Present Status of $g-2$ of Electron and Muon

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Abstract. The current status of the theory of the magnetic moment anomaly of the electron and muon is reviewed with particular emphasis on the on-going effort to reduce the statistical and non-statistical uncertainties in the numerical evaluation of the QED contribution.

1 Introduction

In the University-of-Washington experiment [1] the anomalous magnetic moment is measured for an electron bound in a Penning trap. Such a bound state may be regarded as an exotic atom (which they named *geonium*) in which electron confinement is achieved by a homogeneous magnetic field and an electric quadrupole field instead of the Coulomb potential. The muon storage ring may also be regarded as an exotic bound state in highly excited Landau orbits.

In both cases, the magnetic field is so weak that the binding effect can be ignored and leptons can be regarded as free to a high degree. This enables us to avoid complications associated with the calculation of relativistic bound states and go to high orders in QED (quantum electrodynamics) , and explore the strong and weak interaction effects. As a matter of fact, the magnetic moment anomaly is the only calculable property of leptons in free space since other basic observables such as masses and charges are not calculable and must be treated as external parameters within the context of QED (or, more precisely, the Standard Model). This means that the lepton $g-2$ is the ideal quantity to put the theory of Standard Model to the most stringent experimental test.

I will discuss the current status of theoretical work on the magnetic moment anomalies of the electron and muon, with a particular emphasis on the on-going effort to reduce substantially the statistical and non-statistical uncertainties generated by the adaptive-iterative Monte-Carlo integration routine VEGAS [2] in the numerical evaluation of the QED contribution.

2 Electron Magnetic Moment Anomaly

Breit was the first to suggest [3] that the unanticipated results of hyperfine measurements [4,5] may be explained if the electron g value deviates slightly from 2, the Dirac value. This observation was soon confirmed by the experiment of Kusch and Foley [6]. Together with the discovery of the Lamb shift in the spectrum of hydrogen atom , this provided a timely stimulus for the renormalization

theory of quantum electrodynamics (QED) , which was just being developed. Schwinger demonstrated the power of QED by calculating the electron magnetic moment anomaly $a_e \equiv (g_e - 2)/2$ in the second-order covariant perturbation theory [7].

This was the beginning of a long series of measurements and theoretical calculations over 40 years in which the precision has been improved from 10^{-3} to 10^{-12}. The latest results for the electron and positron anomalies obtained in Penning trap experiments are [1]

$$\begin{aligned} a_{e^-}(\text{exp}) &= 1\ 159\ 652\ 188.4\ (4.3) \times 10^{-12}, \\ a_{e^+}(\text{exp}) &= 1\ 159\ 652\ 187.9\ (4.3) \times 10^{-12}, \end{aligned} \tag{1}$$

where e^- and e^+ refer to the electron and the positron, respectively. The numbers within parentheses stand for measurement uncertainties of $\pm\ 4.3 \times 10^{-12}$, which consists of several parts: the statistical error of 0.62×10^{-12}, the systematic error of 1.3×10^{-12} due to the uncertainty in a residual microwave power shift, and a large uncertainty of 4×10^{-12} assigned to a potential cavity-mode shift. This last error arises from a shift in the cyclotron frequency of the electron associated with image charges induced in the metallic Penning trap, an effect which depends on the cavity frequency modes and on the electron cyclotron frequency [8].

Studies to improve the experimental precision for a_e focus on the understanding and control of this cavity influence on the cyclotron frequency. One approach is to reduce the Q of the cavity [9]. In another attempt Mittleman *et al*. produced and studied a many-electron (kiloelectron) cluster in the trap, which magnifies the shift of the cyclotron frequency [10]. Gabrielse *et al*. are studying the use of a cylindrical cavity where the cyclotron frequency shift can be calculated analytically and is hence under better control [11]. The eventual reduction of experimental uncertainty by about an order of magnitude is the goal of these experiments.

The Standard Model prediction of a_e may be written as the sum of electromagnetic, hadronic, and weak interaction contributions:

$$a_e(\text{theory}) = a_e(\text{QED}) + a_e(\text{had}) + a_e(\text{weak}). \tag{2}$$

The QED contribution can be written as

$$\begin{aligned} a_e(\text{QED}) = {} & A_1 + A_2(m_e/m_\mu) + A_2(m_e/m_\tau) \\ & + A_3(m_e/m_\mu, m_e/m_\tau). \end{aligned} \tag{3}$$

The term A_1 is mass-independent while A_2 and A_3 are functions of indicated mass ratios. A_i, $i = 1, 2, 3$, can be expanded in powers of α:

$$A_i = A_i^{(2)} \left(\frac{\alpha}{\pi}\right) + A_i^{(4)} \left(\frac{\alpha}{\pi}\right)^2 + A_i^{(6)} \left(\frac{\alpha}{\pi}\right)^3 + \ldots. \tag{4}$$

The first 4 terms of mass-independent term A_1 are [12,13]

$$A_1^{(2)} = 0.5,$$

$$A_1^{(4)} = -0.328\ 478\ 965\ 579\ \ldots,$$
$$A_1^{(6)} = 1.181\ 241\ 456\ldots,$$
$$A_1^{(8)} = -1.509\ 8\ (384). \tag{5}$$

The analytic values of $A_1^{(2)}$ and $A_1^{(4)}$ have been known for a long time. The analytic value of $A_1^{(6)}$ has been obtained recently [14]. It is in excellent agreement with the latest numerical result [15]

$$1.181\ 259\ (40), \tag{6}$$

which was obtained shortly before the analytic result became available.

The α^4 term requires evaluation of 891 four-loop Feynman diagrams. They fall naturally into five (gauge-invariant) groups according to the way closed electron loops (of vacuum-polarization (v-p) type and light-by-light (l-l) scattering type) appear in them.

Group I. Second-order vertex diagrams containing v-p loops of second, fourth, and sixth orders. Twenty five Feynman diagrams belong to this group.

Group II. Fourth-order vertex diagrams containing v-p loops of second and fourth orders. Fifty four diagrams belong to this group.

Group III. Sixth-order vertex diagrams containing a v-p loop of second order. One hundred fifty diagrams belong to this group.

Group IV. Vertex diagrams containing an l-l scattering subdiagram with further radiative corrections and/or v-p loop insertions. One hundred forty four diagrams belong to this group.

Group V. Eighth-order vertex diagrams of pure radiative corrections with no closed electron loop. Five hundred eighteen diagrams belong to this group.

Numerical evaluation of the first three groups is relatively easy and had been carried out to a reasonably high precision. Some are also known analytically. Numerical evaluation of *Group IV* diagrams was more difficult but gave results accurate enough for comparison with the measurement (1).

The size and complexity of *Group V* diagrams make their analytic evaluation prohibitively difficult even with the help of the fastest computers. Thus far only a small number of *Group V* diagrams have been evaluated analytically [16,17]. Crude numerical evaluations of all eighth-order integrals began around 1980 [18]. It is only in the last few years that the calculation of this term began to move from a "qualitative" to a "quantitative" stage thanks to the development of massively-parallel computers. In Sect. 4 and the Appendix I will discuss problems encountered in previous numerical works and the current effort to resolve them and obtain higher precision results.

As for A_2, it was shown long time ago by numerical integration that contributions of lowest order terms of A_2 to a_e are very small [19]:

$$A_2^{(4)}(m_e/m_\mu)(\alpha/\pi)^2 = \ 2.804 \times 10^{-12},$$
$$A_2^{(4)}(m_e/m_\tau)(\alpha/\pi)^2 = \ 0.010 \times 10^{-12}. \tag{7}$$

Smallness of $x = m_e/m_\mu$ and $y = m_e/m_\tau$ enables us to evaluate $A_2(x)$, $A_2(y)$, and $A_3(x, y)$, which vanish at $x = 0$ and/or $y = 0$, by expanding them in power series around $x = 0$ and/or $y = 0$. In this manner it was found that [20,21,22,23,24]

$$\begin{aligned} A_2^{(4)}(m_e/m_\mu) &= \ \ 5.197\ 387\ 62\ (32) \times 10^{-7}, \\ A_2^{(4)}(m_e/m_\tau) &= \ \ 1.837\ 50\ (60) \times 10^{-9}, \\ A_2^{(6)}(m_e/m_\mu) &= -7.373\ 942\ 53\ (33) \times 10^{-6}, \\ A_2^{(6)}(m_e/m_\tau) &= -6.5815\ (19) \times 10^{-8}. \end{aligned} \tag{8}$$

where the standard uncertainties are due to the uncertainties of the 1998 recommended values of mass ratios [25]. The first two lines confirm the results (7). The contributions of the remaining two to a_e are

$$\begin{aligned} A_2^{(6)}(m_e/m_\mu)(\alpha/\pi)^3 &= -0.092 \times 10^{-12}, \\ A_2^{(6)}(m_e/m_\tau)(\alpha/\pi)^3 &= -0.001 \times 10^{-12}. \end{aligned} \tag{9}$$

Some coefficients of $A_2^{(8)}$ and $A_2^{(10)}$ have been evaluated precisely in terms of a series expansion in powers of x and/or y [26,27].

The $A_3^{(6)}$ term is even smaller [20,28]:

$$A_3^{(6)}(m_e/m_\mu, m_e/m_\tau) = 1.91 \times 10^{-13}, \tag{10}$$

and

$$A_3^{(6)}(m_e/m_\mu, m_e/m_\tau)(\alpha/\pi)^3 = 2.4 \times 10^{-21}. \tag{11}$$

Finally we must include the non-QED part of the Standard Model contribution, namely, the hadronic and weak interaction terms. The hadronic term is small but not negligible [29,30,31]:

$$a_e(\text{had}) = 1.645\ (42) \times 10^{-12}, \tag{12}$$

being of the same order of magnitude as the current experimental uncertainty in (1). The weak interaction contribution is much smaller:

$$a_e(\text{weak}) = 0.0297\ (7) \times 10^{-12}. \tag{13}$$

To compare the theory of a_e with experiment, it is necessary to know the value of α, which has been measured in diverse branches of physics. Currently best values of α, with relative standard uncertainty of 1×10^{-7} or less, are those based on the quantum Hall effect [32], the ac Josephson effect [25], the neutron de Broglie wavelength [33], the muonium hyperfine structure [34,35], and an absolute optical frequency measurement of the Cesium D_1 line [36]:

$$\begin{aligned} \alpha^{-1}(\text{q.Hall}) &= 137.036\ 003\ 7(33) \quad [2.4 \times 10^{-8}], \\ \alpha^{-1}(\text{acJ}) \ \ &= 137.035\ 988\ 0(51) \quad [3.7 \times 10^{-8}], \\ \alpha^{-1}(\text{h/m}_\text{n}) &= 137.036\ 011\ 9\ (51) \quad [3.7 \times 10^{-8}], \\ \alpha^{-1}(\mu\text{hfs}) \ \ &= 137.035\ 993\ 2\ (83) \quad [6.0 \times 10^{-8}], \\ \alpha^{-1}(\text{C}_\text{s}\text{D}_1) &= 137.035\ 992\ 4\ (41) \quad [3.0 \times 10^{-8}], \end{aligned} \tag{14}$$

where the numerals enclosed in the parentheses are the standard uncertainties and numbers within the brackets are the fractional precisions.

A preliminary result from measurements based on the atom beam interferometry of the Cesium atom has also been reported [37]. The He atom fine structure will become another source of high precision α when the current theoretical work is completed [38,39,40,41,42]. It is fortunate that so many independent ways are available for obtaining high precision α. Precision of some measurements may exceed 1 in 10^8 in the near future. Even higher precision might be achieved by techniques based on the atom interferometry [43] and the single electron tunneling [44].

If one uses α(q.Hall), the best in (14), the theoretical value of a_e becomes

$$a_{\rm e}(\text{q.Hall}) = 1\ 159\ 652\ 153.4\ (1.2)\ (28.0) \times 10^{-12}, \tag{15}$$

where the numbers enclosed in parentheses are the uncertainty in the numerical integration result and that of α from [32], respectively.

Note that the intrinsic theoretical uncertainty in (15) is already very small. Thus the overall uncertainty is dominated by the uncertainty of α(q.Hall). In other words, the insufficient precision of this α prevents us from testing the validity of QED by full exploitation of the precision attained by the theory and measurement of a_e. This means, in turn, that comparison of theory and measurement of a_e will give a more precise value of α. The value of α determined from the average of a_{e^-} and a_{e^+} and the theory is [13]

$$\begin{aligned} \alpha^{-1}(a_{\rm e}) &= 137.035\ 999\ 58\ (14)\ (50) \\ &= 137.035\ 999\ 58\ (52) \qquad [3.8 \times 10^{-9}], \end{aligned} \tag{16}$$

where the uncertainties on the first line are from the α^4 term and the measurement uncertainty of a_e given in (1), respectively.

Implications of high precision $\alpha(a_{\rm e})$ to physics will be discussed in Sect. 5.

3 Muon Magnetic Moment Anomaly

The muon $g-2$ value has been determined in a series of experiments at CERN [45,46]. The primary purpose of the new muon $g-2$ experiment at Brookhaven National Laboratory is to improve the precision of the experiment by about a factor 20 and verify the presence of the electroweak effect which has been evaluated to two loop orders in the Standard Model. In this experiment, polarized muons from pion decays are captured in a storage ring with a uniform magnetic field and a weak-focusing electric quadrupole field. For a muon momentum of 3.09 GeV/c and $\gamma = 29.3$ the muon spin motion is unaffected by the electric quadrupole field and the difference frequency ω_a is given by

$$\omega_a = \omega_s - \omega_c = \frac{eB}{mc} a_\mu, \tag{17}$$

in which ω_s is the spin precession frequency and ω_c the orbital cyclotron frequency. Measurements of ω_a and B thus determine a_μ.

The stored μ^+ in the ring decay to e^+ via the parity-violating weak decay $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$, and high energy e^+ are emitted preferentially in the direction of the muon spin. Decay e^+ are detected with lead/scintillator detectors as a function of time after π (or μ) injection. The time spectrum for the e^+ counts is given by

$$N_e = N_0 e^{-t/\gamma\tau_0}[1 + A\cos(\omega_a t + \phi)], \tag{18}$$

in which τ_0 is the muon lifetime at rest, γ is the relativistic time dilation factor, and A and ϕ are fitting parameters. The exponential muon decay is modulated at the frequency ω_a, which are determined from the fit of (18) to the data. The storage ring field B is measured by NMR.

The important advances for the Brookhaven experiment are:

(1) An increase in primary proton-beam intensity by a factor of 200 as compared with the CERN experiment.

(2) A superferric magnet storage ring that provides a magnetic field of excellent stability and homogeneity, and an NMR system capable of field measurement to 0.1 ppm.

(3) A modern Pb/scintillating fiber detector system, incorporating a Loran frequency standard, capable of measuring time intervals with a precision of 20 ps.

(4) Muon as well as pion injection into the storage ring. Muon injection increases the number of stored muons and reduces background in the ring.

The experiment is making a rapid progress and is expected to achieve a precision of about 40×10^{-11}, more than 20 times better than that of the best previous result [46]

$$a_\mu(\text{exp}) = 1\ 165\ 923\ (8.5) \times 10^{-9} \qquad [7 \times 10^{-6}]. \tag{19}$$

Preliminary results obtained at the Brookhaven National Laboratory in the 1997 and 1998 runs are [47]

$$a_\mu^+(\text{exp}) = 11\ 659\ 251\ (150) \times 10^{-10} \qquad [13 \times 10^{-6}] \tag{20}$$

and [48]

$$a_\mu^+(\text{exp}) = 11\ 659\ 191\ (59) \times 10^{-10} \qquad [5 \times 10^{-6}], \tag{21}$$

respectively. More recent result will be made available shortly.

The theory of a_μ is more complicated than that of a_e because the effect of weak and strong interactions are about $(m_\mu/m_e)^2$ times larger than that of a_e. As in the a_e case, the QED contribution can be written as

$$\begin{aligned} a_\mu(\text{QED}) = {} & A_1 + A_2(m_\mu/m_e) + A_2(m_\mu/m_\tau) \\ & + A_3(m_\mu/m_e, m_\mu/m_\tau), \end{aligned} \tag{22}$$

where A_i, $i = 1, 2, 3$, can be expanded in powers of α:

$$A_i = A_i^{(2)}\left(\frac{\alpha}{\pi}\right) + A_i^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + A_i^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + \ldots. \tag{23}$$

A_1 is the same as for a_e. A_2 and A_3 have logarithmic singularities for $m_\mu/m_e \to \infty$.

$A_2^{(4)}(m_\mu/m_e)$ and $A_2^{(4)}(m_\mu/m_\tau)$ have been evaluated very precisely by an asymptotic expansion in m_μ/m_e and by a power series expansion in m_μ/m_τ, respectively [20,21,22]

$$A_2^{(4)}(m_\mu/m_e) = 1.094\ 258\ 2828\ (98),$$
$$A_2^{(4)}(m_\mu/m_\tau) = 7.8059\ (25) \times 10^{-5}. \tag{24}$$

where the uncertainties come from those of the 1998 recommended values of the mass ratios [25].

$A_2^{(6)}(m_\mu/m_e)$ and $A_2^{(6)}(m_\mu/m_\tau)$ have contributions from 24 Feynman diagrams containing vacuum-polarization loops or an l-l scattering subdiagram. They have been evaluated very precisely by an asymptotic expansion and by a power series expansion, respectively [23,24,49]:

$$A_2^{(6)}(m_\mu/m_e) = 22.868\ 379\ 36\ (23),$$
$$A_2^{(6)}(m_\mu/m_\tau) = 36.054\ (21) \times 10^{-5}, \tag{25}$$

where the uncertainties come from those of the 1998 recommended values of the mass ratios [25].

The extraordinarily large size of $A_2^{(6)}(m_\mu/m_e)$ was first discovered by numerical integration [50]. It comes mainly from diagrams containing the l-l scattering subdiagram. Such a large value has been interpreted as the low-energy effect of binding between the positive muon and the electron [51].

The next term $A_2^{(8)}(m_\mu/m_e)$ has contributions from 469 Feynman diagrams containing v-p loops and/or l-l scattering subdiagrams. They falls naturally into four (gauge invariant) groups according to the way closed electron or muon loops appear in them.

Group I^*. Second-order vertex diagrams containing v-p loops of second, fourth, and sixth orders, of which at least one is an electron loop. This group consists of 49 diagrams.

Group II^*. Fourth-order vertex diagrams containing v-p loops of second and fourth orders, of which at least one is an electron loop. This group consists of 90 diagrams.

Group III^*. Sixth-order vertex diagrams containing a v-p electron loop of second order. This group consists of 150 diagrams.

Group IV^*. Vertex diagrams containing a l-l scattering subdiagram with further radiative corrections and/or v-p loop insertions, of which at least one is an electron loop. This group consists of 180 diagrams.

By numerical integration (and some analytic work) $A_2^{(8)}(m_\mu/m_e)$ was found to be very large:

$$A_2^{(8)}(m_\mu/m_e) = 127.50\ (41). \tag{26}$$

The large size comes primarily from the *Group IV** diagrams which contain the l-l scattering subdiagram [53]. In other words, it is a reflection of largeness of the sixth-order term found in (25).

The presence of closed electron loops with small electron mass enables us to explore the properties of $A_2(x)$ and $A_3(x,y)$ for large values of $x = m_\mu/m_e$ by various analytic means. In particular, the leading logarithmic terms in m_μ/m_e can be obtained from the analysis of mass singularity [52] coupled with the renormalization group method [49,53,54] or the Callan-Symanzik equation [55]. An approach based on a Padé approximation is another fruitful method [56]. Some diagrams of *Group I** and *Group II** have been evaluated precisely using the asymptotic expansion in x [26].

It was pointed out in [56] that the contribution to $A_2^{(8)}(m_\mu/m_e)$ cited in [53] from a subset of *Group I** diagrams containing sixth-order vacuum-polarization diagrams seems to be well outside of its error bars. Their result was obtained using a Padé approximant of the sixth-order v-p spectral function which is determined from an exact information at some points near threshold and the asymptotic behavior. Our recent calculation [57] has confirmed the result of [56] and identified the discrepancy as a consequence of limited number of digits available in computer calculation. This is a problem inherent in any computer calculation and gives rise to serious round-off errors in some cases. (See Appendix for details.) The value (26) includes the new results given in [56,57].

Although the QED contribution $a_\mu(\mathrm{QED})$ given in (29) has a relatively small uncertainty, its evaluation is fairly crude and dated. A new and more accurate evaluation is in order. The largest uncertainty in $a_\mu(\mathrm{QED})$ comes from $A_2^{(8)}(m_\mu/m_e)$, in particular, from the diagrams of *Group IV**. The work in progress will reduce the uncertainty by an order of magnitude.

The tenth-order contribution has also been given a crude estimate [53,58,27]:

$$A_2^{(10)}(m_\mu/m_e) = 930\ (170). \tag{27}$$

The lowest-order non-vanishing coefficient of the A_3 terms was first calculated in [53]. If one uses the 1998 recommended values of the mass ratios [25], more recent work [22] on the lowest two terms of A_3 give

$$\begin{aligned} A_3^{(6)}(m_\mu/m_e, m_\mu/m_\tau) &= 0.52763\ (17) \times 10^{-3}, \\ A_3^{(8)}(m_\mu/m_e, m_\mu/m_\tau) &= 0.079\ (3). \end{aligned} \tag{28}$$

Collecting all these results we obtain the Standard Model prediction of $a_\mu(\mathrm{QED})$:

$$a_\mu(\mathrm{QED}) = 116\ 584\ 705.6\ (1.7) \times 10^{-11}, \tag{29}$$

The hadronic correction consists of three parts.

(1) Hadronic vacuum polarization contribution:

$$a_\mu(\mathrm{had.1}) = 6\ 924\ (62) \times 10^{-11}, \tag{30}$$

We quote here only the latest value obtained from the measured hadronic production cross section in e^+e^- collisions as well as the information obtained from the analysis of hadronic tau decay data [29]:

(2) Higher order hadronic vacuum polarization effect [30,59]:

$$a_\mu(\text{had.2}) = -101\ (6) \times 10^{-11}, \tag{31}$$

(3) Hadronic l-l scattering contribution [31]:

$$a_\mu(\text{had.ll}) = -79.2\ (15.4) \times 10^{-11}, \tag{32}$$

Finally, the electroweak contribution of up to two loop orders is given by [60]

$$a_\mu(\text{weak}) = 153\ (3) \times 10^{-11}. \tag{33}$$

The sum of these contributions gives the prediction of the Standard Model:

$$a_\mu(\text{th}) = 116\ 591\ 602\ (65) \times 10^{-11}. \tag{34}$$

This is in good agreement with the measurements (19), (20), and (21). The uncertainty in (34) comes mainly from the hadronic vacuum-polarization contribution (30). It must be improved by at least a factor of two before we can extract useful physical information from the new high precision measurement of a_μ. Fortunately, this contribution can be calculated from the measured value of R $(= \sigma^{total}(e^+e^- \to \text{hadrons})/\sigma^{total}(e^+e^- \to \mu^+\mu^-))$ in e^+e^- collisions. Future measurements of R at VEPP-2M, VEPP-4M, DAΦNE and BEPS as well as analysis of the hadronic tau decay data [61,62,63], is expected to reduce the uncertainty of this contribution to a satisfactory level.

The contribution of the hadronic l-l scattering effect (32) is relatively small but is potentially a source of serious problem because it is difficult to express in terms of experimentally accessible observables; it must be evaluated by theoretical consideration. It has been estimated by two groups, within the framework of chiral perturbation theory and the $1/N_c$ expansion [64,65]. Recently the theory dependence of these calculations has been reduced [31] by improving a part of the calculation incorporating the measurements of the $P\gamma\gamma^*$ form factors [66], where P stands for π^0, η, and η'. The value quoted in (32) is the result of this work. Evaluation of these effects in lattice QCD will be particularly timely and interesting.

Because of the unusually high sensitivity of a precise experimental value of a_μ to possible physics beyond the Standard Model, theoretical predictions of the contributions to a_μ of these theories are of great interest. In general any new particles or interactions which couple to the muon or to the photon contribute to a_μ, whose value then provides a sum rule for physics [53]. In comparison with experimental data from the higher energy colliders (LEP II, Tevatron, LHC), an a_μ value with a precision of 0.35 ppm, as projected for the current BNL experiment, provides a comparable or greater sensitivity to a composite structure

of the muon or W boson and also to the new particles in supersymmetric theories (SUSY). For the muon a composite mass scale $\Lambda = 4$ TeV and for the W boson an anomalous magnetic moment $\kappa = 0.04$ would be detectable. Of course, any observation of physics beyond the Standard Model from a_μ would be indirect and would not by itself determine the process involved.

Of particular interest is possible SUSY contributions which arise from smuon-neutralino and sneutrino-chargino loops [67]. They can be significant if the supersymmetric particles are not too massive and if $\tan\beta \equiv v_2/v_1$ is large. In the large $\tan\beta$ limit the one-loop SUSY effect gives

$$a_\mu^{\rm SUSY} \simeq \frac{\alpha}{8\pi \sin^2\theta_W} \frac{m_\mu^2}{\tilde{m}^2} \tan\beta \tag{35}$$

where $\tilde{m}$ represents a typical SUSY loop mass. If $\tan\beta \simeq 40$, a sensitivity of 1 ppm in the a_μ measurement probes $\tilde{m}$ at the 750 GeV level, which may be competitive with direct high energy collider searches. Other theories discussed thus far include muon internal structure [53], extra space-time dimensions [68] and possible violations of CPT invariance , and Lorentz invariance , [69,70].

4 Improving the α^4 Term of the Electron $g-2$

As is seen from (5), the largest source of uncertainty in $a_{\rm e}$(theory) is the coefficient $A_1^{(8)}$ of the α^4 term. Although it has a sufficient precision for comparison of theory and experiment at present, it will be necessary to improve it further when the measurement improves [9,10,11] and better α's of non-QED origin become available. The α^4 term comes from 891 Feynman diagrams , which can be classified into five gauge-invariant sets, depending on the number and type of closed electron loops. Of these the largest set, the *Group V*, is the source of most uncertainties in the numerical work.

To discuss the results of our calculation with a proper perspective it is necessary to explain how our computation is carried out for Feynman integrals which require point-by-point renormalization. The first simplifying step is to reduce the number (891) of diagrams contributing to $A_1^{(8)}$ to about 120 using the Ward-Takahashi identity. Each integrand thus obtained is an algebraic function of up to 20,000 terms, each term being a product of up to 10 functions defined in a 10-dimensional hypercube:

$$0 \le x_i \le 1, \quad i = 1, 2, \cdots, 10. \tag{36}$$

FORTRAN codes of some of the integrals are as large as 300 kilobytes. VEGAS [2] is the only effective method currently available to integrate such huge integrals.

The theoretical value of $a_{\rm e}$ has been improved steadily in step with the increasing power of computers over the last decade. Some values of the coefficient of the α^4 term reported are [18,71,72,13]

$$D_1 = -1.434(138),$$

$$D_2 = -1.557(70),$$
$$D_3 = -1.4092(384),$$
$$D_4 = -1.5098(384). \quad (37)$$

Thus far D_1 is the only result published as the final value of a calculation. It is basically an order-of-magnitude estimate, and its uncertainty is statistical only. Although it has meager statistics by today's standard and its non-statistical error has not been analyzed, it is at least independent of other results. One of the purpose of this paper is to examines the nature and magnitude of the non-statistical error of D_1 in detail. The result is reported later in this Section as D_{1a}.

D_2, D_3, and D_4 have increasingly larger statistics. But they are not mutually independent. Many integrals in D_2 appear in D_3, and so on. This is due to the circumstance that complete evaluation of each integral takes typically three to six months of continuous run on fast computers. Thus, at any moment, only a small number of these integrals are being improved. D_2, D_3, D_4 are snapshots of this continuously evolving numerical work. Errors of D_2 and D_3 are statistical only. The uncertainty of D_4 is different and requires an explanation. The statistical uncertainty of D_4 is about one-half that of D_3. Also, D_4 has a smaller non-statistical error. But the uncertainty of D_4 given in (37) is identical with that of D_3. This is done to avoid publishing another value which will be superseded before long. It is also large enough to accommodate the systematic uncertainty which was not fully analyzed yet.

With increasing statistics, understanding of systematic uncertainty becomes more and more crucial. The work in progress (D_5) has not only higher statistics than D_4 but also eliminates most of systematic uncertainties.

The difficulty encountered in evaluating these integrals originates primarily from the round-off errors caused by a finite number of bits available (64 bits, 128 bits, etc.) in the computer work. I call it the digit-deficiency (or *d-d*) problem. Although this was the suspected cause from the beginning, it was clearly identified by comparing our numerical work [57] with the result obtained by the Padé approximant method [56].

In order to deal with the *d-d* problem I have developed various methods described in the Appendix, namely, *stretching*, *higher-precision arithmetic*, *freezing*, and *chopping*. Practically all integrals benefit from stretching. Typically several stretchings are tried for each integral. Other methods are used as the need arises. Once the *d-d* problem is under control, the result of integration behaves (nearly) statistically and its error generated by VEGAS improves proportional to $\mathcal{N}^{-1/2}$ as the total number $\mathcal{N}$ of data sampling increases.

In early calculations, because of computing time restriction, we could not use real*16 extensively. Thus we had to rely heavily on *chopping*. The shortage of computing time also made it difficult to analyze the effect of chopping, since it requires that the calculation is repeated for several values of the chopping parameter δ, which is very time-consuming. This is the main reason why error estimates of early calculations were statistical only.

With the availability of faster computers in recent years, we are now able to handle these problems more easily. In particular, we examined the effect of chopping on D_1 by carrying out a similar calculation in real*16. This calculation, D_{1a}, controls the *d-d* problem by real*16 arithmetic, in contrast to D_1 in which it was controlled by chopping.

D_1 and D_{1a} have similar statistics, hence similar statistical errors generated by VEGAS. The total number of samplings $\mathcal{N}$ for each integral ranges from $\sim 10^8$ to $\sim 10^9$. Statistical errors are of order $\mathcal{N}^{-1/2} \sim$ several $\times\, 10^{-4}$. The new result D_{1a} is

$$D_{1a} = -1.386(129), \tag{38}$$

to be compared with the old value (37). About half of 47 integrals [1] of *Group V* contributing to D_1 used chopping with $\delta \sim 10^{-8}$. The effect of chopping for $\delta = 10^{-8}$ may be judged by the scale set by

$$\delta^{1/2}|\ln(\delta)|^2 \sim 0.034, \quad \delta^{1/2}|\ln(\delta)| \sim 0.0019. \tag{39}$$

Although the actual effect of δ depends on the integral, it exceeds the statistical uncertainty in many cases. About one-half of *Group V* integrals contributing to D_1 deviate from those contributing to D_{1a} by more than 2 standard deviations. This is mostly the effect of chopping.

Summing over positive and negative deviations separately, we obtain

$$\sum_i^{(+)} \Delta^{(i)} \sim 1.27, \quad \sum_i^{(-)} \Delta^{(i)} \sim -1.39, \tag{40}$$

where $\Delta^{(i)} \equiv D_{1a}^{(i)} - D_1^{(i)}$ and $D_{1a}^{(i)}$ and $D_1^{(i)}$ are terms contributing to D_{1a} and D_1, respectively. The total effect of chopping is their sum (~ -0.12). This is not the same as the difference (-0.048) of (37) and (38) which includes improvement in the renormalization terms as well as the contributions of *Group I - Group IV*. In computing D_1, the chopping parameter δ was chosen differently for different integrals. If the same δ were used for all integrals, the effect of chopping on D_1 would have been much smaller (~ 0.01). It is interesting nevertheless to note that the net effect of chopping on D_1 is not too severe.

The result (40) enables us to understand why D_2, D_3, D_3 fluctuated so much. This is because the *positive* and *negative* effects of chopping were not balanced at the moment these intermediate results were reported. This is clearly visible in the new calculation D_5 which avoids chopping almost entirely, and in rare cases where chopping is still needed, the dependence on δ is carefully examined. The resulting uncertainty is nearly statistical, as is seen from the fact that individual terms contributing to D_{1a} and D_5, as well as other intermediate results obtained in real*16, are consistent with each other and have the $\mathcal{N}^{-1/2}$ behavior. Furthermore, D_5 allows us to estimate the effect of chopping on D_2, D_3, D_4 without costly direct calculation.

[1] The number of *Group V* diagrams is reduced from 518 to 47 using the Ward-Takahashi identity and time-reversal symmetry of diagrams.

The latest value of D_5 is consistent with D_4, when the latter's systematic error is taken into account. Unfortunately, numerical work on D_5 is not yet finished (due to delay caused by computer system upgrade). If there is no further unexpected delay, the final result D_5 will be obtained within a year.

5 Concluding remarks

In order to enhance the sensitivity of a_μ to the weak effect and beyond, it is important to sharpen the QED and hadronic contributions as much as possible. Improvement of the α^4 term of a_μ by an order of magnitude is in progress and will be completed within a few months. Some α^5 terms will also be improved or newly evaluated in the near future. This puts the QED part of a_μ in a very good shape. Of course the hadronic contribution must be improved further to give the weak interaction contribution to a_μ a very stringent test.

Within the Standard Model, a_e is not sensitive to the short-distance effects. Thus a_e is used primarily to test the renormalization theory of QED. Our current goal for a_e is to calculate the coefficient of the α^4 term to a precision of ~ 0.01. This corresponds to the uncertainty of $\sim 0.3 \times 10^{-12}$ in a_e. With the matching improvement in experiment, this will provide α with a precision of 10^{-9} or better. How far can we go beyond this ? It is certainly feasible and desirable to improve it further by another factor of 4. This is just a matter of computer time. (It will require about 10 million hours of computing time.) However, improving the α^4 term much further will not make sense until the tenth-order term is evaluated or estimated reliably which will be of the order of

$$(\alpha/\pi)^5 \simeq 0.068 \times 10^{-12}. \tag{41}$$

This is a very formidable task indeed.

Finally let me discuss the physical significance of pushing the precision of α further. At present there are several competing measurements of α based on diverse fields of physics, namely, condensed matter physics, atomic physics, nuclear physics, and QED. These measurements are in agreement with each other within the uncertainties of $\sim 10^{-7}$. This is remarkable since it implies that the underlying theories of these measurements are correct to this level of precision even though most of the theories may not be able to justify such a precision because they do not include correction terms such as relativistic and radiative effects which may be as large as 10^{-3} and which must be included to achieve such a precision. The excellent agreement in spite of this must imply that these corrections are much smaller for some reason and these theories are fundamentally correct to this order even though most of them have no explicit theoretical justification for such a claim. Such a remarkable "coincidence" must be attributable to two fact: One is that they are all built on a common foundation, namely quantum mechanics. The other is that the extraordinary precision derives from the general feature of quantum mechanics, such as one-valuedness of wave function and gauge invariance [73,74], which override various approximations introduced in constructing specific theories.

From this viewpoint we must expect that all measurements of α, whether they are based on atomic physics, nuclear physics, or condensed matter physics, must agree with α obtained from QED (or more precisely the Standard Model) when their precisions are improved to $\sim 10^{-9}$, comparable to that of $\alpha(a_e)$. If serious disagreement develops as precision of measurement improves, it might indicate a serious fault in some of these theories, possibly including quantum mechanics itself. See [12] for further discussions.

Acknowledgments

I thank M. Nio, J. Zollweg, R. Sinkovits, R. Leary, T. Tannenbaum, and J. Ballard for assistance in various phases of computation. The bulk of computation has been carried out on the IBM SP2 and SP3 computers at San Diego Supercomputer Center, IBM SP2 at the Center for Parallel Computing (University of Michigan), and the Condor system at the University of Wisconsin, which are made available by the computing award of the National Resource Allocation Committee. Some part of work has been carried out on the IBM SP2 at Cornell Theory Center. This work is supported in part by the U. S. National Science Foundation.

Appendix: VEGAS and Feynman Integral

VEGAS is an adaptive-iterative integration routine based on random sampling of the integrand. In the i-th iteration, the integral is evaluated by sampling it at points chosen randomly according to a distribution ρ_{i-1} (a step function defined by grids) constructed in the $(i-1)$-st iteration. This generates an approximate value I_i of the integral, an estimate of its uncertainty σ_i, and the new distribution function ρ_i to be used in the next iteration.

The distribution ρ_i is constructed in such a way that the grids concentrate more and more in the region where the integrand is large. The construction of ρ_i involves a positive parameter β that controls the speed of "convergence" to a stable configuration. In most cases we chose $\beta = 0.5$. However, we may even choose $\beta = 0$ (no change in ρ) which is useful in some cases.

After several iterations I_i and σ_i are combined under the assumption that all iterations are statistically independent. The combined value and uncertainty are given by

$$I = (\sum_i (I_i/\sigma_i^2))/(\sum_i (1/\sigma_i^2)), \quad \sigma = (\sum_i (1/\sigma_i^2))^{-1/2}. \tag{42}$$

For well-behaved integrals the function ρ_i converges rapidly to a (practically) stable configuration. Once ρ_i is stabilized, the error generated by VEGAS is (nearly) statistical and is proportional to $\mathcal{N}^{-1/2}$, where $\mathcal{N}$ is the total number of data samplings. The problem is that Feynman integrals in general are *not* well-behaved in the sense described below.

The renormalized Feynman integrand is of the form

$$f = f_0 + \cdots + f_r, \tag{43}$$

where f_0 is obtained directly from a Feynman diagram and f_1,..., f_r are terms needed to renormalize ultraviolet and/or infrared divergences of f_0. All terms f_0, ..., f_r are divergent on the surface of the unit cube (36). Of course, by construction, f is mathematically well-defined and integrable.

This does not guarantee, however, that f is well-behaved on the computer. This is because f is expressed on a computer only as accurately as the number of digits available (64 bits, 128 bits, etc.). The integration domain includes regions where f_0, ..., f_r are singular and the sum f loses most or all of significant digits and is severely affected by round-off errors. When this happens, VEGAS gives I_i and σ_i which become unreliable or even divergent.

Note that this problem is *inherent* to our integration method based on Monte-Carlo sampling. Sooner or later one will be caught up by it (which will be called digit-deficiency or *d-d* problem). In order to cope with the *d-d* problem before it upsets the integration, we have developed several strategies.

a. Stretching. The renormalized integrand f defined in (43) may still have weak (integrable) singularities on some boundary surfaces. Since it is renormalized, however, such singularities can be removed by an appropriate change of variable, or mapping. The integrand being extremely complicated, however, it is difficult to find analytically correct mapping. A simple, although not always successful, way to remove or weaken the *d-d* problem is the "stretching" defined as follows: Suppose VEGAS finds after several iterations that the integrand samplings tend to concentrate near an $(n-1)$-dimensional surface defined, say, by $x_1 = 0$ at an end of the x_1 axis. In such a case, if one maps x_1 into x_1' as

$$x_1' = x_1^{a_1}, \tag{44}$$

where a_1 is a real number greater than 1, the domain near $x_1 = 0$ is stretched out and random samplings in the x_1' variable give more attention to this region from the beginning of iteration. Also, the Jacobian $a_1 x^{a_1 - 1}$ of this transformation weakens the singularity of the integrand. Similarly, the singularity at $x_1 = 1$ can be weakened by the stretching

$$x_1' = 1 - (1 - x_1)^{b_1}, \quad b_1 > 1. \tag{45}$$

Stretching is a one-to-one mapping of a unit hypercube onto itself. It may be applied to all variables independently. By an appropriate stretching, one can speed up convergence of ρ considerably. Note also that different stretchings lead to statistically independent samplings of an integral which should give the same answer within their error bars. This flexibility is important in assessing the reliability of results of integration. Of course, stretching does not solve all problems since it disregards analytic structure of the integrand.

b. Higher precision arithmetic. Going from double precision (64 bits or real*8) to quadruple precision (128 bits or real*16) arithmetic is the most effective method

to control the *d-d* problem. One serious obstacle is that it slows down the computation speed by about 30. Thus we were not able to use real*16 extensively until massively parallel computers became readily available.

In many cases real*16 is needed only in a small portion of integration domain. To take advantage of this situation we normally start with evaluation of Feynman integrals in real*8 which has higher speed. If this runs into a *d-d* problem, we split the integration domain into a small (rectangular) part containing the *d-d* domain and the remainder. The difficult region is then evaluated in real*16, while the rest continues in real*8. This strategy has been very successful and many integrals have been evaluated precisely.

Recently, a modified algorithm of VEGAS (called a-p VEGAS) has been developed [2] which enables us to make this splitting local and automatic [75]. In this approach the integrand f is first evaluated at each point in real*8. The result is tested by computing the ratio

$$t = (f_+ + |f_-|)/|f_+ + f_-|, \tag{46}$$

where $f_+(f_-)$ is the sum of positive(negative) terms of $f_0, ...f_r$. If t is larger than a chosen number $t_0(\gg 1)$, it signals possible presence of the *d-d* problem. (Choosing t_0 is by trial and error.) Then the integrand is reevaluated in real*16 at the same spot.

Note that $t > t_0$ is not a necessary or sufficient condition for identifying the *d-d* candidate. It is simply a quick way to find most (but not all) *d-d* problems. In particular, if the integrand f of (43) has no renormalization term (namely, $r = 0$), t defined by (46) is equal to 1 and thus $t > t_0$ cannot be satisfied for $t_0 > 1$. The integral may still suffer from a *d-d* problem, but for a reason entirely unrelated to the renormalization.

If the *d-d* problem is not severe, this method is very efficient and runs much faster than pure real*16. In some cases, it is useful to split the integration domain into two or more parts in the manner as described above, and apply the a-p VEGAS choosing different values of t_0 in different parts. In more difficult cases, however, pure real*16 works faster since it does not require the overhead needed in computing (46).

It must also be emphasized that the a-p VEGAS is designed to deal with the *d-d* problem found in the real*8 calculation. If the *d-d* problem occurs in real*16, it is necessary to go to even higher precision arithmetic. Unfortunately, such a device is not available at present on massively parallel computers. Thus we may be forced to deal with the *d-d* problem in combination with other techniques described in the following.

c. Freezing. Sometimes, iteration procedure runs into the *d-d* problem before it settles down to a (nearly) stable ρ. In such a case, one may freeze ρ by putting

[2] This program was written by R. Sinkovits for MPI parallel processing FORTRAN as one of the projects of the NPACI Strategic Applications Collaboration (member: R. Sinkovits, R. Leary, and T. Kinoshita). It was adapted to DEC α (with slight modifications) by Makiko Nio.

$\beta = 0$ few steps before the *d-d* problem becomes serious. The resulting ρ is not optimal so that it requires longer computing hours to achieve the desired statistical uncertainty.

d. Chopping. If procedures *a*, *b*, *c* fail to solve the *d-d* problem, one may restrict some integration axis from (0, 1) to $(\delta, 1-\delta)$, where $0 \leq \delta \ll 1$, to avoid the dangerous region. This is referred to as *chopping*. The error introduced by *chopping* will be of order $\delta^{1/2}(\ln\delta)^a$, where a is a positive number that can not be fixed exactly without knowing the analytic structure. For our purpose, it is sufficient to find a crude value of a empirically by carrying out integration for several values of δ.

In using *chopping* we must pay attention to the following points:
(i) We must repeat full scale calculation for several δ. This requires a substantial extra computing time.
(ii) Integration becomes more and more difficult as δ gets smaller, making extrapolation to $\delta = 0$ far from straightforward. The difficulty in assessing the effect of chopping was the major source of non-statistical uncertainty in earlier calculations.
(iii) We can choose a much smaller chopping parameter in real*16 compared with that of real*8. This means that we can reduce the error due to chopping substantially by going to real*16.

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