Recent Results in Positronium Theory

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Abstract. We review our recent results on higher order corrections in positronium physics. We discuss a calculation of the recoil $\mathcal{O}(m\alpha^6)$ corrections to the hyperfine splitting [1] and energy levels of a positronium atom [2], $\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ contributions to the positronium S-wave energy levels [3] and $\mathcal{O}(\alpha^2)$ radiative corrections to the parapositronium decay rate [4].

1 Introduction

Spectroscopy of positronium provides a sensitive test of the bound state theory in Quantum Electrodynamics (QED). Because of the small value of the electron mass, the effects of strong interactions are negligible compared to the accuracy achieved in present experiments. For this reason positronium represents a unique system which can, in principle, be described with very high accuracy by means of QED only. Tests of QED predictions are made possible thanks to a high precision of positronium spectroscopic measurements.

There are several quantities in positronium physics where higher order radiative corrections have to be taken into account for the theoretical prediction to match the experimental precision. For the hyperfine splitting of the ground state, $\Delta\nu \equiv E(1^3S_1) - E(1^1S_0)$, the two best experimental values are [5,6]

$$\Delta\nu = 203\,387.5(1.6)\ \text{MHz}, \tag{1}$$

$$\Delta\nu = 203\,389.10(0.74)\ \text{MHz}. \tag{2}$$

Another precisely measured quantity is the energy difference of 2^3S_1 and 1^3S_1 states [7]:

$$E(2^3S_1) - E(1^1S_1) = 1\,233\,607\,216.4(3.2)\ \text{MHz}. \tag{3}$$

Since $m\alpha^6 = 18.658$ MHz, complete calculations of $\mathcal{O}(m\alpha^6)$ corrections to both the hyperfine splitting and the spin independent part of the energy levels are necessary.

A different type of precisely measured quantities are the decay properties of the positronium atoms, e.g. total decay widths of both ortho ($S = 1$) and para ($S = 0$) positronium ground states into three and two photons, respectively. The orthopositronium decay width is a controversial issue because the theoretical prediction differs significantly from the most precise experimental result [8]. The

current status of the orthopositronium decay is summarized in Ref. [9] and we do not discuss it here in any detail.

The parapositronium decay rate into two photons agrees with the theoretical prediction and has attracted less attention. However, it is also measured quite precisely [10],

$$\Gamma^{\rm exp}_{\rm p-Ps}({\rm gas}) = 7990.9(1.7)\ \mu{\rm s}^{-1}. \tag{4}$$

The accuracy of this measurement is high enough to warrant a calculation of $\mathcal{O}(\alpha^2)$ corrections to the decay rate.

Here we intend to summarize several calculations on precision positronium physics which we have recently completed. We will describe the calculation of: recoil corrections to positronium nS energy levels and the hyperfine splitting at $\mathcal{O}(m\alpha^6)$, $\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ corrections to positronium energy levels and second order corrections to parapositronium lifetime. Additional details of those projects can be found in the original papers [1,2,3,4]. For the summary of the experimental situation, we refer to Ref. [11].

2 Calculational methods

2.1 General discussion

What makes bound state calculations difficult is that they are "non-perturbative"; this is because one has to sum up an infinite number of Feynman diagrams to "create" a bound state in perturbation theory.

The electron and positron in positronium move with a typical velocity $\sim \alpha$ and have a momentum $\sim m\alpha$ and energy $\sim m\alpha^2$. If we are interested in a precision level where relativistic effects become important, the usual quantum mechanical treatment of the bound state has to be merged with the complete field theory description. Then, no matter how weak the coupling is, we deal with the problem of bound states in the field theory, which is rather complicated. One of the possible computational approaches (not always the most convenient) is the Bethe-Salpeter equation.

In the case of a weak coupling, a better way to go is to construct an effective theory which, on the one hand, correctly describes the physics at non-relativistic scales, and on the other hand does not have relativistic degrees of freedom [12]. The price to pay for explicitly integrating out relativistic degrees of freedom is that the new Hamiltonian is an infinite sum of operators and is not renormalizable. These are, however, minor problems compensated by significant technical and conceptual advantages.

In order to quantify which degrees of freedom are retained and which are integrated out, one has to introduce a cut-off. All the coefficients in the effective Hamiltonian become cut-off dependent. This cut-off dependence is canceled when physical quantities are computed. A good or bad choice of the cut-off will facilitate or hinder the actual computations.

In order to see what choice of regularization (cut-off) is optimal, one can benefit from the developments in particle physics, where a number of various

regularization schemes has been tried. At present, the most sophisticated calculations are being done using dimensional regularization, since this regularization minimally increases the complexity of the calculation and allows for some special tricks to be used systematically (integration-by-parts). In what follows, we will apply dimensional regularization to calculations in the QED bound state theory.

We begin with an effective NRQED Hamiltonian,

$$H = \frac{\boldsymbol{p}^2}{m} - \frac{\alpha}{r} + H_{\rm Breit} + H_{\rm rel}. \tag{5}$$

To compute relativistic operators $H_{\rm rel}$, one considers scattering amplitudes both in NRQED and in full QED treating Coulomb interaction as a perturbation. By matching these amplitudes, one determines $H_{\rm rel}$. Therefore, the question of how $H_{\rm rel}$ should be computed is related to the question of how full QED amplitudes can be effectively calculated in threshold kinematics.

To enable an effective computation of the perturbative diagrams close to a threshold, one has to answer the question of how a single Feynman diagram can be expanded in relative velocity $\boldsymbol{v}$ close to the threshold. The practical recipe is provided by the threshold expansion method [13] which makes algorithmic calculations possible. One first identifies relevant momenta regions: hard region, $k \sim m$; soft region, $k_0 \sim |\boldsymbol{k}| \sim mv$; potential region, $k_0 \sim mv^2$, $|\boldsymbol{k}| \sim mv$; ultrasoft region, $k \sim mv^2$.

Then, assuming a certain hierarchy for momenta in a given region, one Taylor expands the integrand and in this way obtains a simpler integrand, homogeneous in v. These integrals are evaluated and the contributions of the different momentum regions are summed,

$$I = \sum_{\text{Regions } \gamma} \int \mathrm{d}^d k \ \mathrm{Taylor}_\gamma \left[\text{Integrand}\right]. \tag{6}$$

An important result is that the new operators $H_{\rm rel}$ are generated by the hard momentum contributions only. Hence, as follows from the above formula, computation of new operators requires only a Taylor expansion of full QED amplitudes in external momenta; no matching is required. This approach provides the required operators in dimensional regularization and hence, for consistency, the rest of the problem (quantum mechanics) must be solved using the same (dimensional) regularization.

2.2 Quantum mechanics in $d = 3 - 2\epsilon$ dimensions

We begin by writing down the Schrödinger equation in d dimensions:

$$\boldsymbol{\phi}(\boldsymbol{p}) = \frac{1}{\boldsymbol{p}^2/m - E} \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \frac{4\pi\alpha}{(\boldsymbol{p}-\boldsymbol{k})^2} \phi(\boldsymbol{k}). \tag{7}$$

Although it looks similar to the usual three dimensional case, we note that this similarity persists in momentum space only whereas in the coordinate space

the Coulomb interaction is not given by α/r any longer. We emphasize that, regularizing the Schrödinger equation in momentum space is consistent with the way the regularization is introduced to compute the contributions of the "hard momentum" region.

For $d = 3$, the solution of Eq. (7) is the familiar Coulomb wave function in 3 dimensions. For $d \neq 3$ this is not true. This represents an additional problem since the non-relativistic calculations have to be done without an explicit knowledge of the wave function. Fortunately, cancelation of all divergences can be ensured on the operator level using the Schrödinger equation in d-dimensions. Once divergences are canceled, the limit $d \to 3$ can be taken and a non-trivial (but now finite) matrix elements can be easily computed.

Another tricky point in this approach is the Dirac algebra. The usual prescription in NRQED is: $\{\gamma_i\} \to \{\boldsymbol{\sigma}_i\}$, $[\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j] = 2i\epsilon_{ijk}\boldsymbol{\sigma}_k$. The problem here is that for $d \neq 3$ the last equation is not true. Therefore, one should proceed as with the Dirac matrices, using traces and the identity $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j + \boldsymbol{\sigma}_j\boldsymbol{\sigma}_i = 2\delta_{ij}$ only.

Let us now present an example of simple but striking identity which is valid in dimensionally regularized quantum mechanics but would be wrong in other regularization schemes. Consider the following integral:

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \boldsymbol{p}^2 \phi(\boldsymbol{p}),$$

where $\phi(\boldsymbol{p})$ is the Coulomb wave function, $\phi(\boldsymbol{p}) \sim 1/\boldsymbol{p}^4$ $(\boldsymbol{p} \to \infty)$. Simple power counting shows that the integral is linearly divergent. Nevertheless, using the Schrödinger equation, one derives a finite result,

$$\begin{aligned}\int \frac{\mathrm{d}^d p}{(2\pi)^d} \boldsymbol{p}^2 \phi(\boldsymbol{p}) &= \int \frac{\mathrm{d}^d \boldsymbol{p}\, \mathrm{d}^d \boldsymbol{k}}{(2\pi)^{2d}} \frac{4\pi\alpha \boldsymbol{p}^2}{(\boldsymbol{p}-\boldsymbol{k})^2(\boldsymbol{p}^2/m - E)} \phi(\boldsymbol{k}) \\ &= mE \int \frac{\mathrm{d}^d k}{(2\pi)^d} \phi(\boldsymbol{k}) + \int \frac{\mathrm{d}^d p\, \mathrm{d}^d k}{(2\pi)^{2d}} \frac{4\pi\alpha m}{\boldsymbol{p}^2} \phi(\boldsymbol{k}) = mE\psi(r=0),\end{aligned}$$

where we used the fact that the scale-less integrals in dimensional regularization $\int \mathrm{d}^d k (\boldsymbol{k}^2)^a$ are defined to be zero.

2.3 Hard scale contributions

Hard scale contributions are obtained by Taylor expanding the scattering amplitude in small external momenta. All relevant two-loop integrals (some examples are shown in Fig.1) can be parameterized by nine Feynman propagators:

$$I(a_1, \ldots a_9) = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} S_1^{a_1} S_2^{a_2} S_3^{a_3} S_4^{a_4} S_5^{a_5} S_6^{a_6} S_7^{a_7} S_8^{a_8} S_9^{a_9},$$

where

$$S_1 = \frac{1}{k_1^2},\ S_2 = \frac{1}{k_2^2},\ S_3 = \frac{1}{(k_1 - k_2)^2},\ S_4 = \frac{1}{k_1^2 + 2pk_1},$$

$$S_5 = \frac{1}{k_2^2 + 2pk_2},\ S_6 = \frac{1}{k_1^2 - 2pk_1},\ S_7 = \frac{1}{k_2^2 - 2pk_2},$$
$$S_8 = \frac{1}{(k_1 - k_2)^2 + 2p(k_1 - k_2)},\ S_9 = \frac{1}{(k_1 - k_2)^2 - 2p(k_1 - k_2)},$$

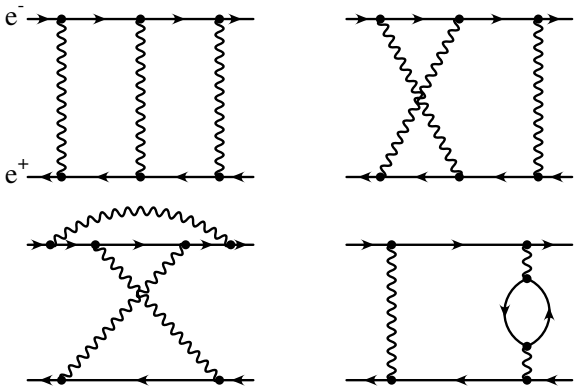

Fig. 1. Examples of "hard scales" contributions

In order to compute these integrals we use the so-called integration-by-parts technique [14], familiar from high energy particle physics. In short, using a set of integration-by-parts identities, one finds a mapping of a function $I(\{a_i\})$ with arbitrary arguments on a few "master" integrals. This mapping procedure is constructed in a completely algebraic way. After the calculation of these master integrals is completed, the problem is essentially solved, since any diagram that belongs to the above class can be computed by using one and the same computer program, which delivers analytic results at the end. A glance at Fig. 1 shows that the "hard scale" diagrams, which we have solved, encompass almost all non-trivial two-loop corrections relevant for the equal mass bound state problems, i.e. recoil, radiative recoil and radiative corrections, as well as (although not shown in Fig. 1) the one-photon annihilation contributions.

3 Results

In this section we present the results of the calculation of the basic positronium properties using theoretical methods described above. We begin with positronium spectroscopy and then discuss the calculation of the second order corrections to the parapositronium decay rate.

3.1 Positronium spectroscopy

Recoil corrections to positronium energy spectrum have been controversial for some time. We have recalculated these corrections within the theoretical framework discussed above. Our results are summarized below.

To begin, we define the average energy and the hyperfine splitting for nS states as:

$$E(J,n) = E_{\rm aver}(n) + \boldsymbol{s}_+\boldsymbol{s}_- E_{\rm hfs}(n),$$

where $\boldsymbol{s}_\pm$ are the spin operators of the positron and electron respectively.

Explicit calculation of the recoil corrections for the ground state $n=1$ gives [1,2]:

$$\begin{aligned}
\Delta_{\rm rec} E_{\rm aver} &= -\frac{m\alpha^6}{8}\left(\frac{299}{192}+\frac{11}{2\pi^2}+\frac{3\zeta(3)}{\pi^2}\right),\\
\Delta_{\rm rec} E_{\rm hfs} &= m\alpha^6\left(-\frac{1}{6}\ln\alpha+\frac{331}{432}-\frac{\ln 2}{4}-\frac{17\zeta(3)}{8\pi^2}+\frac{5}{12\pi^2}\right).
\end{aligned}$$

How can this result be extended to include $n\neq 1$ excited states? The easiest way to accomplish this is to perform the quantum mechanical part of the calculation using arbitrary regularization framework, e.g. by cutting-off all the divergent integrals in momentum or coordinate space. The result has the form

$$\begin{aligned}
\Delta_{\rm rec} E_{\rm aver}(n) &= -\frac{m\alpha^6}{8n^3}\left([div]+\frac{69}{64n^3}-\frac{8}{3n^2}+\frac{2}{n}\right),\\
\Delta_{\rm rec} E_{\rm hfs}(n) &= \frac{m\alpha^6}{n^3}\left\{[div]+\frac{1}{6}\left(-\ln\frac{\alpha}{n}-\Psi(n)-C\right)+\frac{7}{12n}-\frac{1}{2n^2}\right\},
\end{aligned}$$

where C is the Euler constant and $[div]$ denotes a divergent cut-off dependent contribution. It is easy to determine the latter by matching the above result at the expressions for $\Delta_{\rm rec}E_{\rm aver,hfs}$ for $n=1$. The final result for the recoil corrections valid for all values of n reads [2]

$$\begin{aligned}
\Delta_{\rm rec} E_{\rm aver}(n) &= -\frac{m\alpha^6}{8n^3}\left(\frac{55}{48}+\frac{11}{2\pi^2}+\frac{3\zeta(3)}{\pi^2}+\frac{69}{64n^3}-\frac{8}{3n^2}+\frac{2}{n}\right),\\
\Delta_{\rm rec} E_{\rm hfs}(n) &= \frac{m\alpha^6}{n^3}\left\{\frac{1}{6}\left(-\ln\frac{\alpha}{n}-\Psi(n)-C\right)+\frac{7}{12n}-\frac{1}{2n^2}\right.\\
&\quad\left.+\frac{295}{432}-\frac{\ln 2}{4}-\frac{17\zeta(3)}{8\pi^2}+\frac{5}{12\pi^2}\right\}.
\end{aligned}$$

The result for $\mathcal{O}(m\alpha^6)$ recoil corrections was the last controversial piece at that order; all other corrections were computed by more than one group and are known in the analytical form. Our result for the recoil corrections confirms the result by K. Pachucki [15,16]. Adding up known results for many different contributions (see [2] for references to original papers), we derive the following result for $E_{\rm aver}$ and $E_{\rm hfs}$ valid up to $\mathcal{O}(m\alpha^6)$:

$$E(J,n) = E_{\rm aver}(n) + \boldsymbol{s}_+\boldsymbol{s}_- E_{\rm hfs}(n).$$

$$E_{\rm aver}(n) = -\frac{m\alpha^2}{4n^2} + \frac{m\alpha^4}{16n^3}\left(\frac{11}{4n} - 1\right) + \frac{m\alpha^5}{8\pi n^3}\left[-6\ln\alpha - \frac{16}{3}\ln k_0(n,0)\right.$$
$$\left. + \frac{14}{3}\left(\ln\frac{4}{n} + \Psi(n) + C\right) - \frac{37}{45} - 3\ln 2 + \frac{7}{3n}\right] + \frac{m\alpha^6}{32n^3}\left\{-\ln\frac{\alpha}{n} - \Psi(n) - C\right.$$
$$\left. + \frac{141}{4}\frac{\zeta(3)}{\pi^2} + \left(\frac{137}{6} - \frac{68}{\pi^2}\right)\ln 2 + \frac{1421}{27\pi^2} - \frac{2423}{432} - \frac{7}{n} + \frac{17}{12n^2} - \frac{69}{16n^3}\right\}, \qquad (8)$$

$$E_{\rm hfs}(n) = \frac{m\alpha^4}{n^3}\left\{\frac{7}{12} - \frac{\alpha}{\pi}\left(\frac{8}{9} + \frac{1}{2}\ln 2\right) + \alpha^2\left[-\frac{5}{24}\left(\ln\frac{\alpha}{n} + \Psi(n) + C\right)\right.\right.$$
$$\left.\left. + \frac{1367}{648\pi^2} - \frac{4297}{3456} + \left(\frac{221}{144} + \frac{1}{2\pi^2}\right)\ln 2 - \frac{53}{32\pi^2}\zeta(3) + \frac{5}{8n} - \frac{85}{96n^2}\right]\right\}. \qquad (9)$$

Before comparing Eqs. (8,9) to the experimental results, let us say a few words on how higher order effects and therefore the theoretical uncertainty are estimated.

At the moment, it is not possible to compute $m\alpha^7$ corrections completely. Nevertheless, the leading logarithmic correction, $\mathcal{O}(m\alpha^7\ln^2\alpha)$, to positronium energy levels can be computed. This provides an estimate of higher order effects and hence of the uncertainty in the current theoretical prediction. In what follows we briefly describe our calculation of these corrections [3].

Contributions that are logarithmic in α, usually appear as integrals of the form:

$$\int \frac{d^3k}{(2\pi)^3}F(k),$$

with $F(k) \sim k^{-3}$ for the values of k such that $m\alpha \ll k \ll m$ or $m\alpha^2 \ll k \ll m\alpha$ and for this reason can be obtained directly from calculations within quantum mechanics. This is an important point since it shows that genuine relativistic field theoretical calculations are not required to derive those corrections.

Examples of diagrams that lead to double logarithmic corrections are shown in Fig.2. The details of our calculation can be found in [3]. The final result, that agrees with the independent calculation in Refs. [17,18], reads:

$$\Delta E = -\left(\frac{499}{15} + 28\,\boldsymbol{s}_+\boldsymbol{s}_-\right)\frac{m\alpha^7\log^2\alpha}{32\pi n^3}\,\delta_{l0}, \qquad (10)$$

so that numerically we obtain:

$$\Delta E_{\rm triplet} = -\frac{1.3}{n^3}\ \text{MHz}, \quad \Delta E_{\rm singlet} = -\frac{0.4}{n^3}\ \text{MHz}.$$

We are now in position to compare theoretical predictions with experimental results. On the theoretical side we include the shift due to Eq. (10) to our prediction Eqs. (8,9) and also take it as our estimate of the theoretical uncertainty. Then the results read (experimental results are from [6,7]):

$$\Delta\nu_{\rm th} = 203\,392(1)\ \text{MHz},$$
$$\Delta\nu_{\rm ex} = 203\,389.10(0.74)\ \text{MHz};$$

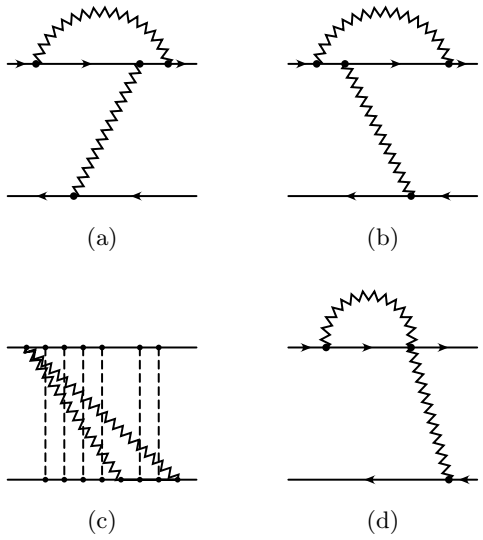

Fig. 2. Examples of diagrams that lead to double logarithmic corrections

$$[E(2^3S_1) - E(1^3S_1)]_{\rm th} = 1\,223\,607\,222(1)\ \text{MHz},$$
$$[E(2^3S_1) - E(1^3S_1)]_{\rm ex} = 1\,223\,607\,216.4(3.2)\ \text{MHz}.$$

We see that there is an approximately 3σ discrepancy between the theoretical prediction and experimental result for the hyperfine splitting. It is unclear at the moment how seriously this discrepancy should be taken. It is perhaps worthwhile to re-measure this quantity and simultaneously continue efforts to compute $\mathcal{O}(m\alpha^7 \ln \alpha)$ corrections in order to estimate the uncertainty in the theoretical prediction for the hyperfine splitting more reliably. As for the $[E(2^3S_1)-E(1^3S_1)]$ shift, there an agreement between theory and experiments is satisfactory; this is so because the experimental result is less accurate than the one for the hyperfine splitting.

3.2 Parapositronium decay to two photons

Another accurately measured quantity is the parapositronium ($S = 0$) decay rate to two photons [10][1].

$$\Gamma^{\rm exp}_{\rm p-Ps} = 7990.9(1.7)\ \mu\text{s}^{-1}. \tag{11}$$

[1] In fact the total decay rate of parapositronium has been measured in [10]. Partial decay rates of parapositronium to four and higher number of photons are, however, negligable

Such experimental accuracy warrants a calculation of $\mathcal{O}(\alpha^2)$ corrections to the decay rate on the theoretical side. To compute them, we again use dimensionally regularized NRQED. In this case, however, finite parts of hard integrals are done numerically because of their complexity; nevertheless, divergent parts of hard integrals are obtained analytically.

For the purpose of presentation, let us parameterize the decay width as:

$$\Gamma_{\rm p-Ps}^{\rm theory} = \Gamma_p^{(0)} \left[1 - 2.53\frac{\alpha}{\pi} + 2\alpha^2 \ln\frac{1}{\alpha} + B_p \left(\frac{\alpha}{\pi}\right)^2\right], \tag{12}$$

with

$$\Gamma_p^{(0)} = \frac{m\alpha^5}{2}.$$

Our aim is the second order non-logarithmic correction B_p.

As usual, B_p receives contributions from both hard and soft scales and for this reason we write it as: $B_p = B_p^{\rm soft} + B_p^{\rm hard}$. We then obtain [4]:

$$\begin{aligned} B_p^{\rm soft} + 2\pi^2 \ln\frac{1}{\alpha} &= \frac{\pi^2}{2\epsilon} + 2\pi^2 \ln\frac{1}{m\alpha} + \frac{107\pi^2}{24}. \\ B_p^{\rm hard} &= -\frac{\pi^2}{2\epsilon} + 2\pi^2 \ln(m) - 40.46(30), \end{aligned} \tag{13}$$

and the final result is $B_p = 5.1(3)$. Note that there is significant cancelation between soft and hard pieces and the final result is almost eight times smaller than the magnitude of the finite constant in the hard scale contribution computed in dimensional regularization.

Using the above result for B_p we arrive at the following result for the decay rate:

$$\Gamma_{\rm p-Ps}^{\rm theory} = 7989.64(2)\ \mu{\rm s}^{-1},$$

which agrees very well with the measured value $\Gamma_{\rm p-Ps}^{\rm exp} = 7990.9(1.7)\ \mu{\rm s}^{-1}$.

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References

1. A. Czarnecki, K. Melnikov and A. Yelkhovsky: Phys. Rev. Lett. **82**, 311 (1999)
2. A. Czarnecki, K. Melnikov and A. Yelkhovsky: Phys. Rev. A **59**, 4316 (1999)

3. K. Melnikov and A. Yelkhovsky: Phys. Lett. B **458**, 143 (1999)
4. A. Czarnecki, K. Melnikov and A. Yelkhovsky: Phys. Rev. Lett. **83**, 1135 (1999); Phys. Rev. A **61**, 052502 (2000); Erratum: *ibid.*, to be published
5. A.P. Mills, Jr.: Phys. Rev. A **27**, 262 (1983);
   A.P. Mills, Jr., and G.H. Bearman: Phys. Rev. Lett. **34**, 246 (1975)
6. M.W. Ritter, P.O. Egan, V.W. Hughes and K.A. Woodle: Phys. Rev. A **30**, 1331 (1984)
7. M.S. Fee, A.P. Mills, Jr., et al.: Phys. Rev. Lett.**70**, 1397 (1993)
8. C.I. Westbrook, D.W. Gidley, R.S. Conti and A. Rich: Phys. Rev. A **40**, 5489 (1989);
   J.S. Nico, D.W. Gidley, A. Rich and P.W. Zitzewitz: Phys. Rev. Lett. **65**, 1344 (1990)
9. G. Adkins: *this edition*, pp. 375–386
10. D. W. Gidley and A.H. Al-Ramadhan: Phys. Rev. Lett. **72** (1994)
11. R. Conti: *this edition*, pp. 103–121
12. W. E. Caswell and G. P. Lepage: Phys. Rev. A **20**, 36 (1979)
13. M. Beneke and V.A. Smirnov: Nucl.Phys. B **522**, 321 (1998)
14. F. Tkachov: Phys. Lett. B **100**, 65 (1981); K. G. Chetyrkin and F. Tkachov: Nucl. Phys. B **192**, 159 (1981)
15. K. Pachucki: Phys. Rev. A **56**, 297 (1997)
16. K. Pachucki: Phys. Rev. Lett. **79**, 4120 (1997)
17. S.G. Karshenboim: JETP **76**, 541 (1993)
18. S.G. Karshenboim and K. Pachucki: Phys. Rev. A **60**, 2792 (1999)