# remarks on arithmetical algorithms and the computation of $\pi$

this is work in progress: thanks for feedback and corrections

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[1]This document is enclosed in the hfloat-package which is online at http://www.jjj.de/hfloat/

**Abstract**

This is a collection of remarks about some arithmetical algorithms. If you think iterating $x \mapsto \frac{1}{2}(x + \frac{d}{x})$ is the best way to compute $\sqrt{d}$ then have a look.

In addition there is a collection of formulas and iterations for the computation of $\pi$. Some of the formulas are highly cryptic and useless. If you think that
$\frac{\pi}{4} = 88 \arctan \frac{1}{192} + 39 \arctan \frac{1}{239} + 100 \arctan \frac{1}{515} - 32 \arctan \frac{1}{1068} - 56 \arctan \frac{1}{173932}$
is fun then stare at more formulas there.

Things are treated pretty superficially, always see the references for details. Do not expect mathematics, expect formulas, ideas and algorithms. Some sections are mere formula buckets or enumerations of names or references, don't panic.

Please report errors and typos !

# Contents

# Chapter 1

# Remarks on arithmetical algorithms

## 1.1 Asymptotics of algorithms

An important feature of an algorithm is the number of operations that must be performed for the completion of a task a certain size $N$. $N$ should be some reasonable quantity that grows strictly with the size of the task, for high precision computations one will take the length of the numbers counted in decimal digits or bits. For computations with square matrices one may take for $N$ the number of the rows. An operation is typically a (machine word) multiplication plus an addition, one could also simply count machine instructions.

An algorithm is said to have some asymptotics $f(N)$ if it needs proportional $f(N)$ operations for a task of size $N$.

Examples:

- Addition of an $N$-digit number needs proportional $N$ operations (here: machine word addition plus some carry operation).
- Ordinary multiplication needs $N^2$ operations.
- The Fast Fourier Transform (FFT) needs $\sim N \log(N)$ operations (a straightforward implementation of the Fourier Transform, i.e. computing $N$ sums each of length $N$ would be $\sim N^2$).
- Matrix multiplication (by the obvious algorithm) is $\sim N^3$ ($N^2$ sums each of $N$ products).

The algorithm with the 'best' asymptotics wins for some, possibly huge, $N$. For smaller $N$ some other algorithm will be superior. For the exact break-even point the constants omitted elsewhere are of course important.

Example: Let the algorithm `mult1` take $1.0 \cdot N^2$ operations, `mult2` take $8.0 \cdot N \log_2(N)$ operations. Then for $N < 64$ `mult1` is faster and for $N > 64$ `mult2` is faster. Often completely different algorithms are optimal for the same task at different problem sizes.

See [62], [16] and [5].

## 1.2 Multiplication of large numbers

Ordinary multiplication is $\sim N^2$. Computing the product of two million digit numbers would require $\approx 10^{12}$ operations, taking (in the order of) 1 day on a machine that does 10 million operations per second. But there are better ways...

### 1.2.1 The Karatsuba algorithm

Split the numbers U and V in two pieces:

$$\begin{aligned} U &= U_0 + U_1 B \\ V &= V_0 + V_1 B \end{aligned} \tag{1.1}$$

Instead of the straightforward multiplication

$$UV = U_0 V_0 + B(U_0 V_1 + V_0 U_1) + B^2 U_1 V_1 \tag{1.2}$$

(4 multiplications with half precision for one multiplication with full precision)

use the relation

$$UV = (1+B)U_0 V_0 + B(U_1 - U_0)(V_0 - V_1) + (B + B^2)U_1 V_1 \tag{1.3}$$

(3 multiplications with half precision for one multiplication with full precision)

recursively.

For Squaring the relation is:

$$U^2 = (1+B)U_0^2 - B(U_1 - U_0)^2 + (B + B^2)U_1^2 \tag{1.4}$$

or

$$U^2 = (1-B)U_0^2 + B(U_1 + U_0)^2 + (-B + B^2)U_1^2 \tag{1.5}$$

The asymptotics of the algorithm is $\sim N^{\log_2(3)} \approx N^{1.585}$.

One can extend the above idea by splitting $U$ and $V$ into more than two pieces each, the resulting algorithm is called Toom Cook algorithm.

Computing the product of two million digit numbers would require $\approx (10^6)^{1.585} \approx 3200 \cdot 10^6$ operations, taking (in the order of) 5 minutes on the 10 Mips machine.

See [5], chapter 4.3.3 ('How fast can we multiply ?').

### 1.2.2 Fast multiplication via FFT

Multiplication of two numbers is essentially a convolution of the sequences of their digits. A convolution $c_k, k = 0...2N-2$ of the two sequences $a_k, b_k, k = 0...N-1$ is defined as

$$c_k := \sum_{i,j=0;i+j=k}^{N-1} a_i b_j. \tag{1.6}$$

A number written (in radix r) as

$$a_P \quad a_{P-1} \quad \ldots \quad a_2 \quad a_1 \quad a_0 \quad . \quad a_{-1} \quad a_{-2} \quad \ldots \quad a_{-p+1} \quad a_{-p}$$

denotes a quantity of

$$\sum_{i=-p}^{P} a_i \cdot r^i = a_P \cdot r^P + a_{P-1} \cdot r^{P-1} + \ldots + a_{-p} \cdot r^{-p}.$$

i.e. digits are coefficients of a polynomial in $r$. (e.g. for decimal numbers $r = 10$ and $123.4 = 1 \cdot 10^2 + 2 \cdot 10^1 + 3 \cdot 10^0 + 4 \cdot 10^{-1}$). The product of two numbers is almost the polynomial product

$$\sum_{k=0}^{2N-2} c_k r^k := \sum_{i=0}^{N-1} a_i r^i \sum_{j=0}^{N-1} b_j r^j \tag{1.7}$$

The $c_k$ are found by comparing coefficients, one gets $c_k = \sum_{i,j=0;i+j=k}^{N-1} a_i b_j$, apparently a convolution.

As the $c_k$ can be greater (or equal) $r$, the carry operations have to be appended: go from right to left, replace $c_k$ by $c_k \% r$ and add $(c_k - c_k \% r)/r$ to its left neighbour.

Here is a tiny decimal example:

| | | | |
|---|---|---|---|
| | .82 | $\times$ | .34 |
| | | 32 | 8 |
| | 24 | 6 | |
| | 24 | 38 | 8 |
| =.2 | $^2$ 7 | $^3$ 8 | 8 |

Convolution can be done effectively using the Fast Fourier Transform (FFT): Convolution is a simple (elementwise array) multiplication in Fourier space (see e.g. [46]). The FFT itself takes $N \cdot \log N$ operations. Instead of the direct convolution ($\sim N^2$) one does

{ FFT; array multiplication; FFT$^{-1}$; }

The operation count is dominated by that of the FFT's (the array multiplication is of course $\sim N$), so the whole fast convolution algorithm takes $\sim N \cdot \log N$ operations. The following carry operation is also $\sim N$ and can therefor be neglected when counting operations.

Well, $N \cdot \log N$ is not really the truth: it has to be $N \cdot \log N \cdot \log\log N$. This is because the sums in the convolutions have to be represented as exact integers. The biggest term $c_{max}$ that can possibly occur (when multiplying two $N$-digit numbers made of all 'nines', i.e. $R-1$) is the central one, it is the sum of $N$ times $(R-1)(R-1)$ or approximately

$$c_{max} \approx NR^2 \tag{1.8}$$

a number with proportional $\log N$ bits. Therefor, working with some fixed radix $R$ one has to do FFTs with $\log N$ bits precision, leading to an operation count of $N \cdot \log N \cdot \log N$. The slightly better $N \cdot \log N \cdot \log\log N$ is obtained by recursive use of FFT multiplies. For realistic applications (where the sums in the convolution all fit into the machine type floating point numbers) it is safe to think of FFT multiplication being proportional $N \cdot \log N$.

Multiplying our million digit numbers will now take only $10^6 \log_2(10^6) \approx 10^6 \cdot 20$ operations, taking (in the order of) 2 seconds on a 10 Mips machine.

See [61].

## 1.3 Division, square root and cube root

### 1.3.1 Division

The ordinary division algorithm is useless for numbers of extreme precision. Instead one replaces does the division $\frac{a}{b}$ as $a \cdot \frac{1}{b}$. The inverse of $b$ is computed by finding a starting approximation $x_0 \approx \frac{1}{b}$ and then iterating

$$x_{k+1} = x_k + x_k(1 - b\,x_k) \tag{1.9}$$

until the desired presision is reached. The convergence is quadratical (2.order), which means that the number of correct digits is doubled with each step: if $x_k = \frac{1}{b}(1+\epsilon)$ then

$$x_{k+1} = \frac{1}{b}(1+\epsilon) + \frac{1}{b}(1+\epsilon)(1 - b\frac{1}{b}(1+\epsilon)) \tag{1.10}$$

$$= \frac{1}{b}(1-\epsilon^2) \tag{1.11}$$

Moreover each step needs only computations with twice the number of digits that were correct at its beginning. Still better: the multiplication $x_k(...)$ needs only to be done with half precision as it computes the 'correcting' digits (which alter only the less significant half of the digits). Thus at each step we have 1.5 multiplications of the 'current' precision. The total work[1] amounts to

$$1.5 \cdot \sum_{n=0}^{n<\infty} \frac{1}{2}^n$$

which is less than 3 full precision multiplications. together with the final multiplication a division costs as much as 4 multiplications. Another nice feature of the algorithm is that it is self-correcting. Cf. figure 1.3.1 for a numerical example.

$$\begin{aligned}
b &:= 3.1415926 && (1.12)\\
x_0 &= 0.31 \qquad \text{2 digit approximation for } 1/b && (1.13)\\
& \quad\; \text{now use 4 digits:} && (1.14)\\
b \cdot x_0 &= 3.141 \cdot 0.3100 = 0.9737 && (1.15)\\
y_0 &:= 1.000 - b \cdot x_0 = 0.02629 && (1.16)\\
x_0 \cdot y_0 &= 0.3100 \cdot 0.02629 = 0.0081(49) && (1.17)\\
x_1 &:= x_0 + x_0 \cdot y_0 = 0.3100 + 0.0081 = 0.3181 && (1.18)\\
& \quad\; \text{now use 8 digits:} && (1.19)\\
b \cdot x_1 &= 3.1415926 \cdot 0.31810000 = 0.9993406 && (1.20)\\
y_1 &:= 1.0000000 - b \cdot x_0 = 0.0006594 && (1.21)\\
x_1 \cdot y_1 &= 0.31810000 \cdot 0.0006594 = 0.0002097(5500) && (1.22)\\
x_2 &:= x_1 + x_1 \cdot y_1 = 0.31810000 + 0.0002097 = 0.31830975 && (1.23)\\
& \quad\; \text{last step with 8 digits ... homework !} && (1.24)
\end{aligned}$$

Figure 1.1: Computation of $1/\pi$ to 8 digits by a second order iteration.

### 1.3.2 Square root extraction

is quite similar: first compute $\frac{1}{\sqrt{d}}$ then a final multiply with $d$ gives $\sqrt{d}$. Find a starting approximation $x_0 \approx \frac{1}{\sqrt{b}}$ then iterate

$$x_{k+1} = x_k + x_k \frac{(1 - d\,x_k^2)}{2} \tag{1.25}$$

until the desired precision is reached. Convergence is again 2.order. Similar considerations as above (with squaring considered as expensive as multiplication[2]) yield an operation count of 4 multiplications for $\frac{1}{\sqrt{d}}$ or 5 for $\sqrt{d}$.

Note that this algorithm is considerably better than the usual one where $x_{k+1} := \frac{1}{2}(x_k + \frac{d}{x_k})$ is iterated, as long divisions are involved.

[1] The asymptotics of the multiplication is set to $\sim N$ (instead of $N\log(N)$) for the estimates made here, this gives a realistic picture for large $N$.

[2] Indeed it costs about $\frac{2}{3}$ of a multiplication.

### 1.3.3 Cube root extraction

Use $d^{1/3} = d\,(d^2)^{-1/3}$, i.e. compute the inverse third root of $d^2$ using the iteration

$$x_{k+1} = x_k + x_k \frac{(1 - d^2\, x_k^3)}{3} \tag{1.26}$$

finally multiply with $d$.

## 1.4 A general procedure for the inverse n-th root

There is a nice general formula that allows to build iterations with arbitrary order of convergence for $d^{-1/a}$ that involve no long division.

One uses the identity

$$d^{-1/a} = x\,(1 - (1 - x^a\, d))^{-1/a} \tag{1.27}$$

$$= x\,(1 - y)^{-1/a} \tag{1.28}$$

where $y := (1 - x^a\, d)$.

Taylor expansion gives

$$d^{-1/a} = x \sum_{k=0}^{\infty} (1/a)^{\bar{k}}\, y^k \tag{1.29}$$

where $z^{\bar{k}} := z(z+1)(z+2)...(z+k-1)$.

$$d^{-1/a} = \tag{1.30}$$

$$= x\left(1 + \frac{y}{a} + \frac{(1+a)\,y^2}{2\,a^2} + \frac{(1+a)(1+2a)\,y^3}{6\,a^3} + \frac{(1+a)(1+2a)(1+3a)\,y^4}{4!\,a^4} + ...\right) \tag{1.31}$$

A $n$-th order iteration for $d^{-1/a}$ is obtained by truncating the above series after the $(n-1)$-th term,

$$\Phi_n(a, x) := x \sum_{k=0}^{n-1} (1/a)^{\bar{k}}\, y^k \tag{1.32}$$

$$x_{k+1} = \Phi_n(a, x_k) \tag{1.33}$$

e.g. second order:

$$\Phi_2(a, x) := x + x\frac{(1 - dx^a)}{a} \tag{1.34}$$

Convergence:

$$\Phi_n(d^{-1/a}(1+\epsilon)) = d^{-1/a}(1 + \epsilon^n + O(\epsilon^{n+1})) \tag{1.35}$$

Examples:

$a = 1$: inverse

$$\frac{1}{d} = x\,\frac{1}{1-y} \tag{1.36}$$
$$= x\,\left(1 + y + y^2 + y^3 + y^4 + ...\right) \tag{1.37}$$

$\Phi_2(1, x)$ was described in the last section.

$a = 2$: inverse square root

$$\frac{1}{\sqrt{d}} = x\,\frac{1}{\sqrt{1-y}} \tag{1.38}$$
$$= x\,\left(y + \frac{y}{2} + \frac{3\,y^2}{8} + \frac{5\,y^3}{16} + \frac{35\,y^4}{128} + ... + \frac{\binom{2k}{k}\,y^k}{4^k} + ...\right) \tag{1.39}$$

$\Phi_2(2, x)$ was described in the last section.

In hfloat, the second order iterations of this type are used. At the second last step the third order correction is used to assure maximum precision at the last step.

## 1.5 n-th root by Goldschmidt's algorithm

Set

$$x_0 := d^a \qquad y_0 := d^b \tag{1.40}$$

then iterate:

$$r_k := \frac{a+1-y_k}{a} \tag{1.41}$$
$$x_{k+1} := x_k \cdot r_k^b \tag{1.42}$$
$$y_{k+1} := y_k \cdot r_k^a \tag{1.43}$$

until $x$ close enough to

$$x_\infty = d^{\frac{a-b}{a}}. \tag{1.44}$$

This is because

$$\frac{x_0^a}{y_0^b} = d^{a-b} \tag{1.45}$$

and

$$\frac{x_{k+1}^a}{y_{k+1}^b} = \frac{(x_k \cdot r^b)^a}{(y_k \cdot r^a)^b} = \frac{x_k^a}{y_k^b} \tag{1.46}$$

e.g. (with $b = 1$)

$$d^{-1/a} = \prod_{k=0}^{\infty} (2 - E_k) \tag{1.47}$$

where $E_0 := d$ and $E_{k+1} := E_k\,\left(\frac{a+1-E_k}{a}\right)^a$.

## 1.6 Trancendental functions & the AGM

### 1.6.1 The AGM

The AGM (arithmetic geometric mean) plays a central role in the (high precision) computation of logarithms and $\pi$.

The $AGM(a, b)$ is defined as the limit of the iteration AGM iteration, cf.1.48 :

$$a_{k+1} = \frac{a_k + b_k}{2} \tag{1.48}$$

$$b_{k+1} = \sqrt{a_k\, b_k} \tag{1.49}$$

starting with $a_0 = a$ and $b_0 = b$. Both of the values converge quadratically to a common limit. The related quantity $c_k$ (needed in many AGM based computations) is defined as

$$c_k^2 = a_k^2 - b_k^2 \tag{1.50}$$

$$= (a_{k-1} - a_k)^2 \tag{1.51}$$

One further defines (cf. [15] p.221)

$$R'(k) := \left[1 - \sum_{n=0}^{\infty} 2^{n-1} c_n^2\right]^{-1} \tag{1.52}$$

where $c_n^2 := a_n^2 - b_n^2$ corresponding to $AGM(1, k)$.

An alternative way of computation for the AGM iteration is

$$c_{k+1} = \frac{a_k - b_k}{2} \tag{1.53}$$

$$a_{k+1} = \frac{a_k + b_k}{2} \tag{1.54}$$

$$b_{k+1} = \sqrt{a_{k+1}^2 - c_{k+1}^2} \tag{1.55}$$

Schönhage gives the most economic variant of the AGM:

$$A_0 = a_0^2 \tag{1.56}$$

$$B_0 = b_0^2 \tag{1.57}$$

$$t_0 = 1 - (A_0 - B_0) \tag{1.58}$$

$$S_k = \frac{A_k + B_k}{4} \tag{1.59}$$

$$b_k = \sqrt{B_k} \tag{1.60}$$

$$a_{k+1} = \frac{a_k + b_k}{2} \tag{1.61}$$

$$A_{k+1} = a_{k+1}^2 \tag{1.62}$$

$$= \left(\frac{\sqrt{A_k} + \sqrt{B_k}}{2}\right)^2 = \frac{A_k + B_k}{4} + \frac{\sqrt{A_k\, B_k}}{2} \tag{1.63}$$

$$B_{k+1} = 2\,(A_{k+1} - S_K) \tag{1.64}$$

$$c_{k+1}^2 = A_{k+1} - B_{k+1} \tag{1.65}$$

$$t_{k+1} = t_k - 2^{k+1}\, c_{k+1}^2 \tag{1.66}$$

Starting with $a_0 = A_0 = 1$, $B_0 = 1/2$ one has $\pi \approx (2\, a_n^2)/t_n$.

Combining two steps of the AGM iteration leads to the 4.order AGM iteration:

$$\alpha_0 = \sqrt{a_0} \tag{1.67}$$

$$\beta_0 = \sqrt{b_0} \tag{1.68}$$

$$\alpha_{k+1} = \frac{\alpha_k + \beta_k}{2} \tag{1.69}$$

$$\beta_{k+1} = \left(\frac{\alpha_k\,\beta_k\,(\alpha_k^2 + \beta_k^2)}{2}\right)^{1/4} \tag{1.70}$$

$$\gamma_k^4 = \alpha_k^4 - \beta_k^4 \qquad = c_{k/2}^2 \tag{1.71}$$

and

$$R'(k) = \left[1 - \sum_{n=0}^{\infty} 4^n \left(\alpha_n^4 - \left(\frac{\alpha_n^2 + \beta_n^2}{2}\right)^2\right)\right]^{-1} \tag{1.72}$$

corresponding to $AGM4(1, \sqrt{k})$ (cf. [15] p.17).
An alternative computation of the 4.order AGM iteration is:

$$\gamma_{k+1} = \frac{\alpha_k - \beta_k}{2} \tag{1.73}$$

$$\alpha_{k+1} = \frac{\alpha_k + \beta_k}{2} \tag{1.74}$$

$$\beta_{k+1} = \left(\alpha_{k+1}^4 - \gamma_{k+1}^4\right)^{1/4} \tag{1.75}$$

$$c_{k/2}^2 + 2\,c_{k/2+1}^2 = \alpha_{k-1}^4 - \left(\alpha_k^2 - \gamma_k^2\right)^2 \tag{1.76}$$

### 1.6.2 log

The (natural) logarithm can be computed using the following relations (cf. [15] p.221)

$$\left|\log(x) - R'(10^{-n}) + R'(10^{-n}\,x)\right| \leq \frac{n}{10^{2(n-1)}} \tag{1.77}$$

that hold for $n \geq 3$ and $x \in ]\frac{1}{2}, 1[$.

### 1.6.3 exp

The exponential function is computed using the log and the iteration that comes from truncating the series

$$\exp(d) = x\,\exp\left(d - \log(x)\right) \tag{1.78}$$

$$= x\left(1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \ldots\right) \tag{1.79}$$

where $y := d - \log(x)$.
A $n$-th oder iteration

$$x_{k+1} = x_k\left(1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \ldots\frac{y^{n-1}}{(n-1)!}\right) \tag{1.80}$$

(where $y := d - \log(x_k)$)

As the computation of one log is expensive one would use a higher (e.g. 8-th) order iteration.

If one had some efficient algorithm for exp one could compute log from exp using

$$\log(d) = x + \log\left(1 + (d\,\exp(-x) - 1)\right) \quad (1.81)$$

$$= x + \left(1 + y + \frac{y^2}{2} + \frac{y^3}{3} + \ldots\right) \quad (1.82)$$

where $y := d\,\exp(-x) - 1$.

### 1.6.4 sin, cos, tan

For the $\arcsin -, \arccos -, \arctan -$ functions use the complex analogue of the AGM. For the $\sin, \cos, \tan$ use the exp iteration above & think complex.

## 1.7 Inverting a function

I am aware of two formulas that produce iterations for $x = f^{-1}(d)$ for general $f()$:

### 1.7.1 Householders formula

$$x_{k+1} \mapsto \Phi_n(x_k) := x_k + (n-1)\,\frac{\left(\frac{1}{f(x_k)}\right)^{(n-2)}}{\left(\frac{1}{f(x_k)}\right)^{(n-1)}} + f(x_k)^{n+1}\,\psi \quad (1.83)$$

(where $n \geq 2$ and $\psi$ is an arbitrary function that is set to zero in what follows, cf. [63])

gives a $n-$th order iteration that converges against $x_*$ so that $f(x_*) = 0$.

For $n = 2$ this is Newton's formula:

$$\Phi_2(x) := x - \frac{f}{f'} \quad (1.84)$$

For $n = 3$ this is Halley's formula:

$$\Phi_3(x) := x - \frac{2ff'}{2f'^2 - ff''} \quad (1.85)$$

$n = 3$ gives:

$$\Phi_4(x) := x - \frac{3f(ff'' - 2f'^2)}{6ff'f'' - 6f'^3 - ff'''} \quad (1.86)$$

Second order 1.83 with $f(x) := \frac{1}{x^a} - d$ gives 1.34, but for higher orders one gets iterations that require long divisions.

### 1.7.2 Schröders formula

$$x_{k+1} \mapsto \Phi_n(x_k) := \sum_{t=0}^{n} (-1)^t\,\frac{f(x_k)^t}{t!}\,\left(\frac{1}{f'(x_k)}\,\partial\right)^{t-1}\,\frac{1}{f'(x_k)} - f(x_k)^{n+1}\,\varphi \quad (1.87)$$

(where $n \geq 2$ and $\varphi$ is an arbitrary function that is set to zero in what follows cf. [14] p.13) gives a $n-$th order iteration that converges against $x_*$ so that $f(x_*) = 0$.
This is, written out,

$$\begin{aligned}
\Phi_n &= x - \frac{f}{1!\,f'} - \frac{f^2}{2!\,f'^3} \cdot f'' - \frac{f^3}{3!\,f'^5} \cdot \left(3f''^2 - f'f'''\right) & (1.88)\\
&-\frac{f^4}{4!\,f'^7} \cdot \left(15f''^3 - 10f'f''f'' + f'^2 f''''\right) \\
&-\frac{f^5}{5!\,f'^9} \cdot \left(105f''^4 - 105f'f''^2 f''' + 10f'^2 f'''^2 + 15f'^2 f'''' - f'^3 f'''''\right) - \ldots
\end{aligned}$$

The second order iteration is the same as the corresponding iteration from 1.83 while all higher order iterations are different.
"If we denote the general term by

$$-\frac{f^a}{a!}\frac{\chi_a}{f^{2a-1}} \tag{1.89}$$

the numbers $\chi_a$ can easily computed by the recurrence

$$\chi_{a+1} = (2a-1)f''\chi_a - f'\partial\chi_a \tag{1.90}$$

." (cited from [14], p.16).
Formula 1.87 with $f(x) := 1/x^a - d$ gives the iteration 1.32 for arbitrary order.
For $f(x) := log(x) - d$ one gets the iteration 1.6.3.
For $f(x) := x^2 - d$ one gets

$$\begin{aligned}
\Phi(x) &= x - \frac{x^2-d}{2x} - \frac{\left(x^2-d\right)^2}{8x^3} - \frac{\left(x^2-d\right)^3}{16x^5} - \frac{5\left(x^2-d\right)^4}{128x^7} - \frac{7\left(x^2-d\right)^5}{256x^9} - \ldots & (1.91)\\
&= x - y - \frac{1}{2x} \cdot y^2 - \frac{2}{(2x)^2} \cdot y^3 - \frac{5}{(2x)^3} \cdot y^4 - \frac{14}{(2x)^4} \cdot y^5 - \ldots \quad \text{where } y := \frac{x^2-d}{2x} & (1.92)
\end{aligned}$$

## 1.8 Computation of $\pi/\log(q)$

For the computation of the natural logarithm one will use

$$\log(m\,r^x) = \log(m) + x\,\log(r) \tag{1.93}$$

where $m$ is the mantissa and $r$ the radix of the internal representation of the floating point numbers.
There is a nice way to compute the value of $\log(r)$ if the value of $\pi$ has been precomputed. We use (cf. [15] p.225)

$$\begin{aligned}
\frac{\pi}{\log(1/q)} &= AGM(\theta_3(q)^2, \theta_2(q)^2) & (1.94)\\
&= -\frac{\pi}{\log(q)} & (1.95)
\end{aligned}$$

Computing $\theta_3(q)$ is easy when $q = 1/r$:

$$\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \tag{1.96}$$

$$= 1 + 2\sum_{n=1}^{\infty} q^{n^2} \tag{1.97}$$

$$= 2\,(1 + \sum_{n=1}^{\infty} q^{n^2}) - 1 \tag{1.98}$$

However, the computation of $\theta_2(q)$ suggests to choose $q = 1/r^4 =: b^4$:

$$\theta_2(q) := \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \tag{1.99}$$

$$= 0 + 2\sum_{n=0}^{\infty} q^{(n+1/2)^2} \tag{1.100}$$

$$= 2\sum_{n=0}^{\infty} b^{4n^2+4n+1} \qquad \text{where} \quad q = b^4 \tag{1.101}$$

$$= 2\,b\sum_{n=0}^{\infty} q^{n^2+n} \tag{1.102}$$

$$= 2\,b\,(1 + \sum_{n=1}^{\infty} q^{n^2+n}) \tag{1.103}$$

An alternative formula given by Gosper is

$$\frac{\pi}{\log(1/q)} = AGM\left(2\,\theta_2(q^2)^2, \frac{\theta_2(q)^4}{2\,\theta_2(q^2)^2}\right) \tag{1.104}$$

## 1.9 The binary splitting algorithm for rational series

The straightforward computation of a series for which each term adds a constant amount of precision[3] to a precision of $N$ digits involves the summation of proportional $N$ terms. To get $N$ bits of precision one has to add proportional $N$ terms of the sum, each term involves one (length-$N$) short division (and one addition). Therefor the total work is proportional $N^2$, which makes it impossible to compute $\pi$ to billions of digits with these linearly convergent series even if they are as 'good' as series 2.139.

Here is an alternative way to evaluate a sum $\sum_{k=0}^{N-1} a_k$ of rational summands: One looks at the ratios $r_k$ of consecutive terms:

$$r_k := a_k/a_{k-1} \tag{1.105}$$

(set $a_{-1} := 1$ to avoid case distinction for $k = 0$)

i.e.

$$\sum_{k=0}^{N-1} a_k =: r_0\,(1 + r_1\,(1 + r_2\,(1 + r_3\,(1 + ...(1 + r_{N-1})...)))) \tag{1.106}$$

Now define

$$r_{m,n} := r_m\,(1 + r_{m+1}\,(...(1 + r_n)...)) \qquad \text{where} \quad m < n \tag{1.107}$$

$$r_{m,m} := r_m \tag{1.108}$$

[3] e.g. arccot series with arguments $> 1$

then

$$r_{m,n} = \frac{1}{a_{m-1}} \sum_{k=m}^{n} a_k \tag{1.109}$$

and especially

$$r_{0,n} = \sum_{k=0}^{n} a_k \tag{1.110}$$

With

$$\begin{aligned} r_{m,n} &= r_m + r_m \cdot r_{m+1} + r_m \cdot r_{m+1} \cdot r_{m+2} + \dots && (1.111) \\ &\quad \dots + r_m \cdot \dots \cdot r_x + r_m \cdot \dots \cdot r_x \cdot [r_{x+1} + \dots + r_{x+1} \cdot \dots \cdot r_n] \\ &= r_{m,x} + \prod_{k=m}^{x} r_k \cdot r_{x+1,n} && (1.112) \end{aligned}$$

The product telescopes, one gets

$$r_{m,n} = r_{m,x} + \frac{a_x}{a_{m-1}} \cdot r_{x+1,n} \tag{1.113}$$

(where $m \leq x < n$).

Now we can formulate the binary splitting algorithm by giving a binsplit function r:

```
function r(function a, int m, int n)
{
    rational ret;

    if m==n then
    {
        ret := a(m)/a(m-1)
    }
    else
    {
        x := floor( (m+n)/2 )
        ret := r(a,m,x) + a(x) / a(m-1) * r(a,x+1,n)
    }

    print( "r:", m,n, "=",ret )

    return ret
}
```

Here `a(k)` must be a function that returns the `k`-th term of the series we wish to compute, in addition one must have `a(-1)`=1. e.g. to compute arctan(1/10)

```
function a(int k)
{
    if k<0 then  return 1
    else         return (-1)^k/((2*k+1)*10^(2*k+1))
}
```

Calling `r(a,0,N)` returns $\sum_{k=0}^{N} a_k$.

Why is this better than the straightforward way ? First, the work for the above procedure is, as pointed out in [54] $O((\log N)^2\, M(N))$, where $M(N)$ is the complexity of one $N$-bit multiplication. This means that sums of linear but sufficient convergence are again candidates for record breaking $\pi$-computations.

Second, the ratio $r_{0,N-1}$ (i.e. the sum of the first $N$ terms) can be reused if one wants to evaluate the sum to a higher, e.g. double, precision than before: $r_{0,2\,N-1} = r_{0,N-1} + a_{N-1} \cdot r_{N,2\,N-1}$. (this is formula 1.113 with $m = 0, x = N-1, n = 2N-1$) Thereby with the appearence of some new computer that can multiply two length $2 \cdot N$ numbers[4] one only needs to combine the two ratios $r_{0,N-1}$ and $r_{N,2N-1}$ that had been precomputed by the last generation of computers. This costs only a few fullsize multiplications on your new and expensive supercomputer (instead of several *hundreds* for the iterative schemes), which means that one gets a new worldrecord almost for free.

If one wants to stare at zillions of decimal digits of the floating point expansion then one division is also needed which costs as much as 4 multiplications (cf. section 1.3).

Of course that would end the era when record computations could be assigned to names ...

Note that this algorithm can trivially be extended (one should say simplified) to infinite products, e.g. matrix products as formula 2.97.

Cf. [54], [64] and [15].

[4]assuming one could multiply length $N$ numbers before

# Chapter 2

# Remarks on the computation of $\pi$

## 2.1 Iterations for $\pi$

For general forms of the examples given here see J. & P. Borwein's book [15] and their papers.

For some iterations an operation count (in units of full precision multiplications) is given. Operations different from multiplication are counted as follows:
1 squaring = 2/3 mult.
1 division = 4 mult.
1 inverse sqrt = 4 mult.
1 sqrt = 5 mult.

full prec mult → efficiency measure

2.order iteration, cf. `src/pi/pi2nd.cc`:

$$y_0 = \frac{1}{\sqrt{2}} \tag{2.1}$$

$$a_0 = \frac{1}{2} \tag{2.2}$$

$$y_{k+1} = \frac{1-(1-y_k^2)^{1/2}}{1+(1-y_k^2)^{1/2}} \quad \to 0+ \tag{2.3}$$

$$= \frac{(1-y_k^2)^{-1/2}-1}{(1-y_k^2)^{-1/2}+1} \tag{2.4}$$

$$a_{k+1} = a_k\,(1+y_{k+1})^2 - 2^{k+1}\,y_{k+1} \quad \to \frac{1}{\pi} \tag{2.5}$$

$$a_k - \pi^{-1} \leq 16 \cdot 2^{k+1}\,e^{-2^{k+1}\,\pi} \tag{2.6}$$

2.4 shows how to save 1 multiplication per step (cf. section 1.3).

Operations per step: 1 inverse sqrt, 1 division, 2 squarings, 1 multiplication.

Quartic (4.order) iterations, cf. `src/pi/pi4th.cc`:
variant $r = 4$:

$$y_0 = \sqrt{2}-1 \tag{2.7}$$

$$a_0 = 6 - 4\sqrt{2} \tag{2.8}$$

$$y_{k+1} = \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \quad \to 0+ \tag{2.9}$$

$$= \frac{(1-y_k^4)^{-1/4}-1}{(1-y_k^4)^{-1/4}+1} \tag{2.10}$$

$$a_{k+1} = a_k\,(1+y_{k+1})^4 - 2^{2k+3}\,y_{k+1}\,(1+y_{k+1}+y_{k+1}^2) \quad \to \frac{1}{\pi} \tag{2.11}$$

$$= a_k\,((1+y_{k+1})^2)^2 - 2^{2k+3}\,y_{k+1}\,((1+y_{k+1})^2 - y_{k+1}) \tag{2.12}$$

$$0 < a_k - \pi^{-1} \le 16\cdot 4^n\,2\,e^{-4^n\,2\,\pi} \tag{2.13}$$

Identities 2.10 and 2.12 show how to save operations.

Operations per step: 1 inverse 4th root, 1 division, 2 squarings, 1 multiplication.

variant $r = 16$:

$$y_0 = \frac{1-2^{-1/4}}{1+2^{-1/4}} \tag{2.14}$$

$$a_0 = \frac{8/\sqrt{2}-2}{(2^{-1/4}+1)^4} \tag{2.15}$$

$$y_{k+1} = \frac{(1-y_k^4)^{-1/4}-1}{(1-y_k^4)^{-1/4}+1} \quad \to 0+ \tag{2.16}$$

$$a_{k+1} = a_k\,(1+y_{k+1})^4 - 2^{2k+4}\,y_{k+1}\,(1+y_{k+1}+y_{k+1}^2) \quad \to \frac{1}{\pi} \tag{2.17}$$

$$0 < a_k - \pi^{-1} \le 16\cdot 4^n\,4\,e^{-4^n\,4\,\pi} \tag{2.18}$$

Same operation count as before, but this variant gives approximately twice as much precision after the same number of steps.

The general form of the quartic iterations (2.7 and 2.14) is

$$y_0 = \sqrt{\lambda^*(r)} \tag{2.19}$$

$$a_0 = \alpha(r) \tag{2.20}$$

$$y_{k+1} = \frac{(1-y_k^4)^{-1/4}-1}{(1-y_k^4)^{-1/4}+1} \quad \to 0+ \tag{2.21}$$

$$a_{k+1} = a_k\,(1+y_{k+1})^4 - 2^{2k+2}\,\sqrt{r}\,y_k\,(1+y_{k+1}+y_{k+1}^2) \quad \to \frac{1}{\pi} \tag{2.22}$$

$$0 < a_k - \pi^{-1} \le \quad 16\cdot 4^n\,\sqrt{r}\,e^{-4^n\,\sqrt{r}\,\pi} \tag{2.23}$$

Cf. [15], p.170f.

AGM variant 1, cf. `src/pi/piagm.cc`:

$$a_0 = 1 \tag{2.24}$$

$$b_0 = \frac{1}{\sqrt{2}} \tag{2.25}$$

$$p_n = \frac{2\,a_{n+1}^2}{1-\sum_{k=0}^{n} 2^k\,c_k^2} \quad \to \pi \tag{2.26}$$

$$\pi - p_n = \frac{\pi^2\,2^{n+4}\,e^{-\pi\,2^{n+1}}}{AGM^2(a_0,b_0)} \tag{2.27}$$

A 4.order version uses 1.67, cf. also `src/pi/piagm.cc`.

AGM variant 3fast, cf. `src/pi/piagm3.cc`:

$$a_0 = 1 \tag{2.28}$$

$$b_0 = \frac{\sqrt{6}+\sqrt{2}}{4} \tag{2.29}$$

$$p_n = \frac{2\,a_{n+1}^2}{\sqrt{3}\,(1-\sum_{k=0}^{n} 2^k\,c_k^2)-1} \quad \to \pi \tag{2.30}$$

$$\pi - p_n < \frac{\sqrt{3}\,\pi^2\,2^{n+4}\,e^{-\sqrt{3}\,\pi\,2^{n+1}}}{AGM^2(a_0,b_0)} \tag{2.31}$$

AGM variant 3slow, cf. `src/pi/piagm3.cc`:

$$a_0 = 1 \tag{2.32}$$

$$b_0 = \frac{\sqrt{6}-\sqrt{2}}{4} \tag{2.33}$$

$$p_n = \frac{6\,a_{n+1}^2}{\sqrt{3}\,(1-\sum_{k=0}^{n} 2^k\,c_k^2)+1} \quad \to \pi \tag{2.34}$$

$$\pi - p_n < \frac{\frac{1}{\sqrt{3}}\,\pi^2\,2^{n+4}\,e^{-\frac{1}{\sqrt{3}}\,\pi\,2^{n+1}}}{AGM(a_0,b_0)^2} \tag{2.35}$$

Derived AGM iteration (2.order), cf. `src/pi/pideriv.cc`:

$$x_0 = \sqrt{2} \tag{2.36}$$

$$p_0 = 2+\sqrt{2} \tag{2.37}$$

$$y_1 = 2^{1/4} \tag{2.38}$$

$$x_{k+1} = \frac{1}{2}\left(\sqrt{x_k}+\frac{1}{\sqrt{x_k}}\right) \quad (k \geq 0) \quad \to 1+ \tag{2.39}$$

$$y_{k+1} = \frac{y_k\,\sqrt{x_k}+\frac{1}{\sqrt{x_k}}}{y_k+1} \quad (k \geq 1) \quad \to 1+ \tag{2.40}$$

$$p_{k+1} = p_k\,\frac{x_k+1}{y_k+1} \quad (k \geq 1) \quad \to \pi + \tag{2.41}$$

$$p_k - \pi = 10^{-2^{k+1}} \tag{2.42}$$

Cubic AGM, from [22], cf. `src/pi/picubagm.cc`:

$$a_0 = 1 \tag{2.43}$$

$$b_0 = \frac{\sqrt{3}-1}{2} \tag{2.44}$$

$$a_{n+1} = \frac{a_n+2\,b_n}{3} \tag{2.45}$$

$$b_{n+1} = \sqrt[3]{\frac{b_n\,(a_n^2+a_n\,b_n+b_n^2)}{3}} \tag{2.46}$$

$$p_n = \frac{3\,a_n^2}{1-\sum_{k=0}^{n} 3^k\,(a_k^2-a_{k+1}^2)} \tag{2.47}$$

Quintic (5. order) iteration, from the article [18] cf. `src/pi/pi5th.cc`:

$$s_0 = 5(\sqrt{5}-2) \tag{2.48}$$

$$
\begin{aligned}
a_0 &= \frac{1}{2} && (2.49)\\
s_{n+1} &= \frac{25}{s_n (z + x/z + 1)^2} \quad \to 1 && (2.50)\\
\text{where } x &= \frac{5}{s_n} - 1 \quad \to 4 && (2.51)\\
\text{and } y &= (x-1)^2 + 7 \quad \to 16 && (2.52)\\
\text{and } z &= \left(\frac{x}{2}\left(y + \sqrt{y^2 - 4x^3}\right)\right)^{1/5} \quad \to 2 && (2.53)\\
a_{n+1} &= s_n^2 a_n - 5^n \left(\frac{s_n^2 - 5}{2} + \sqrt{s_n\,(s_n^2 - 2 s_n + 5)}\right) \quad \to \frac{1}{\pi} && (2.54)\\
a_n - \frac{1}{\pi} &< 16 \cdot 5^n \, e^{-\pi\, 5^n} && (2.55)
\end{aligned}
$$

Btw. the 5th order algorithm is the slowest of the above, because compared to the other iterations much more operations are needed for each step.

Cubic (3. order) iteration, from [25], cf. `src/pi/pi3rd.cc`:

$$
\begin{aligned}
a_0 &= \frac{1}{3} && (2.56)\\
s_0 &= \frac{\sqrt{3} - 1}{2} && (2.57)\\
r_{k+1} &= \frac{3}{1 + 2\,(1 - s_k^3)^{1/3}} && (2.58)\\
s_{k+1} &= \frac{r_{k+1} - 1}{2} && (2.59)\\
a_{k+1} &= r_{k+1}^2\, a_k - 3^k\,(r_{k+1}^2 - 1) \quad \to \frac{1}{\pi} && (2.60)
\end{aligned}
$$

Nonic (9. order) iteration, from [25], cf. `src/pi/pi9th.cc`:

$$
\begin{aligned}
a_0 &= \frac{1}{3} && (2.61)\\
r_0 &= \frac{\sqrt{3} - 1}{2} && (2.62)\\
s_0 &= (1 - r_0^3)^{1/3} && (2.63)\\
t &= 1 + 2\, r_k && (2.64)\\
u &= \left(9\, r_k\,(1 + r_k + r_k^2)\right)^{1/3} && (2.65)\\
v &= t^2 + t\,u + u^2 && (2.66)\\
m &= \frac{27\,(1 + s_k + s_k^2)}{v} && (2.67)\\
a_{k+1} &= m\, a_k + 3^{2\,k-1}\,(1 - m) \quad \to \frac{1}{\pi} && (2.68)\\
s_{k+1} &= \frac{(1 - r_k)^3}{(t + 2\,u)\, v} && (2.69)\\
r_{k+1} &= (1 - s_k^3)^{1/3} && (2.70)
\end{aligned}
$$

## 2.2 Geometric iterations for $\pi$

Let $r_k$ and $R_k$ be the radii of a circles that are inscribed and circumscribed, respectively, to a regular polygon with $2^k$ sides and circumference 2. Then $2\,\pi\,r_k < 2 < 2\,\pi\,R_k$ and the relations

$$r_2 = \frac{1}{4} \tag{2.71}$$

$$R_2 = \frac{1}{\sqrt{8}} \tag{2.72}$$

$$r_{k+1} = \frac{R_k + r_k}{2} \tag{2.73}$$

$$R_{k+1} = \sqrt{R_k\, r_{k+1}} \tag{2.74}$$

allow to compute better and better approximations to $\pi$. This is called Cusanus' method, it was discovered around 1450, cf. [60] pp.155-156. Note the different subscripts on the right hand side of the last equation: if they were equal the above iteration would compute the $AGM(r_2, R_2)$.

Archimedes used the circumferences of regular polygons with $3 \cdot 2^k$ that are inscribed ($s_k$) and circumscribed ($t_k$) the unit circle.

$$t_0 = 2\sqrt{3} \tag{2.75}$$

$$s_0 = 3 \tag{2.76}$$

$$t_{k+1} = \frac{2\,t_k\,s_k}{t_k + s_k} \tag{2.77}$$

$$s_{k+1} = \sqrt{t_{k+1}\,s_k} \tag{2.78}$$

This is also called the Borchard-Pfaff algorithm. Again, if the subscipts on the right hand side of the last equation were the same then one would compute the arithmetic-harmonic mean $AHM(a_0, b_0)$ that also has the quadratic convergence property of the $AGM$. One can identify equation 2.77 as

$$\tan\frac{\phi}{2} = \frac{2\,\tan\phi\,\sin\phi}{\tan\phi + \sin\phi} \tag{2.79}$$

and equation 2.78 as

$$\sin\frac{\phi}{2} = \sqrt{\tan\frac{\phi}{2}\,\sin\phi} \tag{2.80}$$

The quantities in the above algorithms can be identified with (values of) trigonometrical functions and their half-argument relations.

It is easy to contruct one-valued iterations of similar kind. Consider $4 \cdot 2^k$-sided regular polygons circumscribed around the unit circle. The cicumference is used as an approximation for that of the circle: $2\,\pi \approx 8 \cdot 2^k\,x_k$ where $x_k$ is the length of one side of the $4 \cdot 2^k$-gons. One has

$$x_k = \tan\frac{x_0}{2^k} \tag{2.81}$$

Start with the unit square ($x_0 = 1$) and use

$$\tan\frac{\phi}{2} = \frac{\tan\phi}{1 + \sqrt{1 + \tan^2\phi}} \tag{2.82}$$

i.e. iterate

$$x_{k+1} = \frac{x_k}{1 + \sqrt{1 + x_k^2}} \tag{2.83}$$

This can be rewritten in many ways, this one is quite elegant: use

$$\arctan\frac{1}{x} = 2\arctan\frac{1}{x+\sqrt{x^2+1}} \tag{2.84}$$

i.e. iterate

$$\phi_{k+1} = \phi_k + \sqrt{\phi_k^2+1} \tag{2.85}$$

The approximation made in these algorithms is always of the type $\sin\phi \approx \phi$ or $\tan\phi \approx \phi$.
Note that all these iterations converge only linear.
Cf. [3] and [66].

## 2.3 Products for $\pi$

Wallis product

$$\begin{aligned} \frac{\pi}{2} &= \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{8\cdot 8}{7\cdot 9}\cdot\ldots \\ &= \prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1} \end{aligned} \tag{2.86}$$

From

$$\sin 2x = 2\sin x\cos x \tag{2.87}$$

or equivalent

$$\frac{\sin 2x}{2x} = \frac{\sin x}{x}\cos x \tag{2.88}$$

follows by repeated substitution

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\cos\frac{x}{2^n} \tag{2.89}$$

using

$$\cos\frac{x}{2} = \sqrt{\frac{1}{2}+\frac{\cos x}{2}} \tag{2.90}$$

$$= \frac{\sqrt{2+2\cos x}}{2} \tag{2.91}$$

and $x=\frac{\pi}{2}$ one gets

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\cdot\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\frac{1}{2}\cdot\sqrt{\frac{1}{2}}}}\cdot\ldots \tag{2.92}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdot\ldots \tag{2.93}$$

From

$$\frac{\tan x}{x} = \prod_{n=1}^{\infty} \frac{1}{1 - \tan \frac{x}{2^n}} \tag{2.94}$$

and

$$\cos x = \frac{\sin x}{x} \Big/ \frac{\tan x}{x} \tag{2.95}$$

and formula 2.89 follows

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \tan^2 \frac{x}{2^n}\right)^{2^n - 1} \tag{2.96}$$

$$\prod_{k=0}^{\infty} \begin{bmatrix} \frac{2\,(k-\frac{1}{2})\,(k+2)}{27\,(k+\frac{2}{3})\,(k+\frac{4}{3})} & 10 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \pi + 6 \\ 0 & 1 \end{bmatrix} \tag{2.97}$$

R.W.Gosper gave

$$\prod_{k=0}^{\infty} \begin{bmatrix} \frac{(k-\frac{5}{2})\,(k+\frac{3}{2})\,(k+3)\,(k+\frac{7}{2})}{64\,(k+\frac{3}{4})\,(k+\frac{5}{4})\,(k+\frac{9}{4})\,(k+\frac{11}{4})} & 48\,(k+\frac{41}{21}) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 15\,\pi + 32 \\ 0 & 1 \end{bmatrix} \tag{2.98}$$

and

$$\prod_{k=0}^{\infty} \begin{bmatrix} \frac{(k-\frac{19}{6})\,(k-\frac{13}{6})\,(k+1)\,(k+\frac{19}{6})\,(k+\frac{25}{6})}{64\,(k-\frac{7}{12})\,(k-\frac{1}{12})\,(k+\frac{3}{2})\,(k+\frac{31}{12})\,(k+\frac{37}{12})} & (k-\frac{23}{42})\,(k+\frac{1}{3})\,(k+\frac{11}{6}) \\ 0 & 1 \end{bmatrix} \tag{2.99}$$

$$= \begin{bmatrix} \frac{6175}{2268}\,\pi - \frac{325}{378} & \cdot \\ \cdot & \cdot \end{bmatrix}$$

Gosper gives two matrix products for $\arctan(x)$ (and thereby for $\pi$): Define

$$K(k,n) := \begin{bmatrix} \frac{k\,x^2}{(n+k)(x^2+1)} & \frac{x}{x^2+1} \\ 0 & 1 \end{bmatrix} \tag{2.100}$$

and

$$N(k,n) := \begin{bmatrix} -\frac{n\,x^2}{n+k} & x \\ 0 & 1 \end{bmatrix} \tag{2.101}$$

then

$$\arctan(x) = \text{upper-right}\left(K(1,\frac{1}{2}) \cdot K(2,\frac{1}{2}) \cdot K(3,\frac{1}{2}) \cdot K(4,\frac{1}{2}) \cdot \ldots\right) \tag{2.102}$$

$$= \text{upper-right}\left(N(1,\frac{1}{2}) \cdot N(1,\frac{3}{2}) \cdot N(1,\frac{5}{2}) \cdot N(1,\frac{7}{2}) \cdot \ldots\right) \tag{2.103}$$

## 2.4 Arctan formulas

Formulas of the form

$$k\,\frac{\pi}{4} = \sum_{i=1}^{N} m_i \arctan \frac{1}{x_i} \tag{2.104}$$

$(k \in \mathbb{N}, \quad m_i \in \mathbb{Z}, \quad x_i \in \mathbb{N})$

where all $x_i^2 + 1$ factor completely into factors in

$$F := \{2, 5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, 109, 113\} \tag{2.105}$$

(exceptions are formulas 2.116 and 2.118).

For some of the formulas the factors are given in curly braces[1].

All these formulas were built in 1992 with a mixture of C-programs and Computer Algebra. For the known formulas the original authors are given.

Formulas 2.110 and 2.111 were used for the 100,000 digit computation of $\pi$ in 1961, see [33]. Machin's formula (2.109) was often used in $\pi$-computations before 1960. The computations used the expansion

$$\arctan\frac{1}{x} = \frac{1}{x} - \frac{1}{3\,x^3} + \frac{1}{5\,x^5} - \frac{1}{7\,x^7} + - \ldots \tag{2.106}$$

Another expansion is given by

$$\arctan\frac{1}{x} = x\left(\frac{1}{x^2+1} + \frac{2}{3}\,\frac{1}{(x^2+1)^2} + \frac{2\cdot 4}{3\cdot 5}\,\frac{1}{(x^2+1)^3} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\,\frac{1}{(x^2+1)^4} + \ldots\right) \tag{2.107}$$

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3} \tag{2.108}$$

{5} *(Euler, 1706)*

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239} \tag{2.109}$$

{13} *(Machin, 1776)*

$$\frac{\pi}{4} = 6\arctan\frac{1}{8} + 2\arctan\frac{1}{57} + \arctan\frac{1}{239} \tag{2.110}$$

{5,13} *(Størmer, 1896)*

$$\frac{\pi}{4} = 12\arctan\frac{1}{18} + 8\arctan\frac{1}{57} - 5\arctan\frac{1}{239} \tag{2.111}$$

{5,13} *(Gauss)*

$$\frac{\pi}{4} = 44\arctan\frac{1}{57} + 7\arctan\frac{1}{239} - 12\arctan\frac{1}{682} + 24\arctan\frac{1}{12943} \tag{2.112}$$

{5,13,61} *(Størmer, 1896)*

$$\frac{\pi}{4} = 88\arctan\frac{1}{172} + 51\arctan\frac{1}{239} + 32\arctan\frac{1}{682} + +44\arctan\frac{1}{5357} + 68\arctan\frac{1}{12943} \tag{2.113}$$

{5,13,61,97} *(Størmer, 1896)*

$$\frac{\pi}{4} = 88\arctan\frac{1}{192} + 39\arctan\frac{1}{239} + 100\arctan\frac{1}{515} - -32\arctan\frac{1}{1068} - 56\arctan\frac{1}{173932} \tag{2.114}$$

{5,13,73,101}

[1] the 2 is always omitted.

$$\begin{aligned}\frac{\pi}{4} &= 100 \arctan\frac{1}{319} + 127 \arctan\frac{1}{378} + 71 \arctan\frac{1}{557} - \\ &\quad -15 \arctan\frac{1}{1068} + 66 \arctan\frac{1}{2943} + 44 \arctan\frac{1}{478707}\end{aligned} \tag{2.115}$$

*{5,13,17,41,73}*

$$\begin{aligned}\frac{\pi}{4} &= 322 \arctan\frac{1}{577} + 76 \arctan\frac{1}{682} + 139 \arctan\frac{1}{1393} + \\ &\quad +156 \arctan\frac{1}{12943} + 132 \arctan\frac{1}{32807} + 44 \arctan\frac{1}{1049433}\end{aligned} \tag{2.116}$$

*{5,13,61,89,197}*

$$\begin{aligned}\frac{\pi}{4} &= 1074 \arctan\frac{1}{1568} + 657 \arctan\frac{1}{4662} + 183 \arctan\frac{1}{5357} - \\ &\quad -779 \arctan\frac{1}{12943} - 32 \arctan\frac{1}{17923} - 449 \arctan\frac{1}{32807} + \\ &\quad +398 \arctan\frac{1}{390112}\end{aligned} \tag{2.117}$$

*{5,13,17,61,89,97}*

$$\begin{aligned}\frac{\pi}{4} &= 1587 \arctan\frac{1}{2852} + 295 \arctan\frac{1}{4193} + 593 \arctan\frac{1}{4246} + \\ &\quad +359 \arctan\frac{1}{39307} + 481 \arctan\frac{1}{55603} + 625 \arctan\frac{1}{211050} - \\ &\quad -708 \arctan\frac{1}{390112}\end{aligned} \tag{2.118}$$

*{5,13,17,29,97,433}*

$$\begin{aligned}\frac{\pi}{4} &= 1074 \arctan\frac{1}{4246} + 1257 \arctan\frac{1}{5357} + 1731 \arctan\frac{1}{6107} + \\ &\quad +295 \arctan\frac{1}{12943} + 625 \arctan\frac{1}{19703} - 481 \arctan\frac{1}{32807} - \\ &\quad -1042 \arctan\frac{1}{39307} + 398 \arctan\frac{1}{390112}\end{aligned} \tag{2.119}$$

*{...}*

$$\begin{aligned}\frac{\pi}{4} &= 7162 \arctan\frac{1}{12943} + 3796 \arctan\frac{1}{32807} + 2558 \arctan\frac{1}{34208} + \\ &\quad +2729 \arctan\frac{1}{44179} - 708 \arctan\frac{1}{51387} + 2192 \arctan\frac{1}{114669} - \\ &\quad -2805 \arctan\frac{1}{157318} - 3696 \arctan\frac{1}{485298} - 2407 \arctan\frac{1}{24208144}\end{aligned} \tag{2.120}$$

*{...}*

$$\begin{aligned}\frac{\pi}{4} &= 2805 \arctan\frac{1}{5257} - 398 \arctan\frac{1}{9466} + 1950 \arctan\frac{1}{12943} + \\ &\quad +1850 \arctan\frac{1}{34208} + 2021 \arctan\frac{1}{44179} + 2097 \arctan\frac{1}{85353} + \\ &\quad +1484 \arctan\frac{1}{114669} + 1389 \arctan\frac{1}{330182} + 808 \arctan\frac{1}{485298}\end{aligned} \tag{2.121}$$

*{5,13,17,29,37,41,53,61} (Gauss)*

$$\begin{aligned}\frac{\pi}{4} &= 50539 \arctan\frac{1}{51387} + 1555 \arctan\frac{1}{114669} \\ &\quad -6601 \arctan\frac{1}{157318} - 20678 \arctan\frac{1}{390112} - 5617 \arctan\frac{1}{485298} \\ &\quad -64126 \arctan\frac{1}{617427} + 10958 \arctan\frac{1}{1984933} - 30569 \arctan\frac{1}{3449051} \\ &\quad +23407 \arctan\frac{1}{22709274} + 25433 \arctan\frac{1}{24208144} \\ &\quad \{...\}\end{aligned} \tag{2.122}$$

$$\begin{aligned}\frac{\pi}{4} &= 36462 \arctan\frac{1}{390112} + 135908 \arctan\frac{1}{485298} \\ &\quad +274509 \arctan\frac{1}{683982} - 39581 \arctan\frac{1}{1984933} \\ &\quad +178477 \arctan\frac{1}{2478328} - 114569 \arctan\frac{1}{3449051} \\ &\quad -146571 \arctan\frac{1}{18975991} + 61914 \arctan\frac{1}{22709274} \\ &\quad -69044 \arctan\frac{1}{24208144} - 89431 \arctan\frac{1}{201229582} \\ &\quad -43938 \arctan\frac{1}{2189376182} \\ &\quad \{5,13,17,29,37,53,61,89,97,101\}\end{aligned} \tag{2.123}$$

$$\begin{aligned}\frac{\pi}{4} &= 446879 \arctan\frac{1}{683982} + 172370 \arctan\frac{1}{1635786} \\ &\quad -193720 \arctan\frac{1}{1984933} + 369078 \arctan\frac{1}{2478328} \\ &\quad +18231 \arctan\frac{1}{3014557} + 21339 \arctan\frac{1}{3449051} \\ &\quad -154139 \arctan\frac{1}{6225244} - 110109 \arctan\frac{1}{18975991} \\ &\quad +80145 \arctan\frac{1}{22709274} - 223183 \arctan\frac{1}{24208144} \\ &\quad -107662 \arctan\frac{1}{201229582} - 216308 \arctan\frac{1}{2189376182} \\ &\quad \{...\}\end{aligned} \tag{2.124}$$

$$\begin{aligned}\frac{\pi}{4} &= 872408 \arctan\frac{1}{1984933} + 619249 \arctan\frac{1}{2298668} \\ &\quad +369078 \arctan\frac{1}{2478328} + 18231 \arctan\frac{1}{3014557} \\ &\quad -1217159 \arctan\frac{1}{5033696} + 911989 \arctan\frac{1}{6225244} \\ &\quad +783649 \arctan\frac{1}{18975991} - 70886 \arctan\frac{1}{22709274} \\ &\quad -374214 \arctan\frac{1}{24208144} - 1044789 \arctan\frac{1}{168623905} \\ &\quad +339217 \arctan\frac{1}{201229582} - 446879 \arctan\frac{1}{284862638} \\ &\quad +402941 \arctan\frac{1}{2189376182} \\ &\quad \{...\}\end{aligned} \tag{2.125}$$

## 2.5 How to build arctan formulas

For a $n$-term arctan relation of the form

$$\frac{k\,\pi}{4} = m_1 \arctan\frac{1}{a_1} + m_2 \arctan\frac{1}{a_2} + \ldots + m_n \arctan\frac{1}{a_n} \tag{2.126}$$

1. Choose a set $F = \{p_1, ..., p_{n-1}\}$ of primes of the form $4k+1$
2. Find $a_1, ...a_y$ so that $a_i^2+1$ factor completely in $2 \cup F$,
   i.e. $a_i^2 + 1 = 2^{e_{i,0}} \cdot p_1^{e_{i,1}} \cdot p_2^{e_{i,2}} \cdot \ldots \cdot p_{n-1}^{e_{i,n-1}} \qquad i = 1, 2, ..., y$
   (the factor 2 for the odd $x_i$ is ignored in what follows).
3. If you can't find more than $n$ $a_i$, i.e. $y < n$, then restart with another $F$
4. give the $e_{i,j}$ a minus sign if $(a_i \% p_j) > \frac{p_j}{2}$
5. Find the nullspace of the $(n-1) \times y$ matrix $M_{ij} := \{e_{ij}\}$
6. Find linear combinations of the basis vectors of the nullspace that correspond to nice and nontrivial (i.e. $k \neq 0$) arctan relations.

An example: for a 5-term relation

1. Choose $F = \{5, 13, 61, 101\}$
2. Find $\{2, 3, 5, 7, 8, 18, ..., 57, ..., 111, 239, 515, 682, 12943\}$
   (from which i choose the 5 largest $111, 239, 515, 682, 12943$)
3. $y >= 5$ ok.
4. and
5. $M =$

| | 5 | 13 | 61 | 101 |
|---|---|---|---|---|
| 111 | 0 | 0 | -1 | +1 |
| 239 | 0 | +4 | 0 | 0 |
| 515 | 0 | -1 | 0 | 2 |
| 682 | +3 | 0 | +2 | 0 |
| 12943 | -4 | -3 | +1 | 0 |

   (e.g. $239^2 + 1 = 2 \cdot 13^4$ and $239\%13 = 5 < 13/2$)
6. we were lucky:

$$\begin{aligned}\frac{1\,\pi}{4} = {} & 88 \arctan\frac{1}{111} + 7 \arctan\frac{1}{239} - 44 \arctan\frac{1}{515} + \\ & +32 \arctan\frac{1}{682} + 24 \arctan\frac{1}{12943}\end{aligned} \tag{2.127}$$

Often one ends up with a trivial relation, i.e.

$$0 = m_1 \arctan \frac{1}{a_1} + m_2 \arctan \frac{1}{a_2} + ... + m_n \arctan \frac{1}{a_n} \tag{2.128}$$

To build formulas with terms of the type $\arctan \frac{a}{b}$ use the factors of $a^2 + b^2$.

Open questions:

1. What is the upper bound for the $a_i$ for a certain set of factors $F$ ?
2. Is there an algorithm that (in subexponential running time) finds for a certain set $A = \{a_i\}, i = 1, ..., y$ (for which $a_i^2 + 1$ factor completely in $F$) the arctan relations also for the subsets of $A$ that contain only $a_i$ for which $a_i^2 + 1$ factor completely in a subset of ?
3. Is there a better algorithm to find the $a_i$ for a certain $F$ than these:
   (a) tree search over all products $\prod p_j^{e_{ij}}$
   (checking for each product if the product minus one is square)
   (b) brute force checking all $a = 2...\infty$ if $a^2 + 1$ factors in $F$

## 2.6 Ramanujan type formulas

Some nice explicit 'Ramanujan-type' formulas for $1/\pi$ follow. For more formulas and explanation cf. [15] and [17] but don't see appendix A.

The explicit formulas were made in 1994 using Mathematica (for numerical computation of the quantities in the general formulas and for finding the minimal polynomials) and MapleV (for solving the polynomials and numerical verification of the results).

The 'huge' formulas here are given rather for fun than for the computation of pi.

The 'type X' are the same as in [17].

It is

$$(z)_k := z^{\bar{k}} := z(z+1)(z+2)...(z+k-1) \tag{2.129}$$

in what follows.

### 2.6.1 Type 1 $n = 58$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3} \left(2\sqrt{2}\,(1103 + 26390\,n)\right) \frac{1}{\left(99^2\right)^{2n+1}} \tag{2.130}$$

$$= \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!\,(1103 + n\,26390)}{(n!)^4\,396^{4n}} \tag{2.131}$$

(Ramanujan) about 8 correct digits per term

### 2.6.2 Type 1 $n = 862$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3} \frac{A + n\,B}{X^{2n+1}} \tag{2.132}$$

(type 1, $n = 862$)

$$\begin{aligned} A := & \Big[4521962731044058367634998271455136035/4 + \\ & +7993776278485234586059121251125632 34\sqrt{2} + \\ & +12\Big(1775012755290923520301237736918207934527539078119087387065649126105721 9 + \\ & +1255123555958829884236839904476079251826408616374387198187634303258534\sqrt{2}\Big)^{1/2}\Big]^{1/2} \end{aligned} \tag{2.133}$$

$$\begin{aligned} B := & \Big[96177613950889534859154440913076361060 00 + \\ & +68007843023015886866162539734297821544 00\sqrt{2} \\ & +52003425600\sqrt{2}\Big(3420456658672290315173107253751646913664067204719883059296 3 + \\ & +2418628098101856660655230981125577585184945651021683039952 2\sqrt{2}\Big)^{1/2}\Big]^{1/2} \end{aligned} \tag{2.134}$$

$$\begin{aligned} X := & 1670141896514232075 + 1180968660568974600\sqrt{2} + \\ & +2736\sqrt{2}\Big(37262720186501774634179156460 3 + \\ & +263487221293322577155951514850\sqrt{2}\Big)^{1/2} \end{aligned} \tag{2.135}$$

(each term adds 37 digits)

### 2.6.3 Type 2 $n = 37$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3} \frac{(1123 + 21460\,n)}{4} \frac{1}{882^{2n+1}} \tag{2.136}$$

(Ramanujan)

### 2.6.4 Type 3a $n = 7 \ldots 163$

$$\frac{1}{\pi} = \frac{1}{\sqrt{-1728\,J}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^3} \frac{A + n\,B}{J^n} \tag{2.137}$$

$$= \frac{1}{\sqrt{-1728\,J}} \sum_{n=0}^{\infty} \frac{(6n)!}{12^{3n}\,(3n)!\,(n!)^3} \frac{A + n\,B}{J^n} \tag{2.138}$$

| n | A | B | J | correct digits per term |
|---|---|---|---|---|
| 7 | 24 | 189 | -125/64 | $\leq 1$ |
| 11 | 60 | 616 | -512/27 | 1 |
| 19 | 300 | 4104 | -512 | 3 |
| 27 | 1116 | 18216 | -64000/9 | 4 |
| 43 | 9468 | 195048 | $-80^3$ | 6 |
| 67 | 122124 | 3140242 | $-440^3$ | 8 |
| 163 | 163096908 | 6541681608 | $-53360^3$ | 15 |

The last ($n = 163$) is known as Chudnovsky's formula:

$$\frac{1}{\pi} = \frac{6541681608}{\sqrt{640320}^3} \sum_{k=0}^{\infty} \left( \frac{13591409}{545140134} + k \right) \left( \frac{(6k)!}{(k!)^3 \, (3k)!} \frac{(-1)^k}{640320^{3k}} \right) \tag{2.139}$$

$$= \frac{12}{\sqrt{640320}^3} \sum_{k=0}^{\infty} (-1)^k \frac{(6k)!}{(k!)^3 \, (3k)!} \frac{13591409 + k\, 545140134}{(640320)^{3k}} \tag{2.140}$$

### 2.6.5 Type 3c $n = 1555$

$$\frac{1}{\pi} = \frac{1}{\sqrt{-12\,J}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^3} \frac{A + n\,B}{J^n} \tag{2.141}$$

$$= \frac{1}{\sqrt{-12\,J}} \sum_{n=0}^{\infty} \frac{(6n)!}{12^n \, (3n)! \, (n!)^3} \frac{A + n\,B}{J^n} \tag{2.142}$$

(type 3c, $n = 1555$)

$$\begin{aligned} A := {} & 5280419026080999965452185 + 2361475178400070170568800\sqrt{5} + \\ & +32\sqrt{5}\big(10891728551171178200467436212395209160385656017 + \\ & +4870929086578810225077338534541688721351255040\sqrt{5}\big)^{1/2} \end{aligned} \tag{2.143}$$

$$\begin{aligned} B := {} & 6541592044580522675241457 50 + 2925488898550776690804672 00\sqrt{5} + \\ & +209664\sqrt{3110}\Big(6260208323789001636993322654444020882161 + \\ & +2799650273060444296577206890718825190235\sqrt{5}\Big)^{1/2} \end{aligned} \tag{2.144}$$

$$\begin{aligned} J := {} & -\Big[178977495886260 20 + 8004116944887336\sqrt{5} + \\ & +108\sqrt{5}\big(1098523457946355032371331847 3 + \\ & +4912746253692362754607395912\sqrt{5}\big)^{1/2}\Big]^3 \end{aligned} \tag{2.145}$$

(each term adds 50 correct digits)

$$\begin{aligned} 0 = {} & 910569653371944388150731586242 25 + \\ & +921418726536039039180892700310 0\,A + \\ & +672035320036821921804675631270\,A^2 - \end{aligned} \tag{2.146}$$

$$-2112167610432399986180874 0\, A^3 + A^4$$

$$\begin{aligned} 0 \quad = \quad & 1751418001813738732656513138904579560 0 + \\ & +1937569475857433007251933220133800 00\, B + \\ & +2063419786805410130433556462222680\, B^2 - \\ & -2616636817832209070096583000\, B^3 + B^4 \end{aligned} \tag{2.147}$$

$$\begin{aligned} 0 \quad = \quad & 27192565672854630400 + \\ & +49698245345181030400\, U + \\ & +2288545308972778272 0\, U^2 - \\ & -7159099835450408 0\, U^3 + U^4 \end{aligned} \tag{2.148}$$

$$J = -U^3 \tag{2.149}$$

### 2.6.6 Type 3b $n = 190$

$$\frac{1}{\pi} = \frac{1}{\sqrt{3J}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^3} \frac{A + n\,B}{J^n} \tag{2.150}$$

(type 3b, $n = 190$)

$$\begin{aligned} A \quad := \quad & 21242668516504965 + \\ & +15020834958518500\sqrt{2} + \\ & +2\sqrt{5}\Big(451250964275865682516456101416 59 + \\ & +319082616856433129021735854342 50\sqrt{2}\Big)^{1/2} \end{aligned} \tag{2.151}$$

$$\begin{aligned} B \quad := \quad & 18397793537034219 00 + \\ & +1300920456890691000\sqrt{2} + \\ & +24337404\sqrt{10}\Big(1142912476713024496667 + \\ & +808161162586491705750\sqrt{2}\Big)^{1/2} \end{aligned} \tag{2.152}$$

$$\begin{aligned} J \quad := \quad & \Big[71864175655 + 22725423252\sqrt{10} + \\ & +2808\sqrt{5}\left(261993316778681 + 82849561276216\sqrt{10}\right)^{1/2}\Big]^3 \end{aligned} \tag{2.153}$$

(each term adds 34 digits)

$$\begin{aligned} 0 \quad = \quad & 18983882886895192207942622025 - \\ & -22331544571858356551863737 00\, A + \\ & +57049989022950294432409 90\, A^2 - \\ & 84970674066019860\, A^3 + A^4 \end{aligned} \tag{2.154}$$

$$\begin{aligned} 0 &= 1131604728750730378510589191731 8400 - \\ &\quad -1435640467917904306324392329 28000\, B + \\ &\quad +2139623589886529111302499 8560\, B^2 - \\ &\quad -7359117414813687600\, B^3 + B^4 \end{aligned} \tag{2.155}$$

$$\begin{aligned} 0 &= 1860185517864501025 - 1262383694834359900\, U + \\ &\quad +4449869714512 0230\, U^2 - 287456702620\, U^3 + U^4 \end{aligned} \tag{2.156}$$

$$J = -U^3 \tag{2.157}$$

## 2.7 How to build Ramanujan type formulas

Here is how to build 'Ramanujan-type' formulas like those in section 2.6:

1. Read the definitions of the general formulas.
2. Pick out the necessary definitions from the messy appendix A or from Borwein's book ([15]).
3. Write a mupad-package that implements all the needed quantities (steal from the mathematica package `src/pi/bucket/piram.m`).
4. For each formula and $n$ do
   (a) Get numeric approximations for the quantities you need (e.g. $f_n$ and $J_n$), compute 500 digits or so.
   (b) Find the minimal polynomials for those quantities.
   (c) Solve the polynomials & beautify the results.
   (d) Check the symbolic results by comparing them to the quantities they were made of, using a higher precision than before.

For more formulas and explanation cf. [17] and [15].

## 2.8 Approximations for $\pi$

In what follows $\pi(n)$ denotes an $n-$digit approximation for $\pi$.

$$\pi(2) = \frac{22}{7} = 3.1428... \tag{2.158}$$

$$\pi(6) = \frac{355}{113} = 3.14159292... \tag{2.159}$$

The last approximation tells us that it is a particularly bad idea to use $\pi$ as an irrational value e.g. in chaos theoretic programs: it is almost up to floating point (single-) precision a rational number with pretty small denominator. Use $\phi = \frac{\sqrt{5}-1}{2}$ instead, cf. section E.

$$\pi(2) = \sqrt{3} + \sqrt{2} = 3.1462... \tag{2.160}$$

Ramanujan [28] gives in his paper (among many other):

$$\pi(5) = \frac{12}{\sqrt{22}} \log\left(\sqrt{2}+2\right) \tag{2.161}$$

$$\pi(15) = \frac{12}{\sqrt{130}} \log\left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+2\right)\left(\sqrt{13}+3\right)\right) \tag{2.162}$$

$$\pi(16) = \frac{24}{\sqrt{142}} \log\left(\frac{1}{2}\sqrt{7\sqrt{2}+10}+\sqrt{11\sqrt{2}+10}\right) \tag{2.163}$$

$$\pi(18) = \frac{12}{\sqrt{190}} \log\left(\left(\sqrt{10}+3\right)\left(2\sqrt{2}+\sqrt{10}\right)\right) \tag{2.164}$$

$$\pi(22) = \frac{12}{\sqrt{310}} \log\left( \left(\sqrt{2}+2\right)\left(\sqrt{5}+3\right)\left(2\sqrt{10}+\sqrt{20\sqrt{10}+61}+5\right)\right) \tag{2.165}$$

$$\pi(31) = \frac{4}{\sqrt{522}} \log\left(\frac{1}{256}\sqrt{2}\left(\sqrt{29}+5\right)^3\left(11\sqrt{6}+5\sqrt{29}\right)\left(\sqrt{3\sqrt{6}+5}+\sqrt{3}\sqrt{\sqrt{6}+3}\right)^6\right) \tag{2.166}$$

From the definition of the $J$-function

$$\begin{aligned} J(\tau) \;:=\;& 1/q + 744 + 196884\,q + 21493760\,q^2 + 864299970\,q^3 + 20245856256\,q^4 + \\ +\;& 333202640600\,q^5 + 4252023300096\,q^6 + 44656994071935\,q^7 + 401490886656000\,q^8 + \\ +\;& 3176440229784420\,q^9 + 22567393309593600\,q^{10} + \ldots \end{aligned} \tag{2.167}$$

(where $q := e^{2\,i\,\pi\,\tau}$) it is possible to give approximations to $\pi$ for certain values of $\tau$ (cf. [11]).
E.g.

$$\pi(17) = \frac{\log\left(5280^3+744\right)}{\sqrt{67}} \tag{2.168}$$

$$\pi(30) = \frac{\log\left(640320^3+744\right)}{\sqrt{163}} \tag{2.169}$$

for $\tau = 67$ and $\tau = 163$, respectively.

As $q$ is close to an integer for these values, one can take more terms from the above series to get (approximating $q$ by $[q]$) better approximations of the same type. With $z := -J(\frac{1+i\sqrt{163}}{2}) + 744 = 640320^3 + 744$ one gets

$$\pi(46) = \frac{\log\left(z - 196884/z\right)}{\sqrt{163}} \tag{2.170}$$

$$\pi(60) = \frac{\log\left(z - 196884/z + 21493760/z^2\right)}{\sqrt{163}} \tag{2.171}$$

Using more terms doesn't improve the accuracy anymore because of the approximation made for $q$.

## 2.9 Continued fractions for $\pi$

### 2.9.1 The simple continued fraction for $\pi$

$$\pi = 3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}}} \tag{2.172}$$

there is no pattern in the occuring numbers,

### 2.9.2 other continued fractions for $\pi$

$$\frac{4}{\pi} = 1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cfrac{9^2}{2+\cfrac{11^2}{2+\ldots}}}}}} \tag{2.173}$$

(Brouncker, 1658)

$$\frac{4}{\pi} = 1+\cfrac{1^2}{3+\cfrac{2^2}{5+\cfrac{3^2}{7+\cfrac{4^2}{9+\cfrac{5^2}{11+\ldots}}}}} \tag{2.174}$$

(cf. [59])

$$\frac{6}{\pi^2-6} = 1+\cfrac{1^2}{1+\cfrac{1\cdot 2}{1+\cfrac{2^2}{1+\cfrac{2\cdot 3}{1+\cfrac{3^2}{1+\cfrac{3\cdot 4}{1+\cfrac{4^2}{1+\ldots}}}}}}} \tag{2.175}$$

$$\frac{12}{\pi^2} = 1+\cfrac{1^4}{3+\cfrac{2^4}{5+\cfrac{3^4}{7+\cfrac{4^4}{9+\cfrac{5^4}{+\ldots}}}}} \tag{2.176}$$

## 2.10 Series for $\pi$

$$\begin{aligned}\frac{\pi}{4} &= \arctan 1 &(2.177)\\ &= 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}+-... &(2.178)\end{aligned}$$

(Gregory, 1671). The error of the truncated series is about one half of the first neglected term.
Applying the Euler transform (cf. [53] p.253-255) to 2.177 gives the series

$$\begin{aligned}\frac{\pi}{2} &= 1+\frac{1}{3}+\frac{1\cdot 2}{3\cdot 5}+\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}+... &(2.179)\\ &= 1+\frac{1}{3}\left(1+\frac{2}{5}\left(1+\frac{3}{7}\left(1+...\right)\right)\right) &(2.180)\end{aligned}$$

which is used in the spigot algorithm, cf. [12].
In [1] the following acceleration of the arctan-series is given:

$$\arctan x = \sum_{k=0}^{n-1}(-1)^k\,\frac{x^{2k+1}}{2k+1}+(-1)^n\,\frac{1}{2}\,x^{2n-1}\sum_{k=0}^{\infty}\left(\frac{2x^2}{x^2+1}\right)^{k+1}\frac{k!}{((2n+1))_{k+1}} \qquad (2.181)$$

where $((2n+1))_{k+1}$ denotes $(2n+1)(2n+3)...(2n+2k+1)$. The error of the truncated series is less than $|1+x|$ times the first neglected term.
For $x=1$ (formula 2.177) and $n=500$ this is

$$\begin{aligned}\frac{\pi}{4} &= 1-\frac{1}{3}+\frac{1}{5}-+...-\frac{1}{999}+ &(2.182)\\ &+ \frac{1}{2}\left(\frac{1}{1001}+\frac{1!}{1001\cdot 1003}+\frac{2!}{1001\cdot 1003\cdot 1005}+\frac{3!}{1001\cdot 1003\cdot 1005\cdot 1007}+...\right)\end{aligned}$$

$$\begin{aligned}\frac{\pi}{6} &= \arcsin\frac{1}{2} &(2.183)\\ &= \frac{1}{2}+\frac{1}{2}\left(\frac{1}{3\cdot 2^3}\right)+\frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{5\cdot 2^5}\right)+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{1}{7\cdot 2^7}\right)+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\left(\frac{1}{9\cdot 2^9}\right)+... &(2.184)\end{aligned}$$

(Newton, 1665)

$$\frac{\pi}{4} = \frac{3}{4}+\frac{1}{2\cdot 3\cdot 4}-\frac{1}{4\cdot 5\cdot 6}+\frac{1}{6\cdot 7\cdot 8}-\frac{1}{8\cdot 9\cdot 10}+\frac{1}{10\cdot 11\cdot 12}-+... \qquad (2.185)$$

$$\frac{4}{\pi} = 1+\frac{1}{4}+\left(\frac{1\cdot 1}{2\cdot 4}\right)^2+\left(\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\right)^3+\left(\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\right)^4+... \qquad (2.186)$$

(Gauss)

$$\frac{4}{\pi} = 1+\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}\right)^2+\frac{1}{2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2+\frac{1}{2}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2+... \qquad (2.187)$$

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 - 13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + - \ldots \tag{2.188}$$

(Ramanujan, cf. p.7 in [67])

$$\pi = 3\sqrt{3}\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!\,n} \tag{2.189}$$

$$\pi = 3\sqrt{3}\sum_{n=0}^{\infty}\frac{1}{(3n+1)\,(3n+2)} \tag{2.190}$$

Using the identity

$$\pi = \sum_{i=0}^{\infty}\frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right) \tag{2.191}$$

it is possible to compute some hexadecimal digits of $\pi$ *without* computing any of the preceding digits. See the article [42] for the algorithm.

A similar series is

$$\pi = \sum_{i=0}^{\infty}\frac{(-1)^i}{4^i}\left(\frac{2}{4i+1} + \frac{2}{4i+2} + \frac{1}{4i+3}\right) \tag{2.192}$$

$$\pi = 3 + \frac{1}{60}\left(8 + \frac{2\cdot 3}{7\cdot 8\cdot 3}\left(13 + \frac{3\cdot 5}{10\cdot 11\cdot 3}\left(18 + \frac{4\cdot 7}{13\cdot 14\cdot 3}\,(23 + \ldots)\right)\right)\right) \tag{2.193}$$

(Gosper, cf. [82])

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sum_{k=1}^{n} k^2} = 6\,(\pi - 3) \tag{2.194}$$

(from [15], p.101)

F.Bellard gives

$$740025\,\pi + 20379280 = \sum_{n=1}^{\infty}\frac{3\,P(n)}{\binom{7\,n}{2\,n}\,2^{n-1}} \tag{2.195}$$

where

$$\begin{aligned} P(n) := {} & -885673181\,n^5 + 3125347237\,n^4 - 2942969225\,n^3 \\ & +1031962795\,n^2 - 196882274\,n + 10996648 \end{aligned} \tag{2.196}$$

$$\pi = \log\frac{1}{x} - 2\,x^4 - 13\,x^8 - \frac{368}{3}\,x^{12} - \ldots \qquad \text{where} \quad x = \frac{1}{2}\,\frac{2^{1/4}-1}{2^{1/4}+1} \tag{2.197}$$

## 2.11 Miscellaneous formulas for $\pi$

$$e^{\sqrt{-1}\,\pi} = -1 \tag{2.198}$$

(Euler)

In [2] integrals of the form

$$I_{n,m} = \int_0^1 \frac{x^m\,(1-x)^n}{(1+x^2)} \tag{2.199}$$

examples are

$$I_{4,4} = \int_0^1 \frac{x^4\,(1-x)^4}{(1+x^2)} = \frac{22}{7} - \pi \tag{2.200}$$

$$I_{2,4} = \pi - \frac{47}{15} \tag{2.201}$$

$$I_{6,12} = 16\,\pi - \frac{153966181}{3063060} \tag{2.202}$$

$$I_{32,32} = 16384\,\pi - \frac{31694514838868667276634759966 4}{61576400213688659766216 75} \tag{2.203}$$

The fractions on the rhs. are, as shown in the paper, approximations for $\pi$, the last gives 23 correct digits.

$$\pi = \frac{426880\,\sqrt{10005}}{13591409\,\left({}_3F_2(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;A) - B\,{}_3F_2(\frac{7}{6},\frac{3}{2},\frac{11}{6};2,2;A)\right)} \tag{2.204}$$

where

$$A := -\frac{1}{151931373056000} \tag{2.205}$$

$$\tag{2.206}$$

$$B := \frac{30285563}{1651969144908540723200} \tag{2.207}$$

(Chudnovskys), this is from [45].

The

From the Poisson summation formula (cf. [46]) follows:

$$\sum_{x=-\infty}^{+\infty} \left(\frac{\sin x}{x}\right) = \pi \tag{2.208}$$

$$\sum_{x=-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^2 = \pi \tag{2.209}$$

$$\sum_{x=-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^3 = \frac{4}{3}\,\pi \tag{2.210}$$

$$\sum_{x=-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^4 = \frac{2}{3}\,\pi \tag{2.211}$$

(cf. appendix C).

## 2.12 A bit recursion for $1/\pi$

$$a_0 := \tan(1) \tag{2.212}$$

$$a_{k+1} := \frac{2\,a_k}{1-a_k^2} \tag{2.213}$$

$$b(x) := \begin{cases} 1 & \text{if} \quad x<0 \\ 0 & \text{else} \end{cases} \tag{2.214}$$

then

$$\sum_{k=0}^{\infty} \frac{b(a_k)}{2^{k+1}} = \frac{1}{\pi} \tag{2.215}$$

$$\sum_{k=0}^{\infty} \frac{b(a_k)}{2^{k+1}} = \frac{\arctan(a_0)}{\pi} \tag{2.216}$$

See  and the paper [51].

## 2.13 A self correcting iteration for $\pi$

Use

$$\sin\frac{\pi}{2} = 0 \tag{2.217}$$

therefor the iteration defined by

$$x_0 = \frac{\pi}{2} + \epsilon_0 \tag{2.218}$$

$$x_{k+1} = x_k + \sin x_k \tag{2.219}$$

converges towards $\pi$.
Convergence is of third order: if $x_k = \frac{\pi}{2} + \epsilon_k$ then

$$\epsilon_{k+1} \approx \frac{\epsilon_k^3}{6} \tag{2.220}$$

Of course, this is not an 'efficient' iteration as the computation of a sine function is required. Similar iterations exist for cos() and tan(), see [57].

# Appendix A

# How to build Ramanujan type formulas

NOTE: this section may be detrimental to your health, do not read it.

$$\left(\frac{2\,K}{\pi}(k)\right)^2 = m(k)\,F(\Phi(k)) \tag{A.1}$$

$$= \frac{1}{1+k^2}\,{}_3F_2\left(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;\left(\frac{g^{12}+g^{-12}}{2}\right)^{-2}\right) \tag{A.2}$$

$$= \frac{1}{k'^2-k^2}\,{}_3F_2\left(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;-\left(\frac{G^{12}-G^{-12}}{2}\right)^{-2}\right) \tag{A.3}$$

$$= \frac{1}{\sqrt{1-k^2\,k'^2}}\,{}_3F_2\left(\frac{1}{6},\frac{5}{6},\frac{1}{2};1,1;J^{-1}\right) \tag{A.4}$$

1. Series in $x_N$ $N \geq 3$:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3}\, d_n(N)\, x_N^{2n+1} \tag{A.5}$$

$$x_N := \left(\frac{g_N^{12}+g_N^{-12}}{2}\right)^{-1} \tag{A.6}$$

$$= \frac{4\,k(N)\,k'^2(N)}{\left(1+k^2(N)\right)^2} \tag{A.7}$$

$$d_n := \left(\frac{\alpha(N)\,x_N^{-1}}{1+k^2(N)} - \frac{\sqrt{N}}{4}\,g_N^{-12}\right) + n\,\sqrt{N}\,\left(\frac{g_N^{12}-g_N^{-12}}{2}\right) \tag{A.8}$$

2. Series in $y_N$ $N \geq 4$:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n\,\frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3}\, e_n(N)\, y_N^{2n+1} \tag{A.9}$$

$$y_N := \left(\frac{G_N^{12}-G_N^{-12}}{2}\right)^{-1} \tag{A.10}$$

$$= \frac{4\,k(N)\,k'(N)}{1-\left(2\,k(N)\,k'(N)\right)^2} \tag{A.11}$$

$$e_n := \left( \frac{\alpha(N)\, y_N^{-1}}{k'^2(N) - k^2(N)} + \frac{\sqrt{N}}{2}\, k^2(N)\, G_N^{12} \right) + n\,\sqrt{N} \left( \frac{G_N^{12} + G_N^{-12}}{2} \right) \tag{A.12}$$

3. Series in $J_N^{-1}$ $N \geq 2$:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^3} f_n(N) \left(J_N^{-1/2}\right)^{2n+1} \tag{A.13}$$

$$J_N^{-1} := \frac{27\, G_N^{24}}{\left(4\, G_N^{24} - 1\right)^3} \tag{A.14}$$

$$= \frac{27\, g_N^{24}}{\left(4\, g_N^{24} + 1\right)^3} \tag{A.15}$$

$$f_n := \frac{1}{3\sqrt{3}} \left( \sqrt{N}\, \sqrt{1 - G_N^{-24}} + 2\, \left(\alpha(N) - \sqrt{N}\, k^2(N)\right)\, \left(4G_N^{24} - 1\right) \right) + \tag{A.16}$$

$$+ n\,\sqrt{N}\, \frac{2}{3\sqrt{3}} \left( \left(8G_N^{24} + 1\right)\, \sqrt{1 - G_N^{-24}} \right) \tag{A.17}$$

Elliptic integral of the first kind:

$$K(k) := \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) = \frac{\pi}{2} \sum_{i=0}^{\infty} \left( \frac{(2\,i-1)!!}{2^i\, i!} \right)^2 k^{2i} \tag{A.18}$$

$$= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2\,t^2)}} = \int_0^{\pi/2} \frac{d\Theta}{\sqrt{1 - k^2\, sin^2(\Theta)}} \tag{A.19}$$

$$= \frac{\pi}{2\, AGM(1, k')} \tag{A.20}$$

$$= \frac{\pi}{2} \Theta_3^2(q) \qquad q = e^{-\pi K'(k)/K(k)} \tag{A.21}$$

Elliptic integral of the second kind:

$$E(k) := \frac{\pi}{2}\, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right) = \frac{\pi}{2} \left( 1 - \sum_{i=0}^{\infty} \left( \frac{(2\,i-1)!!}{2^i\, i!} \right)^2 \frac{k^{2i}}{2i-1} \right) \tag{A.22}$$

$$= \int_0^1 \frac{\sqrt{1-k^2\,t^2}}{\sqrt{1-t^2}} dt = \int_0^{\pi/2} \sqrt{1 - k^2\, sin^2(\Theta)}\, d\Theta \tag{A.23}$$

Derivatives:

$$\frac{dK}{dk} = \frac{E - k'^2\, K}{k\, k'} \tag{A.24}$$

$$\frac{dE}{dk} = \frac{E - K}{k} \tag{A.25}$$

Differential equation for K and E:

$$0 = \left(k^3 - k\right) \frac{d^2 y}{dk^2} + \left(3\, k^2 - 1\right) \frac{dy}{dk} + k\, y \tag{A.26}$$

( also satisfied by $AGM(1, k)^{-1}$ and $AGM(1 + k, 1 - k)^{-1}$ )

$$k' := \sqrt{1 - k^2} \tag{A.27}$$

$$K'(k) := K(k') \tag{A.28}$$

$$E'(k) := E(k') \tag{A.29}$$

$$\Theta_2(q) := \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \tag{A.30}$$

$$\Theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \tag{A.31}$$

$$\Theta_4(q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \tag{A.32}$$

$$k(q) := k = \frac{\Theta_2^2(q)}{\Theta_3^2(q)} \tag{A.33}$$

$$k'(q) := k' = \frac{\Theta_4^2(q)}{\Theta_3^2(q)} \tag{A.34}$$

$$q = e^{-\frac{K'(k)}{K(k)}} \tag{A.35}$$

$$E = k'^2 K + k k'^2 \frac{dK}{dk} \tag{A.36}$$

$$E' K + K' E - K' K = \frac{\pi}{2} \tag{A.37}$$

singular value function:

$$k(N): \qquad \frac{K'}{K}(k(N)) = \sqrt{N} \qquad k(N) := \left(\frac{\Theta_2(q)}{\Theta_3(q)}\right)^2, \text{ where } q := e^{-\pi\sqrt{N}} \tag{A.38}$$

$$k(0) = \infty, \quad k(1) = \frac{1}{\sqrt{2}}, \quad k(\infty) = 0 \qquad k(N) \quad k(n)\text{algebraic for } N \text{ rational} \tag{A.39}$$

$$l(N): \qquad \frac{K'}{K}(l(N)) = \frac{L'}{L} = \frac{1}{\sqrt{N}} \qquad l(N) := \left(\frac{\Theta_2(r)}{\Theta_3(r)}\right)^2, \text{ with} \tag{A.40}$$

$$r := e^{-\pi/\sqrt{N}} = q^{1/N} \tag{A.41}$$

$$\frac{K'}{K} = N\frac{L'}{L} \tag{A.42}$$

$$u := k^{1/4} \qquad v := l^{1/4} \tag{A.43}$$

singular value function of the second kind:

$$\alpha(N) := \frac{E'}{K} - \frac{\pi}{4K^2} \quad (\text{where } k := k(N)) \tag{A.44}$$

$$= \sqrt{N}\frac{E'}{K'} - \frac{N\pi}{4K'^2} = \frac{\pi}{4K^2} - \sqrt{N}\left(\frac{E}{K} - 1\right) \tag{A.45}$$

$$= \frac{\pi^{-1} - \sqrt{N}\, 4q\frac{\dot\Theta_4}{\Theta_4}}{\Theta_3^4} \quad (\text{where } q = e^{-\pi\sqrt{N}}) \tag{A.46}$$

$$\alpha(1) = \frac{1}{2}, \quad \alpha(\infty) = \frac{1}{\pi} \tag{A.47}$$

recursions:

$$\alpha(4N) = \frac{4\alpha(N) - 2\sqrt{N}\,k^2(N)}{(1 + k'(N))^2} \tag{A.48}$$

$$= \left(1 + y^2\right)\alpha(4N) - 2\sqrt{N}y \tag{A.49}$$

where

$$y := k(4N) = \frac{1-k'(N)}{1+k'(N)} \tag{A.50}$$

$$\alpha(16N) = (1+y)^4\alpha(N) - 4\sqrt{N}y(1+y+y^2) \tag{A.51}$$

$$\text{where} \qquad y := k(16N) = \left(\frac{1-\sqrt[4]{1-k^2(n)}}{1+\sqrt[4]{1-k^2(n)}}\right)^2 \tag{A.52}$$

$$\alpha(N^{-1}) = \frac{\sqrt{N}-\alpha(N)}{N} \tag{A.53}$$

G,g:

$$G := (2k\,k')^{-1/12}, \quad g := \left(\frac{2k}{k'^2}\right)^{-1/12} \tag{A.54}$$

recursions:

$$9 = \left(1+2\sqrt{2}\frac{G_{9n}^3}{G_n^9}\right)\left(1+2\sqrt{2}\,\frac{G_n^3}{G_{9n}^9}\right) \tag{A.55}$$

$$9 = \left(1+2\sqrt{2}\frac{g_{9n}^3}{g_n^9}\right)\left(1-2\sqrt{2}\,\frac{g_n^3}{g_{9n}^9}\right) \tag{A.56}$$

$$G_{81n}^3 = G_{9n}\,\frac{\sqrt{2}G_{9n}+G_n^3}{\sqrt{2}G_n^3-G_{9n}} \tag{A.57}$$

$$g_{81n}^3 = -g_{9n}\,\frac{\sqrt{2}g_{9n}+g_n^3}{\sqrt{2}g_n^3-g_{9n}} \tag{A.58}$$

$$J := \frac{\left(4\,G^{24}-1\right)^3}{27\,G^{24}} \tag{A.59}$$

$$= \frac{\left(4\,g^{24}+1\right)^3}{27\,g^{24}} \tag{A.60}$$

$$= \frac{4}{27}\,\frac{\left(1-k^2(N)+k^4(N)\right)^3}{k^4(N)\,\left(1-k^2(N)\right)^2} \tag{A.61}$$

$$= \frac{4}{27}\,\frac{\left(1-k^2(N)k'^2(N)\right)^3}{k^4(N)k'^4(N)} \tag{A.62}$$

$$j := 1728\,J \tag{A.63}$$

multiplier:

$$M_N(l,k) := \frac{\Theta_3^2(q)}{\Theta_3^2(q^{1/N})} = \frac{K}{L} \tag{A.64}$$

$$\frac{1}{\pi} = \sqrt{N}\,k\,k'^2\,\frac{4K\dot{K}}{\pi^2} + \left(\alpha(N)-\sqrt{N}\,k^2\right)\,\frac{4K^2}{\pi^2} \tag{A.65}$$

$$\frac{1}{K} = \sqrt{N}\, k\, k'^2\, \frac{4\dot{K}}{\pi} + \left(\alpha(N) - \sqrt{N}\, k^2\right) \frac{4K}{\pi} \tag{A.66}$$

$$\Theta_1(z,q) := \Theta_1(z,t) := 2q^{1/4}\sin z - 2q^{9/4}\sin 3z + 2q^{25/4}\sin 5z - 2q^{49/4}\sin 7z + \ldots \tag{A.67}$$
$$\Theta_2(z,q) := \Theta_2(z,t) := 2q^{1/4}\cos z + 2q^{9/4}\cos 3z + 2q^{25/4}\cos 5z + 2q^{49/4}\cos 7z + \ldots \tag{A.68}$$
$$\Theta_3(z,q) := \Theta_3(z,t) := 1 + 2q\cos 2z + 2q^4\cos 4z + 2q^9\cos 6z + 2q^{16}\cos 8z + \ldots \tag{A.69}$$
$$\Theta_4(z,q) := \Theta_4(z,t) := 1 - 2q\cos 2z + 2q^4\cos 4z - 2q^9\cos 6z + 2q^{16}\cos 8z - \ldots \tag{A.70}$$
$$\text{where} \quad q := e^{\pi i t} \tag{A.71}$$

$$\Theta_1(z,q) := \Theta_1(z,t) := \sum_{n\in Z}(-1)^n q^{((2n+1)/2)^2}\sin{(2n+1)z} \tag{A.72}$$
$$\Theta_2(z,q) := \Theta_2(z,t) := \sum_{n\in Z} q^{((2n+1)/2)^2}\cos{(2n+1)z} \tag{A.73}$$
$$\Theta_3(z,q) := \Theta_3(z,t) := \sum_{n\in Z} q^{(2n)^2}\cos{(2n)z} \tag{A.74}$$
$$\Theta_4(z,q) := \Theta_4(z,t) := \sum_{n\in Z}(-1)^n q^{(2n)^2}\cos{(2n)z} \tag{A.75}$$

$$\Theta_1(z) = -\Theta_2(z+\frac{\pi}{2}) = -iM\Theta_3(z+\frac{\pi}{2}+\frac{\pi t}{2}) = -iM\Theta_4(z+\frac{\pi t}{2}) \tag{A.76}$$
$$\Theta_2(z) = M\Theta_3(z+\frac{\pi t}{2}) = M\Theta_4(z+\frac{\pi}{2}+\frac{\pi t}{2}) = \Theta_1(z+\frac{\pi}{2}) \tag{A.77}$$
$$\Theta_3(z) = \Theta_4(z+\frac{\pi}{2}) = M\Theta_1(z+\frac{\pi}{2}+\frac{\pi t}{2}) = M\Theta_2(z+\frac{\pi t}{2}) \tag{A.78}$$
$$\Theta_4(z) = -iM\Theta_1(z+\frac{\pi t}{2}) = iM\Theta_2(z+\frac{\pi}{2}+\frac{\pi t}{2}) = \Theta_3(z+\frac{\pi}{2}) \tag{A.79}$$
$$\text{where} \quad M := q^{1/4}e^{iz} \tag{A.80}$$

$$\Theta_1(z) = \Theta_1(z+\pi) = -qe^{2iz}\Theta_1(z+\pi t) \tag{A.81}$$
$$\Theta_2(z) = \Theta_2(z+\pi) = +qe^{2iz}\Theta_2(z+\pi t) \tag{A.82}$$
$$\Theta_3(z) = \Theta_3(z+\pi) = +qe^{2iz}\Theta_3(z+\pi t) \tag{A.83}$$
$$\Theta_4(z) = \Theta_4(z+\pi) = -qe^{2iz}\Theta_4(z+\pi t) \tag{A.84}$$

$$\frac{\partial^2\Theta(z,t)}{\partial z^2} = \frac{4i}{\pi}\frac{\partial\Theta(z,t)}{\partial t} \tag{A.85}$$
$$\Theta_3^2 = 1 + 4\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}} \tag{A.86}$$

Let

$$T(x,q) := \sum_{-\infty}^{\infty} x^n q^{n^2} \tag{A.87}$$

then:

$$T(x,q^a)T(y,q^b) = \sum_{m,n} x^m y^n q^{am^2+bn^2} \quad (A.88)$$

$$= \sum_{k=0}^{a+b-1} y^k q^{bk^2} T(xyq^{2bk}, q^{a+b}) T(y^a x^{-b} q^{2abk}, q^{ab(a+b)}) \quad (A.89)$$

$$Q_0 := \prod_{n=1}^{\infty} 1 - q^{2n} \quad (A.90)$$

$$Q_1 := \prod_{n=1}^{\infty} 1 + q^{2n} \quad (A.91)$$

$$Q_2 := \prod_{n=1}^{\infty} 1 + q^{2n-1} \quad (A.92)$$

$$Q_3 := \prod_{n=1}^{\infty} 1 - q^{2n-1} \quad (A.93)$$

$$Q_0 Q_1 = Q_0(q^2) \quad (A.94)$$

$$Q_0 Q_3 = Q_0(q^{1/2}) \quad (A.95)$$

$$Q_2 Q_3 = Q_3(q^2) \quad (A.96)$$

$$Q_1 Q_2 = Q_1(q^{1/2}) \quad (A.97)$$

$$\sum_{n=-\infty}^{\infty} (\pm 1)^n q^{kn^2+ln} = \prod_{n=0}^{\infty} (1 \pm q^{2kn+k-l})(1 \pm q^{2kn+k+l})(1 - q^{2kn+2k}) \quad (A.98)$$

$$e.g. :1 = Q_1 Q_2 Q_3 \quad (A.99)$$

$$\Theta_3(q) = Q_0 Q_2^2 \qquad \Theta_4(q) = Q_0 Q_3^2 \qquad \Theta_2(q) = 2q^{1/4} Q_0 Q_1^2 \quad (A.100)$$

(A.101)

$k = 3/2$ and $l = 1/2$ gives: (A.102)

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{-\infty}^{\infty} (-1)^n q^{(3n+1)n/2} \quad (A.103)$$

triple-product identity:

$$\prod_{n=1}^{\infty} (1 - q^{2n})(1 + xq^{2n-1})(1 + x^{-1} q^{2n-1}) = \sum_{n=-\infty}^{\infty} x^n q^{n^2} \quad (A.104)$$

quintuple-product identity:

$$\prod_{n=1}^{\infty} (1 - q^n)(1 - xq^n)(1 - x^{-1} q^{n-1})(1 - x^2 q^{2n-1})(1 - x^{-2} q^{2n-1}) \quad (A.105)$$

$$= \sum_{n=-\infty}^{\infty} (x^{3n} - x^{-3n-1}) q^{n(3n+1)/2} \quad (A.106)$$

equivalent:

$$\prod_{n=1}^{\infty} (1 - q^{2n})(1 - xq^{2n-1})(1 - x^{-1} q^{2n-1})(1 - x^2 q^{4n-4})(1 - x^{-2} q^{4n-4}) \quad (A.107)$$

$$= \sum_{n=-\infty}^{\infty} q^{3n^2-2n}[(x^{3n}+x^{-3n})-(x^{3n-2}+x^{-(3n-2)})] \qquad (A.108)$$

# Appendix B

# More arctan formulas

In what follows let $\mathbf{A}\frac{1}{x}$ denote $\arctan\frac{1}{x}$.

$$\arctan\frac{1}{x} = \arctan\frac{1}{x+d} + \arctan\frac{1}{x+\frac{x^2+1}{d}} \tag{B.1}$$

$$\arctan\frac{1}{x} + \arctan\frac{1}{y} = \arctan\frac{1}{\frac{x\,y-1}{x+y}} \tag{B.2}$$

$$\arctan\frac{1}{z} = \arctan\frac{1}{x_1} + \arctan\frac{1}{x_2} + ... + \arctan\frac{1}{x_n} \tag{B.3}$$

$$\iff z = \frac{Re(\prod_{k=1}^{n}(x_k+i))}{Im(\prod_{k=1}^{n}(x_k+i))} \tag{B.4}$$

$$\iff (z^2+1) = \frac{\prod_{k=1}^{n}(x_k^2+1)}{\left(Im(\prod_{k=1}^{n}(x_k+i))\right)^2} \tag{B.5}$$

$$\arctan\frac{1}{x} = \arctan\frac{1}{y} + \arctan\frac{1}{z} \tag{B.6}$$

$$\iff (x^2+1) = \frac{(y^2+1)(z^2+1)}{(y+z)^2} = (y-x)(z-x) \tag{B.7}$$

$$\frac{k\,\pi}{4} = \arctan\frac{b_1}{a_1} + \arctan\frac{b_2}{a_2} + ... + \arctan\frac{b_n}{a_n} \tag{B.8}$$

$$\iff (a_1+i\,b_1)(a_2+i\,b_2)...(a_n+i\,b_n)(1-i)^k \in \mathbb{R} \tag{B.9}$$

$$\frac{k\,\pi}{4} = \arctan\frac{1}{a_1} + \arctan\frac{1}{a_2} + ... + \arctan\frac{1}{a_n} \tag{B.10}$$

$$\iff (a_1+i)(a_2+i)...(a_n+i)(1-i)^k \in \mathbb{R} \tag{B.11}$$

$$\frac{k\,\pi}{4} = m_1\arctan\frac{1}{a_1} + m_2\arctan\frac{1}{a_2} + ... + m_n\arctan\frac{1}{a_n} \tag{B.12}$$

$$\iff (a_1+i)^{m_1}(a_2+i)^{m_2}\dots(a_n+i)^{m_n}(1-i)^k \in \mathbb{R} \tag{B.13}$$

Let

$$a > b, \qquad gcd(a,b) = 1, \quad a\,\alpha + b\,\beta = 1 \tag{B.14}$$

then

$$\mathbf{A}\frac{1}{a}b = \mathbf{A}\frac{1}{-\beta}\alpha + \arctan\frac{1}{a\,\beta - b\,\alpha} \tag{B.15}$$

because if

$$r := a\,\beta - b\,\alpha \tag{B.16}$$

$$\arctan\frac{1}{z} := \arctan\frac{1}{a/b} + \arctan\frac{1}{r} \tag{B.17}$$

then

$$z = \frac{(a/b)\,r - 1}{(a/b) + r} \tag{B.18}$$

$$= \frac{a(\alpha\,b - \beta\,a) - b}{a + b(\alpha\,b - \beta\,a)} \tag{B.19}$$

$$= \frac{a(\alpha\,b - \beta\,a) - b(1)}{a(1) + b(\alpha\,b - \beta\,a)} \tag{B.20}$$

$$= \frac{a(\alpha\,b - \beta\,a) - b(\alpha\,a + \beta\,b)}{a(\alpha\,a + \beta\,b) + b(\alpha\,b - \beta\,a)} \tag{B.21}$$

$$= \frac{-a^2\,\beta - b^2\,\beta}{\alpha\,a^2 + \alpha\,b^2} \tag{B.22}$$

$$= \frac{-\beta}{\alpha} \tag{B.23}$$

$$\mathbf{A}\frac{1}{x} = \mathbf{A}\frac{1}{x+1} + \mathbf{A}\frac{1}{x^2+x+1} \tag{B.24}$$

$$\frac{\pi}{4} = \mathbf{A}\frac{1}{1} = \sum_{x=1}^{\infty}\mathbf{A}\frac{1}{x^2+x+1} = \tag{B.25}$$

$$= \mathbf{A}\frac{1}{3} + \mathbf{A}\frac{1}{7} + \mathbf{A}\frac{1}{13} + \mathbf{A}\frac{1}{21} + \mathbf{A}\frac{1}{31} + \mathbf{A}\frac{1}{43} + \mathbf{A}\frac{1}{57} + ... \tag{B.26}$$

$$\mathbf{A}\frac{1}{2x+1} = \mathbf{A}\frac{1}{2x+3} + \mathbf{A}\frac{1}{2x^2+4x+2} \tag{B.27}$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{2n^2} \tag{B.28}$$

$$= \mathbf{A}\frac{1}{2} + \mathbf{A}\frac{1}{8} + \mathbf{A}\frac{1}{18} + \mathbf{A}\frac{1}{50} + \mathbf{A}\frac{1}{98} + \mathbf{A}\frac{1}{128} + \ldots \tag{B.29}$$

define

$$
\begin{aligned}
jt(a_0,a_1,m_1,m_2,0) &:= a0 && \text{(B.30)}\\
jt(a_0,a_1,m_1,m_2,1) &:= a1 && \text{(B.31)}\\
jt(a_0,a_1,m_1,m_2,n) &:= m_1\, jt(a_0,a_1,m_1,m_2,n-1) + m_2\, jt(a_0,a_1,m_1,m_2,n-2) && \text{(B.32)}
\end{aligned}
$$

then

$$
\begin{aligned}
jt(a_0,a_1,m_1,m_2,n) &= C_1\, x_1^n + C_2\, x_2^n \qquad \text{where:} && \text{(B.33)}\\
x_{1,2}^2 &= m_1\, x_{1,2} + m_2, \quad C_1 + C_2 = a_0, \quad C_1\, x_1 + C_2\, x_2 = a_1 && \text{(B.34)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\pi}{8} &= \sum_{n=1}^{\infty} (-1)^n \mathbf{A}\frac{1}{jt(0,1,2,1,2n)} = \mathbf{A}\frac{1}{2} - \mathbf{A}\frac{1}{12} + \mathbf{A}\frac{1}{70} - \mathbf{A}\frac{1}{408} + \mathbf{A}\frac{1}{2378} - + \dots && \text{(B.35)}\\
\frac{\pi}{8} &= \sum_{n=1}^{\infty} \mathbf{A}\frac{1}{jt(1,1,2,1,2n)} = && \text{(B.36)}\\
&= \sum_{n=1}^{\infty} (-1)^n\, \mathbf{A}\frac{1}{(1-\sqrt{2})^{2n} + (1+\sqrt{2})^{2n}} && \text{(B.37)}\\
&= \mathbf{A}\frac{1}{3} + \mathbf{A}\frac{1}{17} + \mathbf{A}\frac{1}{99} + \mathbf{A}\frac{1}{577} + \mathbf{A}\frac{1}{3363} + \dots && \text{(B.38)}
\end{aligned}
$$

$$
\begin{aligned}
f(n) &:= jt(0,1,2,1,2n)\, jt(0,1,2,1,2n+1)^2 + jt(1,1,2,1,2n+1) && \text{(B.39)}\\
&\quad \text{eg.} \quad f(0) = 2\cdot 5^2 + 7 = 57, \quad f(1) = 12\cdot 29^2 + 41 = 10133, \quad \dots && \text{(B.40)}\\
\frac{\pi}{8} &= \sum_{n=1}^{\infty} 3^n\, \mathbf{A}\frac{1}{jt(0,1,2,1,2n+1)} - \sum_{n=1}^{\infty} 3^n\, \mathbf{A}\frac{1}{f(n)} && \text{(B.41)}\\
&= \mathbf{A}\frac{1}{5} + 3\left(\mathbf{A}\frac{1}{29} + 3\left(\mathbf{A}\frac{1}{169} + 3\left(\mathbf{A}\frac{1}{5741} + \dots\right)\right)\right) - && \text{(B.42)}\\
&- \left(\mathbf{A}\frac{1}{57} + 3\left(\mathbf{A}\frac{1}{10133} + 3\left(\mathbf{A}\frac{1}{1999509} + \dots\right)\right)\right)\\
\mathbf{A}\frac{1}{P_n} &= 4\,\mathbf{A}\frac{1}{Q_{n+2}} + \mathbf{A}\frac{1}{P_{n+4}} - 2\,\mathbf{A}\frac{1}{q(Q_{n+2})} (\text{n odd}) && \text{(B.43)}\\
\frac{\pi}{4} &= \mathbf{A}\frac{1}{P_1} = 4\left(\mathbf{A}\frac{1}{5} + \mathbf{A}\frac{1}{169} + \mathbf{A}\frac{1}{5741} + \dots\right) - && \text{(B.44)}\\
&- 2\left(\mathbf{A}\frac{1}{q(5)} + \mathbf{A}\frac{1}{q(169)} + \mathbf{A}\frac{1}{q(5741)} + \dots\right)
\end{aligned}
$$

$$
\begin{aligned}
\text{with } g(n) &:= jt(0,1,2,1,n+3)\, jt(0,1,2,1,n+2)^2 + jt(1,1,2,1,n+2) && \text{(B.45)}\\
\mathbf{A}\frac{1}{jt(1,1,2,1,n)} &= 5\,\mathbf{A}\frac{1}{jt(1,1,2,1,n+2)} + 2\,\mathbf{A}\frac{1}{jt(1,1,2,1,n+3)} && \text{(B.46)}\\
&- \mathbf{A}\frac{1}{q(jt(0,1,2,1,n+2))} + 2\,\mathbf{A}\frac{1}{g(n)} \quad (\text{n odd})\\
\frac{\pi}{4} &= 2\,\mathbf{A}\frac{1}{3} + \mathbf{A}\frac{1}{7} = 2\,\mathbf{A}\frac{1}{3} + 2\,\mathbf{A}\frac{1}{17} + \mathbf{A}\frac{1}{41} = && \text{(B.47)}
\end{aligned}
$$

$$= \left(12\,\mathbf{A}\frac{1}{17}+5\,\mathbf{A}\frac{1}{41}\right)-2\,\mathbf{A}\frac{1}{70}-4\,\mathbf{A}\frac{1}{307} \tag{B.48}$$

$$= \left(70\,\mathbf{A}\frac{1}{99}+29\,\mathbf{A}\frac{1}{239}\right)-2\,\mathbf{A}\frac{1}{70}-4\,\mathbf{A}\frac{1}{307}-12\,\mathbf{A}\frac{1}{12238}-24\,\mathbf{A}\frac{1}{58911} \tag{B.49}$$

$$= \left(408\,\mathbf{A}\frac{1}{577}+169\,\mathbf{A}\frac{1}{1393}\right)- \tag{B.50}$$

$$- \quad (2\,\mathbf{A}\frac{1}{70}+4\,\mathbf{A}\frac{1}{307}+12\,\mathbf{A}\frac{1}{12238}+24\,\mathbf{A}\frac{1}{58911}+70\,\mathbf{A}\frac{1}{...}+140\,\mathbf{A}\frac{1}{...})$$

$$\frac{\pi}{4} = 2\,(\sqrt{2}-1)-\sum_{n=1}^{\infty}\left(jt(0,1,2,1,2n)\,\mathbf{A}\frac{1}{q(jt(0,1,2,1,2n+1))}+\right. \tag{B.51}$$

$$+ \quad \left. 2\,jt(0,1,2,1,2n)\,\mathbf{A}\frac{1}{g(2n+1)}\right)$$

$$\frac{\pi}{4} = 3\sum_{n=1}^{\infty}\mathbf{A}\frac{1}{jt(1,1,3,1,2n)}-\sum_{n=1}^{\infty}\mathbf{A}\frac{1}{jt(1,1,3,1,2n)} \tag{B.52}$$

$$= 3\left(\mathbf{A}\frac{1}{4}+\mathbf{A}\frac{1}{43}+\mathbf{A}\frac{1}{469}+\mathbf{A}\frac{1}{5116}+\ldots\right) \tag{B.53}$$

$$- \quad \left(\mathbf{A}\frac{1}{q(4)}+\mathbf{A}\frac{1}{q(43)}+\mathbf{A}\frac{1}{q(469)}+\mathbf{A}\frac{1}{q(5116)}+\ldots\right)$$

$$= 3\left(\mathbf{A}\frac{1}{4}+\mathbf{A}\frac{1}{43}+\mathbf{A}\frac{1}{469}+\mathbf{A}\frac{1}{5116}+\ldots\right) \tag{B.54}$$

$$- \quad \left(\mathbf{A}\frac{1}{38}+\mathbf{A}\frac{1}{39818}+\mathbf{A}\frac{1}{51581558}+\ldots\right)$$

$$q(n) := \frac{3n+n^3}{2} \tag{B.55}$$

$$F_n := jt(0,1,1,1,n) \tag{B.56}$$

$$= Fibonacci(n) = 1,1,2,3,5,8,13,21,34,55,... \tag{B.57}$$

$$\mathbf{A}\frac{1}{x} = 2\,\mathbf{A}\frac{1}{2x}-\mathbf{A}\frac{1}{3x+4x^3} \tag{B.58}$$

$$\frac{\pi}{4} = 1-\sum_{n=0}^{\infty}2^n\,\mathbf{A}\frac{1}{3\,2^n+4\,2^{3n}} = 1-\sum_{n=0}^{\infty}2^n\,\mathbf{A}\frac{1}{q(2^{n+1})} \tag{B.59}$$

$$= 1-\left(\mathbf{A}\frac{1}{7}+2\,\mathbf{A}\frac{1}{38}+4\,\mathbf{A}\frac{1}{268}+8\,\mathbf{A}\frac{1}{2072}+16\,\mathbf{A}\frac{1}{16432}+\ldots\right) \tag{B.60}$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{F_{2n+1}} \tag{B.61}$$

$$= \mathbf{A}\frac{1}{2}+\mathbf{A}\frac{1}{5}+\mathbf{A}\frac{1}{13}+\mathbf{A}\frac{1}{34}+\mathbf{A}\frac{1}{89}+\mathbf{A}\frac{1}{233}+\ldots$$

$$\mathbf{A}\frac{1}{F_{2n}} = 3\,\mathbf{A}\frac{1}{F_{2n+2}}-\mathbf{A}\frac{1}{F_{2n+4}}-\mathbf{A}\frac{1}{q(F_{2n+2})} \Rightarrow \tag{B.62}$$

$$\frac{\pi}{4} = \frac{3\sqrt{5}-5}{2}-\sum_{n=1}^{\infty}F_{2n}\,\mathbf{A}\frac{1}{q(F_{2n+2})} \tag{B.63}$$

$$= \frac{3\sqrt{5}-5}{2}-\left(1\,\mathbf{A}\frac{1}{18}+3\,\mathbf{A}\frac{1}{268}+8\,\mathbf{A}\frac{1}{4662}+21\,\mathbf{A}\frac{1}{83270}+\ldots\right) \tag{B.64}$$

$$\begin{aligned}
\mathbf{A}\frac{1}{x+\sqrt{x^2-4}} &= \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{x\,jt(0,1,x,-1,n)^2} && \text{(B.65)}\\
\text{for } x=2: & \\
\mathbf{A}\frac{1}{2+\sqrt{2^2-4}} &= \mathbf{A}\frac{1}{1} && \text{(B.66)}\\
&= \frac{\pi}{4}=\sum_{n=1}^{\infty}\mathbf{A}\frac{1}{2\,n^2} && \text{(B.67)}\\
\text{for } x=4: & \\
\mathbf{A}\frac{1}{4+\sqrt{4^2-4}} &= \mathbf{A}\frac{1}{2+\sqrt{3}} && \text{(B.68)}\\
&= \frac{\pi}{12}=\sum_{n=1}^{\infty}\mathbf{A}\frac{1}{4\,jt(0,1,4,-1,n)^2} && \text{(B.69)}\\
&= \mathbf{A}\frac{1}{4}+\mathbf{A}\frac{1}{64}+\mathbf{A}\frac{1}{900}+\mathbf{A}\frac{1}{12544}+\mathbf{A}\frac{1}{174724}+\mathbf{A}\frac{1}{2433600}+... && \text{(B.70)}\\
\mathbf{A}\frac{1}{2x^2} &= 2\,\mathbf{A}\frac{1}{4x^2-2x+1}-\mathbf{A}\frac{1}{4x^2+2x+1}\Rightarrow && \text{(B.71)}\\
\frac{\pi}{4} &= \sum_{n=1}^{\infty}\left[\mathbf{A}\frac{1}{4n^2-2n+1}+\mathbf{A}\frac{1}{4n^2+2n+1}\right] && \text{(B.72)}\\
\mathbf{A}\frac{1}{2x^2} &= 2\,\mathbf{A}\frac{1}{4x^2+2x+1}-\mathbf{A}\frac{1}{2x^3+x}+\mathbf{A}\frac{1}{4x^3+3x} && \text{(B.73)}\\
\mathbf{A}\frac{1}{2x^2} &= 2\,\mathbf{A}\frac{1}{4x^2-2x+1}+\mathbf{A}\frac{1}{2x^3+x}-\mathbf{A}\frac{1}{4x^3+3x}\Rightarrow && \text{(B.74)}\\
\frac{\pi}{4} &= \sum_{n=1}^{\infty}\left[\mathbf{A}\frac{1}{4n^2+2n+1}+\mathbf{A}\frac{1}{2n^3+n}-\mathbf{A}\frac{1}{4n^3+3n}\right] && \text{(B.75)}\\
\frac{\pi}{6} &= \sum_{n=0}^{\infty}\mathbf{A}\frac{1}{2\,jt(1,3,4,-1,n)^2} && \text{(B.76)}\\
&= \mathbf{A}\frac{1}{2}+\mathbf{A}\frac{1}{18}+\mathbf{A}\frac{1}{242}+\mathbf{A}\frac{1}{3362}+\mathbf{A}\frac{1}{46818}+\mathbf{A}\frac{1}{652082}+... && \text{(B.77)}
\end{aligned}$$

$$\begin{aligned}
\frac{\pi}{4} &= \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{2n^2}=\text{(by changing the order of summation)} && \text{(B.78)}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)2^{2n+1}}\zeta(4n+2) && \text{(B.79)}
\end{aligned}$$

# Appendix C

# Sums and integrals of $sin(x)/x$

For $k \in \mathbb{Z}$

$$\sum_{x=-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^k = \int_{-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^k dx \tag{C.1}$$

$$= r_k\, \pi \tag{C.2}$$

where $r_k$ is a rational number. The equality of the integral with the sum follows from the Euler-MacLaurin summation formula (cf. [7] section 9.5 and [8] section 0.7.1.3). That the sums are equal to $\pi$ times a rational follows from the Poisson summation formula (cf. [46]) and that

$$\frac{sin(x)}{x} = \int_{-\infty}^{+\infty} q(x) \tag{C.3}$$

where

$$q(x) := \begin{cases} \pi & \text{if } -\frac{1}{2} < x < +\frac{1}{2} \\ 0 & \text{else} \end{cases} \tag{C.4}$$

(i.e. $\frac{sin(x)}{x}$ is the Fourier transform of $q(x)$). $\left(\frac{sin(x)}{x}\right)^k$ is the fourier transform of the $k$-1 – fold self convolution of $q(x)$.

The first values of $r_k$ are:

| $k$ | $r_k$ |
|---|---|
| 1 | 1 |
| 2 | 1 |
| 3 | 3 / 4 |
| 4 | 2 / 3 |
| 5 | 115 / 192 |
| 6 | 11 / 20 |
| 7 | 5887 / 11520 |
| 8 | 151 / 315 |
| 9 | 259723 / 573440 |
| 10 | 15619 / 36288 |
| 11 | 381773117 / 928972800 |
| 12 | 655177 / 1663200 |
| 13 | 20646903199 / 54499737600 |
| 14 | 27085381 / 74131200 |
| 15 | 467168310097 / 1322526965760 |

Paul Zimmerman found an order 2 recursion for the more general integral

$$c(k,j) \quad = \quad \int_{-\infty}^{\infty} \frac{\sin(x)^k}{x^j} \tag{C.5}$$

It is

$$c(k,j) \quad = \quad k\,\frac{(k-1)\,c(k-2,j-2) - k\,c(k,j-2)}{(j-1)\,(j-2)} \tag{C.6}$$

with

$$c(k,j) \quad = \quad \binom{k-j}{k/2-j/2}\frac{\pi}{2^{k-j}} \quad \text{for} \quad j=1,2 \tag{C.7}$$

# Appendix D

# The magic sumalt algorithm

The following algorithm is due to Cohen, , see [81].

Pseudo code to compute an estimate of $\sum_{k=0}^{\infty} x_k$ using the first $n$ summands. The $x_k$ summands are expected in `x[0,1,...,n-1]`.

```
function sumalt(x[],n)
{
    d := (3+sqrt(8))^n
    d := (d+1/d)/2

    b := 1
    c := d
    s := 0

    for k:=0 to n-1
    {
        c := c - b
        s := s + c * x[k]
        b := b * (2*(n+k)*(n-k)) / ((2*k+1)*(k+1))
    }

    return s/d
}
```

With alternating sums the accuracy of the estimate will be $(3+\sqrt{8})^{-n} \approx 5.82^{-n}$.

As an example let us explicitely write down the estimate for the $4 \cdot \arctan(1)$ using the first 8 summands:

$$\begin{aligned} \pi &\approx 4 \cdot \left( \frac{665856}{1} - \frac{665728}{3} + \frac{663040}{5} - \frac{641536}{7} \right. \\ &+ \left. \frac{557056}{9} - \frac{376832}{11} + \frac{163840}{13} - \frac{32768}{15} \right) / 665857 \\ &= 4\,\frac{3365266048}{4284789795} \\ &= 3.141592665... \end{aligned} \tag{D.1}$$

This gives already 7 correct digits of $\pi$, whereas the first 8 'unmassaged' terms of the arctan series give 3.017.... Note that all the values occuring in the computation are integers.

An alternative computation avoids the computation of $(3+\sqrt{8})^n$:

```
function sumalt(x[],n)
{
    b := 2**(2*n-1)
    c := b
    s := 0

    for k:=n-1 to 0 step -1
    {
        s := s + c * x[k]
        b := b * ((2*k+1)*(k+1)) / (2*(n+k)*(n-k))
        c := c + b
    }

    return s/c
}
```

Pseudo code to compute an estimate of $\sum_{k=0}^{\infty} x_k$ using the first $n$ partial sums. The partial sums $p_k = \sum_{j=0}^{k} x_j$ are expected in `p[0,1,...,n-1]`.

```
function sumalt(p[],n)
{
    d := (3+sqrt(8))^n
    d := (d+1/d)/2

    b := 1
    c := d
    s := 0

    for k:=0 to n-1
    {
        s := s + b * p[k]
        b := b * (2*(n+k)*(n-k)) / ((2*k+1)*(k+1))
    }

    return s/d
}
```

The alternative computation that avoids the computation of $(3+\sqrt{8})^n$ is:

```
function sumalt(p[],n)
{
    b := 2**(2*n-1)
    s := 0

    for k:=n-1 to 0 step -1
    {
        s := s + b * p[k]
        b := b * ((2*k+1)*(k+1)) / (2*(n+k)*(n-k))
    }

    return s/c
}
```

# Appendix E

# Continued fractions

Set

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ldots}}}} \tag{E.1}$$

For $k > 0$ let $\frac{p_k}{q_k}$ be the value of the above fraction if $a_{k+1}$ is set to zero (set $\frac{p_{-1}}{q_{-1}} := \frac{1}{0}$ and $\frac{p_0}{q_0} := \frac{b_0}{1}$). Then

$$p_k = b_k\, p_{k-1} + a_k\, p_{k-2} \tag{E.2}$$

$$q_k = b_k\, q_{k-1} + a_k\, q_{k-2} \tag{E.3}$$

(Simple continued fractions are those with $a_k = 1\ \forall k$).
Pseudo code for a procedure that computes the $p_k, q_k \quad k = -1 \ldots n$ of a continued fraction :

```
procedure ratios_from_contfrac(a[0..n], b[0..n], n, p[-1..n], q[-1..n])
{
    p[-1] := 1
    q[-1] := 0

    p[0] := b[0]
    q[0] := 1

    for k:=1 to n
    {
        p[k] := b[k] * p[k-1] + a[k] * p[k-2]
        q[k] := b[k] * q[k-1] + a[k] * q[k-2]
    }
}
```

Pseudo code for a procedure that fills the first `n` terms of the simple continued fraction of (the floating point number) `x` into the array `cf[]`:

```
procedure continued_fraction(x, n, cf[0..n-1])
{
    for k:=0 to n-1
    {
        xi := floor(x)
```

```
        cf[k] := xi
        x := 1 / (x-xi)
    }
}
```

Pseudo code for a function that computes the numerical value of a number `x` from (the leading `n` terms of) its simple continued fraction representation:

```
function number_from_contfrac(cf[0..n-1], n)
{
    x := cf[n-1]

    for k:=n-2 to 0 step -1
    {
        x := 1/x + cf[k]
    }

    return x
}
```

(cf. [58], [59], [7], [8]).

# Appendix F

# A modulo multiplication trick

The following trick allows easy multiplication modulo some modulus that has more bits than a halfword. It uses the fact that integer multiplication gets the least significant bits of the result whereas float multiplication gets most significant bits of the result.

The C-code given here assumes that you have 64 bit integer types `sint64` (signed) and `uint64` (unsigned) and a floating point type with 64 bit mantissa, `float64` (typically `long double`).

```
sint64  modulus;
float64 invmod=(float64)1.0/modulus;

sint64 mul(const sint64 a, const sint64 b)
{
    sint64 quot = (sint64)((float64)a*(float64)b*invmodulus+(float64)0.5);

    uint64 ab = (uint64)a*(uint64)b;
    uint64 mq = (uint64)modulus*(uint64)quot;

    sint64 rem = (sint64)(ab-mq);

    if ( rem<0 )  // correction
    {
        rem += modulus;
        quot--;
    }

    // quot == (a*b)/modulus
    return rem;  // rem == (a*b) % modulus
}
```

The code works if $0 <=$ `a, b` $<$ modulus $< 2^{63}$. Note that for fixed modulus the division for the inverse modulus `invmod` needs only be done once, so the routine avoids any division.

I found this trick in a documentation file (the file 'PROJECTS' in version 2.02) of the gmp package ([55]) where it is ascribed to Peter Montgomery.

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