which do not depend on the order parameter  $M$ . As these contributions are of no interest for the discussions which are to follow the  $\tilde{F}_0$  term will be dropped from all further formulae.

Using the above form of the free energy the magnetic moment  $M$  is obtained by minimisation of  $F(M)$ :

$$\frac{\partial \left( \frac{F_0}{N} \right)}{\partial M} = 0 \Rightarrow AM + CM^3 - (B_0)_{\text{eff}} = 0 \quad (23)$$

For small fields the magnetic moment increases linearly with field. With  $M \cong \frac{1}{A} (B_0)_{\text{eff}}$  and compared to  $M \cong \chi_M (B_0)_{\text{eff}}$  identifies the coefficient  $A$  as the inverse magnetic susceptibility (with respect to the field  $(B_0)_{\text{eff}}$ ):

$$A = \frac{1}{\chi_M} \quad (24)$$

For a magnetic moment in an effective field the coefficient  $A$  is proportional to temperature.

As the field strength is increased the non-linear contributions become more important. Using Cardan's formula [79B1] for solving the third order polynomial, the magnetisation is obtained as a function of the effective field as

$$M((B_0)_{\text{eff}}) = \sqrt[3]{\frac{(B_0)_{\text{eff}}}{2C} + \sqrt{\left(\frac{A}{3C}\right)^3 + \left(\frac{(B_0)_{\text{eff}}}{2C}\right)^2}} + \sqrt[3]{\frac{(B_0)_{\text{eff}}}{2C} - \sqrt{\left(\frac{A}{3C}\right)^3 + \left(\frac{(B_0)_{\text{eff}}}{2C}\right)^2}} \quad (25)$$

Expanding this formula into a power series in  $(B_0)_{\text{eff}}$  and comparing the coefficient of third order to  $\chi_M^{(3)}$  yields

$$\chi_M^{(3)} = -\frac{C}{A^4} \quad (26)$$

with  $C$  being proportional to temperature.

Thus the explicit form of the coupling coefficients and their temperature dependencies are given by

$$A = 3 \frac{k_B T}{J(J+1)} \quad C = \frac{27}{5} \cdot \frac{2J^2 + 2J + 1}{J^3(J+1)^3} \cdot k_B T \quad (27)$$

With the knowledge of these coefficients and the form of the effective magnetic field the free energy takes the form

$$\frac{F}{N} = \frac{1}{2}A M^2 + \frac{1}{4}C M^4 - M(B_0)_{\text{eff}} + J_M M^2 \quad (28)$$

With

$$(B_0)_{\text{eff}} = (B_0)_{\text{external}} + b_0 = (B_0)_{\text{external}} + 2J_M M^2 \quad (29)$$

the temperature dependence of the 'A' coefficient is renormalised:

$$A(T) \Rightarrow \tilde{A}(T) = \frac{3}{J(J+1)} (k_B T - k_B \Theta_M) \quad (30)$$

Here  $\Theta_M$  is the paramagnetic Curie temperature defined as

$$k_B \Theta_M = \frac{2}{3} J_M \cdot J(J+1) \quad (31)$$

The free energy  $F$  as a function of magnetic order parameter  $M$  and temperature  $T$  takes the form

$$\frac{F}{N} = \frac{3}{2} \frac{1}{J(J+1)} k_B (T - \Theta_M) M^2 + \frac{27}{20} \frac{2J^2 + 2J + 1}{J^3(J+1)^3} k_B T M^4 - M(B_0)_{\text{ext}} \quad (32)$$

This form of the free energy is suitable for the analysis of a ferromagnetic phase transition within a Landau approach.

For zero external magnetic field the minimisation of  $F$  results in

$$\frac{\partial \left( \frac{F(M)}{N} \right)}{\partial M} = 0 \quad \Rightarrow \quad M(\tilde{A} + CM^2) = 0 \quad (33)$$

The above equation has two solutions:

$$(I) \quad M = 0, \quad (II) \quad M = \sqrt{-\frac{\tilde{A}(T)}{C}} \quad (34)$$

In order for  $F$  to be a minimum, i.e.

$$\frac{\partial^2 \left( \frac{F(M)}{N} \right)}{\partial M^2} \geq 0 \quad (35)$$

solution (I) in eq. (34) has to be chosen for  $\tilde{A}(T)$  positive (i.e. for  $T > \Theta_M$  where  $\Theta_M$  is the transition temperature sometimes denoted by  $T_C$ ), while solution (II) in eq. (34) will minimise the free energy for  $\tilde{A}(T)$  negative (i.e. for  $T < \Theta_M$ ). Thus one obtains the magnetic moment as a function of temperature:

$$M(T) = \begin{cases} \sqrt{-\frac{\tilde{A}(T)}{C(T)}} & \text{for } T < \Theta_M \\ 0 & \text{for } T > \Theta_M \end{cases} \quad (36)$$

Thus the temperature dependence and the change of sign of the coefficient  $A$  in the general form of the free energy  $F$  as given in eq. (32) will lead to a magnetic phase transition from a paramagnetic state for  $T > \Theta_M$  to a magnetically ordered one for  $T < \Theta_M$ . For purely magnetic interactions the  $C$  coefficient in the free energy expansion is positive. If, due to coupling to other degrees of freedom,  $C$  turns out to be negative for any real material, the series expansion of the free energy in eq. (32) has to be carried to higher orders in  $M$ .

The temperature dependence of the magnetic moment within the ordered state and without an external magnetic field applied is given by

$$M(T) \propto \sqrt{\frac{\Theta_M - T}{T}} \quad (37)$$

(Note that in the ordered state  $T < \Theta_M$ ). As the temperature decreases towards zero, the size of the magnetic moment increases without limit. Such a behaviour signals the inapplicability of the expressions for low temperatures and within the magnetically ordered state. This result is not surprising as the free energy has been obtained from a high temperature expansion of the partition function. At low temperatures when  $\beta$  is large (i.e. the energy scale as determined by temperature  $k_B T$  is of the order of the magnetic energies  $\frac{1}{\beta} = k_B T \propto \varepsilon(m)$ ) a finite power series expansion in  $\beta$  is not a valid description of either the partition function or the free energy. Thus a different approximation has to be sought for the low temperature region. The new formulation should take proper account of the saturation of the magnetic moment at low temperatures and it should give a reasonable description of the temperature dependence of the magnetic moment compared to experimental observations.

At  $T = 0$  and for ferromagnetic order the ground state is characterised by a parallel arrangement of magnetic moments along one direction. Due to the Zeeman contribution to the Hamiltonian the internal (and, if present, the external) magnetic field completely remove the  $2J+1$  fold degeneracy of the magnetic state of the free atom. The energy level which is occupied at zero temperature is the one with the maximal projection of the magnetic moment onto the axis of quantisation which, for convenience, is taken to be the  $z$  axis. With this choice the axis of quantisation is oriented parallel to the (effective) magnetic field direction. This physical picture suggests the following approximation to the partition function: the lowest energy level of the  $2J+1$  fold Zeeman split multiplet is the most populated one. At low temperatures the probability of occupying any higher level is small, making an expansion of the partition function in terms of the average occupation factors of the excited levels an attractive possibility.

An explicit summation of the partition function (14) results in

$$Z(\beta, J, (B_0)_{\text{eff}}) = \frac{\sinh\left(\beta\mu_B \frac{2J+1}{2J} (B_0)_{\text{eff}}\right)}{\sinh\left(\frac{\beta\mu_B}{2J} (B_0)_{\text{eff}}\right)} \quad (38)$$

With the above form of the partition function the free energy is obtained, from which in turn the magnetic moment is readily calculated as

$$\langle M_z \rangle = g_J \mu_B J \left( \frac{2J+1}{2J} \coth\left(\frac{2J+1}{2J} \beta g_J \mu_B J (B_0)_{\text{eff}}\right) - \frac{1}{2J} \coth\left(\frac{g_J \mu_B}{2J} \beta J (B_0)_{\text{eff}}\right) \right) \quad (39)$$

Here the magnetic moment has to be determined self consistently as the effective magnetic field  $(B_0)_{\text{eff}}$  is itself a function of the average magnetic moment. The function on the right hand side of eq. (39) is known as the Brillouin function.

With the magnetic moment as given in eq. (39) and for low temperatures (i.e. large  $\beta$ ) and/or large effective magnetic fields  $(B_0)_{\text{eff}}$  one indeed obtains saturation of the magnetic moment. For large, positive values of  $x$  the  $\coth(x)$  function may be written as

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1 + e^{-2x}}{1 - e^{-2x}} \cong (1 + e^{-2x})(1 + e^{-2x}) \cong 1 + 2e^{-2x} \quad (40)$$

resulting in the approximation (with  $M_0 = g_J \mu_B J$ )

$$M = M_0 \left( 1 + \frac{2J+1}{2J} e^{-\beta g_J \mu_B \frac{2J+1}{J} (B_0)_{\text{eff}}} - \frac{1}{2J} e^{-\beta g_J \mu_B (B_0)_{\text{eff}}} \right) \quad (41)$$

As expected, this formulation describes the saturation of the magnetic moment at low temperatures and for all magnetic fields. However, the renormalisation of the magnetic moment with temperature

is given by an exponential correction term. This is in contrast to experimental observation for a ferromagnet for which the low temperature decrease of the magnetic moment is given by a power law of the form  $M \propto T^{3/2}$ . The deviation of mean field model and experiment is to be sought in the mean field approximation which neglects magnetic fluctuations. At low temperatures the magnetic fluctuations in the form of spin waves determine the thermodynamic properties. As these are not included in the present formulation of the theory the mean field treatment is incomplete and insufficient at low temperatures. It can not be used for a faithful representation of the experimental observations of a ferromagnet at low temperatures

Therefore in order to remedy the situation while still using the Landau form of the free energy as given in eq. (32), one can resort to the procedure of fixing the coefficient  $C$  at a convenient temperature. Depending on the range for which the expansion is intended to be used the coefficient may be either fixed to its value at the transition temperature, resulting in

$$C = \frac{18}{5} \frac{2J^2 + 2J + 1}{J^2(J+1)^2} J_M \quad (42)$$

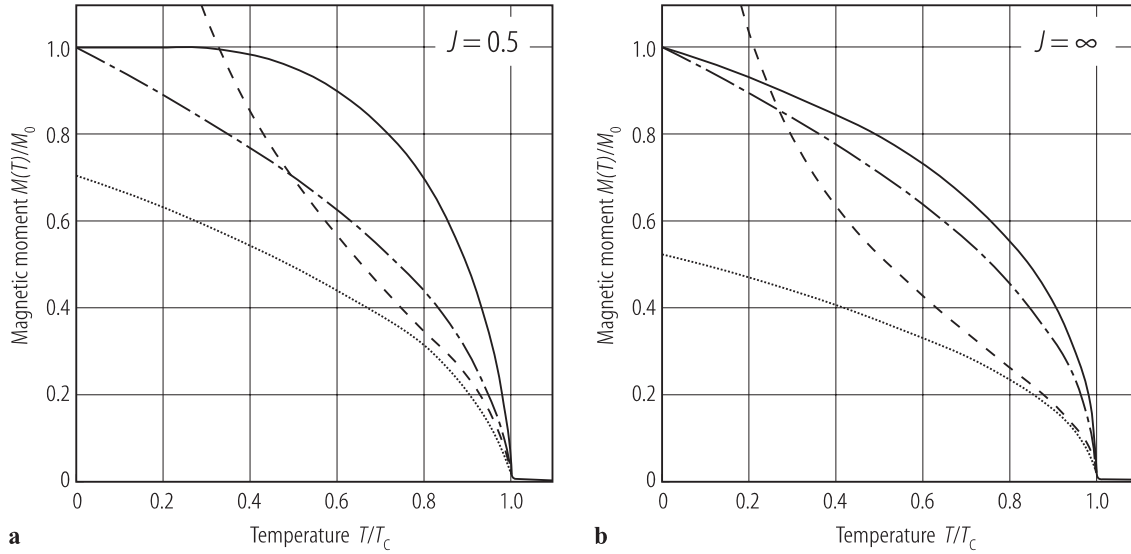
or, alternatively, the  $C$  coefficient may be set to a constant in such a way as to ensure the correct saturation moment at  $T = 0$ . The later form results in a  $C$  coefficient given by

$$C = 2 \frac{J_M}{J^2} \quad (43)$$

For both approximations the  $C$  coefficient is taken to be temperature independent. The thermodynamics as given by the resulting form of the Landau free energy will obviously depend on the choice of coefficient. This is demonstrated in Fig. 62 where for  $T \leq T_C$  the temperature dependence of a magnetic moment has been plotted using the low temperature expansion (dotted line), the high temperature expansion (dashed line) and the "mean field solution" (full line) which is obtained by solving numerically the self consistency equation for the magnetic moment. The temperature dependence of the magnetic susceptibility within the paramagnetic state is only affected in third order in the applied magnetic field with no change at all of the low field magnetic susceptibility.

As seen in Fig. 62, the Landau expansion in powers of the magnetic order parameter is a poor approximation to the results of a mean field calculation for the temperature range which corresponds to the magnetically ordered state. It is therefore not possible to obtain a faithful representation of the mean field results while employing the high temperature series expansion of the free energy as discussed above. In order to improve the agreement with either the mean field calculation or experimental data, it is necessary to either abandon the description in favour of a more appropriate model or to renormalize the coefficients  $A$  and  $C$ .

However, the simplicity of the Landau description of the free energy in terms of the magnetic order parameters is an appealing feature, and for this simple reason it is used for the description of experimental data. Such a simple model offers to the experimentalist a means of qualitatively understanding and modelling the magnetic behaviour. Therefore the Landau model can be used if the coefficients are determined empirically by, for example, comparison with experiment. The point of view taken here is that for an experimental physicist it is the experimentally determined set of coefficients which is the most useful one for the initial investigation of magnetic phenomena or new materials. Therefore the relationship has to be identified between the  $A$  and  $C$  coefficients of the free energy and experimentally observable entities. This connection will be developed in the next section.



**Fig. 62.** Magnetisation for two values of  $J$  and for various approximations. Full line: mean field model, dashed line: high temperature series expansion with coefficients  $A$  and  $C$  temperature dependent, dotted

line: high temperature expansion with  $C$  set equal to its value at  $T_c$ , dash-dotted line: high temperature expansion with  $C$  fixed such that the magnetisation at  $T = 0$  is equal to the mean field value.

### 1.5.5.3.2.3 Magnetisation and Arrott plots

Using a temperature independent  $C$  coefficient, the magnetisation is obtained as shown in Fig. 63. The above analysis of the  $\tilde{A}$  and  $C$  coefficients has shown that  $\tilde{A}$  is related to the magnetic susceptibility (see eq. (24)) and that the coefficient  $C$  is determined by the higher order corrections. These corrections are connected with the deviation from linearity of the induced magnetic moment as a function of the external magnetic field. Thus the expansion coefficients of the free energy may be obtained from experiment by using the low and high field part of the magnetisation curve to extract the values of  $\tilde{A}$  and  $C$ , respectively. An alternative means of analysis, which is both more direct and physically more transparent, has been proposed by Arrott [57A1]. He noticed that eq. (23) may be written in the form

$$M^2 = \frac{1}{C} \frac{(B_0)_{\text{ext}}}{M} - \frac{\tilde{A}}{C} \quad (44)$$

which resembles the equation for a straight line as given by  $y = mx + b$ . Thus, plotting the magnetisation for a given temperature as a function of field, and using as the unit on the x axis the ratio of applied magnetic field divided by the observed magnetic moment and on the y axis the square of the observed magnetic moment will yield straight lines. The graphical representation is given in Fig. 64. According to eq. (44) the lines have a slope proportional to  $1/C$ . They intersect the x axis at  $\tilde{A}$ . As temperature is varied (and  $\tilde{A}(T)$  changes as a function of temperature) the lines are displaced parallel (for a temperature independent  $C$  coefficient) with respect to one another. The critical temperature is defined by the line which goes through the origin of the Arrott plot. For temperatures  $T < T_c$  the lines cut the y axis at positive values of  $M^2$ , thus yielding the size of the spontaneous magnetic moment in zero external magnetic field. The intercept with the y axis is given by the ratio  $-\tilde{A}/C = M^2$  ( $> 0$  as  $A$  is negative for  $T < T_c$ ). This result is consistent with the analysis of the free energy as given above and with the size of the magnetic moment as determined in eq. (42) or (43).