Hyperfine Structure in Muonic Hydrogen

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Abstract. We consider the hyperfine structure of the $1s$ and $2s$ states in muonic hydrogen and muonic deuterium. We put emphasis on two particular topics: a possibility to measure the *hfs* interval in the ground state and a calculation of a specific difference $E_{\rm hfs}(1s) - 8 \cdot E_{\rm hfs}(2s)$. Such a measurement and the calculations are of interest in connection with an upcoming experiment at PSI in which different $2s-2p$ transitions in muonic hydrogen shall be determined. Together all these investigations will improve the knowledge of the internal structure of proton and deuteron.

1 Introduction

Muonic hydrogen is one of the most fundamental atomic systems. It is of particular present actuality due to an upcoming experiment at the Paul Scherrer Institute (PSI) in Villigen, Switzerland, which aims for a measurement of the Lamb shift in this system [1], i.e the energy difference between several $2s$ and $2p$ fine and hyperfine structure (*hfs*) levels [2]. The eventual target of the experiment is a determination of the mean square proton charge radius. This proton feature limits now the accuracy of Quantum Electrodynamics (QED) calculations of the Lamb shift in natural hydrogen atoms. Recently it has been found that the precision of proton radius measurements by electron scattering had been overestimated. This has now resulted in an uncertainty of about 7-10 ppm for the hydrogen Lamb shift [3,4,5], while progress in calculations [6,7] reduced the QED part of the uncertainty to 2 ppm. In the PSI experiment [1] the $2s$ hyperfine structure will also be measured. In this work we consider a possibility to measure the hyperfine structure interval in the $1s$ ground state and to calculate specific differences $\Delta_{12} = E(1s) - 8 \cdot E(2s)$ of hyperfine separations.

The experiment was considered in part in Refs. [8,9]. It requires laser radiation at 6800 nm which could be obtained by optical difference frequency generation. Using an intense polarized slow beam of negative muons and a low density target one might expect the formation of polarized muonic hydrogen atoms in the ground state. In a similar fashion polarized muonic atoms had been produced before, e.g. polarized muonic helium [10]. A hyperfine transition signal from the muonic atom can then be detected in a straight forward way through a change in the spatial anisotropy of the electrons appearing after the

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muon decay in analogy with measurements of the muonium *hfs* [12] and muonic helium *hfs* [11].

The energy difference Δ_{12} of the two hyperfine levels in the ground state is considered in detail for the normal hydrogen atom in Ref. [13]. The calculations for the muonic atom differ from the case of natural hydrogen predominantly because of the vacuum polarization effects. Evaluation of the leading term does not present a problem. We use for the $1s$ and $2s$ states in muonic hydrogen the results found in semi-analytic form in Ref. [13]. This is considered adequate for calculating the difference with an uncertainty at a level of 10^{-3} of the Fermi energy. Calculations of higher order terms are in progress.

The proposed experiment on $1s$ *hfs* will test the accuracy and verify the validity of the PSI experiment on the excited states. With improved accuracy it may yield a result which can be used as a sum rule for the magnetic form factor of the proton. The uncertainty of this is on the same level as the proton polarizability contribution.

2 Theory for $\boldsymbol{\Delta_{12}}$ for muonic hydrogen and deuterium HFS structure.

The hyperfine splitting of s-state is determined in the leading non-relativistic approximation by[1]

$$E_{\rm hfs}(ns) = \frac{E_F}{n^3}\,, \tag{1}$$

where the Fermi energy E_F is defined as

$$E_F = \frac{16}{3}\mu_\mu\mu\,\frac{2I+1}{2I}\,\left(Z\alpha m_R\right)^3\,. \tag{2}$$

Here m_μ and M are the masses of the muon and the nucleus, μ_μ and μ are their magnetic moments, and I is the nuclear spin (1/2 for hydrogen and 1 for deuterium). The reduced mass factor is

$$\frac{m_R}{m_\mu} = \frac{M}{m_\mu + M} \simeq \begin{cases} 0.90 \text{ for hydrogen}\,, \\ 0.95 \text{ for deuterium}\,. \end{cases}$$

We are interested in a value of a specific difference in muonic hydrogen and deuterium atoms

$$\Delta_{12} = E_{\rm hfs}(1s) - 8\cdot E_{\rm hfs}(2s)\,.$$

This vanishes in the leading non-relativistic approximation (1) and so it is sensitive to state-dependent corrections to the hyperfine structure.

In the case of normal hydrogen [14] the difference is mainly determined by a relativistic contribution of order $(Z\alpha)^2E_F$ (so-called Breit term [15]). In muonic atoms the leading effect is due to vacuum polarization and it is of order αE_F.

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[1] We work in relativistic units in which $\hbar = c = 1$. The splitting are presented either in the energy units (eV) for ΔE or in the frequency units (Hz) for $\Delta E/h$.

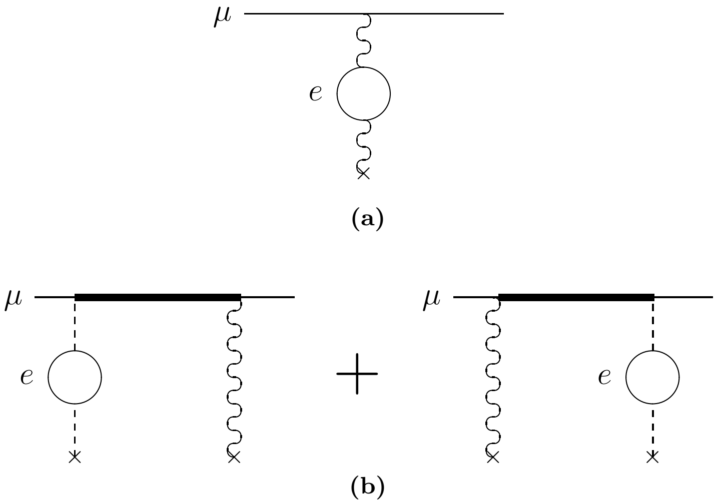

Fig. 1. Electronic vacuum polarization diagrams: one-potential **(a)** and two-potential **(b)** contributions. The bold line is for the non-relativistic reduced Green function of a muon in the Coulomb field of nucleus

2.1 Vacuum polarization term Δ^{VP}

The vacuum polarization correction to the hyperfine splitting consists of two parts: one- (E_{TU}) and two-potential ($E_{U\cdot T}$) contributions (see Fig. 1a and 1b)

$$E_{TU} = \frac{\alpha}{\pi}\,E_F\,R_{TU}(nl) \tag{3}$$

and

$$E_{U\cdot T} = \frac{\alpha}{\pi}\,E_F\,R_{U\cdot T}(nl) \tag{4}$$

and it was numerously considered [16]. We follow paper [13].

For the two lowest s-states we have

$$R_{TU}(1s) = -\frac{\pi}{3\kappa_1^3} + \frac{\kappa_1^2+6}{9\kappa_1^2} + \frac{2\kappa_1^4-\kappa_1^2+2}{3\kappa_1^3}\,\mathcal{A}(\kappa_1)\,, \tag{5}$$

$$\begin{aligned} R_{TU}(2s) = &-\frac{\pi}{3\kappa_2^3} + \frac{22\kappa_2^6-29\kappa_2^4-44\kappa_2^2+24}{36\kappa_2^2(\kappa_2^2-1)^2} + \\ &\frac{8\kappa_2^8-20\kappa_2^6+33\kappa_2^4-20\kappa_2^2+8}{12\kappa_2^3(\kappa_2^2-1)^2}\,\mathcal{A}(\kappa_2)\,, \end{aligned} \tag{6}$$

$$R_{U\cdot T}(1s) = \pi\frac{\kappa_1^2-2}{2\kappa_1^3} - \frac{5\kappa_1^4-8\kappa_1^2+6}{3\kappa_1^2(\kappa_1^2-1)} + \frac{2\kappa_1^6-3\kappa_1^4+4\kappa_1^2-2}{\kappa_1^3(\kappa_1^2-1)}\,\mathcal{A}(\kappa_1) + J(\kappa_1)\,, \tag{7}$$

and

$$R_{U\cdot T}(2s) = \pi\frac{3\kappa_2^2-26}{3\kappa_2^3} - \frac{44\kappa_2^8-195\kappa_2^6+894\kappa_2^4-920\kappa_2^2+312}{18\kappa_2^2(\kappa_2^2-1)^3}$$

$$+\frac{12\kappa_2^{10}-42\kappa_2^8+309\kappa_2^6-506\kappa_2^4+376\kappa_2^2-104}{6\kappa_2^3(\kappa_2^2-1)^3}\,\mathcal{A}(\kappa_2)+K(\kappa_2)\,. \tag{8}$$

Here:

$$J(\kappa_1)=-\frac{2\kappa_1^2}{3}\int_0^1 dy\,\frac{y\sqrt{1-y^2}(y^2+2)}{(1+\kappa_1 y)^2}\log\frac{\kappa_1 y}{1+\kappa_1 y}\,, \tag{9}$$

$$K(\kappa_2)=-\frac{4\kappa_2^2}{3}\int_0^1 dy\,\frac{y\sqrt{1-y^2}(y^2+2)(\kappa_2^2y^2+2)}{(1+\kappa_2 y)^4}\log\frac{\kappa_2 y}{1+\kappa_2 y}\,, \tag{10}$$

and

$$\mathcal{A}(t)=\begin{cases}\frac{\arccos t}{\sqrt{1-t^2}}\,, & t\le 1\,,\\ \frac{\log\left(t+\sqrt{t^2-1}\right)}{\sqrt{t^2-1}}\,, & t>1\,.\end{cases}$$

The parameter

$$\kappa_n=\frac{Z\alpha\, m_R}{n\, m_e}$$

is the ratio of the typical atomic momentum to the inverse radius of the effective potentials. This is induced by inserting electronic vacuum polarization into a Coulomb or magnetic photon. This ratio is about 1.4 for the $1s$ state in muonic hydrogen and deuterium and 0.7 in the $2s$ state. This means that the calculation involves essential bound-state effects and in particular the muon propagator in Fig. 1b is a full Coulomb Green function.

The value of the nuclear charge $Z=1$ is useful to classify the corrections.

Numerically we obtain

$$\begin{aligned}\varDelta^{\rm VP}&=\varDelta^{(TU)}+\varDelta^{(U\cdot T)}\\ &=\frac{\alpha}{\pi}E_F\left[(0.883\,041-0.910\,260)+(1.731\,152-1.404\,246)\right]\\ &=\frac{\alpha}{\pi}E_F[-0.027\,219+0.326\,906]=0.299\,687\,\frac{\alpha}{\pi}E_F\end{aligned}$$

for the muonic hydrogen atom, and

$$\begin{aligned}\varDelta^{\rm VP}&=\varDelta^{(TU)}+\varDelta^{(U\cdot T)}\\ &=\frac{\alpha}{\pi}E_F\left[(0.910\,476-0.938\,587)+(1.801\,155-1.452\,300)\right]\\ &=\frac{\alpha}{\pi}E_F[-0.028\,111+0.348\,855]=0.320\,744\,\frac{\alpha}{\pi}E_F\end{aligned}$$

for muonic deuterium.

The contributions of the correction are presented in Table 1.

Table 1. Contributions of electronic vacuum polarization to the difference $E_{\rm hfs}(1s) - 8 \cdot E_{\rm hfs}(2s)$. Units are $10^{-4} E_F$

	$\Delta^{(TU)}$	$\Delta^{(U \cdot T)}$	$\Delta^{\rm VP}$
μH	-0.632248	7.59343	6.96119
μD	-0.652964	8.10328	7.45031

2.2 Relativistic corrections

The leading relativistic correction for $1s$ and $2s$ was calculated by Breit [15] and it contributes to the energy difference

$$\Delta^{\rm rel} = E_F (Z\alpha)^2 \left(-\frac{5}{8} + \mathcal{O}(Z\alpha)^2 \right) . \tag{11}$$

In contrast to normal atoms this calculation is not accurate enough because of essential recoil effects. The leading relativistic recoil term for the $1s$ state is of order $(Z\alpha)^2 m/M$ and it depends on the nuclear structure. The difference is free of nuclear influence and the result is [16]

$$\Delta^{\rm rec} = E_F \, (Z\alpha)^2 \, \frac{m}{M}$$
$$\times \left[-\frac{9}{8} + \left(-\frac{7}{32} + \frac{1}{2} \log 2 \right) \left(1 - \frac{1}{x} \right) - \left(\frac{145}{128} - \frac{7}{8} \log 2 \right) x \right] ,$$

where

$$x = \frac{\mu M}{\mu_\mu m_\mu} \frac{1}{Z \, I} .$$

2.3 Nuclear-structure effects

Nuclear-structure effects lead in part to contributions to Δ similar to the ones for normal hydrogen [17,14]

$$\Delta^I(\text{strong}) = (Z\alpha)^2 \left(\psi(n+1) - \psi(2) - \log n - \frac{(n-1)(n+9)}{4n^2} \right)$$
$$\times \Delta\nu_{1s}(\text{Zemach} + \text{polarizability}) - \frac{4}{3} (Z\alpha)^2 \left(\psi(n) - \psi(1) - \log n \right.$$
$$\left. + \frac{n-1}{n} - \left(\frac{R_M}{R_E} \right)^2 \frac{n^2 - 1}{4n^2} \right) \left(m R_E \right)^2 E_F ,$$

where the Zemach correction and the nuclear polarizability terms are for the ground state. The corrections are of the same shape but they have a larger influence because of the effective parameter $R_N m$ is now larger by the factor m_μ / m_e.

The other state-dependent corrections originate from accurate calculations of the Zemach term, which is proportional to the value of the wave function at the origin ($\psi(0)$). The wave function affected by the Uehling potential leads to a correction

$$\Delta^{II}(\text{strong}) = 2\nu_1(\text{strong}) \times \left(\frac{\delta\psi_{1s}(0)}{\psi_{1s}(0)} - \frac{\delta\psi_{2s}(0)}{\psi_{2s}(0)}\right) = \nu_1(\text{strong}) \times \frac{\alpha}{\pi} R_{U\cdot T}\,.$$

The strong interaction corrections are not higher than $10^{-5}E_F$.

2.4 Numerical results

The magnetic moment of a muon is used in the form

$$\mu_\mu = \left(1 + \frac{\alpha}{2\pi} + 0.766\,\frac{\alpha^2}{\pi^2}\right)\frac{e}{2m_\mu}\,,$$

which is sufficient for our purposes. For our calculations we use the values of constants in Table 2 and the numerical values of all discussed corrections are collected in Table 3.

Table 2. Parameters in numerical calculations for muonic hydrogen and deuterium

Parameter	Muonic hydrogen	Muonic deuterium
M/m_μ	8.880 244 1(3)	17.751 674 5(5)
μ/μ_N	2.792 847 34(3)	0.857 438 228(9)
E_F, MHz	441 660 11(3)	95 065 119(5)
E_F, μeV	182 655.93(1)	393 157.70(6)
κ_1	1.356 145 98(4)	1.428 395 53(4)
x	5.585 694 674(9)	0.857 012 72(1)

A very important point is the overall uncertainty of our calculations and we consider here particularly the contribution of the QED part to the uncertainty. The second order vacuum polarization effects (δ^{VP2}) are of order $(\alpha^2/\pi^2)E_F$ (cf. relativistic corrections are of order $(Z\alpha)^2E_F$)). Another source of uncertainty arises from higher order recoil effects (δ^{rec2}) which can be estimated as $\delta^{\text{rec2}} = m/M|\Delta^{\text{rec}}|$.

3 Experimental considerations

Beams of slow negative muons have been developed and are for example employed in the muonic hydrogen Lamb shift experiment at PSI. Other developments are under way at the Rutherford Appleton Laboratory (RAL) in Chilton,

Table 3. Numerical values of different contributions to Δ_{12} in muonic hydrogen and deuterium

	Muonic hydrogen			Muonic deuterium		
	μeV	GHz	$10^{-4}\,E_F$	μeV	GHz	$10^{-4}\,E_F$
Δ^{VP}	127.15	30.745	6.961	292.91	70.826	7.450
Δ^{rel}	−6.08	−1.470	−0.333	−52.34	−12.656	−1.331
Δ^{rec}	−4.34	−1.049	−0.237	−7.54	−1.822	−0.192
δ^{VP2}	±1.0	±0.24	±0.054	±2.1	±0.51	±0.054
δ^{rec2}	±0.5	±0.12	±0.027	±0.4	±0.10	±0.011
$\Delta_{12}^{\mathrm{QED}}$	117(1)	28.2(3)	6.39(6)	233(2)	56.3(6)	5.93(6)

United Kingdom, using techniques similar to ultra slow muon production by cold moderators. Intense sources of cold negative muons will be indispensable for muon colliders and corresponding preparatory work is in progress worldwide at various accelerator centers. As an example, some 3-4 orders of magnitude higher muon fluxes compared to present facilities can be expected from the PRISM beam line which is discussed in connection with the Japanese Hadron Project (JHP) facilities [18]. Similar fluxes could also be obtained from a muon facility in connection with plans to set up a 60 GeV proton/30 GeV heavy ion synchrotron facility [19] at the Gesellschaft für Schwerionenforschung (GSI) in Darmstadt, Germany. If cold slow muons are stopped in a low density gas target one might expect to produce polarized muonic hydrogen atoms in the ground state. A moderate polarization would be already sufficient [8,9] as could be shown for muonic helium [11].

The excitation of the transition at 43.9 THz requires laser radiation at 6.8 μm. Such could be obtained by optical difference frequency generation and is in the reach of present laser technology, particularly for pulsed laser devices. Therefore the experiment would fit nicely into the environment of a pulsed muon facility with intense pulses of up to 1 μs duration and an ideal pulse separation given by the tolerable laser repetition rate which may be up to kHz.

A transition signal can then be detected in a straight forward way through a change in the spatial anisotropy of electrons from the decay $\mu^- \to e^- + \nu_\mu + \overline{\nu}_e$. This would be in full analogy with the detection of the muonium *hfs* transitions in a recent experiment at the Los Alamos Meson Physics Facility (LAMPF) in Los Alamos, USA [12].

Of course, prior to any realization of a laser setup, the formation of polarized muonic hydrogen needs to be verified in a first experimental phase. Therefore a slow polarized muon beam would be required. These steps could start already at low intensity muon sources such as PSI or RAL where depending on the achievable polarization and fluxes also a first experiment (presumably with low statistics yet) could take place. Given the importance and impact of such an experiment, this research should be started soon.

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References

1. R. Pohl *et al.*: *this edition*, pp. 454–466; F. Kottmann et al.: PSI proposal R-98.03.1 (1998)
2. K. Pachucki: Phys. Rev. **A 60**, 3593 (1999)
3. S. G. Karshenboim: invited talk at ICAP 2000, to be published, e-print hep-ph/0007278
4. S. G. Karshenboim: invited talk at MPLP 2000, to be published, e-print physics/0008215
5. S. G. Karshenboim: Can. J. Phys. **77**, 241 (1999).
6. U. D. Jentschura, P. J. Mohr and G. Soff: Phys. Rev. Lett. **82**, 53 (1999); *presented at this conference* (unpublished)
7. K. Melnikov and T. van Ritbergen: *this edition*, pp. 344–351; Phys. Rev. Lett., **84**, 1673 (2000)
8. K. Jungmann: Z. Phys. C**56**, S59 (1992)
9. M. G. Boshier *et al.*: Comm. At. Mol. Phys. **33**, 17 (1996)
10. P.A. Souder *et al.*: Phys. Rev. Lett. **34**, 1417 (1975); Phys. Rev. **A22**, 33 (1980)
11. H. Orth *et al.*: Phys. Rev. Lett. **45**, 1483 (1980)
12. W. Liu *et al.*, Phys. Rev. Lett. **82**, 711 (1999); K. Jungmann: *this edition*, pp. 81–102
13. S. G. Karshenboim *et al.*: Euro. Phys. J. D **2**, 209 (1998)
14. S. G. Karshenboim: *this edition*, pp. 335–343
15. G. Breit: Phys. Rev. **35**, 1477 (1930)
16. M. Sternheim: Phys. Rev. **138**, 430 (1965)
17. S. G. Karshenboim: Phys. Lett. A **225**, 97 (1997)
18. Y. Kuno: Proceedings of the HISMUS Workshop, KEK, Tsukuba 1999, *in print*
19. W. Henning: *private communication* (2000)